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Twisted Alexander polynomial for the Lawrence-Krammer representation

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TWISTED ALEXANDER POLYNOMIAL FOR THE LAWRENCE-KRAMMER REPRESENTATION

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Abstract. In this paper, we prove that the twisted Alexander polynomial for the Lawrence-Krammer representation of the braid group B_4 is trivial. This gives an answer for the problem whether the twisted Alexander polynomial for given faithful representations is always non-trivial.

1. Introduction

Twisted Alexander polynomial for finitely presentable groups was introduced by Wada in [W]. As a notable application, it is shown that the twisted Alexander polynomial can tell Kinoshita-Terasaka knot from Conway's 11-crossing knot.

In [M], the twisted Alexander polynomial for Jones representations of the braid group $B_n\,(n \geq 3)$ is studied. One of the main results of [M] is the twisted Alexander polynomial for the Burau representation is not trivial for $n = 3$ and trivial for $n \geq 4$. We know that the Burau representation is faithful for $n = 3$, not faithful for $n \geq 5$ and the faithfulness is still open for the case $n = 4$. Then it is mentioned in [M] that *it would be interesting to study a relation between the faithfulness of the Burau representation and the twisted Alexander polynomial*. In other words,

Problem 1.1. *If a given representation is faithful, is the twisted Alexander polynomial non-trivial?*

In this paper, we present the answer for this question.

Krammer constructed in [K1] a representation of the braid group, which is now called the Lawrence-Krammer representation, and showed that it is faithful for $n = 4$. Moreover, Bigelow [B] and Krammer [K2] proved that the Lawrence-Krammer representation is faithful for all n. Then we may show a relation between the faithfulness of a representation and the twisted Alexander polynomial as a consequence

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of an explicit calculation of the twisted Alexander polynomial for the Lawrence-Krammer representation.

In this paper, we show the following. (See Section 3 for the precise statement.)

Theorem 1.2. *The twisted Alexander polynomial for the Lawrence-Krammer representation of the braid group B_4 is trivial.*

This gives the negative answer for Problem 1.1.

In Section 2, we briefly recall the definition of the Lawrence-Krammer representation of the braid group B_4. In Section 3, the twisted Alexander polynomial of B_4 is computed and we prove Theorem 1.2.

2. Lawrence-Krammer representation of B_4

Let B_n be the braid group of n strings, $B_n \to \mathbb{Z} \simeq \langle x \rangle$ the Abelianisation and LK the Lawrence-Krammer representation

$$LK : B_n \longrightarrow GL\left(n(n-1)/2; \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]\right).$$

In this paper, we treat the case $n = 4$, then we argue the definition of the braid group and the Lawrence-Krammer representation for only this case. The braid group B_4 admits the presentation:

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3,\ \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle.$$

The Lawrence-Krammer representation of B_4 is defined as follows (see [B],[K1] and [K2] for general cases).

$$LK(\sigma_1) = \begin{pmatrix} tq^2 & 0 & 0 & 0 & 0 & 0 \\ tq(q-1) & 0 & 0 & q & 0 & 0 \\ tq(q-1) & 0 & 0 & 0 & q & 0 \\ 0 & 1 & 0 & 1-q & 0 & 0 \\ 0 & 0 & 1 & 0 & 1-q & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$LK(\sigma_2) = \begin{pmatrix} 1-q & q & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & tq^2(q-1) & 0 & 0 \\ 0 & 0 & 1 & tq(q-1)^2 & 0 & 0 \\ 0 & 0 & 0 & tq^2 & 0 & 0 \\ 0 & 0 & 0 & tq(q-1) & 0 & q \\ 0 & 0 & 0 & 0 & 1 & 1-q \end{pmatrix}$$

$$LK(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q & q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & tq^3(q-1) \\ 0 & 0 & 0 & 1-q & q & 0 \\ 0 & 0 & 0 & 1 & 0 & tq^2(q-1) \\ 0 & 0 & 0 & 0 & 0 & tq^2 \end{pmatrix}$$

3. Twisted Alexander polynomial

In this section, we compute the twisted Alexander polynomial. All notations are the same as ones used in [M] unless we state it.

First, we obtain a denominator in the twisted Alexander polynomial by an explicit calculation.

Lemma 3.1.

$$\det\left(I_6 - xLK(\sigma_3)\right) = (1-x)^3\,(1+qx)^2\,(1-q^2tx).$$

Next, we calculate a numerator in the twisted Alexander polynomial. In our case, we have the 18×12-matrix M_3 which is obtained from the Alexander matrix removing the third column. The numerator which we need is the greatest common divisor of $\det M_3^I$ for all the choices of the indices I. Here $I = (i_1, i_2, \ldots, i_{12})$ and M_3^I denotes the square matrix consisting of the i_k-th rows of the matrix M_3, where $1 \le i_1 < \cdots < i_{12} \le 18$.

Lemma 3.2. *For any index I, $\det M_3^I$ has a common divisor $(1-x)^3(1+qx)^2(1-q^2tx)$.*

Proof. For a given 18×12-matrix A, we denote by $A(i; a_1, \ldots, a_{12})$ the matrix obtained from A adding a_j times the j-th column to the i-th column. We note that

$$\det A(i; a_1, \ldots, a_{12})^I = (1 + a_i)\det A^I.$$

1. First, we consider

$$M^{(1)} = M_3(4; -1+q^2t, p, p, 0, 1, 0, 0, 0, 0, 0, 0, 0),$$

where $p = -1 - qt + q^2t$. Then we can take a term $1-x$ as a common divisor from the fourth column. Next, we observe

$$M^{(2)} = M^{(1)}(12; 0, 0, 0, 0, 0, 0, q^2, pq, (1-q)^2qt, -1+q^2t, p, 0)$$

and

$$M^{(3)} = M^{(2)}(8; -1+q^2t, (-1+q)qt, (-1+q)qt, 0, 0, 0, -q, 0, 0, 0, 0, 0).$$

Therefore the eighth and the twelfth columns have common divisors $1-x$ and $\det M_3^I$ has a divisor $(1-x)^3$ for any index I.

2. Similarly, it can be considered

$$\begin{aligned} M^{(4)} &= M_3(12; 0, 0, 0, 0, 0, 0, \\ &\qquad q^2, pq^2, -1 + q^3t - q^4t + pq, -q^2(1 + qt), -pq, 0) \end{aligned}$$

and

$$M^{(5)} = M^{(4)}(5; 0, -q^2, q, -q, 0, 0, -q^2, -q^2, 1 + q, 0, 0, 0).$$

Then the fifth and the twelfth columns have common divisors $1 + qx$ and $\det M_3^I$ has a divisor $(1 + qx)^2$ for any index I.

3. Finally, we set

$$\begin{aligned} M^{(6)} &= M_3(12; 0, q^3t(1 - q)(1 - q^2t), \\ &\quad q^2t(-1 + q)(1 - q^2t + q^4t^2 + pq), q^2t(1 - q)(1 - q^2t), \\ &\quad qt(-1 + q)(1 - q^2t + q^4t^2 + pq), (1 + qt)(1 - q^2t)^2, \\ &\quad (1 - q)q^4t, (-1 + q)q^4t^2, q^2t(-1 + q)(1 - q - qt + q^4t^2), \\ &\quad 0, q(1 + qt - q^2t)(1 - q^3t^2), (1 - q - q^2t)(1 - q^3t^2)). \end{aligned}$$

The twelfth column of $M^{(6)}$ has a common divisor $1 - q^2tx$. We need to note that the determinant of this matrix $M^{(6)I}$ is different from that of M_3^I. More precisely,

$$\det M^{(6)I} = \big(1 + (1 - q - q^2t)(1 - q^3t^2)\big) \det M_3^I.$$

However, the greatest common divisor of two polynomials $1+(1-q-q^2t)(1-q^3t^2)$ and $1-q^2tx$ is a unit, that is, they are relatively prime. This deduces that $\det M_3^I$ has a divisor $1 - q^2tx$ for any index I. Then it completes the proof.

□

Lemma 3.3. *There exist indices I_1, I_2 such that*

$$\gcd(\det M_3^{I_1}, \det M_3^{I_2}) = (1 - x)^3(1 + qx)^2(1 - q^2tx).$$

Proof. We select

$$I_1 = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12),$$

$$I_2 = (2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 15, 17)$$

and calculate $\det M_3^{I_1}, \det M_3^{I_2}$ explicitly, then we get the conclusion. □

The above two lemmas deduce that $\det M_3^I$ has a common divisor $(1-x)^3(1+qx)^2(1-q^2tx)$ and does not have any other common divisor, then the numerator is settled. It follows by the definition that

Theorem 3.4. *The twisted Alexander polynomial $\Delta_{B_4,LK}(x)$ for the Lawrence-Krammer representation with the Abelianisation $B_4 \to \mathbb{Z} \simeq \langle x \rangle$ is given by*

$$\Delta_{B_4,LK}(x) = 1.$$

Remark 3.5. *The twisted Alexander polynomial for the Lawrence-Krammer representation is not always trivial for n. In fact, we get $\Delta_{B_3,LK}(x) = 1 + q^3tx^3$ by an easy calculation.*

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