UTMS 2004–5 February 5, 2004

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Abstract. We define and study the Magnus representation for homology cylinders generalizing the work of Kirk, Livingston and Wang [KLW] which treats the case of string links. Using this, we give a factorization formula of Alexander polynomials for three dimensional manifolds obtained by closing homology cylinders. We also show a relationship between the Gassner representation for string links and the Magnus representation for homology cylinders.

1. Introduction

Let $\mathcal{P}_g$ be the pure braid group of g strands. The braid group appears in various contexts of mathematics as well as in the knot theory, so that it is important to understand this group. In general, we can obtain a lot of informations about the structure of a given group by considering its representations. As for the braid group, a series of Magnus representations such as the Burau representation and the Gassner representation, are well known. We refer to Birman's book [Bi] for the general theory of the Magnus representation. The Gassner representation has the following form:

$$g : \mathcal{P}_g \longrightarrow GL(g, \mathbb{Z}H_1 D_g)$$

where D_g is the unit disk in the Euclidean plane with g punctures.

As a generalization of braids, we have string links whose difference from braids are typically seen in the following Figure 1.

An example of braids An example of string links

Figure 1

We will recall the precise definition of string links in the next section. The set of isotopy classes of pure string links $\mathcal{L}_g$ has a natural monoid structure by defining its product as in the case of the

2000 *Mathematics Subject Classification.* Primary 57M05, Secondary 20F34; 57N05; 57M27.

Key words and phrases. homology cylinder; Magnus representation; mapping class group; Alexander polynomial; string link.

braid group. Moreover, the equivalence relation of concordances gives $\mathcal{L}_g$ the group structure. We denote this group by $\mathcal{S}_g$. $\mathcal{P}_g$ can be considered to be a subgroup of $\mathcal{L}_g$ and $\mathcal{S}_g$:

$$\begin{array}{ccc} \mathcal{P}_g & \hookrightarrow & \mathcal{L}_g \\ & \searrow & \downarrow \\ & & \mathcal{S}_g \end{array}$$

This enlargement of braids to string links leads the enlargement of Gassner and Burau representations. For example, the enlarged Gassner representation has the following form:

$$g : \mathcal{S}_g \longrightarrow GL(g, (\mathbb{Z}H_1 D_g)_S)$$

where $(\mathbb{Z}H_1D_g)_S$ denotes some local ring obtained from $\mathbb{Z}H_1D_g$. It was first done by Le Dimet in [LD] by using some algebraic devices, such as the algebraic closure of groups and the universal localization of rings with augmentations. After that, a simpler description of this representation was done by Kirk, Livingston and Wang in [KLW] by using the cohomology of some local coefficient system.

On the other hand, let $\Sigma_{g,1}$ be a compact connected oriented surface of genus g with one boundary component and let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$ relative to the boundary. Namely $\mathcal{M}_{g,1}$ is the group of all isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which fix the boundary pointwise. The mapping class group has also been studied by many people in various fields of mathematics.

When we consider the counterpart of the enlargement of braids in the context of the mapping class group, homology cylinders over $\Sigma_{g,1}$ give its answer. Homology cylinders over $\Sigma_{g,1}$, which we simply call them homology cylinders from now on, are some kinds of three manifolds with boundary. We refer to [Ha], [GL] and [Le] for their origin and generalities. We recall their definitions in Section 3. The set of diffeomorphism classes of them has a natural semi-group structure. We denote this semi-group by $\mathcal{C}_{g,1}$. We can shift $\mathcal{C}_{g,1}$ to the group $\mathcal{H}_{g,1}$ by taking a quotient with respect to the equivalence relation of homology cobordisms. Then we have an embedding of $\mathcal{M}_{g,1}$ into $\mathcal{C}_{g,1}$ and $\mathcal{H}_{g,1}$:

$$\begin{array}{ccc} \mathcal{M}_{g,1} & \hookrightarrow & \mathcal{C}_{g,1} \\ & \searrow & \downarrow \\ & & \mathcal{H}_{g,1} \end{array}$$

In this paper, we define the Magnus representation for the group of homology cylinders and study their applications by generalizing the work of Kirk, Livingston and Wang [KLW]. The Magnus representation for the mapping class group:

$$r_0 : \mathcal{M}_{g,1} \longrightarrow GL(2g, \mathbb{Z}\pi_1\Sigma_{g,1})$$

was first used by Morita in his theory of characteristic classes of surface bundles. Note that this representation is *not* a homomorphism but a crossed homomorphism. To obtain a genuine homomorphism, we need to restrict it to the Torelli group $\mathcal{I}_{g,1}$, which is a subgroup of $\mathcal{M}_{g,1}$, and reduce the coefficients to $\mathbb{Z}H_1\Sigma_{g,1}$. Then we have the Magnus representation for the Torelli group

$$r_0 : \mathcal{I}_{g,1} \longrightarrow GL(2g, \mathbb{Z}H_1\Sigma_{g,1}),$$

which is the counterpart of the Gassner representation. We generalize this representation and obtain a representation of the following form:

$$r : \mathcal{H}_{g,1} \longrightarrow GL(2g, (\mathbb{Z}H_1\Sigma_{g,1})_S)$$

which is a crossed homomorphism.

Below we describe the organization of this paper.

In Section 2, we quickly review the theory of the enlarged Gassner representation for the group of string links by Kirk, Livingston and Wang.

In Section 3, we first recall definitions of homology cylinders and related groups. Then we define the Magnus representation for the group of homology cylinders and show that it actually generalizes the one for the mapping class group.

In Section 4, some fundamental properties of the representation are mentioned. This section contains the way of computing the representation. The operation of closing the homology cylinder is also given. This corresponds to the operation which makes a link from a braid by closing .

In Section 5, we restrict our attention to the semi-group $\mathcal{IC}_{g,1}$ of Torelli homology cylinders, which is a sub semi-group of $\mathcal{C}_{g,1}$. Then we define the Alexander rational function of the Torelli homology cylinder by using the Magnus representation and show a factorization formula of Alexander polynomials for manifolds obtained by closing homology cylinders.

In Section 6, we see a connection between the Gassner representation for string links and the Magnus representation for homology cylinders. We show that the Magnus representation contains the information of the Gassner representation completely. From this, we see that $\mathcal{M}_{g,1}$ is not normal in $\mathcal{H}_{g,1}$ by using the example of string links given in [KLW].

In Section 7, we give some examples of calculations which contain the proof of the non-triviality of Alexander rational functions defined in Section 5.

The author would like to express his gratitude to Professor Shigeyuki Morita for his encouragement and helpful suggestions. He also would like to thank Masaaki Suzuki for variable discussions and advice.

2. The brief review of the Gassner representation for string links

We begin by reviewing the definition of the Gassner representation for string links by Kirk, Livingston and Wang. We refer to Sections 2, 4, 5 in [KLW] for details. For simplicity, we only treat the case of pure string links. Let D be the unit disk in the Euclidean plane. Given a positive integer g, we take g points $p_1, \ldots, p_g$ in D, where $p_i = (-1/(i+1), 0)$. We denote by D_g the unit disk with g punctures, namely $D_g = D \setminus \{p_1, \ldots, p_g\}$. We fix a system of generators $\beta_1, \ldots, \beta_g$ of $\pi_1 D_g$ as shown in Figure 2 where we take $p = (0, -1) \in \partial D_g$ as a base point. A *pure string link L of g strands* is a smooth proper embedding

$$L : \coprod_{i=1}^{g} I_{(i)} \longrightarrow D \times I$$

which maps 0 in $I_{(i)}$ to $(p_i, 0)$ and 1 in $I_{(i)}$ to $(p_i, 1)$ where $I_{(i)}$ is a copy of the unit interval $I = [0, 1]$. We denote by $\mathcal{L}_g$ the set of isotopy classes of pure string links. $\mathcal{L}_g$ has a natural monoid structure by defining the product $L_1 \cdot L_2$ of two string links L_1, L_2 as shown in Figure 3.

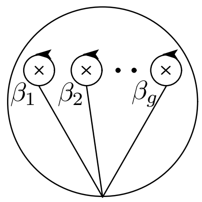

Figure 2

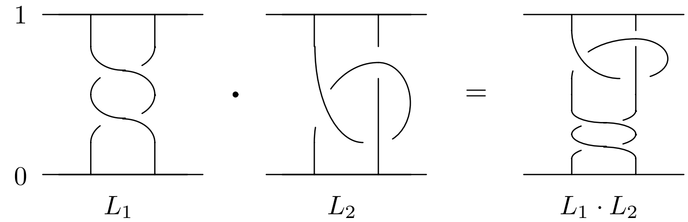

Figure 3

Let $\mathcal{P}_g$ be the pure braid group of g strands. For the general theory of braids, we refer to [Bi]. We also refer to it for the Fox's free calculus. $\mathcal{P}_g$ is naturally embedded into $\mathcal{L}_g$ as a unit subgroup.

Given a string link L of g strands, let X denote its complement $D \times I \setminus L$. Then we have two inclusion maps

$$i_0, i_1 : D_g \longrightarrow X$$

defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$. i_0 and i_1 induce the same isomorphism between homology groups $H_1 D_g$ and $H_1 X$ so that we can identify them. Abelianization maps $\pi_1 D_g \to H_1 D_g$ and $\pi_1 X \to H_1 X$ induce actions on the field $F = \mathrm{Frac}(\mathbb{Z}H_1 D_g) = \mathrm{Frac}(\mathbb{Z}H_1 X)$ by multiplications. Here, in general, we denote by $\mathrm{Frac}\, A$ the fraction field of an integral domain A and we write the additional operation of an abelian group G multiplicatively in the group ring $\mathbb{Z}G$ or its fraction field $\mathrm{Frac}(\mathbb{Z}G)$. These actions are compatible with i_{0*} and i_{1*} so that we can consider F to be a locally coefficient system on D_g and X.

Let $P \subset X$ denote the arc $\{p\} \times I$. We often identify the first ordinary cohomology of a given space with the group cohomology of its fundamental group. As for the group cohomology, we use the standard complex $(C^*(\pi, M), \delta)$ for a group π and a left $\mathbb{Z}\pi$ module M.

Lemma 2.1. (1) *$H^1(D_g, p; F) \cong H^1(\pi_1 D_g, \{1\}; F) \cong Z^1(\pi_1 D_g; F) \cong F^g$ where the correspondence is given as follows:*

$$\begin{array}{ccc} Z^1(\pi_1 D_g; F) & \overset{\cong}{\longrightarrow} & F^g \\ \cup\!\!\!\! \rule{0pt}{0pt} \Psi & & \Psi \\ f & \longmapsto & \begin{pmatrix} f(\beta_1) \\ \vdots \\ f(\beta_g) \end{pmatrix} \end{array}$$

(2) *$i_0^*, i_1^* : H^1(X, P; F) \to H^1(D_g, p; F)$ are both isomorphisms.*

Proof. (1) Note that 1-cocycles $Z^1(\pi_1 D_g; F)$ are given by crossed homomorphisms

$$f : \pi_1 D_g \longrightarrow F$$

and determined by images of free generators of $\pi_1 D_g$ by f.

(2) See Lemma 2.1 and Proposition 2.1 in [KLW]. $\square$

Using these isomorphisms, we define the Gassner representation g. Notice that our convention for the definition of the Gassner representation is slightly different from the one in [KLW]. For later use, we first define an anti-homomorphism $\widetilde{g}$.

Definition 2.2. (1) The *Gassner anti-representation* is an anti-homomorphism which assigns the representation matrix $\widetilde{g}(L) \in GL(g, F)$ of the isomorphism

$$F^g \cong H^1(D_g, p; F) \xrightarrow[(i_0^*)^{-1}]{\cong} H^1(X, P; F) \xrightarrow[i_1^*]{\cong} H^1(D_g, p; F) \cong F^g$$

to a string link $L \in \mathcal{L}_g$.

(2) The *Gassner representation* $g : \mathcal{L}_g \longrightarrow GL(g, F)$ is a homomorphism which assigns $\widetilde{g}^{-1}(L)$ to a string link $L \in \mathcal{L}_g$.

The monoid $\mathcal{L}_g$ can be shifted to a group $\mathcal{S}_g$ by taking a quotient with respect to the concordance. Here we omit the details about this equivalence relation. An important result of Le Dimet [LD] is that $\widetilde{g}$ and g factor through $\mathcal{S}_g$ so that we have a group anti-homomorphism $\widetilde{g}$ and a group homomorphism g:

$$\widetilde{g}, g : \mathcal{S}_g \longrightarrow GL(g, F).$$

The Gassner representation for string links can be computed as follows. Given a string link $L \in \mathcal{L}_g$, we can calculate the fundamental group of its complement $X = D \times I \setminus L$ by the Wirtinger presentation. This presentation has the following form:

$$\pi_1 X \cong \langle i_1(\beta_1), \ldots, i_1(\beta_g), z_1, \ldots, z_l, i_0(\beta_1), \ldots, i_0(\beta_g) \mid r_1, \ldots r_{g+l} \rangle.$$

We simply write $\gamma_1, \ldots, \gamma_{2g+l}$ for these ordered generators. Let A, B and C be matrices of sizes $(g+l) \times g$, $(g+l) \times l$ and $(g+l) \times g$ defined by the equality

$$\begin{pmatrix} A & B & C \end{pmatrix} = {}^{\rho}\!\left(\frac{\partial r_i}{\partial \gamma_j} \right)_{i,j}$$

where the right hand side is obtained by applying the map ρ induced from the composite of the quotient map and the abelianization map $\mathfrak{a}$:

$$F_{2g+l} = \langle \gamma_1, \ldots, \gamma_{2g+l} \rangle \longrightarrow \pi_1 X \xrightarrow{\mathfrak{a}} H_1 X$$

on each entry of the matrix given by the Fox calculus. Then we have the following proposition:

Proposition 2.3. (1) *There exists a $(l \times g)$ matrix Z which satisfies the equality*

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \widetilde{g}(L) \\ Z \end{pmatrix} = -C.$$

(2) $(A \quad B) \in GL(g+l, (\mathbb{Z}H_1D_g)_S)$ *where* $(\mathbb{Z}H_1D_g)_S$ *is the localization of* $\mathbb{Z}H_1D_g$ *obtained by inverting all elements in the set*

$$S = \{f \in \mathbb{Z}H_1D_g \mid \mathfrak{t}(f) = \pm 1 \in \mathbb{Z}\{1\}\}.$$

(3) $\widetilde{g}(L) \in GL(g, (\mathbb{Z}H_1D_g)_S)$.

Proof. We only give the sketch of the proof. We refer to Section 4 in [KLW] for details.

(1) $f \in C^1(\pi_1 X, F)$ is a 1-cocycle if and only if

$$\sum_{j=1}^{2g+l} {}^{\rho}\!\left(\frac{\partial r_i}{\partial \gamma_j}\right) f(\gamma_j) = 0$$

for all $i = 1, \ldots, g+l$. Then (1) follows from the correspondence of 1-cocycles

$$\begin{array}{ccccc} Z^1(\pi_1 D_g; F) & \xleftarrow[i_0^*]{\cong} & Z^1(\pi_1 X; F) & \xrightarrow[i_1^*]{\cong} & Z^1(\pi_1 D_g; F) \\ \cup & & \cup & & \cup \\ \widetilde{g}(L)_{\cdot j} & \longleftarrow & \begin{pmatrix} \widetilde{g}(L)_{\cdot j} \\ Z_{\cdot j} \\ e_j \end{pmatrix} & \longmapsto & e_j \end{array}$$

where $\widetilde{g}(L)_{\cdot j} = {}^t(\widetilde{g}(L)_{1j}, \ldots, \widetilde{g}(L)_{gj})$, $Z_{\cdot j} = {}^t(Z_{1j}, \ldots, Z_{lj})$ and e_j is the unit column vector whose j-th entry is 1 and the others are 0.

(2) This follows from the form of relations obtained by the Wirtinger presentation.

(3) This follow from (1) and (2). $\square$

This proposition gives a way to compute the Gassner representation for string links.

3. The definition of the Magnus representation for homology cylinders

In this section, we define the Magnus representation for the group of homology cobordism classes of homology cylinders. First we recall some definitions. Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus g with one boundary component. We fix a system of generators

$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of $\pi_1\Sigma_{g,1}$ as shown in Figure 4. Let a_i (resp. b_i) be the image of α_i (resp. β_i) by the abelianization map $\pi_1\Sigma_{g,1} \to H$ where $H = H_1\Sigma_{g,1}$.

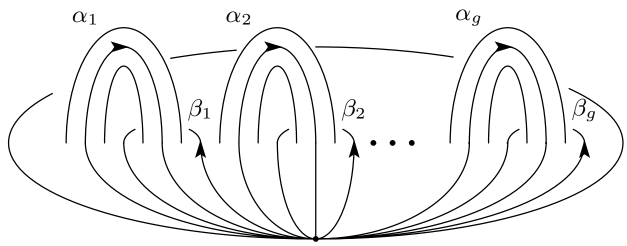

Figure 4

A *homology cylinder over* $\Sigma_{g,1}$ is a compact connected oriented 3-manifold M equipped with two embeddings $i_+, i_- : \Sigma_{g,1} \to \partial M$ satisfying that

1. i_+ is orientation-preserving and i_- is orientation-reversing,
2. $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$ and $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$,
3. $i_+, i_- : \Sigma_{g,1} \to M$ are homology isomorphisms.

We refer to [Ha], [GL] and [Le] for the origin and generalities of homology cylinders. We take a common base point p on $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1})$. We write a homology cylinder by (M, i_+, i_-) or simply M if any confusion will not occur.

Given two homology cylinders $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$, we define the multiplication $M \cdot N$ of M and N by identifying $i_+(\Sigma_{g,1})$ and $j_-(\Sigma_{g,1})$, namely

$$M \cdot N = (M \cup_{j_- \circ (i_+)^{-1}} N, j_+, i_-).$$

Then the set $\mathcal{C}_{g,1}$ of orientation-preserving diffeomorphism classes of homology cylinders becomes a monoid with an identity element defined by $(\Sigma_{g,1} \times I, i_+ = \mathrm{id} \times 1, i_- = \mathrm{id} \times 0)$ where collars of $i_+(\Sigma_{g,1})$ and $i_-(\Sigma_{g,1})$ are stretched half-way along $\partial\Sigma_{g,1} \times I$.

We can inject the mapping class group $\mathcal{M}_{g,1}$ into $\mathcal{C}_{g,1}$ so that $\mathcal{M}_{g,1}$ can be considered to be a unit subgroup of $\mathcal{C}_{g,1}$ when we define a homology cylinder $M_\varphi \in \mathcal{C}_{g,1}$ by $M_\varphi = (\Sigma_{g,1} \times I, \varphi \times 1, \mathrm{id} \times 0)$ for $\varphi \in \mathcal{M}_{g,1}$.

For a homology cylinder (M, i_+, i_-), the composite $(i_{-*})^{-1} \circ i_{+*}$ gives an isomorphism of H which preserves the intersection form on H. This correspondence gives a homomorphism

$$\begin{array}{cccccc} \|\cdot\| & : & \mathcal{C}_{g,1} & \longrightarrow & Sp(2g, \mathbb{Z}) \\ & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & & (M, i_+, i_-) & \longmapsto & (i_{-*})^{-1} \circ i_{+*} \end{array}$$

which coincides with the classical representation $\mathcal{M}_{g,1} \to Sp(2g, \mathbb{Z})$ when we restrict the domain of this homomorphism to $\mathcal{M}_{g,1}$. For later use, we denote by $\mathcal{IC}_{g,1}$ the kernel of $\|\cdot\|$. An element of $\mathcal{IC}_{g,1}$ is called a *Torelli homology cylinder*.

We can convert the monoid $\mathcal{C}_{g,1}$ into a *group* $\mathcal{H}_{g,1}$ *of homology cobordism classes of homology cylinders* as follows. Two homology cylinders $M=(M,i_+,i_-)$ and $N=(N,j_+,j_-)$ are *homology cobordant* if there exists a compact 4-manifold W satisfying

$$\partial W = M \cup (-N)/(i_+(x)=j_+(x),\, i_-(x)=j_-(x)) \quad x \in \Sigma_{g,1}$$

and inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ which are homology isomorphisms. This manifold W are called the *homology cobordism* between M and N. It is easily checked that the homomorphism $\|\cdot\|$ factors through $\mathcal{H}_{g,1}$.

Now we define the Magnus representation for homology cylinders. Given a homology cylinder $M=(M,i_+,i_-)$, we denote by F the field $\mathrm{Frac}(\mathbb{Z}H_1M)$. The abelianization map $\mathfrak{a}:\pi_1M \to H_1M$ induces an action on F by multiplications. This action defines a locally coefficient system F on M. Through $i_\pm$, We also have locally coefficient systems $i_\pm^*F$ on $\Sigma_{g,1}$. In general, $\pi_1\Sigma_{g,1}$ acts on F differently, so that i_+^*F and i_-^*F are different locally coefficient systems on $\Sigma_{g,1}$. As in the case of string links, we obtain the following lemma.

Lemma 3.1. (1) $H^1(\Sigma_{g,1},p;i_\pm^*F) \cong H^1(\pi_1\Sigma_{g,1},\{1\};i_\pm^*F) \cong Z^1(\pi_1\Sigma_{g,1};i_\pm^*F) \cong F^{2g}$ *where the correspondence is given as follows:*

$$\begin{array}{ccc} Z^1(\pi_1\Sigma_{g,1};i_\pm^*F) & \xrightarrow{\cong} & F^{2g} \\ \cup & & \cup \\ f & \longmapsto & \begin{pmatrix} f(\alpha_1) \\ \vdots \\ f(\alpha_g) \\ f(\beta_1) \\ \vdots \\ f(\beta_g) \end{pmatrix} \end{array}$$

(2) $i_\pm^*: H^1(M,p;i_\pm^*F) \to H^1(\Sigma_{g,1},p;i_\pm^*F)$ *are both isomorphisms.*

Proof. The proof is the same as that of Lemma 2.1. □

Definition 3.2. (1) The *Magnus anti-representation for homology cylinders* is a map which assigns a matrix $\widetilde{r}(M) \in GL(2g,\mathrm{Frac}(\mathbb{Z}H))$ to a homology cylinder $M=(M,i_+,i_-) \in \mathcal{C}_{g,1}$ where the matrix $\widetilde{r}(M)$ is obtained from the representation matrix $\widehat{r}(M) \in GL(2g,F)$ of the isomorphism

$$F^{2g} \cong H^1(\Sigma_{g,1},p;i_-^*F) \xrightarrow[(i_-^*)^{-1}]{\cong} H^1(M,p;F) \xrightarrow[i_+^*]{\cong} H^1(\Sigma_{g,1},p;i_+^*F) \cong F^{2g}$$

by applying $(i_{-*})^{-1}: F \to \mathrm{Frac}(\mathbb{Z}H)$ on each entry.

(2) The *Magnus representation for homology cylinders* $r: \mathcal{C}_{g,1} \longrightarrow GL(2g,\mathrm{Frac}(\mathbb{Z}H))$ is a map which assigns ${}^t\overline{\widetilde{r}(L)}$ to a homology cylinder $M=(M,i_+,i_-) \in \mathcal{C}_{g,1}$ where ${}^tA$ means the

transpose of a matrix A and $\bar{\cdot} : \mathrm{Frac}(\mathbb{Z}H) \to \mathrm{Frac}(\mathbb{Z}H)$ is the anti-automorphism induced by the map $x \mapsto x^{-1}$.

The reader may feel the above definition clumsy. However, this definition makes it easier to see that it is a generalization of the Magnus representation for the mapping class group $\mathcal{M}_{g,1}$. We refer to [Mo] for this representation.

Proposition 3.3. *Let $M_\varphi = (\Sigma_{g,1} \times I, \varphi \times 1, id \times 0)$ be a homology cylinder contained in the mapping class group $\mathcal{M}_{g,1}$ which is considered to be a subgroup of homology cylinder semi-group $\mathcal{C}_{g,1}$. Then*

$$r(M_\varphi) = r_0(\varphi)$$

where r_0 is the Magnus representation for the mapping class group.

Proof. It is easily checked when we use Proposition 4.4 (2) mentioned later. $\square$

Note that the Magnus representation for homology cylinder is *not* a homomorphism. The following proposition shows that it is actually a crossed homomorphism.

Theorem 3.4. *Let $M_1 = (M_1, i_{1+}, i_{1-})$ and $M_2 = (M_2, i_{2+}, i_{2-})$ be homology cylinders. Then*

$$r(M_1 \cdot M_2) = r(M_1) \cdot {}^{\|M_1\|}r(M_2)$$

where ${}^{\|M_1\|}r(M_2)$ is the matrix obtained from $r(M_2) \in GL(2g, \mathrm{Frac}(\mathbb{Z}H))$ by applying the map induced from $\|M_1\| : H \to H$ on each entry.

Proof. Let $j_1 : M_1 \hookrightarrow M_1 \cdot M_2$ and $j_2 : M_2 \hookrightarrow M_1 \cdot M_2$ be natural inclusions. We denote $H_1(M_1 \cdot M_2)$ by F and denote $j_1^* F$ (resp. $j_2^* F$) by F_1 (resp. F_2). Then

$$\begin{array}{rccc}
 & \widehat{r}(M_1 \cdot M_2) & = & {}^{j_2}\widehat{r}(M_2) \cdot {}^{j_1}\widehat{r}(M_1) \\
\Longrightarrow & {}^{(j_1 i_{1-})^{-1}}\widehat{r}(M_1 \cdot M_2) & = & {}^{(j_1 i_{1-})^{-1} j_2}\widehat{r}(M_2) \cdot {}^{(j_1 i_{1-})^{-1} j_1}\widehat{r}(M_1) \\
\Longrightarrow & {}^{(j_1 i_{1-})^{-1}}\widehat{r}(M_1 \cdot M_2) & = & {}^{i_{1-}^{-1} j_1^{-1} j_2}\widehat{r}(M_2) \cdot {}^{i_{1-}^{-1}}\widehat{r}(M_1) \\
\Longrightarrow & \widetilde{r}(M_1 \cdot M_2) & = & {}^{i_{1-}^{-1} i_{1+} i_{2-}^{-1}}\widehat{r}(M_2) \cdot \widetilde{r}(M_1) \\
\Longrightarrow & \widetilde{r}(M_1 \cdot M_2) & = & {}^{\|M_1\|}\widetilde{r}(M_2) \cdot \widetilde{r}(M_1) \\
\Longrightarrow & r(M_1 \cdot M_2) & = & r(M_1) \cdot {}^{\|M_1\|}r(M_2)
\end{array}$$

where matrices of the form ${}^{\varphi}A$ are ones obtained from A by applying the homomorphism φ on each entry. $\square$

Theorem 3.5. *The Magnus representation $r : \mathcal{C}_{g,1} \to GL(2g, \mathrm{Frac}\,\mathbb{Z}H)$ factors through the group $\mathcal{H}_{g,1}$ of homology cobordism classes of homology cylinders.*

Proof. Let $M_1 = (M_1, i_{1+}, i_{1-})$ and $M_2 = (M_2, i_{2+}, i_{2-})$ be homology cylinders and assume that they are homology cobordant. We denote the homology cobordism by W and write $j_1 : M_1 \hookrightarrow W$ and $j_2 : M_2 \hookrightarrow W$ for natural inclusions. From the definition, $j_1 \circ i_{1+} = j_2 \circ i_{2+}$ and

$j_1 \circ i_{1-} = j_2 \circ i_{2-}$ hold and we denote these homomorphisms by I_+ and I_-, for short. Using j_1 and j_2, we identify H_1M_1, H_1M_2 and H_1W. Then we have the following commutative diagram

$$\begin{tikzcd}
F^{2g} \cong H^1(\Sigma_{g,1},p;I_-^*F) \arrow[r, equal] \arrow[d, "(i_{1-}^*)^{-1}"'] & H^1(\Sigma_{g,1},p;I_-^*F) \arrow[r, equal] \arrow[d, "(I_-^*)^{-1}"'] & H^1(\Sigma_{g,1},p;I_-^*F) \cong F^{2g} \arrow[d, "(i_{2-}^*)^{-1}"] \\
H^1(M_1,p;j_1^*F) \arrow[r, equal] \arrow[d, "i_{1+}^*"'] & H^1(W,p;F) \arrow[r, equal] \arrow[d, "I_+^*"'] & H^1(M_2,p;j_2^*F) \arrow[d, "i_{2+}^*"] \\
F^{2g} \cong H^1(\Sigma_{g,1},p;I_+^*F) \arrow[r, equal] & H^1(\Sigma_{g,1},p;I_+^*F) \arrow[r, equal] & H^1(\Sigma_{g,1},p;I_+^*F) \cong F^{2g}
\end{tikzcd}$$

where $F = \mathrm{Frac}(\mathbb{Z}W)$. The left vertical map gives $\widehat{r}(M_1)$ and the right one gives $\widehat{r}(M_2)$. Hence we obtain $r(M_1) = r(M_2)$. □

From above arguments, we obtain the Magnus representation for $\mathcal{H}_{g,1}$:

$$r : \mathcal{H}_{g,1} \longrightarrow GL(2g, \mathrm{Frac}(\mathbb{Z}H))$$

which is a crossed homomorphism.

4. Some fundamental properties of the Magnus representation for homology cylinders

In this section, we will give some fundamental properties of the Magnus representation for homology cylinders containing a method for computations. Suzuki showed in [Su] that the Magnus representation for the Torelli group:

$$r_0 : \mathcal{I}_{g,1} \longrightarrow GL(2g, \mathbb{Z}H_1\Sigma_{g,1})$$

is *not* faithful for $g \geq 2$. Hence our representation is *not* injective for $g \geq 2$. One of big differences between string links and homology cylinders is the existence of the Wirtinger presentation. We need a substitute for this presentation to have the argument similar to the one in [KLW]. Since homology cylinders are all compact manifolds, their fundamental groups are finitely presentable.

Definition 4.1. A finite presentation of $\pi_1 M$ of a homology cylinder M is called *admissible* if this presentation is given by a system of ordered generators $i_+(\alpha_1), \ldots, i_+(\beta_g), z_1, \ldots, z_l$, $i_-(\alpha_1), \ldots, i_-(\beta_g)$ and $(2g+l)$ relations.

Note that $i_+(\alpha_1), \ldots, i_+(\beta_g)$ means $i_+(\alpha_1), \ldots, i_+(\alpha_g), i_+(\beta_1), \ldots, i_+(\beta_g)$. We often use this notation for simplicity.

Lemma 4.2. *There exists an admissible presentation for every homology cylinder.*

We will give two different proofs of this lemma. One is given by using the deficiency of finitely presentable groups as mentioned below and the other is given by studying the homotopy type of homology cylinders.

Proof. First we recall about the deficiency of finitely presentable groups. The deficiency of a finite presentation $P = \{x_1, \dots, x_n \mid r_1, \dots, r_m\}$ of a group G is defined to be $n - m$. Then we define the deficiency of G to be the maximum of the deficiency of P over all possible finite presentations. From Epstein's results [Ep], we can easily see that the deficiency of $\pi_1 M$ is $2g$. Hence there exists a finite presentation of the form

$$\pi_1 M = \{z_1, \dots, z_{2g+k} \mid r_1, \dots r_k\}.$$

Then we have an admissible presentation by adding $4g$ generators $i_+(\alpha_1), \dots, i_+(\beta_g), i_-(\alpha_1), \dots, i_-(\beta_g)$ and $4g$ relations to introduce them. □

Definition 4.3. An admissible presentation is called *standard* if it is constructed as in the proof of Lemma 4.2.

Now we deduce a formula for computing the Magnus representation for homology cylinders as in the case of the Gassner representation for string links. Given a homology cylinder $M \in \mathcal{C}_{g,1}$ and an admissible presentation

$$\pi_1 M \cong \langle i_+(\alpha_1), \dots, i_+(\beta_g), z_1, \dots, z_l, i_-(\alpha_1), \dots, i_-(\beta_g) \mid r_1, \dots r_{2g+l}\rangle,$$

we simply write $\gamma_1, \dots, \gamma_{4g+l}$ for these ordered generators. Let A, B and C be matrices of sizes $(2g+l) \times 2g$, $(2g+l) \times l$ and $(2g+l) \times 2g$ defined by the equality

$$\begin{pmatrix} A & B & C \end{pmatrix} = {}^{\rho}\!\left(\frac{\partial r_i}{\partial \gamma_j}\right)_{i,j}$$

where ρ is the composite of

$$F_{4g+l} = \langle \gamma_1, \dots, \gamma_{4g+l}\rangle \longrightarrow \pi_1 X \xrightarrow{\mathfrak{a}} H_1 X.$$

Then we have the following proposition where $\mathfrak{t} : H_1 M \to \{1\}$ is the trivialization map.

Proposition 4.4. (1) $\begin{pmatrix} A & B \end{pmatrix} \in GL(2g+l, \mathbb{Z}H_1M_S) \subset GL(2g+l, F)$ *where* $\mathbb{Z}H_1M_S$ *is the localization of* $\mathbb{Z}H_1M$ *obtained by inverting all elements in the set*

$$S = \{f \in \mathbb{Z}H_1M \mid \mathfrak{t}(f) = \pm 1 \in \mathbb{Z}\{1\}\}.$$

(2) *There exists a* $(l \times 2g)$ *matrix* Z *which satisfies the equality*

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \widehat{r}(M) \\ Z \end{pmatrix} = -C.$$

(3) $\widehat{r}(M) \in GL(2g, \mathbb{Z}H_1M_S)$.

Proof. From the general theory, ${}^t(A \quad B \quad C)$ gives a presentation matrix of H_1M, namely we have an exact sequence

$$\mathbb{Z}^{2g+l} \xrightarrow{{}^t(A \quad B \quad C)\cdot} \mathbb{Z}^{4g+l} \longrightarrow H_1M \longrightarrow 0\,.$$

Then ${}^t(A \quad B)$ gives a presentation matrix of H_1M/I_- where I_- is a subgroup of H_1M generated by $i_{-*}(a_1), \ldots, i_{-*}(b_g)$. The reader can consult [Fo] for this fact through the concept of *presentations of a pair of groups*. By definition, $H_1M/I_- = 0$. Hence we have an exact sequence

$$\mathbb{Z}^{2g+l} \xrightarrow{{}^t(A \quad B)\cdot} \mathbb{Z}^{2g+l} \longrightarrow H_1M/I_- = 0\,.$$

From the Hopfian property of $\mathbb{Z}^{2g+l}$, we see that $\det {}^t(A \quad B) = \pm 1$. Therefore (1) follows. (2) follows from the same argument as Proposition 2.3. (3) follows from (1) and (2). □

This proposition gives a way to compute the Magnus representation for homology cylinders. Some examples will be given in Section 7.

As a corollary of this proposition, we see that the group of homology cobordism classes of homology 3-spheres is in the kernel of the Magnus representation. This group can be embedded in $\mathcal{H}_{g,1}$ by considering the connected sum $X\sharp(\Sigma_{g,1} \times I)$ for a homology 3-sphere X. Then $\pi_1(X\sharp(\Sigma_{g,1} \times I))$ has an admissible presentation of the form

$$\left\langle \begin{array}{c} i_+(\alpha_1), \ldots, i_+(\beta_g) \\ z_1, \ldots, z_l \\ i_-(\alpha_1), \ldots, i_-(\beta_g) \end{array} \middle| \begin{array}{c} i_+(\alpha_j) = i_-(\alpha_j) \\ i_+(\beta_j) = i_-(\beta_j) \\ r_1, \ldots, r_l \end{array} \right\rangle \cong \pi_1(\Sigma_{g,1} \times I) * \pi_1 X$$

which shows that the corresponding Magnus matrix is equal to I_{2g}.

Next we consider the operation on homology cylinders similar to the one which gives a link by closing a braid. Let $M \in \mathcal{C}_{g,1}$ be a homology cylinder with an admissible presentation

$$\pi_1 M \cong \langle i_+(\alpha_1), \ldots, i_+(\beta_g), z_1, \ldots, z_l, i_-(\alpha_1), \ldots, i_-(\beta_g) \mid r_1, \ldots r_{2g+l} \rangle$$

of $\pi_1 M$ and let $(A \quad B \quad C)$ be the matrix constructed from this presentation as above.

Definition 4.5. For a homology cylinder $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$, we define its *closing* $\widehat{M}$ by

$$\widehat{M} = M/(i_+(x) = i_-(x)) \quad x \in \Sigma_{g,1}$$

which is a closed manifold.

Let $q : M \to \widehat{M}$ be the natural quotient map. Using van-Kampen's theorem, we can easily compute $\pi_1 \widehat{M}$ from an admissible presentation. Namely if

$$\pi_1 M = \langle i_+(\alpha_1), \ldots, i_+(\beta_g), z_1, \ldots, z_l, i_-(\alpha_1), \ldots, i_-(\beta_g) \mid r_1, \ldots r_{2g+l} \rangle,$$

then

$$\pi_1 \widehat{M} = \langle i(\alpha_1), \ldots, i(\beta_g), z_1, \ldots, z_l \mid r_1', \ldots r_{2g+l}' \rangle$$

where $i(\alpha_j) = q_* i_+(\alpha_j) = q_* i_-(\alpha_j)$, $i(\beta_j) = q_* i_+(\beta_j) = q_* i_-(\beta_j)$ and r_i' are obtained from r_i by replacing $i_\pm(\alpha_j)$ (resp. $i_\pm(\beta_j)$) by $i(\alpha_j)$ (resp. $i(\beta_j)$). We simply write $\gamma_1, \ldots, \gamma_{2g+l}$ for these ordered generators. Then we assign to this presentation the *Alexander matrix* defined by

$$V_{\widehat{M}} = {}^{\rho}\left(\frac{\partial r_i'}{\partial \gamma_j}\right)_{i,j}$$

where ρ is the composite of

$$F_{2g+l} = \langle \gamma_1, \ldots, \gamma_{2g+l} \rangle \longrightarrow \pi_1 \widehat{M} \xrightarrow{\mathfrak{a}} H_1 \widehat{M}\,.$$

As in [KLW], we obtain the following equalities.

Proposition 4.6. *The Alexander matrix for the manifold $\widehat{M}$ obtained by closing M can be written by using the matrix $(A \quad B \quad C)$ as follows:*

$$V_{\widehat{M}} = {}^{q}(A + C \quad B) = {}^{q}(A \quad B)\,{}^{q}\begin{pmatrix} I_{2g} - \widehat{r}(M) & O \\ -Z & I_l \end{pmatrix}.$$

where Z is the matrix obtained from $(A \quad B \quad C)$ as in Proposition 4.4 (2).

Proof. Recall that $C = -(A \quad B)\begin{pmatrix} \widehat{r}(M) \\ Z \end{pmatrix}$. The proposition follows from this. □

5. Alexander polynomials for Torelli homology cylinders

In this section, we restrict our attention to the monoid of Torelli homology cylinders $\mathcal{IC}_{g,1}$ and examine a relationship with the Alexander polynomial. The goal of this section is to deduce a factorization formula for the Alexander polynomial by using our Magnus representation. Note that for every Torelli homology cylinder (M, i_+, i_-), two inclusions i_+ and i_- induce the same isomorphism between homology groups $H_1\Sigma_{g,1}$ and $H_1 M$ so that we can naturally identify them. We can also identify $H_1 M$ with $H_1 \widehat{M}$. In the argument below, the following lemma which is slightly extended from the one in [KLW] has an important role. For an $n \times n$ matrix A, we denote by $A_{(i,j)}$ the matrix obtained from A by removing its i-th row and j-th column.

Lemma 5.1. *Let A be an $n \times n$ matrix over a domain R. Let $u = (u_1, \ldots, u_n)$ and $w = {}^t(w_1, \ldots, w_n)$ be a row and column vector so that*

$$uA = 0 \quad \textit{and} \quad Aw = 0.$$

Then

(1) *If $u_i = 0$ or $w_j = 0$, then* $\det M_{(i,j)} = 0$.

(2) *The element in the fraction field*

$$(-1)^{i+j} \frac{\det M_{(i,j)}}{u_i w_j}$$

is independent of the choice of i and j satisfying $u_i \neq 0$ and $w_j \neq 0$.

Proof. We can prove it by the same argument as Lemma 6.2 in [KLW]. □

From this lemma, we can define some invariant from the Magnus matrix $\widetilde{r}(M)$ if we find two vectors u and v satisfying the above condition. The following lemma gives an answer to it.

Lemma 5.2. *Let M be a Torelli homology cylinder of genus $g \geq 1$.*

(1) $$(I_{2g} - \widetilde{r}(M)) \begin{pmatrix} 1-a_1 \\ \vdots \\ 1-a_g \\ 1-b_1 \\ \vdots \\ 1-b_g \end{pmatrix} = 0.$$

(2) $$(b_1 - 1, \ldots, b_g - 1, 1-a_1, \ldots, 1-a_g)(I_{2g} - \widetilde{r}(M)) = 0.$$

Proof. For simplicity, we put $u = {}^t(1-a_1, \ldots, 1-a_g, 1-b_1, \ldots, 1-b_g)$ and $w = (b_1-1, \ldots, b_g-1, 1-a_1, \ldots, 1-a_g)$.

(1) Take a 0-cochain $f = 1 \in F \cong C^0(\pi_1 M; F)$. By definition, $\delta f(x) = 1 - [x]$ holds for every $x \in \pi_1 M$ where $[x]$ is the homology class of x. Then we see the correspondence of 1-cocycles

$$\begin{array}{ccccc} Z^1(\pi_1\Sigma_{g,1}; i_-^* F) & \cong & Z^1(\pi_1 M; F) & \cong & Z^1(\pi_1\Sigma_{g,1}; i_+^* F) \\ \cup & & \cup & & \cup \\ u & \leftrightarrow & \delta f & \leftrightarrow & u = \widetilde{r}(M)u \end{array}$$

so that (1) follows.

(2) Let ζ be the Dehn twist map along the simple closed curve which is parallel to the boundary of $\Sigma_{g,1}$. It is easy to see that

$$\widetilde{r}(\zeta) = I_{2g} + uw.$$

When we see ζ as an element of $\mathcal{H}_{g,1}$, it is in the center of $\mathcal{H}_{g,1}$. Therefore for every Torelli homology cylinder M, we have

$$\begin{array}{rccc} & \widetilde{r}(\zeta)^{-1}\widetilde{r}(M)\widetilde{r}(\zeta) & = & \widetilde{r}(M) \\ \Longrightarrow & \widetilde{r}(M)\widetilde{r}(\zeta) & = & \widetilde{r}(\zeta)\widetilde{r}(M) \\ \Longrightarrow & \widetilde{r}(M)(I_{2g} + uw) & = & (I_{2g} + uw)\widetilde{r}(\zeta) \\ \Longrightarrow & \widetilde{r}(M)uw & = & uw\widetilde{r}(M). \end{array}$$

From (1), we obtain that $uw = uw\widetilde{r}(M)$. When we compare first rows, we have the equality $(1-a_1)w = (1-a_1)w\widetilde{r}(M)$. (2) follows from this. □

Lemmas 5.1, 5.2 allow us to define the following rational function.

Definition 5.3. Let M be a Torelli homology cylinder of genus $g \geq 1$, the *Alexander rational function of M* is the rational function

$$\Delta_M(a_1, \ldots, b_g) = -\frac{\det\left((I_{2g} - \widetilde{r}(M)_{(1,1)}\right)}{(1-a_1)(1-b_1)} \in F$$

where $\widetilde{r}(M)$ is the Magnus matrix of M.

Note that Δ_M is a homology cobordism invariant since $\widetilde{r}(M)$ is. We need not use the matrix $\widetilde{r}(M)_{(1,1)}$ but we can use $\widetilde{r}(M)_{(i,j)}$ for arbitrary i and j because Δ_M are independent of the choice of them by Lemma 5.1. Now u and w in the above lemma do not have their entries which are equal to 0.

Next we briefly recall the definition of the Alexander polynomial for finitely presentable groups. Given a finitely presentable group G with a finite presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_m\rangle$, we can obtain an Alexander matrix A defined by

$$A = {}^{\rho}\left(\frac{\partial r_i}{\partial x_j}\right)_{i,j} \quad \in M(m \times n, \mathbb{Z}H)$$

where ρ is the composite of

$$F_n = \langle x_1, \dots, x_n\rangle \longrightarrow G \xrightarrow{\ \mathfrak{a}\ } H = H_1 G\,.$$

Note that $\mathbb{Z}H$ is a UFD. Then the Alexander polynomial of G is

$$\Delta_G \doteq \gcd(E_1(A)) \in \mathbb{Z}H$$

where $E_1(A)$ is the first elementary ideal of A generated by all $n-1$ minors of it. Δ_G is uniquely determined up to units in $\mathbb{Z}H$ and it is independent of the choice of finite presentations of G, so that we use $\doteq$ in equalities of Alexander polynomials.

For a Torelli homology cylinder M, we define $\Delta_{\widehat{M}} = \Delta_{\pi_1\widehat{M}}$. Now we deduce an explicit formula of the Alexander polynomial $\Delta_{\widehat{M}}$.

Proposition 5.4. *Let M be a Torelli homology cylinder of genus $g \geq 1$ and $V_{\widehat{M}}$ be the Alexander matrix for the presentation of $\pi_1\widehat{M}$ obtained from a standard admissible presentation of $\pi_1 M$. Then*

$$\Delta_{\widehat{M}}(a_1, \dots, b_g) \doteq -\frac{\det\left(V_{\widehat{M}\,(1,1)}\right)}{(1-a_1)(1-b_1)} \in \mathbb{Z}H_1\widehat{M}.$$

Proof. Recall that a standard admissible presentation of $\pi_1 M$ is an admissible presentation which has the following form

$$\pi_1 M \cong \left\langle\ i_+(\alpha_1), \dots, i_+(\beta_g), z_1, \dots, z_{2g+k}, i_-(\alpha_1), \dots, i_-(\beta_g)\ \middle|\ \begin{array}{c} i_+(\alpha_1)\varphi_1, \dots, i_+(\beta_g)\varphi_{2g} \\ r_1, \dots r_k \\ i_-(\alpha_1)\psi_1, \dots, i_-(\beta_g)\psi_{2g} \end{array} \right\rangle$$

where φ_i, r_i and ψ_i are words in $z_1, \dots, z_{2g+k}$. Let $V_{\widehat{M}}$ be the Alexander matrix for the presentation of $\pi_1\widehat{M}$ obtained from this. Then $V_{\widehat{M}} = (A + C \quad B)$ has the form of

$$V_{\widehat{M}} = \begin{pmatrix} I_{2g} & * \\ O & * \\ I_{2g} & * \end{pmatrix}.$$

Now we seek a non-zero row vector u and a non-zero column vector w satisfying $uV_{\widehat{M}} = 0$ and $V_{\widehat{M}}w = 0$. By the fundamental formula of free calculus in [Bi], we obtain

$$V_{\widehat{M}}\,{}^t(1-a_1,\dots,1-b_g,1-\rho(z_1),\dots,1-\rho(z_{2g+k}),1-a_1,\dots,1-b_g) = 0$$

so that we can take ${}^t(1-a_1,\dots,1-b_g,1-\rho(z_1),\dots,1-\rho(z_{2g+k}),1-a_1,\dots,1-b_g)$ as the vector w. To seek the vector u, we recall Lemma 5.2 and Proposition 4.6 that

$$V_{\widehat{M}} = (A \quad B)\begin{pmatrix} I_{2g}-\widetilde{r}(M) & O \\ -Z & I_{2g+k} \end{pmatrix},$$

$$(b_1-1,\dots,b_g-1,1-a_1,\dots,1-a_g)(I_{2g}-\widetilde{r}(M)) = 0.$$

Therefore we obtain

$$(b_1-1,\dots,b_g-1,1-a_1,\dots,1-a_g,0,\dots,0)(A \quad B)^{-1}V_{\widehat{M}} = 0.$$

Thus we can take $(b_1-1,\dots,b_g-1,1-a_1,\dots,1-a_g,0,\dots,0)(A \quad B)^{-1}$ as the vector u. From the form of $(A \quad B)$ with respect to a standard admissible presentation, we have

$$u = (b_1-1,\dots,b_g-1,1-a_1,\dots,1-a_g,*,\dots,*).$$

Now we deduce the greatest common divisor of the first elementary ideal of $V_{\widehat{M}}$ which gives the Alexander polynomial $\Delta_{\widehat{M}}$. By Lemma 5.1,

$$h = (-1)^{i+j}\frac{\det\left(V_{\widehat{M}\,(i,j)}\right)}{u_i w_j} \in F$$

is independent of the choice of i and j satisfying $u_i \neq 0$ and $w_j \neq 0$. If h is in $\mathbb{Z}H_1\widehat{M}$, it gives the greatest common divisor. To show that suppose $h = h_1/h_2$ where $h_1 \in \mathbb{Z}H_1\widehat{M}$ and $h_2 \in \mathbb{Z}H_1\widehat{M}-\{0\}$ are relatively prime. When $(i,j)=(g+1,1)$, we have $(-1)^g\det\left(V_{\widehat{M}\,(g+1,1)}\right) = h_1(1-a_1)^2/h_2 \in \mathbb{Z}H_1\widehat{M}$. When $(i,j)=(1,g+1)$, we have $(-1)^{g+1}\det\left(V_{\widehat{M}\,(1,g+1)}\right) = h_1(b_1-1)^2/h_2 \in \mathbb{Z}H_1\widehat{M}$. Hence h_2 is a common divisor of $(1-a_1)^2$ and $(b_1-1)^2$ which are relatively prime so that h_2 is in units of $\mathbb{Z}H_1\widehat{M}$. Therefore the proposition follows. □

Finally, we show the relationship between the Alexander rational function and the Alexander polynomial. We use the *Milnor torsion* of a homology cylinder to describe it. For this torsion, we refer to [Tu].

Let $M=(M,i_+,i_-)$ be a Torelli homology cylinder and Σ_- be the image of $\Sigma_{g,1}$ by the embedding i_-. By Lemma 3.1 (2), we obtain $H^1(M,\Sigma_-;F)=0$, so that the cochain complex $C^*(M,\Sigma_-;F)$ with respect to some cell structure is acyclic. We write $\tau(M)$ for the torsion of this complex. $\tau(M) \in F^*$ is well defined up to multiplication by units H in $\mathbb{Z}H$. One important property of this torsion is that it is invariant under homotopy equivalences. In particular, it does not depend on the choice of the cell structure on M.

Theorem 5.5. *Let M be a Torelli homology cylinder and $\widehat{M}$ be its closing. Then the Alexander polynomial of $\widehat{M}$ is the product of the torsion $\tau(M)$ and the Alexander rational function Δ_M of the homology cylinder:*

$$\Delta_{\widehat{M}} \doteq \tau(M) \cdot \Delta_M.$$

The proof of this theorem is divided into three parts, Lemmas 5.6, 5.7 and 5.8. Given a Torelli homology cylinder $M = (M, i_+, i_-)$ and an admissible presentation of $\pi_1 M$, we calculate the matrix $(A \quad B \quad C)$ as before.

Lemma 5.6. $\Delta_{\widehat{M}} \doteq \det(A \quad B) \cdot \Delta_M.$

Proof. Note that $\det(A \quad B)$ up to units in $\mathbb{Z}H$ is independent of the choice of admissible presentations. Indeed when we consider the admissible presentation to be a presentation of the pair of groups

$$(\quad \pi_1 M \quad , \quad \langle i_-(\alpha_1), \ldots, i_-(\beta_g) \rangle \,/\, R \quad)$$

with distinguished generators $i_-(\alpha_1), \ldots, i_-(\beta_g)$ where R is the normal closure of relations $r_1, \ldots, r_l$, then $\det(A \quad B)$ gives the generator of 0-th elementary ideal, which is principal, so that it is invariant under Tietze transformations. We refer to [Fo] for the presentation of a pair of groups.

The rest of the proof of this lemma is almost the same as in [KLW]. Since $\Delta_{\widehat{M}}$, $\det(A \quad B)$ and Δ_M are independent of the choice of admissible presentations, so that we can take a standard admissible presentation

$$\pi_1 M \cong \left\langle \; i_+(\alpha_1), \ldots, i_+(\beta_g), z_1, \ldots, z_{2g+k}, i_-(\alpha_1), \ldots, i_-(\beta_g) \;\middle|\; \begin{array}{c} i_+(\alpha_1)\varphi_1, \ldots, i_+(\beta_g)\varphi_{2g} \\ r_1, \ldots r_k \\ i_-(\alpha_1)\psi_1, \ldots, i_-(\beta_g)\psi_{2g} \end{array} \right\rangle$$

to calculate them. Then

$$(A \quad B) = \begin{pmatrix} I_{2g} & * \\ O & * \end{pmatrix}.$$

Recall that $\det(A \quad B) \neq 0$. Therefore by making use of the following row operations

- Add a multiple by an element of F of the i-th row to the j-th row for $2 \leq i, j \leq 4g + k$
- Interchange the i-th row and the j-th row $2 \leq i, j \leq 4g + k$

we can transform $(A \quad B)$ into the matrix

$$\widetilde{D} = \begin{pmatrix} 1 & 0 & * \\ 0 & I_{2g-1} & O \\ 0 & O & D \end{pmatrix}$$

where D is a diagonal matrix whose determinant is equal to $\pm\det(A \quad B)$. Then the transformation matrix E is of the form

$$E = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}$$

where X is a $(4g+k-1)\times(4g+k-1)$ matrix whose determinant is equal to ± 1. Thus

$$\begin{aligned}
\det\left(V_{\widehat{M}\,(1,1)}\right) &= \pm\det\left(X\cdot V_{\widehat{M}\,(1,1)}\right)\\
&= \pm\det\left((E\cdot V_{\widehat{M}})_{(1,1)}\right)\\
&= \pm\det\left\{\left(E\cdot(A \quad B)\begin{pmatrix} I_{2g}-\widetilde{r}(M) & O \\ -Z & I_{2g+k}\end{pmatrix}\right)_{(1,1)}\right\}\\
&= \pm\det\begin{pmatrix} I_{2g-1}\cdot(I_{2g}-\widetilde{r}(M))_{(1,1)} & O \\ Z' & D\end{pmatrix}\\
&= \pm\det D\cdot\det\left((I_{2g}-\widetilde{r}(M))_{(1,1)}\right)\\
&= \pm\det(A \quad B)\cdot\det\left((I_{2g}-\widetilde{r}(M))_{(1,1)}\right).
\end{aligned}$$

Hence the lemma follows after dividing them by $(1-a_1)(1-b_1)$. □

Lemma 5.7. *(M,Σ_-) is homotopy equivalent to (M',R'_{2g}) as the pair of CW-complexes where M' is a certain two dimensional CW-complex with only one 0-cell and R'_{2g} is a one dimensional CW-subcomplex of M' with $2g$ 1-cells.*

Proof. We deform (M,Σ_-) into (M',R'_{2g}) step by step.

Step 1. First we give a standard cell structure to $\partial M\cong\Sigma_{2g}$ as in Figure 5, where we draw pictures only in the case of $g=1$ since the cases of higher genera are similar. Let R_{2g} be the CW-subcomplex of ∂M whose 1-cells are given by $i_-(\alpha_1),\dots,i_-(\beta_g)$.

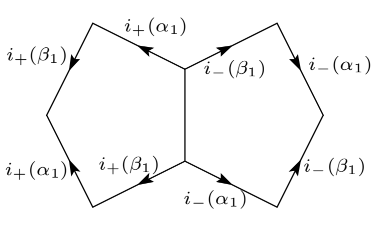

Figure 5

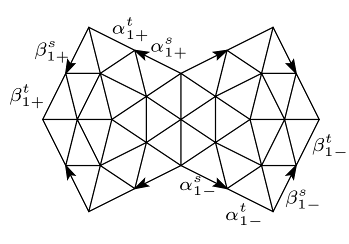

Figure 6

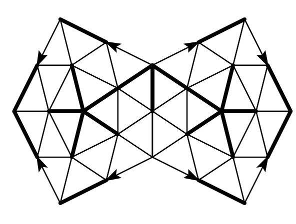
Figure 7

Take a triangulation which is a refinement of the cell decomposition as in Figure 6. By this refinement, for example, $i_+(\alpha_j)$ is divided into two edges α^s_{j+} and α^t_{j+}. Then we can extend this triangulation to the whole of M by a theorem of Cairns and J. H. C. Whitehead. It is easy to see that we can deform (M,Σ_-) into (M,R_{2g}) by some homotopy equivalence which is not necessary a cell map.

Step 2. Starting from a 3-simplex of M facing the boundary, we can deform M onto a 2-dimensional subcomplex M''. In this process, 1-skeleton of M is kept invariant, so that the pair (M, R_{2g}) is homotopy equivalent to (M'', R_{2g}).

Step 3. Take a maximal tree T of the 1-skeleton of M'' containing $\alpha_{1+}^t, \ldots, \beta_{g+}^t, \alpha_{1-}^t, \ldots, \beta_{g-}^t$ as in Figure 7, where T is drawn by thick lines, and collapse T to a point. Then we obtain a pair of CW-complexes $(M', {R'}_{2g}) = (M''/T, R_{2g}/T)$ which is homotopy equivalent to (M'', R_{2g}). Notice that $(M', {R'}_{2g})$ has all the property we want. Hence the lemma follows. $\square$

As a corollary of this lemma, we see the existence of a standard admissible presentation of $\pi_1 M$ from the cell structure of M'.

Lemma 5.8. $\tau(M) \doteq \det(A \quad B)$.

Proof. Consider the exact sequence

$$0 \longrightarrow C^*(M', {R'}_{2g}; F) \longrightarrow C^*(M'; F) \longrightarrow C^*({R'}_{2g}; F) \longrightarrow 0$$

of cochain complexes. Counting the number of cells, we have the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F^n & \longrightarrow & F^n & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & {\scriptstyle\delta}\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F^n & \longrightarrow & F^{2g+n} & \longrightarrow & F^{2g} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & F & \longrightarrow & 0
\end{array}$$

where we use the fact that the Euler characteristic of M' is equal to $1 - 2g$. By definition, $\tau(M)$ is the determinant of the differential δ. Then we see that $\delta = (A \quad B)$ from the construction of M'. Thus the lemma follows. $\square$

From Lemmas 5.6, 5.7 and 5.8, Theorem 5.5 follows. $\square$

6. String links and homology cylinders

It is known that the pure braid group $\mathcal{P}_g$ can be embedded into the mapping class group $\mathcal{M}_{g,1}$ through the embedding of the disk with g holes denoted by D_g^0. In fact, $\mathcal{P}_g$ is a subgroup of the *framed pure braid group* $\mathcal{P}_g^{fr} \cong \mathcal{P}_g \times \mathbb{Z}^g$, which is naturally isomorphic to the mapping class group of D_g^0, and we have an injective homomorphism $\mathcal{P}_g^{fr} \to \mathcal{M}_{g,1}$ for each suitable embedding of D_g^0. Now we fix an embedding of D_g^0 as in Figure 8. In [Le], these homomorphisms are extended to the whole of the group of pure string links, so that we have an injective homomorphism $\Phi : \mathcal{S}_g \to \mathcal{H}_{g,1}$ which is a composite of injective homomorphisms

$$\mathcal{S}_g \longrightarrow \mathcal{S}_g^{fr} \longrightarrow \mathcal{H}_{g,1}.$$

where $\mathcal{S}_g^{fr} \cong \mathcal{S}_g \times \mathbb{Z}^g$ is the group of concordance classes of pure framed string links.

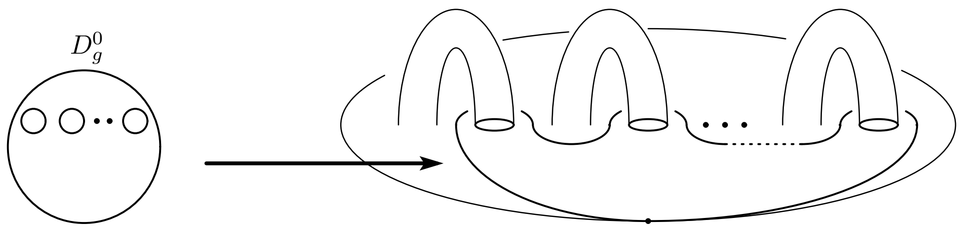

Figure 8

Now we briefly review this homomorphism. Let L be a pure framed string link and C be the complement of an open tubular neighborhood of L in $D_g^0 \times I$. We have a canonical identification of ∂C with $\partial(D_g^0 \times I)$ by using the framing of L which decides the way to identify their meridians. Then we make a new homology cylinder M_L by removing $D_g^0 \times I$ from $\Sigma_{g,1} \times I$ and replacing it with C by using this identification. This homology cylinder M_L gives $\Phi(L)$.

We have two representations for string links, one is the Gassner representation $g : \mathcal{S}_g \to GL(2g, \mathrm{Frac}(\mathbb{Z}H_1 D_g))$ and the other is the restriction $r|_{\mathcal{S}_g} : \mathcal{S}_g \to GL(2g, \mathrm{Frac}(\mathbb{Z}H_1\Sigma_{g,1}))$ of Magnus representation for homology cylinders. These representations have the following relationship.

Theorem 6.1. *Let $L \in \mathcal{S}_g$ be (a concordance class of) a pure string link. Then*

$$\widetilde{r}(\Phi(L)) = \left(\begin{array}{c|c} I_g & * \\ \hline O & \widetilde{g}(L) \end{array}\right)$$

where $\widetilde{r}$ is the Magnus anti-representation and $\widetilde{g}$ is the Gassner anti-representation mentioned before.

We mention two remarks about the above theorem. First we identify homology groups $H_1(D_g) \cong H_1(D_g^0)$ with the subgroup of $H_1(\Sigma_{g,1})$ generated by $b_1, \ldots, b_g$. Second, the homomorphism Φ has ambiguity with respect to framings. However we can see below that the lower right part of $\widetilde{r}(\Phi(L))$ does not depend on the choice of framings, so that the statement of the theorem makes sense.

Proof. All we have to do is to give a suitable admissible presentation of $\pi_1 M_L$ where M_L is the homology cylinder which corresponds to the element $\Phi(L) \in \mathcal{H}_{g,1}$. To use van-Kampen's theorem, we divide M_L into two parts B and C as follows.

We take g points $q_1, \ldots, q_g$ and g paths l_j from the base point p to q_j as in Figure 9.

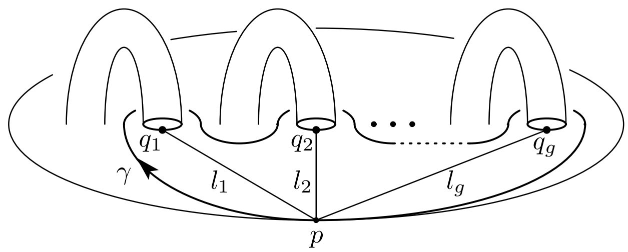

Figure 9

Let B be the union of $\overline{\Sigma_{g,1} \times I \setminus D_g^0 \times I}$ and $2g$ paths $i_+(l_j)$ and $i_-(l_j)$ $(j = 1, \ldots, g)$. We denote by C the complement of an open tubular neighborhood of L in $D_g^0 \times I$ as before. We glue C to B by using some fixed framing. Then

$$\pi_1 B \cong \left\langle \begin{array}{c} i_+(\widetilde{\alpha_1}), \ldots, i_+(\widetilde{\alpha_g}) \\ i_+(\beta_1), \ldots, i_+(\beta_g) \\ i_-(\widetilde{\alpha_1}), \ldots, i_-(\widetilde{\alpha_g}) \\ \delta_1, \ldots, \delta_g, i_+(\gamma) \end{array} \middle| \begin{array}{c} i_+(\widetilde{\alpha_1}) = i_-(\widetilde{\alpha_1})\delta_1 \\ \vdots \\ i_+(\widetilde{\alpha_g}) = i_-(\widetilde{\alpha_g})\delta_g \end{array} \right\rangle$$

where $\widetilde{\alpha_j} = [\alpha_1, \beta_1] \cdots [\alpha_{j-1}, \beta_{j-1}]\alpha_j$, γ is the loop corresponding to a boundary of D_g^0, and δ_j are loops which are composite of paths $i_-(l_j)$, $\overrightarrow{i_-(q_j) i_+(q_j)}$ and $i_+^{-1}(l_j)$. On the other hand,

$$\pi_1 C \cong \langle i_+(\beta_1), \ldots, i_+(\beta_g), z_1, \ldots, z_l, i_-(\beta_1), \ldots, i_-(\beta_g) \mid r_1, \ldots r_{g+l} \rangle$$

is given by the Wirtinger presentation of $D \times I \setminus L$ and

$$\pi_1(B \cap C) \cong \langle i_+(\beta_1), \ldots, i_+(\beta_g), \delta_1, \ldots, \delta_g, i_+(\gamma) \rangle.$$

Using the above decomposition, we obtain

$$\pi_1 M_L \cong \left\langle \begin{array}{c} i_+(\widetilde{\alpha_1}), \ldots, i_+(\widetilde{\alpha_g}) \\ i_+(\beta_1), \ldots, i_+(\beta_g) \\ z_1, \ldots, z_l \\ i_-(\widetilde{\alpha_1}), \ldots, i_-(\widetilde{\alpha_g}) \\ i_-(\beta_1), \ldots, i_-(\beta_g) \end{array} \middle| \begin{array}{c} i_+(\widetilde{\alpha_1}) = i_-(\widetilde{\alpha_1})\widetilde{\delta_1} \\ \vdots \\ i_+(\widetilde{\alpha_g}) = i_-(\widetilde{\alpha_g})\widetilde{\delta_g} \\ r_1, \ldots r_{g+l} \end{array} \right\rangle$$

where

$$\langle i_+(\beta_1), \ldots, i_+(\beta_g), z_1, \ldots, z_l, i_-(\beta_1), \ldots, i_-(\beta_g) \mid r_1, \ldots r_{g+l} \rangle$$

coincides with the Wirtinger presentation of $D \times I \setminus L$ and $\widetilde{\delta_i}$ are words in $i_+(\beta_1), \ldots, i_+(\beta_g), z_1, \ldots, z_l$, $i_-(\beta_1), \ldots, i_-(\beta_g)$ which depends on the framing. After we rewrite the above presentation by using $i_+(\alpha_j)$ and $i_-(\alpha_j)$, we see that the theorem follows from Proposition 4.4 (2). Note that this rewriting does not affect generators $i_\pm(\beta_j)$, z_j and relations r_j. $\square$

Corollary 6.2. *The mapping class group $\mathcal{M}_{g,1}$ is not normal in the group $\mathcal{H}_{g,1}$ of homology cobordism classes of homology cylinders*

Proof. In [KLW], they give an example of string links L_5 and L_6 of 3 strands which have the condition that L_5 is a pure braid, while the conjugate $L_6 L_5 L_6^{-1}$ is not. To show that $L_6 L_5 L_6^{-1}$ is not a pure braid, they use the fact that $g(L_6 L_5 L_6^{-1})$ has the entry which is not a Laurent polynomial of b_1, b_2 and b_3. Then the corollary follows from Theorem 6.1 with respect to this example. □

7. Some examples

Finally, we give some examples of computations. Examples 7.1 and 7.2 show that the Alexander rational function defined in Section 5 is actually non-trivial. Examples 7.3, 7.4 and 7.5 treat the relationship between the Gassner representation for string links and the Magnus representation for homology cylinders observed in Section 6.

Example 7.1. Assume that $g = 1$. Let ζ be the Dehn twist map along the simple closed curve which is parallel to the boundary of $\Sigma_{g,1}$. This map gives a Torelli homology cylinder M_ζ. It is easy to see that

$$\widetilde{r}(M_\zeta) = \begin{pmatrix} a_1 + b_1 - a_1 b_1 & 1 - 2a_1 + a_1^2 \\ -1 + 2b_1 - b_1^2 & 2 - a_1 - b_1 + a_1 b_1 \end{pmatrix}.$$

Then $\Delta_{M_\zeta} = 1 \in \mathbb{Z}H$ which is non-trivial.

Example 7.2. Assume that $g = 2$. Let τ_α, τ_β and τ_γ be Dehn twist maps along the simple closed curves α, β and γ as in Figure 10. We define $\varphi = \tau_\alpha \tau_\beta^{-1}$ and $\psi = \tau_\gamma$. They give Torelli homology cylinders. For simplicity, we also write φ and ψ for corresponding homology cylinders.

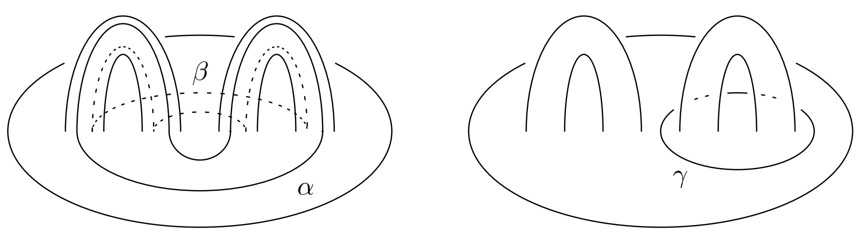

Figure 10

Then

$$\widetilde{r}(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_2 + b_2 - a_2 b_2 & 0 & 1 - 2a_2 + a_2^2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 + 2b_2 - b_2^2 & 0 & 2 - a_2 - b_2 + a_2 b_2 \end{pmatrix} \quad \text{and} \quad \Delta_\psi = 0.$$

$$\widetilde{r}(\varphi) = \begin{pmatrix} a_2^{-1} & -a_2^{-1} + a_1 a_2^{-1} & 0 & 0 \\ a_1^{-1} - a_1^{-1}a_2^{-1} & 1 - a_2^{-1} + a_1^{-1}a_2^{-1} & 0 & 0 \\ \varphi_1 & \varphi_2 & 1 - a_1^{-1} + a_1^{-1}a_2^{-1} & a_1^{-1}a_2 - a_1^{-1} \\ \varphi_3 & \varphi_4 & a_2^{-1} - a_1^{-1}a_2^{-1} & a_1^{-1} \end{pmatrix}$$

$$\begin{aligned} \varphi_1 &= -a_1^{-1}a_2^{-1} + a_1^{-1} - a_1^{-1}b_1 + a_1^{-1}a_2^{-1}b_1 \\ \varphi_2 &= -a_2^{-1} + a_1^{-1} - a_1^{-1}b_2 + a_2^{-1}b_1 \\ \varphi_3 &= -a_1^{-1}a_2^{-1}b_1 + a_1^{-1}a_2^{-1}b_2 \\ \varphi_4 &= -a_2^{-1} + a_1^{-1}a_2^{-1} - a_1^{-1}a_2^{-1}b_2 + a_2^{-1}b_2 \end{aligned}$$

and $\Delta_\varphi = 0$. However we can obtain

$$\widetilde{r}(\varphi\psi) = \widetilde{r}(\psi)\widetilde{r}(\varphi) \quad \text{and} \quad \Delta_{\varphi\psi} = \frac{(a_2-1)^2}{a_1 a_2^2}.$$

Examples 7.3, 7.4 and 7.5 treat string links through the injective homomorphism $\Phi : \mathcal{S}_g \to \mathcal{H}_{g,1}$. To define this homomorphism, we need to give the framing of string links. Now we adopt the convention of black board framings. We use Proposition 4.4 to compute the Magnus matrix. Given a string link L, let $M_L = (M_L, i_+, i_-)$ be the homology cylinder corresponding to $\Phi(L)$. We identify $H_1(M_L)$ and $H_1(\Sigma_{g,1})$ through i_-. We write α_{j+} and β_{j+} for $i_+(\alpha_j)$ and $i_+(\beta_j)$, for short. Similarly α_{j-} and β_{j-} mean $i_-(\alpha_j)$ and $i_-(\beta_j)$.

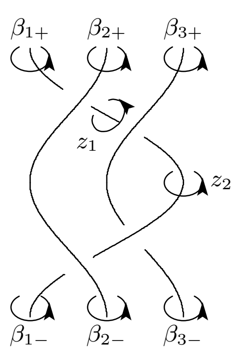

Figure 11

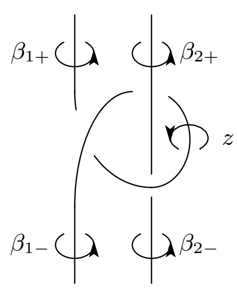

Figure 12

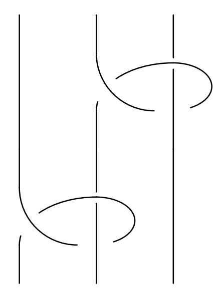

Figure 13

Example 7.3. Let L_5 be a pure braid of 3 strands as depicted in Figure 11. Then the presentation of $\pi_1 M_{L_5}$ is given by

$$\pi_1 M_{L_5} \cong \left\langle \begin{matrix} \alpha_{1+}, \dots, \beta_{3+} \\ z_1, z_2 \\ \alpha_{1-}, \dots, \beta_{3-} \end{matrix} \;\middle|\; \begin{matrix} \alpha_{1+}\beta_{2+}^{-1}\beta_{3+}^{-1}\beta_{2+}\alpha_{1-}^{-1}, \quad [\alpha_{1+}, \beta_{1+}]\alpha_{2+}\beta_{2+}\alpha_{2-}^{-1}[\beta_{1-}, \alpha_{1-}] \\ [\alpha_{1+}, \beta_{1+}][\alpha_{2+}, \beta_{2+}]\alpha_{3+}z_2^{-1}\alpha_{3-}^{-1}[\beta_{2-}, \alpha_{2-}][\beta_{1-}, \alpha_{1-}] \\ \beta_{2+}\beta_{1+}^{-1}\beta_{2+}^{-1}z_1, \quad \beta_{3+}z_1^{-1}\beta_{3+}^{-1}z_2 \\ \beta_{3-}z_2\beta_{3+}^{-1}z_2^{-1}, \quad \beta_{2+}\beta_{1-}\beta_{2+}^{-1}z_2^{-1}, \quad \beta_{2-}\beta_{2+}^{-1} \end{matrix} \right\rangle.$$

From this presentation, we obtain

$$(A \quad B) = \begin{pmatrix} 1 & 0 & 0 & 0 & -a_1 b_2^{-1} b_3 + a_1 b_2^{-1} & -a_1 b_2^{-1} & 0 & 0 \\ 1-b_1 & 1 & 0 & a_1 b_3 - 1 & a_2 b_2^{-1} & 0 & 0 & 0 \\ 1-b_1 & 1-b_2 & 1 & a_1 b_3 - 1 & a_2 b_2^{-1} - 1 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & -b_1^{-1} b_2 & 1 - b_1^{-1} & 0 & b_1^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 - b_1^{-1} & -b_1^{-1} b_3 & b_1^{-1} \\ 0 & 0 & 0 & 0 & 0 & -b_1 & 0 & b_3 - 1 \\ 0 & 0 & 0 & 0 & 1 - b_1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ b_1 - 1 & -1 & 0 & 1 - a_1 & 0 & 0 \\ b_1 - 1 & b_2 - 1 & -1 & 1 - a_1 & 1 - a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence $\widetilde{r}(\Phi(L_5))$

$$= \begin{pmatrix} 1 & 0 & 0 & a_1 b_3 b_1^{-1} - a_1 b_1^{-1} & a_1 b_3 b_1^{-1} b_2^{-1} - a_1 b_1^{-1} b_2^{-1} & a_1 b_1^{-1} b_2^{-1} \\ 0 & 1 & 0 & b_1^{-1} b_3^{-1} - b_1^{-1} & b_2^{-1}((b_1^{-1} - 1)(b_3^{-1} - 1) - a_2) & b_2^{-1} b_3^{1} - b_1^{-1} b_2^{-1} b_3^{1} \\ 0 & 0 & 1 & b_1^{-1} b_2 (b_3^{-1} + a_3 b_1 - 1) & (b_1^{-1} - 1)(b_3^{-1} + a_3 b_1 - 1) & b_3^{-1} - b_1^{-1} b_3^{-1} \\ 0 & 0 & 0 & 1 - b_1^{-1} + b_1^{-1} b_3^{-1} & b_2^{-1}(1 - b_1^{-1})(1 - b_3^{-1}) & b_2^{-1} b_3^{-1} - b_1^{-1} b_2^{-1} b_3^{-1} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b_1^{-1} b_2 (b_3 - 1) & (b_1^{-1} - 1)(b_3 - 1) & b_1^{-1} \end{pmatrix}.$$

Example 7.4. Let L be a string link of 2 strands as depicted in Figure 12. Then the presentation of $\pi_1 M_L$ is given by

$$\pi_1 M_L \cong \left\langle \begin{array}{c} \alpha_{1+}, \alpha_{2+}, \beta_{1+}, \beta_{2+} \\ z \\ \alpha_{1-}, \alpha_{2-}, \beta_{1-}, \beta_{2-} \end{array} \middle| \begin{array}{l} \alpha_{1+} \beta_{1-}^{-1} \beta_{2+} \alpha_{1-}^{-1}, \quad [\alpha_{1+}, \beta_{1+}] \alpha_{2+} z \alpha_{2-}^{-1} [\beta_{1-}, \alpha_{1-}] \\ \beta_{2+} \beta_{1-} \beta_{2+}^{-1} z^{-1}, \quad \beta_{1-} \beta_{1+}^{-1} \beta_{1-}^{-1} z, \quad \beta_{2-} z^{-1} \beta_{2+}^{-1} z \end{array} \right\rangle.$$

From this presentation, we obtain

$$(A \quad B) = \begin{pmatrix} 1 & 0 & 0 & a_1 b_2^{-1} & 0 \\ 1 - b_1 & 1 & a_1 b_1 b_2^{-1} & 0 & a_2 b_1^{-1} \\ 0 & 0 & 0 & 1 - b_1 & -1 \\ 0 & 0 & -1 & 0 & b_1^{-1} \\ 0 & 0 & 0 & -b_1^{-1} & -b_1^{-1} b_2 + b_1^{-1} \end{pmatrix}, \; C = \begin{pmatrix} -1 & 0 & -a_1 b_2^{-1} & 0 \\ b_1 - 1 & -1 & 1 - a_1 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 1 - b_1^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\widetilde{r}(\Phi(L)) = \begin{pmatrix} 1 & 0 & \frac{-a_1b_1b_2^{-1}+a_1b_1-a_1b_2}{b_1b_2-b_1-b_2} & \frac{a_1b_1b_2^{-1}}{b_1b_2-b_1-b_2} \\ 0 & 1 & \frac{b_1+a_2b_2b_1^{-1}+b_2-b_1b_2}{b_1b_2-b_1-b_2} & \frac{a_2-a_2b_1}{b_1b_2-b_1-b_2} \\ 0 & 0 & \frac{b_1b_2-b_1-2b_2+1}{b_1b_2-b_1-b_2} & \frac{b_1-1}{b_1b_2-b_1-b_2} \\ 0 & 0 & \frac{b_2^2-b_2}{b_1b_2-b_1-b_2} & \frac{-b_1}{b_1b_2-b_1-b_2} \end{pmatrix}.$$

Example 7.5. Let L_6 be a string link of 3 strands as depicted in Figure 13. We consider the conjugate $L = L_6L_5L_6^{-1}$. From the somewhat long computation, we obtain the corresponding Magnus matrix which has complicated entries. Hence we write only one entry of this matrix.

$$\widetilde{r}(\Phi(L))_{6,6} = \frac{b_1b_2 - b_1b_2^2 + b_1b_3 - b_2b_3 - 2b_1b_2b_3 + b_1b_2^2b_3 - b_1b_3^2 + b_1b_2b_3^2}{b_1(b_2+b_3-1)(b_2b_3-b_2-b_3)}.$$

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