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Abstract

We study the q-difference analogue of the sixth Painlevé equation (q-P_{VI}) by means of tau functions associated with affine Weyl group of type D_5. We prove that a solution of q-P_{VI} coincides with a self-similar solution of the q-UC hierarchy. As a consequence, we obtain in particular algebraic solutions of q-P_{VI} in terms of the universal character which is a generalization of Schur polynomial attached to a pair of partitions.

Contents

Introduction

The *sixth q-Painlevé equation* (q-P_{VI}) is equivalent to the following system of q-difference equations (see [4]):

$$f\overline{f} = b_7 b_8 \frac{(g+b_5)(g+b_6)}{(g+b_7)(g+b_8)}, \quad g\underline{g} = b_3 b_4 \frac{(f+b_1)(f+b_2)}{(f+b_3)(f+b_4)}. \tag{0.1}$$

Here $f = f(\boldsymbol{a})$ and $g = g(\boldsymbol{a})$ are the unknown functions in variables $\boldsymbol{a} = (a_0, \dots, a_5)$ with $a_0 a_1 {a_2}^2 {a_3}^2 a_4 a_5 = q$; and b_i's are the parameters given by (1.1) below; the symbols $\overline{f}$ and $\underline{g}$ stand for $f(\dots, qa_2, q^{-1}a_3, \dots)$ and $g(\dots, q^{-1}a_2, qa_3, \dots)$, respectively. Notice that a_2/a_3 plays the roll of the independent variable and other a_i's ($i \neq 2, 3$) constant parameters of (0.1). This system satisfies the singularity confinement criterion which is a discrete counterpart of the Painlevé property (see [18]), and actually goes to the sixth Painlevé differential equation through a certain limiting procedure as $q \to 1$.

We have known at least two important aspects of nature of the sixth q-Painlevé equation. First, q-P_{VI} is closely related to the *generalized* Riemann–Hilbert problem (see [1]), as analogous to the case of continuous one; it was shown by Jimbo–Sakai [4] that q-P_{VI} governs the connection preserving deformation of a linear q-difference equation. The second is algebraic geometry of rational surfaces due to Sakai [19]; he presented a class of discrete Painlevé equations defined by the group of Cremona transformations on certain rational surfaces associated with affine root systems; *cf.* [2]. Among them, q-P_{VI} corresponds to the surface with affine Weyl group symmetry of type D_5, that is the same surface as studied by Looijenga [10].

The aim of the present work is to provide yet another formulation of q-P_{VI}, from the viewpoint of infinite-dimensional integrable systems. An extension of the KP hierarchy, called the *UC hierarchy*, was proposed in [23]. This hierarchy is considered as an integrable system characterized by the *universal character* (see [9]) which is a generalization of Schur polynomial attached to a pair of partitions. Also a q-difference analogue of the hierarchy (q-UC hierarchy) was studied in [26]. In this paper we prove, by using *tau functions*, that q-P_{VI} coincides with a certain similarity reduction of the q-UC hierarchy. Consequently, we obtain in particular a class of algebraic solutions of q-P_{VI} in terms of the universal character.

In Sect. 1, we present the geometric formulation of q-P_{VI} by means of tau functions; then we obtain a birational representation of affine Weyl group of type D_5 (see Theorem 1.4). By virtue of this representation, we transform q-P_{VI} equivalently into a system of bilinear equations among tau functions in Sect. 2 (see Theorem 2.3). We sum up, in Sect. 3, some results concerning the universal character and the q-UC hierarchy. Finally we see that the bilinear equations of q-

P_{VI} coincide with a similarity reduction of the q-UC hierarchy, thus obtain an expression of the solution of q-P_{VI} in terms of that of the hierarchy in Sect. 4 (see Theorem 4.1). Since the q-UC hierarchy admits the universal character as its homogeneous solution, we have immediately a class of algebraic solutions of q-P_{VI} in terms of the universal character (see Theorem 4.2). Sect. 5 is devoted to the verification of Theorem 4.1.

Recall that the sixth Painlevé equation can be deduced from q-P_{VI} as a continuous limit and so are all the other Painlevé equations. Hence the above relation between q-P_{VI} and the q-UC hierarchy gives a natural explanation why the universal character appears in the solutions of the Painlevé equations; see [13, 14].

Remark 0.1. We refer to the result [26] where the higher order q-Painlevé equations also turn out to be certain similarity reductions of the q-UC hierarchy; *cf.* [24]. It is still an interesting open question why the universal character solves the Garnier system; see [25].

Note. We use the following conventions throughout this paper.
q-shifted factrials:

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \quad (a;p,q)_n = \prod_{i,j=0}^{n-1}(1-ap^iq^j). \tag{0.2}$$

We use also the notation $(a_1,\dots,a_r;q)_n = (a_1;q)_n\cdots(a_r;q)_n$, and so on.
Jacobi's theta function:

$$\theta(a;q) = \left(a, qa^{-1}; q\right)_\infty . \tag{0.3}$$

Elliptic gamma function:

$$\Gamma(a;p,q) = \frac{\left(pqa^{-1};p,q\right)_\infty}{(a;p,q)_\infty}. \tag{0.4}$$

We have

$$\frac{\Gamma(qa;q,q)}{\Gamma(a;q,q)} = \theta(a;q), \tag{0.5}$$

and

$$\frac{\theta(qa;q)}{\theta(a;q)} = -a^{-1}. \tag{0.6}$$

1 The sixth q-Painlevé equation

In this section we present, by means of *tau functions*, the geometric formulation of q-$P_{\rm VI}$; *cf.* [19]. Let $(f,g)=(f_0/f_1, g_0/g_1)$ denote the inhomogeneous coordinate of $\mathbb{P}^1\times\mathbb{P}^1$. Consider the eight points, p_i $(1\le i\le 8)$, given as follows:

$$\begin{aligned} &p_1=(-b_1,0),\quad p_2=(-b_2,0),\quad p_3=(-b_3,\infty),\quad p_4=(-b_4,\infty),\\ &p_5=(0,-b_5),\quad p_6=(0,-b_6),\quad p_7=(\infty,-b_7),\quad p_8=(\infty,-b_8),\end{aligned}$$

where

$$\begin{aligned} &b_1={a_3}^2{a_4}^{-1}a_5,\quad b_2={a_3}^2{a_4}^3a_5,\quad b_3={a_3}^{-2}{a_4}^{-1}a_5,\quad b_4={a_3}^{-2}{a_4}^{-1}{a_5}^{-3},\\ &b_5={a_0}^{-1}a_1{a_2}^{-2},\quad b_6={a_0}^{-1}{a_1}^{-3}{a_2}^{-2},\quad b_7={a_0}^{-1}a_1{a_2}^2,\quad b_8={a_0}^3a_1{a_2}^2,\end{aligned}\tag{1.1}$$

and $a_i\in\mathbb{C}^\times$ being constant parameters such that $a_0a_1{a_2}^2{a_3}^2a_4a_5=q$. Let $\varepsilon: X=X_{\boldsymbol{a}}\to\mathbb{P}^1\times\mathbb{P}^1$ be the blowing-up at eight points p_i $(1\le i\le 8)$; let $e_i=\varepsilon^{-1}(p_i)$ be the exceptional divisor and let $h_0=\{0\}\times\mathbb{P}^1$, $h_1=\mathbb{P}^1\times\{0\}$. We thus have the Picard lattice of X:

$$\mathrm{Pic}(X)=\mathbb{Z}h_0+\mathbb{Z}h_1+\sum_{1\le i\le 8}\mathbb{Z}e_i,$$

equipped with the intersection form (symmetric bilinear form), $(\ |\ )$, defined by

$$(h_i|h_j)=1-\delta_{i,j},\quad (e_i|e_j)=-\delta_{i,j},\quad (h_i|e_j)=0.$$

First we shall see that the (extended) affine Weyl group $\widetilde{W}(D_5^{(1)})$ acts on $\mathrm{Pic}(X)$ as the group of *Cremona isometries* of rational surface X. Here recall that an automorphism σ of $\mathrm{Pic}(X)$ is said to be a *Cremona isometry* (see [10, 19]) iff σ preserves the intersection form $(\ |\ )$, the canonical divisor $\mathcal{K}_X$, and effectiveness of each effective divisor of $\mathrm{Pic}(X)$.

The anti-canonical divisor $-\mathcal{K}_X$ is uniquely decomposed into prime divisors:

$$-\mathcal{K}_X=2h_0+2h_1-\sum_{1\le i\le 8}e_i=\mathcal{D}_0+\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3,$$

where $\mathcal{D}_0=h_1-e_1-e_2$, $\mathcal{D}_1=h_0-e_5-e_6$, $\mathcal{D}_2=h_1-e_3-e_4$ and $\mathcal{D}_3=h_0-e_7-e_8$. Let $(-\mathcal{K}_X)^\perp=\{v\in\mathrm{Pic}(X)\,|\,(v|\mathcal{D}_i)=0 \text{ for all } i\}$, then we have the

Lemma 1.1 (see [19]**).** $(-\mathcal{K}_X)^\perp\simeq Q(D_5^{(1)})$: *root lattice of type* $D_5^{(1)}$.

We have the canonical root basis $B=\{\alpha_0,\alpha_1,\dots,\alpha_5\}$ given as follows:

$$\begin{aligned} &\alpha_0=e_7-e_8,\quad \alpha_1=e_5-e_6,\quad \alpha_2=h_1-e_5-e_7,\\ &\alpha_3=h_0-e_1-e_3,\quad \alpha_4=e_1-e_2,\quad \alpha_5=e_3-e_4.\end{aligned}\tag{1.2}$$

The intersection matrix multiplied by -1 actually coincides with the Cartan matrix of $D_5^{(1)}$:

$$(C_{ij}) = -((\alpha_i|\alpha_j)) = \begin{pmatrix} 2 & & -1 & & & \\ & 2 & -1 & & & \\ -1 & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & \\ & & & -1 & & 2 \end{pmatrix}.$$

Now define the action of simple reflection s_i on $\mathrm{Pic}(X)$ corresponding to α_i as

$$s_i(v) = v + (v|\alpha_i)\alpha_i \quad \text{for} \quad v \in \mathrm{Pic}(X),$$

and also that of Dynkin diagram automorphism σ_i as

$$\begin{aligned} &\sigma_1 : h_{\{0,1\}} \mapsto h_{\{1,0\}}, \quad e_{\{1,2,3,4,5,6,7,8\}} \mapsto e_{\{5,6,7,8,1,2,3,4\}}, \\ &\sigma_2 : e_{\{5,6,7,8\}} \mapsto e_{\{7,8,5,6\}}. \end{aligned}$$

We have in fact the fundamental relations (see *e.g.* [5]):

$$s_i^2 = 1, \quad s_is_j = s_js_i \quad (\text{if } C_{ij} = 0), \quad s_is_js_i = s_js_is_j \quad (\text{if } C_{ij} = -1);$$

and $\sigma_1 \circ s_{\{0,1,2,3,4,5\}} = s_{\{5,4,3,2,1,0\}} \circ \sigma_1$, $\sigma_2 \circ s_{\{0,1\}} = s_{\{1,0\}} \circ \sigma_2$. One can immediately verify that each action of s_i and σ_i is a Cremona isometry. Denote by $\mathrm{Cr}(X)$ the group of Cremona isometries of X.

Proposition 1.2 (see [10, 19]**).** $\mathrm{Cr}(X) = \langle s_0, \ldots, s_5, \sigma_1, \sigma_2 \rangle \simeq \widetilde{W}(D_5^{(1)})$.

In parallel, we let the action of $\widetilde{W}(D_5^{(1)})$ on the multiplicative root variables $\boldsymbol{a} = (a_0, \ldots, a_5)$ be as follows:

$$\begin{aligned} &s_i(a_j) = a_ja_i^{-C_{ij}}, \\ &\sigma_1(a_{\{0,1,2,3,4,5\}}) = a_{\{5,4,3,2,1,0\}}{}^{-1}, \quad \sigma_2(a_{\{0,1,2,3,4,5\}}) = a_{\{1,0,2,3,4,5\}}{}^{-1}. \end{aligned} \tag{1.3}$$

Secondly we shall realize the action of each element $w \in \mathrm{Cr}(X)$ as an isomorphism between rational surfaces $X_{\boldsymbol{a}}$ and $X_{w(\boldsymbol{a})}$. To this end, we now introduce *tau functions*. Consider the field $\mathcal{L} = \mathbb{K}(\tau_1, \ldots, \tau_8)$ of rational functions in indeterminants τ_i $(1 \leq i \leq 8)$ with the coefficient field $\mathbb{K} = \mathbb{C}(a_0{}^{1/2}, \ldots, a_5{}^{1/2})$. Take a sub-lattice $M = \bigcup_{i=0,1,2,3} M_i$ of $\mathrm{Pic}(X)$, where

$$M_i = \left\{ v \in \mathrm{Pic}(X) \,\middle|\, (v|v) = (v|\mathcal{D}_i) = -1,\ (v|\mathcal{D}_j) = 0\ (j \neq i) \right\}.$$

Definition 1.3 (*cf.* [6]**).** *A function $\tau : M \to \mathcal{L}$ is said to be a tau function iff it satisfies the following conditions*:
(i) $\tau(w.v) = w.\tau(v)$ *for any* $v \in M$ *and* $w \in \mathrm{Cr}(X) \simeq \widetilde{W}(D_5^{(1)})$;
(ii) $\tau(e_i) = \tau_i$ $(1 \le i \le 8)$.

We can determine such functions and the action of $\mathrm{Cr}(X)$ on them as follows. Suppose

$$f_0 = \tau(e_5)\tau(e_6), \quad f_1 = \tau(e_7)\tau(e_8), \quad g_0 = \tau(e_1)\tau(e_2), \quad g_1 = \tau(e_3)\tau(e_4), \tag{1.4}$$

and

$$\begin{aligned}
&{b_1}^{-\frac{1}{2}} f_0 + {b_1}^{\frac{1}{2}} f_1 = \tau(h_0 - e_1)\tau(e_1), \quad {b_2}^{-\frac{1}{2}} f_0 + {b_2}^{\frac{1}{2}} f_1 = \tau(h_0 - e_2)\tau(e_2),\\
&{b_3}^{-\frac{1}{2}} f_0 + {b_3}^{\frac{1}{2}} f_1 = \tau(h_0 - e_3)\tau(e_3), \quad {b_4}^{-\frac{1}{2}} f_0 + {b_4}^{\frac{1}{2}} f_1 = \tau(h_0 - e_4)\tau(e_4),\\
&{b_5}^{-\frac{1}{2}} g_0 + {b_5}^{\frac{1}{2}} g_1 = \tau(h_1 - e_5)\tau(e_5), \quad {b_6}^{-\frac{1}{2}} g_0 + {b_6}^{\frac{1}{2}} g_1 = \tau(h_1 - e_6)\tau(e_6),\\
&{b_7}^{-\frac{1}{2}} g_0 + {b_7}^{\frac{1}{2}} g_1 = \tau(h_1 - e_7)\tau(e_7), \quad {b_8}^{-\frac{1}{2}} g_0 + {b_8}^{\frac{1}{2}} g_1 = \tau(h_1 - e_8)\tau(e_8).
\end{aligned} \tag{1.5}$$

Notice that $s_2(e_5) = h_1 - e_7$ and $s_2(e_7) = h_1 - e_5$, then we obtain the action of s_2 on $\mathcal{L}$. Similarly we obtain the action of s_3 by $s_3(e_1) = h_0 - e_3$ and $s_3(e_3) = h_0 - e_1$. The action of each element s_i $(i \neq 2, 3)$ and σ_j $(j = 1, 2)$ is realized as just a permutation of τ_i's $(1 \le i \le 8)$. Summarizing above, we thus have a realization of $\mathrm{Cr}(X) \simeq \widetilde{W}(D_5^{(1)})$ on $\mathcal{L}$.

Theorem 1.4. *Let*

$$\begin{aligned}
&s_0(\tau_{\{7,8\}}) = \tau_{\{8,7\}}, \quad s_1(\tau_{\{5,6\}}) = \tau_{\{6,5\}}, \quad s_4(\tau_{\{1,2\}}) = \tau_{\{2,1\}}, \quad s_5(\tau_{\{3,4\}}) = \tau_{\{4,3\}},\\
&s_2(\tau_5) = \left({b_7}^{-\frac{1}{2}}\tau_1\tau_2 + {b_7}^{\frac{1}{2}}\tau_3\tau_4\right){\tau_7}^{-1}, \quad s_2(\tau_7) = \left({b_5}^{-\frac{1}{2}}\tau_1\tau_2 + {b_5}^{\frac{1}{2}}\tau_3\tau_4\right){\tau_5}^{-1},\\
&s_3(\tau_1) = \left({b_3}^{-\frac{1}{2}}\tau_5\tau_6 + {b_3}^{\frac{1}{2}}\tau_7\tau_8\right){\tau_3}^{-1}, \quad s_3(\tau_3) = \left({b_1}^{-\frac{1}{2}}\tau_5\tau_6 + {b_1}^{\frac{1}{2}}\tau_7\tau_8\right){\tau_1}^{-1},\\
&\sigma_1(\tau_{\{1,2,3,4,5,6,7,8\}}) = \tau_{\{5,6,7,8,1,2,3,4\}}, \quad \sigma_2(\tau_{\{5,6,7,8\}}) = \tau_{\{7,8,5,6\}},
\end{aligned} \tag{1.6}$$

where b_i being the parameters given by (1.1). *Then* (1.6) *with* (1.3) *gives a representation of* $\widetilde{W}(D_5^{(1)}) = \langle s_0, \dots, s_5, \sigma_1, \sigma_2 \rangle$ *on* $\mathcal{L}$.

Hence we obtain, by virtue of (1.4), also birational actions of $\widetilde{W}(D_5^{(1)})$ on variables $(f, g) = (f_0/f_1, g_0/g_1)$.

Theorem 1.5 (see [19]**).** *Let*

$$\begin{aligned}
&s_2(f) = f\frac{{b_7}^{-\frac{1}{2}} g + {b_7}^{\frac{1}{2}}}{{b_5}^{-\frac{1}{2}} g + {b_5}^{\frac{1}{2}}}, \quad s_3(g) = g\frac{{b_3}^{-\frac{1}{2}} f + {b_3}^{\frac{1}{2}}}{{b_1}^{-\frac{1}{2}} f + {b_1}^{\frac{1}{2}}},\\
&\sigma_1(f) = g, \quad \sigma_1(g) = f, \quad \sigma_2(f) = f^{-1},
\end{aligned} \tag{1.7}$$

under (1.1). *Then* (1.7) *with* (1.3) *realizes the actions of* $\widetilde{W}(D_5^{(1)})$ *on* $\mathbb{C}(a_0, \dots, a_5)(f, g)$. *Moreover, each element* $w \in \widetilde{W}(D_5^{(1)})$ *gives an isomorphism from* $X_{\boldsymbol{a}}$ *to* $X_{w(\boldsymbol{a})}$.

Finally we regard the birational action of a translation in $\widetilde{W}(D_5^{(1)})$ as the sixth q-Painlevé equation (see [19]):

$$\begin{aligned}
q\text{-}P_{\mathrm{VI}} &= (\sigma_3 \circ s_3 \circ s_5 \circ s_4 \circ s_3) \circ (\sigma_2 \circ s_2 \circ s_0 \circ s_1 \circ s_2) : \\
&\quad (a_0, a_1, a_2, a_3, a_4, a_5; f, g) \mapsto \left(a_0, a_1, qa_2, q^{-1}a_3, a_4, a_5; \overline{f}, \overline{g}\right), \\
&\quad f\overline{f} = b_7 b_8 \frac{(g + b_5)(g + b_6)}{(g + b_7)(g + b_8)}, && (1.8a) \\
&\quad g\underline{g} = b_3 b_4 \frac{(f + b_1)(f + b_2)}{(f + b_3)(f + b_4)}. && (1.8b)
\end{aligned}$$

Here we let $\sigma_3 = \sigma_1\sigma_2\sigma_1$ and recall that $a_0 a_1 a_2{}^2 a_3{}^2 a_4 a_5 = q$. This system goes to the sixth Painlevé equation through a certain limiting procedure as $q \to 1$, in fact; see [4].

Remark 1.6. Theorem 1.5 asserts that q-P_{VI} acts on the family of surfaces $\mathcal{X} = \cup_{\boldsymbol{a}} X_{\boldsymbol{a}}$ as an automorphism; we call $\mathcal{X}$ the *defining variety* of q-P_{VI}. Each fiber $X_{\boldsymbol{a}}$ of $\mathcal{X}$ is called the *space of initial conditions* of q-P_{VI}. This is a counterpart of Okamoto's space of initial conditions for the sixth Painlevé equation; *cf.* [17].

2 Bilinear relations among tau functions

This section is devoted to obtain bilinear equations satisfied by tau functions. We will use them later in the following sections to clarify the internal relation between q-P_{VI} and the q-UC hierarchy.

Let r_i $(0 \le i \le 3)$ and π be the elements of $\widetilde{W}(D_5^{(1)}) = \langle s_0, \ldots, s_5, \sigma_1, \sigma_2 \rangle$ defined by

$$r_0 = s_5 s_3 s_5, \quad r_1 = s_1 s_2 s_1, \quad r_2 = s_4 s_3 s_4, \quad r_3 = s_0 s_2 s_0, \quad \pi = \sigma_2 \sigma_1.$$

We can easily verify the relations among them: $r_i{}^2 = 1$, $(r_i r_{i\pm1})^3 = 1$, $(r_i r_j)^2 = 1$ $(j \neq i, i \pm 1)$, and $\pi r_i = r_{i+1}\pi$, where we regard the suffix i of r_i as an element of $\mathbb{Z}/(4\mathbb{Z})$. So that $\langle r_0, \ldots, r_3, \pi \rangle \simeq \widetilde{W}(A_3^{(1)})$. Note that the action of diagram automorphism π of order four is given as

$$\begin{aligned}
\pi : &(a_0, a_1, a_2, a_3, a_4, a_5; f, g, \tau_{\{1,2,3,4,5,6,7,8\}}) \\
&\mapsto \left(a_4, a_5, a_3, a_2, a_1, a_0; \frac{1}{g}, f, \tau_{\{5,6,7,8,3,4,1,2\}}\right).
\end{aligned}$$

Now we compute the relations among the chain of tau functions connected with the action of π:

$$\cdots \to \tau_2 \to \tau_6 \to \tau_4 \to \tau_8 \to \tau_2 \to \cdots \tag{2.1}$$

Let us consider an element

$$\ell = \sigma_2\sigma_3 s_0 s_1 s_4 s_5 s_2 s_3 s_2 \in \widetilde{W}(D_5^{(1)}), \tag{2.2}$$

which acts on the parameters as

$$\ell : (a_0, a_1, a_2, a_3, a_4, a_5) \mapsto \left(\frac{1}{a_1 a_2 a_3}, \frac{1}{a_0 a_2 a_3}, a_0 a_1 {a_2}^2 a_3, a_2 {a_3}^2 a_4 a_5, \frac{1}{a_2 a_3 a_5}, \frac{1}{a_2 a_3 a_4}\right). \tag{2.3}$$

Notice that ℓ commutes with each action of $\widetilde{W}(A_3^{(1)})$ and particularly that of q-P_{VI}; *cf.* [22]. By using the birational actions of $\widetilde{W}(D_5^{(1)})$ on tau functions given in Theorem 1.4, we therefore obtain the bilinear relation between τ_2 and τ_6.

Lemma 2.1. *The following formula holds*:

$$ {a_3}^{-1}\ell(\tau_2)\ell^{-1}(\tau_6) + \left((a_2 a_3)^2 - (a_2 a_3)^{-2}\right)\left(\frac{a_0 a_4}{a_1 a_5}\right)^{\frac{1}{2}} \tau_2\tau_6 - a_2 \ell^{-1}(\tau_2)\ell(\tau_6) = 0. \tag{2.4}$$

Other relations among the chain, (2.1), can be obtained via the action of π. We can verify also that

$$\ell(\tau_{\{1,5,3,7\}}) = \tau_{\{4,8,2,6\}}, \tag{2.5}$$

by straightforward computations.

Let $\xi = a_2 a_3$ and use the notation $\hat{a}_i = \ell(a_i)$ for convenience. Consider the variables $\tilde{\tau}_i = \tau_i/\psi_i$ where

$$\psi_2 = \Gamma(q^2\xi, a_2, \hat{a}_2, a_3, q^{-1}\hat{a}_3; q, q) \times \left(-q^2\xi^2, -q^2{a_2}^2, -q^2{\hat{a}_2}^2, -q^2{a_3}^2, -{\hat{a}_3}^2; q^2, q^2\right)_\infty, \tag{2.6}$$

and other ψ_i's are also determined by applying π to this formula. We have from Lemma 2.1 the

Proposition 2.2. *The following formula holds*:

$$\begin{aligned}&\left(\frac{a_2}{a_3} + \frac{1}{a_2 a_3}\right)\ell(\tilde{\tau}_2)\ell^{-1}(\tilde{\tau}_6) + \left(a_2 a_3 - \frac{1}{a_2 a_3}\right)\left(\frac{a_0 a_4}{a_1 a_5}\right)^{\frac{1}{2}} \tilde{\tau}_2\tilde{\tau}_6 \\ &-\left(a_2 a_3 + \frac{a_2}{a_3}\right)\ell^{-1}(\tilde{\tau}_2)\ell(\tilde{\tau}_6) = 0.\end{aligned} \tag{2.7}$$

Proof. First we notice the formulae

$$\ell(\xi) = q\xi, \quad \ell(a_2) = q\ell^{-1}(a_2), \quad \ell(a_3) = q\ell^{-1}(a_3),$$

and the fact that π and ℓ commute with each other. We have (see (0.5))

$$\begin{aligned}\frac{\ell(\psi_2)}{\psi_2} &= \frac{\Gamma(q^3\xi, \hat{a}_2, qa_2, \hat{a}_3, a_3; q, q)}{\Gamma(q^2\xi, a_2, \hat{a}_2, a_3, q^{-1}\hat{a}_3; q, q)} \frac{\left(-q^4\xi^2, -q^2\hat{a}_2{}^2, -q^4a_2{}^2, -q^2\hat{a}_3{}^2, -q^2a_3{}^2; q^2, q^2\right)_\infty}{(-q^2\xi^2, -q^2a_2{}^2, -q^2\hat{a}_2{}^2, -q^2a_3{}^2, -\hat{a}_3{}^2; q^2, q^2)_\infty} \\ &= \frac{\theta(q^2\xi, a_2, q^{-1}\hat{a}_3; q)}{(-q^2\xi^2, -q^2a_2{}^2, -\hat{a}_3{}^2; q^2)_\infty}.\end{aligned}$$

Applying the action of π to this, we get

$$\frac{\ell(\psi_6)}{\psi_6} = \frac{\theta(q^2\xi, a_3, q^{-1}\hat{a}_2; q)}{(-q^2\xi^2, -q^2a_3{}^2, -\hat{a}_2{}^2; q^2)_\infty}.$$

Accordingly we have (see (0.6))

$$\begin{aligned}\frac{\ell(\psi_2)\ell^{-1}(\psi_6)}{\psi_2\psi_6} &= \frac{\theta(q^2\xi, a_2, q^{-1}\hat{a}_3; q)}{\theta(q\xi, q^{-1}\hat{a}_3, q^{-1}a_2; q)} \frac{\left(-\xi^2, -\hat{a}_3{}^2, -a_2{}^2; q^2\right)_\infty}{(-q^2\xi^2, -q^2a_2{}^2, -\hat{a}_3{}^2; q^2)_\infty} \\ &= (-q\xi)^{-1}(-q^{-1}a_2)^{-1}(1+\xi^2)(1+a_2{}^2) \\ &= (\xi+\xi^{-1})(a_2+a_2{}^{-1}). \end{aligned} \tag{2.8}$$

Applying π, we have also

$$\frac{\ell^{-1}(\psi_2)\ell(\psi_6)}{\psi_2\psi_6} = (\xi+\xi^{-1})(a_3+a_3{}^{-1}). \tag{2.9}$$

Substituting $\tau_i = \psi_i\tilde{\tau}_i$ in (2.4), we arrive at (2.7) via (2.8) and (2.9). □

Let us rename tau functions as follows:

$$\rho_{\{0,1,2,3\}} = \tilde{\tau}_{\{2,6,4,8\}}, \tag{2.10}$$

so that we have (see (2.5))

$$\ell^{-1}(\rho_{\{0,1,2,3\}}) = \tilde{\tau}_{\{3,7,1,5\}}. \tag{2.11}$$

Introduce the parameters $d_i \in \mathbb{C}$ such that

$$\pi^{2j}(A) = q^{d_{2j+1}-d_{2j}}, \quad \pi^{2j-1}(A) = q^{d_{2j-1}-d_{2j}}, \quad A = \left(\frac{a_0a_4}{a_1a_5}\right)^{\frac{1}{2}}. \tag{2.12}$$

Since $\pi^2(A) = 1/A$, we have $d_0 - d_1 + d_2 - d_3 = 0$. Let

$$\xi = a_2a_3 \quad \text{and} \quad \eta = \frac{a_2}{a_3}. \tag{2.13}$$

Note that $\pi(\xi) = \xi$ and $\pi(\eta) = \eta^{-1}$. We then obtain, from Proposition 2.2, the following bilinear equations satisfied by the quartet of tau functions ρ_i $(i = 0, 1, 2, 3)$:

$$\begin{aligned}
&(\eta + \xi^{-1})\ell(\rho_{2j})\ell^{-1}(\rho_{2j+1}) + q^{d_{2j+1}-d_{2j}}(\xi - \xi^{-1})\rho_{2j}\rho_{2j+1} \\
&-(\xi + \eta)\ell^{-1}(\rho_{2j})\ell(\rho_{2j+1}) = 0, && \text{(2.14a)} \\
&(\eta^{-1} + \xi^{-1})\ell(\rho_{2j-1})\ell^{-1}(\rho_{2j}) + q^{d_{2j-1}-d_{2j}}(\xi - \xi^{-1})\rho_{2j-1}\rho_{2j} \\
&-(\xi + \eta^{-1})\ell^{-1}(\rho_{2j-1})\ell(\rho_{2j}) = 0. && \text{(2.14b)}
\end{aligned}$$

Conversely, we can obtain a solution of q-P_{VI} from that of the above system of bilinear equations. In summary, we have the

Theorem 2.3. *Let* $\{\rho_i(\boldsymbol{a})\}_{i=0,1,2,3}$ *be a solution of* (2.14), *then the pair*

$$f(\boldsymbol{a}) = \frac{\ell^{-1}(\rho_3)\rho_1}{\ell^{-1}(\rho_1)\rho_3}, \quad g(\boldsymbol{a}) = \frac{\ell^{-1}(\rho_2)\rho_0}{\ell^{-1}(\rho_0)\rho_2}, \tag{2.15}$$

satisfies q-P_{VI}, (1.8).

Remark 2.4. If $a_0a_1 = a_4a_5$, then ℓ leaves the variable $\eta = a_2/a_3$ invariant; see (2.3). In this case, (2.14) is equivalent to the bilinear equations of the fifth q-Painlevé equation (q-P_{V}); see [7, 12, 26]. We cite the result [22] by Takenawa; he showed that the defining variety of q-P_{V} is actually included in that of q-P_{VI}.

3 Universal character and q-UC hierarchy

In this section we briefly review the notion of the universal character and the q-UC hierarchy; see [9, 23, 26].

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\lambda_i = 0$ for all sufficiently large i. For a pair of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{l'})$, the *universal character* $S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y})$ is a polynomial in $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, y_1, y_2, \ldots)$ defined as follows (see [9, 23]):

$$S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = \det \begin{pmatrix} p_{\mu_{l'-i+1}+i-j}(\boldsymbol{y}), & 1 \leq i \leq l' \\ p_{\lambda_{i-l'}-i+j}(\boldsymbol{x}), & l'+1 \leq i \leq l+l' \end{pmatrix}_{1\leq i,j\leq l+l'}. \tag{3.1}$$

Here p_n are the polynomials defined by the generating function:

$$\sum_{n=0}^{\infty} p_n(\boldsymbol{x})z^n = \exp\left(\sum_{n=1}^{\infty} x_n z^n\right), \tag{3.2}$$

and $p_n(\boldsymbol{x}) = 0$ for $n < 0$; note that it can be explicitly written as

$$p_n(\boldsymbol{x}) = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{{x_1}^{k_1}{x_2}^{k_2}\cdots{x_n}^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{3.3}$$

The Schur polynomial S_λ (see *e.g.* [11]) is regarded as a special case of the universal character:

$$S_\lambda(\boldsymbol{x}) = \det\left(p_{\lambda_i-i+j}(\boldsymbol{x})\right) = S_{[\lambda,\emptyset]}(\boldsymbol{x},\boldsymbol{y}).$$

If we count the degree of each variable x_n and y_n ($n = 1, 2, \ldots$) as

$$\deg x_n = n, \quad \deg y_n = -n,$$

then $S_{[\lambda,\mu]}$ is a (weighted) homogeneous polynomial of degree $|\lambda| - |\mu|$, where we let $|\lambda| = \lambda_1 + \cdots + \lambda_l$. It is known that the universal character $S_{[\lambda,\mu]}$ describes the irreducible character of a rational representation of the general linear group $GL(n;\mathbb{C})$ corresponding to a pair of partitions $[\lambda,\mu]$, while the Schur polynomial S_λ does that of a polynomial representation corresponding to a partition λ; see [9], in detail.

The *UC hierarchy*, introduced in [25], is an extension of the KP hierarchy and is an infinite-dimensional integrable system characterized by the universal character in a similar sense that the KP hierarchy is characterized by the Schur polynomial; see [15, 20, 21]. The *q-UC hierarchy* is a q-difference analogue of the UC hierarchy defined as follows. Consider finite subsets $I \subset \mathbb{Z}_{>0}$ and $J \subset \mathbb{Z}_{<0}$. Let t_i ($i \in I \cup J$) be the independent variable and $T_{i;q}$ be its q-shift operator defined as follows:

$$T_{i;q}(t_i) = \begin{cases} qt_i & (i \in I), \\ q^{-1}t_i & (i \in J), \end{cases} \tag{3.4}$$

and $T_{i;q}(t_j) = t_j$ ($i \neq j$). We use also the notation $T_{i_1i_2\ldots i_n;q} = T_{i_1;q}T_{i_2;q}\cdots T_{i_n;q}$ for the sake of simplicity.

Definition 3.1 (see [26]**).** *The following system of q-difference equations is called the q-UC hierarchy*:

$$\begin{aligned}&(t_i - t_j)T_{ij;q}(\omega_0)T_{k;q}(\omega_1) + (t_j - t_k)T_{jk;q}(\omega_0)T_{i;q}(\omega_1)\\ &+(t_k - t_i)T_{ki;q}(\omega_0)T_{j;q}(\omega_1) = 0,\end{aligned} \tag{3.5}$$

for $i, j, k \in I \cup J$.

Recall that the q-UC hierarchy includes the q-KP hierarchy (see [7]) as a special case.

In what follows we extend, for convenience, the universal character $S_{[\lambda,\mu]}$, (3.1), to be defined for a pair of arbitrary sequences of integers $[\lambda,\mu]$. Note that for any pair of sequences of integers $[\lambda,\mu]$, we have a unique pair of partitions $[\tilde{\lambda},\tilde{\mu}]$ such that $S_{[\lambda,\mu]} = \pm S_{[\tilde{\lambda},\tilde{\mu}]}$. Now let us consider the function $s_{[\lambda,\mu]} = s_{[\lambda,\mu]}(\boldsymbol{t})$ in t_i $(i \in I \cup J)$ defined by

$$s_{[\lambda,\mu]}(\boldsymbol{t}) = S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}), \tag{3.6}$$

with

$$\begin{aligned} x_n &= \frac{\sum_{i\in I} t_i{}^n - q^n \sum_{j\in J} t_j{}^n}{n(1-q^n)}, &(3.7a)\\ y_n &= \frac{\sum_{i\in I} t_i{}^{-n} - q^{-n} \sum_{j\in J} t_j{}^{-n}}{n(1-q^{-n})}. &(3.7b)\end{aligned}$$

Note that we have an expression of $H_n(\boldsymbol{t}) = p_n(\boldsymbol{x})$ by the following generating function:

$$\sum_{n=0}^{\infty} H_n(\boldsymbol{t}) z^n = \prod_{i\in I,\, j\in J} \frac{(qt_j z;q)_\infty}{(t_i z;q)_\infty}. \tag{3.8}$$

The universal characters solve the q-UC hierarchy in the sense of the

Proposition 3.2 (see [26]**).** *For every integer m and pair of sequences of integers* $[\lambda,\mu]$,

$$\omega_0 = s_{[\lambda,\mu]}(\boldsymbol{t}), \quad \omega_1 = s_{[(m,\lambda),\mu]}(\boldsymbol{t}), \tag{3.9}$$

satisfy the q-UC hierarchy (3.5).

4 The solutions of the sixth q-Painlevé equation

In this section we see that a solution of q-P_{VI} coincides with a self-similar solution of the q-UC hierarchy. As a consequence, we obtain in particular a class of algebraic solutions of q-P_{VI} in terms of the universal character.

Let $I = \{1,2\}$, $J = \{-1,-2\}$ and replace the base q with q^2. Consider the following chain of the q-UC hierarchy:

$$\begin{aligned} &(t_1 - t_{-2}) T_{1,-2;q^2}(\omega_{2j}) T_{-1;q^2}(\omega_{2j+1}) + (t_{-2} - t_{-1}) T_{-2,-1;q^2}(\omega_{2j}) T_{1;q^2}(\omega_{2j+1})\\ &+(t_{-1} - t_1) T_{-1,1;q^2}(\omega_{2j}) T_{-2;q^2}(\omega_{2j+1}) = 0, &(4.1a)\\ &\left(\frac{1}{t_1} - \frac{1}{t_{-2}}\right) T_{-1;q^2}(\omega_{2j-1}) T_{1,-2;q^2}(\omega_{2j}) + \left(\frac{1}{t_{-2}} - \frac{1}{t_{-1}}\right) T_{1;q^2}(\omega_{2j-1}) T_{-2,-1;q^2}(\omega_{2j})\\ &+\left(\frac{1}{t_{-1}} - \frac{1}{t_1}\right) T_{-2;q^2}(\omega_{2j-1}) T_{-1,1;q^2}(\omega_{2j}) = 0, &(4.1b)\end{aligned}$$

with the four-periodic condition:

$$\omega_{i+4} = \omega_i. \tag{4.2}$$

Let $\{\omega_i = \omega_i(t_{-2}, t_{-1}, t_1, t_2)\}_{i=0,1,2,3}$ be a solution of (4.1) satisfying the similarity condition:

$$\omega_i(qt_{-2}, qt_{-1}, qt_1, qt_2) = q^{d_i}\omega_i(t_{-2}, t_{-1}, t_1, t_2), \tag{4.3}$$

where $d_i \in \mathbb{C}$ being the parameters given by (2.12). Let

$$\xi = a_2a_3, \quad \eta = \frac{a_2}{a_3}, \quad \zeta = \frac{a_0a_1a_2}{a_3a_4a_5}q. \tag{4.4}$$

Define the functions $F_i(\xi, \eta, \zeta)$ by

$$F_i(\xi, \eta, \zeta) = \omega_i(t_{-2}, t_{-1}, t_1, t_2), \tag{4.5}$$

under the substitution

$$t_1 = \eta, \quad t_2 = \zeta, \quad t_{-1} = -q^{-2}\xi, \quad t_{-2} = -q^{-2}\xi^{-1}. \tag{4.6}$$

We have the following expression of the solution of q-P_{VI} in terms of that of the q-UC hierarchy.

Theorem 4.1. *The quartet*

$$(\rho_0, \rho_1, \rho_2, \rho_3) = \left(F_0(\xi, \eta, \zeta), F_1(\xi, \eta, q^{-2}\zeta), F_2(\xi, \eta, \zeta), F_3(\xi, \eta, q^{-2}\zeta)\right), \tag{4.7}$$

solves the system of bilinear equations of q-P_{VI}, (2.14). *Consequently the pair*

$$f = \frac{\ell^{-1}(\rho_3)\rho_1}{\ell^{-1}(\rho_1)\rho_3}, \quad g = \frac{\ell^{-1}(\rho_2)\rho_0}{\ell^{-1}(\rho_0)\rho_2}, \tag{4.8}$$

satisfies q-P_{VI}, (1.8).

The proof of the theorem is given in the next section.

Finally, we shall present algebraic solutions of q-P_{VI}. Introduce a function $R_{[\lambda,\mu]}(\xi, \eta, \zeta)$, for each pair of partitions $[\lambda, \mu]$, defined by

$$R_{[\lambda,\mu]}(\xi, \eta, \zeta) = s_{[\lambda,\mu]}(\boldsymbol{t}) = S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}), \tag{4.9}$$

under the substitution (4.6), or

$$x_n = \frac{\eta^n + \zeta^n - (-\xi)^n - (-\xi)^{-n}}{n(1 - q^{2n})}, \tag{4.10a}$$

$$y_n = \frac{\eta^{-n} + \zeta^{-n} - (-\xi)^n - (-\xi)^{-n}}{n(1 - q^{-2n})}. \tag{4.10b}$$

Recall that the universal character $S_{[\lambda,\mu]}$ is a homogeneous solution of the q-UC hierarchy whose degree equals $|\lambda| - |\mu|$ (see [26], or Proposition 3.2). Hence we can immediately obtain from Theorem 4.1 a class of algebraic solutions of q-P_{VI} in terms of the universal character; *cf.* [24, 26].

Theorem 4.2. *For every $m, n \in \mathbb{Z}$, the quartet*

$$\begin{aligned}
&(\rho_0, \rho_1, \rho_2, \rho_3) \\
&= \left(R_{[(m-1)!,(n-1)!]}(\xi,\eta,\zeta), R_{[m!,(n-1)!]}(\xi,\eta,q^{-2}\zeta), R_{[m!,n!]}(\xi,\eta,\zeta), R_{[(m-1)!,n!]}(\xi,\eta,q^{-2}\zeta)\right),
\end{aligned} \tag{4.11}$$

satisfies (2.14) *with $d_1 - d_0 = d_2 - d_3 = m$ and $d_2 - d_1 = d_3 - d_0 = -n$. Consequently the pair*

$$f = \frac{\ell^{-1}(\rho_3)\rho_1}{\ell^{-1}(\rho_1)\rho_3}, \quad g = \frac{\ell^{-1}(\rho_2)\rho_0}{\ell^{-1}(\rho_0)\rho_2}, \tag{4.12}$$

gives an algebraic solution of q-P_{VI}, (1.8)*, when*

$$\frac{a_0}{a_1} = q^{m-n} \quad \text{and} \quad \frac{a_4}{a_5} = q^{m+n}. \tag{4.13}$$

Here $n!$ stands for the two-core partition $(n, n-1, \ldots, 1)$; also we let $n! = (-1-n)!$ for $n < 0$.

Remark 4.3. The algebraic solutions obtained in Theorem 4.2 are reduced to those of the sixth Painlevé equation, in parallel with the continuous limit from q-P_{VI} to the equation; *cf.* [13].

Remark 4.4. Let us consider the function $R_{[(n),\emptyset]}(\xi,\eta,\zeta) = R_n(\xi,\eta,\zeta) = p_n(\boldsymbol{x})$ under the substitution (4.10), where $p_n(\boldsymbol{x})$ is the polynomial given by (3.2), or (3.3). We then have the following expression by means of the generating function (see (3.8)):

$$\sum_{n=0}^{\infty} R_n(\xi,\eta,\zeta) z^n = \frac{\left(-\xi z, -\xi^{-1} z; q^2\right)_\infty}{(\eta z, \zeta z; q^2)_\infty}. \tag{4.14}$$

Hence we obtain

$$R_n = \zeta^n \frac{(-\xi^{-1}\zeta^{-1}; q^2)_n}{(q^2;q^2)_n} {}_2\phi_1 \left(\begin{matrix} q^{-2n}, -\xi\eta^{-1} \\ -q^{2-2n}\xi\zeta \end{matrix} \middle| q^2; -q^2\xi\eta \right), \tag{4.15}$$

where ${}_2\phi_1$ denotes the basic hypergeometric series (see *e.g.* [3]):

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| q; x \right) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} x^n.$$

We remark also that $R_n(\xi,\eta,\zeta)$ is equivalent to a kind of q-orthogonal polynomial called the *Al-Salam–Chihara polynomial*; see [8].

Example 4.5. Let us consider the polynomial

$$\begin{aligned}
&P_{[\lambda,\mu]}(\xi,\eta,\zeta) \\
&\quad = \xi^{|\lambda|+|\mu|}\eta^{|\mu|}\zeta^{|\mu|}q^{-2|\nu|} \prod_{(i,j)\in\lambda} \left(1 - q^{2h(i,j)}\right) \prod_{(k,l)\in\mu} \left(q^{2h(k,l)} - 1\right) R_{[\lambda,\mu]}(\xi,\eta,\zeta).
\end{aligned}$$

Here we denote by $h(i,j)$ the *hook-length*, that is, $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ (see [11]); let $\nu = (\nu_1, \nu_2, \ldots)$ be a sequence of integers defined by $\nu_i = \max\{0, \mu'_i - \lambda_i\}$. It is interesting that $P_{[\lambda,\mu]}$ seems to be a polynomial whose coefficients are all positive integers. Some examples of the polynomial are given as follows:

λ	μ	$P_{[\lambda,\mu]}(\xi,\eta,\zeta)$
$\emptyset$	$\emptyset$	1
(1)	$\emptyset$	$1+\xi^2+(\eta+\zeta)\xi$
(2)	$\emptyset$	$q^2(1+\xi^4)+(1+q^2)(\eta+\zeta)(\xi+\xi^3)+((1+q^2)(1+\eta\zeta)+\eta^2+\zeta^2)\xi^2$
$(1,1)$	$\emptyset$	$1+\xi^4+(1+q^2)(\eta+\zeta)(\xi+\xi^3)+((1+q^2)(1+\eta\zeta)+q^2(\eta^2+\zeta^2))\xi^2$
$\emptyset$	(1)	$\eta\zeta(1+\xi^2)+(\eta+\zeta)\xi$
$\emptyset$	(2)	$\eta^2\zeta^2(1+\xi^4)+(1+q^2)\eta\zeta(\eta+\zeta)(\xi+\xi^3)+(q^2(\eta^2+\zeta^2)+(1+q^2)\eta\zeta(1+\eta\zeta))\xi^2$
(1)	(1)	$q^2\eta\zeta(1+\xi^4)+q^2(\eta+\zeta)(1+\eta\zeta)(\xi+\xi^3)+((1+q^2)^2\eta\zeta+q^2(\eta^2+\zeta^2))\xi^2$
(1)	(2)	$q^2\eta^2\zeta^2(1+\xi^6)+q^2\eta\zeta(\eta+\zeta)(1+q^2+\eta\zeta)(\xi+\xi^5)$ $+((1+3q^2+2q^4+q^6)\eta^2\zeta^2+q^2(q^2(\eta^2+\zeta^2)+(1+q^2)\eta\zeta(1+\eta^2+\zeta^2)))(\xi^2+\xi^4)$ $+(\eta+\zeta)((1+q^2)(1+q^2+q^4)\eta\zeta+q^4(\eta^2+\zeta^2)+q^2(1+q^2)\eta^2\zeta^2)\xi^3$

We regard this polynomial as a q-analogue of the *Umemura polynomial* associated with algebraic solutions of the sixth Painlevé equation; see [13, 16].

5 Verification of Theorem 4.1

In order to prove the theorem, we shall see that the system of bilinear equations of q-P_{VI}, (2.14), arises from that of the q-UC hierarchy, (4.1), through a certain similarity reduction.

Proof of Theorem 4.1. We have from (2.3) and (4.4) that

$$\ell^{\pm 1}(\xi) = q^{\pm 1}\xi, \quad \ell^{\pm 1}(\eta) = q^{-1}\zeta, \quad \ell^{\pm 1}(\zeta) = q\eta. \tag{5.1}$$

Notice that function $\omega_i = \omega_i(t_{-2}, t_{-1}, t_1, t_2)$ can be regarded as a function in variables $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, y_1, y_2, \ldots)$ via

$$x_n = \frac{{t_1}^n + {t_2}^n - q^{2n}({t_{-1}}^n + {t_{-2}}^n)}{n(1-q^{2n})}, \tag{5.2a}$$

$$y_n = \frac{{t_1}^{-n} + {t_2}^{-n} - q^{-2n}({t_{-1}}^{-n} + {t_{-2}}^{-n})}{n(1-q^{-2n})}. \tag{5.2b}$$

Let us consider the substitution of variables (see (4.6)):

$$t_1 = \eta, \quad t_2 = \zeta, \quad t_{-1} = -q^{-2}\xi, \quad t_{-2} = -q^{-2}\xi^{-1}.$$

We verify by using (5.1) that

$$\begin{aligned} T_{1,-2;q^2}(x_n) &= \frac{q^{2n}t_1{}^n + t_2{}^n - q^{2n}(t_{-1}{}^n + q^{-2n}t_{-2}{}^n)}{n(1-q^{2n})} \\ &= \frac{q^{2n}\eta^n + \zeta^n - (-\xi)^n - q^{-2n}(-\xi)^{-n}}{n(1-q^{2n})} \\ &= q^{-n}\frac{(q^3\eta)^n + (q\zeta)^n - (-q\xi)^n - (-q\xi)^{-n}}{n(1-q^{2n})} \\ &= q^{-n}\ell\left(\frac{(q^2\zeta)^n + (q^2\eta)^n - (-\xi)^n - (-\xi)^{-n}}{n(1-q^{2n})}\right); \end{aligned}$$

and similarly

$$T_{1,-2;q^2}(y_n) = q^n\ell\left(\frac{(q^2\zeta)^{-n} + (q^2\eta)^{-n} - (-\xi)^n - (-\xi)^{-n}}{n(1-q^{-2n})}\right).$$

Combine this with the similarity condition (4.3), we obtain

$$T_{1,-2;q^2}(\omega_{2j}) = q^{-d_{2j}}\ell(F_{2j}(\xi, q^2\eta, q^2\zeta)).$$

One can verify in the same way that $T_{-1;q^2}(\omega_{2j+1}) = q^{-d_{2j+1}}\ell^{-1}(F_{2j+1}(\xi, q^2\eta, \zeta))$; and also

$$\begin{aligned} &T_{-2,-1;q^2}(\omega_{2j}) = q^{-2d_{2j}}F_{2j}(\xi, q^2\eta, q^2\zeta), \quad T_{1;q^2}(\omega_{2j+1}) = F_{2j+1}(\xi, q^2\eta, \zeta); \\ &T_{-1,1;q^2}(\omega_{2j}) = q^{-d_{2j}}\ell^{-1}(F_{2j}(\xi, q^2\eta, q^2\zeta)), \quad T_{-2;q^2}(\omega_{2j+1}) = q^{-d_{2j+1}}\ell(F_{2j+1}(\xi, q^2\eta, \zeta)). \end{aligned}$$

Substitute (4.6) and the above formulae into the bilinear equation of q-UC hierarchy, (4.1a); replacing η and ζ with $q^{-2}\eta$ and $q^{-2}\zeta$, respectively, hence we obtain

$$\begin{aligned} &\left(\eta + \xi^{-1}\right)\ell(F_{2j}(\xi, \eta, \zeta))\ell^{-1}(F_{2j+1}(\xi, \eta, q^{-2}\zeta)) \\ &+q^{d_{2j+1}-d_{2j}}\left(\xi - \xi^{-1}\right)F_{2j}(\xi, \eta, \zeta)F_{2j+1}(\xi, \eta, q^{-2}\zeta) \\ &-(\xi + \eta)\,\ell^{-1}(F_{2j}(\xi, \eta, \zeta))\ell(F_{2j+1}(\xi, \eta, q^{-2}\zeta)) = 0. \end{aligned} \tag{5.3}$$

In parallel, we have also from (4.1b) that

$$\begin{aligned} &\left(\eta^{-1} + \xi^{-1}\right)\ell(F_{2j-1}(\xi, \eta, q^{-2}\zeta))\ell^{-1}(F_{2j}(\xi, \eta, \zeta)) \\ &+q^{d_{2j-1}-d_{2j}}\left(\xi - \xi^{-1}\right)F_{2j-1}(\xi, \eta, q^{-2}\zeta)F_{2j}(\xi, \eta, \zeta) \\ &-\left(\xi + \eta^{-1}\right)\ell^{-1}(F_{2j-1}(\xi, \eta, q^{-2}\zeta))\ell(F_{2j}(\xi, \eta, \zeta)) = 0. \end{aligned} \tag{5.4}$$

These formulae, (5.3) and (5.4), coincide with the system of bilinear equations of q-P_{VI}, (2.14). The proof is now complete. □

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