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# The distribution of firm size

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**Abstract**

In this paper, we studies the distribution of firm size by using a model based on Sato's paper in 1970, and proved the static distribution of firm size satisfies Pareto distribution in its upper tail.

## 1 Introduction

Studies on empirical size distributions have a long history and attracted many scientists' interest in the past years, since these distributions were frequently used to describe sociological, biological and economic phenomena (e.g.[4]). These empirical size distributions include (1) distributions of incomes by size, (2) distributions of words in prose samples by their frequency of occurrence, (3) distributions of scientists by number of papers published, (4) distributions of cities by population, (5) distributions of biological gene by number of species, and (6) distributions of firms by size.

More than one hundred years ago, Pareto[2] reported that personal income distribution follows a power law with a universal exponent 1.5 approximately. Due to Pareto's contributions to this field, the distributions with the form of $f(i) = \frac{ab^i}{i^k}$ are called Pareto distribution, where $a$, $b$ and $k$ are constants. In 1924, Yule[5] constructed a probability model with $f(i) = cB(i, \rho + 1)$ as its limiting distribution, in order to explain the distributions of biological genera by numbers of species, where $B$ is the Beta function and $c$ is a constant. This distribution is called Yule distribution. Actually, Yule distribution can be approximated in its upper tail by Pareto distribution. In 1955, Simon[4] constructed a stochastic model to describe the distribution of words by their frequency of occurrence and obtained Yule distribution as the stationary solution of the stochastic process. In his model, he supposed that the probability of absolute growth of a variable is proportional to its size, and relative growth or the growth rate is stochastically independent of size. It is called the law of Proportional Effect. In 1970, Sato[3] studied the size distributions which follow the law of nonproportional effect. Sato derived steady-state distributions for a few specific forms of the size-growth relation.

In the present paper, Sato's model is introduced and been used to describe the growth of firms size-growth. The empirical results are proved by using a strict mathematical method.

# 2 Stochastic Model and Main Result

Let $(\Omega, \mathscr{F}, P; \{\mathscr{F}_n\}_{n=1}^{\infty})$ be a filtered probability space. Let $\alpha \in (0,1)$, $a \in (0, \dfrac{1}{1-\alpha})$ and $b = \dfrac{1-a}{\alpha}$. Note that $a + b > 0$. Let $N_n,\ S_{n,i},\ i = 1,\ 2,\ \ldots,\ N_n$, be $\mathscr{F}_n-$measurable random variables, for each $n = 1,\ 2,\ \ldots$, satisfying the following assumptions.

(A-1) $\quad P(N_{n+1} = N_n, S_{n+1,i} = S_{n,i} + 1, S_{n+1,j} = S_{n,j},\ j \neq i | \mathscr{F}_n) = (1-\alpha)\dfrac{aS_{n,i} + b}{\sum_{k=1}^{N_n}(aS_{n,k} + b)}$, $i = 1, 2, \ldots, N_n$.

(A-2) $\quad P(N_{n+1} = N_n + 1, S_{n+1,\ N_{n+1}} = 1, S_{n+1,m} = S_{n,m}, m = 1, 2, \ldots, N_n | \mathscr{F}_n) = \alpha$.

(A-3) $\quad N_1 = 1, S_{1,1} = 1$.

This model can be used to explain a system of developing firms, as described in Figure 1.

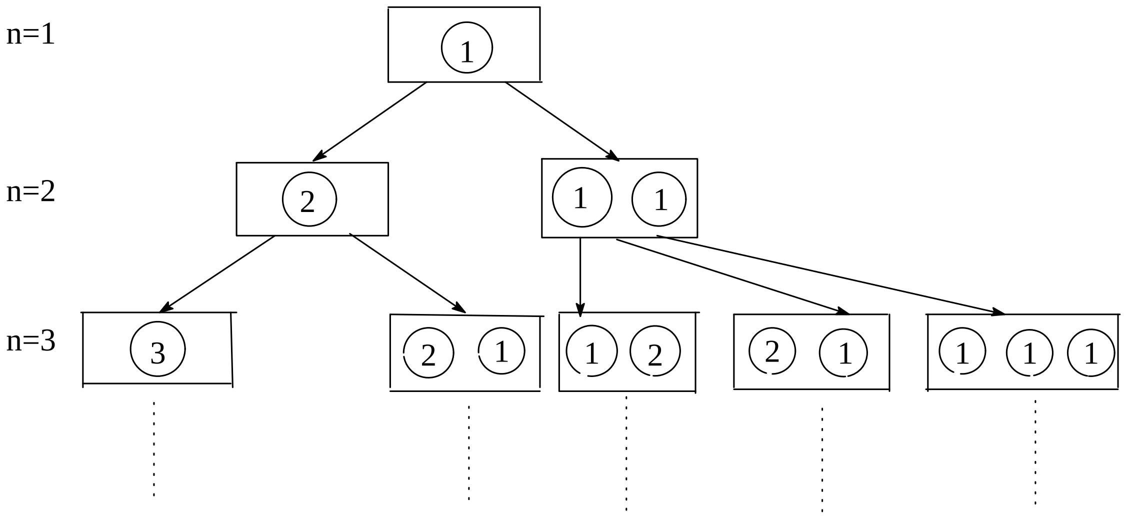

Figure 1. The schematic of size increase in developing firms.

At time 1, there is only one firm with size 1. Let $N_n,\ n = 1,\ 2,\ \ldots,$ denote the total number of firms at time $n$, and $S_{n,i}$ be the size of the $i$-th firm at time $n$, $1 \le i \le N_n$. For all $n$, $N_n$ and $S_{n,i}$ satisfy the above assumptions. Actually the above assumptions indicate the following. When the total size of the firms increases 1, the size of $i$-th firm increase 1 at time $n+1$, with probability $(1-\alpha)(aS_{n,i}+b)/(\sum_{k=1}^{N_n} aS_{n,k}+b)$, and a new firm is created with size of 1 with probability $\alpha$.

Let $f_{n,k}$ be the number of the firms with size of $k$ at time $n$, that is, $f_{n,k}$ is the cardinal number of the set $\{i;\ S_{n,i} = k,\ i = 1, 2, \ldots, N_n\}$. $f_{n,k} = 0$ when $k > n$. Our main result is the following,

**Theorem 1** *Let $c_k$, $k = 1,\ 2,\ \cdots$, be defined by*

$$c_k = \frac{\alpha}{1+(a+b)(1-\alpha)} \frac{B(k+\frac{b}{a}, 1+\gamma)}{B(1+\frac{b}{a}, 1+\gamma)},$$

*where $B(x,y)$ is the Beta function, $\gamma = \frac{1}{a(1-\alpha)}$. Then, $n^{-\frac{1}{2}+\varepsilon}(f_{n,k} - nc_k) \to 0$, almost surely, as $n \to \infty$, for any $\varepsilon > 0$ and $k \ge 1$.*

*In particular $\dfrac{f_{n,k}}{n} \to c_k$, almost surely, as $n \to \infty$, for any $k \ge 1$.*

**Remark 2** (1) *The limit distribution of firm sizes satisfies Yule distribution, i.e.* $c_k(k+\frac{b}{a})^{1+\gamma}$ *converges to a constant as* $k\to\infty$*, here* $\gamma=\dfrac{1}{a(1-\alpha)}$ *is called the Pareto coefficient.*

(2) *When* $a=1,\ b=0$*, Sato's model is simplified into Simon's model. In this case, the probability that the size of i-th firm increases* 1 *at time* $n+1$ *is* $(1-\alpha)S_{n,i}/n$*, when the total size of the firms increases* 1*. Also* $c_k=\dfrac{\alpha}{2-\alpha}B(k,1+\dfrac{1}{1-\alpha})$*, and the Pareto coefficient is* $\dfrac{1}{1-\alpha}$.

# 3 Some Analysis About the Model

Let us make some preparations.

**Lemma 3** *For any* $n\geq 1$ *and* $k\geq 1$*, we have the following.*

(1) $P(f_{n+1,k+1}=f_{n,k+1}+1, f_{n+1,k}=f_{n,k}-1, f_{n+1,j}=f_{n,j}, j\neq k, N_{n+1}=N_n|\mathscr{F}_n)$
$=\dfrac{f_{n,k}(ak+b)(1-\alpha)}{an+bN_n}.$

(2) $P(f_{n+1,1}=f_{n,1}+1, f_{n+1,k}=f_{n,k}, k=2,\ldots n, N_{n+1}=N_n+1|\mathscr{F}_n)=\alpha.$

*Proof*. By Assumption 1, we have

$$\begin{aligned}
&P(f_{n+1,k+1}=f_{n,k+1}+1,\ f_{n+1,k}=f_{n,k}-1,\ f_{n+1,j}=f_{n,j}\ j\neq q,\ N_{n+1}=N_n|\mathscr{F}_n)\\
=&P(\{\exists i,\ S_{n,i}=k,\ S_{n+1,i}=S_{n,i}+1\}|\mathscr{F}_n)=E(f_{n,k}1_{\{S_{n,i}=k,\ S_{n+1,i}=S_{n,i}+1,\ N_{n+1}=N_n\}}|\mathscr{F}_n)\\
=&f_{n,k}P(S_{n,i}=k,\ S_{n+1,i}=S_{n,i}+1,\ N_{n+1}=N_n|\mathscr{F}_n)=\frac{(ak+b)(1-\alpha)}{an+bN_n}f_{n,k}.
\end{aligned}$$

So we have the assertion (1).

The assertion (2) follows from (A-2).

□

**Proposition 4** *Suppose that* $X_n,\ n=1,2,\cdots$*, are random variables which satisfy*

$$E[\max_{1\leq k\leq n}|X_k|^2]\leq Cn^{\delta},\qquad n=1,\ 2,\ \cdots, \tag{1}$$

*for some constants* $C > 0$ *and* $\delta > 0$. *Then* $n^{-\frac{\delta}{2}-\varepsilon}X_n \to 0$ *a.s. for any* $\varepsilon > 0$.

*Proof*. For any $l \in \mathbb{N}$, we have

$$E[\max_{2^l \le k \le 2^{l+1}} \left(\frac{|X_k|}{k^{\frac{\delta}{2}+\varepsilon}}\right)^2] \le 2^{-l(\frac{\delta}{2}+\varepsilon)} E[\max_{2^l \le k \le 2^{l+1}} X_k^2] \le C 2^{\delta - 2l\varepsilon}.$$

So we have

$$E[\sum_{l=1}^{\infty} \max_{2^l \le k \le 2^{l+1}} \left(\frac{|X_k|}{k^{\frac{\delta}{2}+\varepsilon}}\right)^2] \le C \sum_{l=1}^{\infty} 2^{\delta - 2l\varepsilon} < \infty,$$

which implies that $\max_{2^l \le k \le 2^{l+1}} \frac{|X_k|}{k^{\frac{\delta}{2}+\varepsilon}} \to 0$ a.s., as $l \to \infty$ .

This completes the proof.

□

Using this proposition, we get the evaluation about $N_n$, $n = 1,\ 2,\ \cdots$ .

**Lemma 5** $\frac{1}{n^{\frac{1}{2}+\varepsilon}} |N_n - n\alpha| \to 0,\ a.s.\ as\ \ n \to \infty,\ \ for\ \ any\ \ \varepsilon > 0.$

*Proof*. Notice that $N_n$ satisfies

$$P(N_{n+1} = N_n + 1 | \mathscr{F}_n) = \alpha,$$

$$P(N_{n+1} = N_n | \mathscr{F}_n) = 1 - \alpha.$$

Therefore we have,

$$E[\max_{1 \le k \le n} (N_k - k\alpha)^2] \le 4E[(N_n - n\alpha)^2] = 4[n(\alpha - \alpha^2) + 2\alpha^2 - 3\alpha + 1].$$

By Proposition 4, we have our assertion.

□

# 4 Proof of The Main Result

For $n,\ k \geq 1$, let $X_{n,k} = f_{n,k} - nc_k$, $d_{n,k} = X_{n,k} - E[X_{n,k}|\mathscr{F}_{n-1}]$,$C_{n,k} = \begin{cases} \dfrac{(ak+b)(1-\alpha)}{an+bN_n}, & n \geq k, \\ 0, & n < k, \end{cases}$

and $u_k = \dfrac{(ak+b)(1-\alpha)}{a+b\alpha}$. Note that $f_{n,k} = 0$ for $n \leq k-1$, then by the Lemma 3, we have, for any $k$ and $n \geq 1$

$$E[f_{n+1,k}|\mathscr{F}_n] = f_{n,k} + C_{n,k-1} f_{n,k-1} - C_{n,k} f_{n,k}, k \geq 2 \tag{2}$$

$$E[f_{n+1,1}|\mathscr{F}_n] = f_{n,1} + \alpha - C_{n,1} f_{n,1}. \tag{3}$$

Following Equations (2) and (3), we have

$$E[X_{n+1,k}|\mathscr{F}_n] = (1 - C_{n,k})X_{n,k} + C_{n,k-1} f_{n,k-1} - c_k - C_{n,k} nc_k. \tag{4}$$

$$E[X_{n+1,1}|\mathscr{F}_n] = (1 - C_{n,1})X_{n,1} + \alpha - nc_1 C_{n,1} - c_1. \tag{5}$$

For each integer $n \geq 1$ and $k \geq 1$, we denote $\gamma_{n,k} = 1 - C_{n,k}$. Let $\varepsilon_{n,1} = \alpha - nc_1 C_{n,1} - c_1$ and $\varepsilon_{n,k} = C_{n,k-1} f_{n,k-1} - c_k - C_{n,k} nc_k,\ k \geq 2$. Then we have, $X_{n+1,k} = d_{n+1,k} + \gamma_{n,k} X_{n,k} + \varepsilon_{n,k}$.

We see that

$$(\prod_{j=1}^{n-1} \gamma_{j,k})^{-1} X_{n,k} = M_{n,k} + A_{n,k},$$

where $M_{n,k} = \sum_{l=1}^{n-1} (\prod_{j=1}^{l} \gamma_{j,k})^{-1} d_{l+1,k}$ is a martingale in $n$ and $A_{n,k} = \sum_{l=1}^{n-1} (\prod_{j=1}^{l} \gamma_{j,k})^{-1} \varepsilon_{l,k}$.

**Lemma 6** *Let $c = \min\{a, a+b\}$. Then for each $k \geq 1$, there are constants $r_k$ and $s_k$ such that*

(1) $(\prod_{j=1}^{n} \gamma_{j,k})^{-1} \leq r_k \exp(\frac{u_k|b|}{c} \sum_{j=1}^{n} \frac{1}{j} |\alpha - \frac{N_j}{j}|) n^{u_k}, \qquad n \geq 1.$
(2) $(\prod_{j=1}^{n} \gamma_{j,k})^{-1} \geq s_k \exp(-\frac{u_k|b|}{c} \sum_{j=1}^{n} \frac{1}{j} |\alpha - \frac{N_j}{j}|) n^{u_k}, \qquad n \geq 1.$

*Proof*. For each $k \geq 1$, let $i_k = \min\{n \geq 1, \dfrac{(ak+b)(1-\alpha)}{cn} < 1\} \vee k$. Then we have

$$\log((\prod_{j=i_k}^{n} \gamma_{j,k})^{-1}) = -\sum_{j=i_k}^{n} \log(\gamma_{j,k}) = -\sum_{j=i_k}^{n} \log(1-C_{j,k}) = I_1 + I_2, \qquad n > i_k,$$

where $I_1 = \sum_{j=i_k}^{n} C_{j,k}$ and $I_2 = \sum_{j=i_k}^{n} \sum_{l=2}^{\infty} \dfrac{1}{l}(C_{j,k})^l$. Then we have

$$I_1 = \sum_{j=i_k}^{n} \frac{1}{j} u_k + \sum_{j=i_k}^{n} (C_{j,k} - \frac{1}{j} u_k) \leq u_k(1+\log n) + \frac{u_k|b|}{c} \sum_{j=i_k}^{n} \frac{1}{j} |\alpha - \frac{N_j}{j}|,$$

and

$$\begin{aligned}
I_2 &= \sum_{m=2}^{\infty} \frac{1}{m} \sum_{j=i_k}^{n} \frac{1}{j^2} (\frac{(ak+b)(1-\alpha)}{a+b\frac{N_j}{j}})^2 (C_{j,k})^{m-2} \\
&\leq \sum_{m=2}^{\infty} \frac{1}{m} (\frac{(ak+b)(1-\alpha)}{c})^2 \left(\frac{(ak+b)(1-\alpha)}{i_k c}\right)^{m-2} \sum_{j=i_k}^{n} \frac{1}{j^2} \\
&\leq \sum_{m=1}^{\infty} \frac{2i_k^2}{m} \left(\frac{(ak+b)(1-\alpha)}{i_k c}\right)^m \\
&\leq 2i_k^2 \log\left(1 - \frac{(ak+b)(1-\alpha)}{i_k c}\right).
\end{aligned}$$

So we have

$$(\prod_{j=i_k}^{n} \gamma_{j,k})^{-1} \leq \left(1 - \frac{(ak+b)(1-\alpha)}{i_k c}\right)^{2i_k^2} \exp(u_k + \frac{u_k|b|}{c} \sum_{j=i_k}^{n} \frac{1}{j} |\alpha - \frac{N_j}{j}|) n^{u_k}.$$

On the other hand,

$$\log((\prod_{j=i_k}^{n} \gamma_{j,k})^{-1}) \geq I_1 \geq u_k \log n - \sum_{j=1}^{i_k} \frac{1}{j} u_k - \frac{u_k|b|}{c} \sum_{j=i_k}^{n} \frac{1}{j} |\alpha - \frac{N_j}{j}|,$$

and so,

$$(\prod_{j=i_k}^{n} \gamma_{j,k})^{-1} \geq \exp(-\sum_{j=1}^{i_k} \frac{1}{j} u_k - \frac{u_k|b|}{c} \sum_{j=i_k}^{n} \frac{1}{j} |\alpha - \frac{N_j}{j}|) n^{u_k}.$$

Since $\alpha \leq \gamma_{n,k} \leq 1$, then we have our assertion.

$\square$

Next, we evaluate martingale $\{M_{n,k}\}_{n=1}^{\infty}$ and the remain part $\{A_{n,k}\}_{n=1}^{\infty}$. Let $\tau=\tau_t$ be the stopping time defined by $\tau=\inf\{n,|N_n-n\alpha|\geq tn^{\frac{3}{4}}\}$, $t>0$. Then we have $|N_{\tau\wedge n}-(\tau\wedge n)\alpha|\leq t(\tau\wedge n)^{\frac{3}{4}}+1\leq(t+1)(\tau\wedge n)^{\frac{3}{4}}$

**Proposition 7** *For each $k\in\mathbb{N}$ and $t>0$, there exist some constant $\tilde{C}_{t,k}$ such that $E[\max_{1\leq m\leq n}(M_{m\wedge\tau,k})^2]\leq \tilde{C}_{t,k}n^{2u_k+1}$, $n>1$.*

*Proof*. Note that

$$|d_{n,k}|=|X_{n,k}-X_{n-1,k}-E[X_{n,k}-X_{n-1,k}|\mathscr{F}_{n-1}|\leq 2(1+c_k). \tag{6}$$

Therefore we have,

$$\begin{aligned}E[\max_{1\leq m\leq n}M_{m\wedge\tau,k}^2]&\leq 4E[|M_{n\wedge\tau,k}|^2]\leq 8E[\sum_{l=1}^{(n-1)\wedge\tau}(\prod_{j=1}^{l}\gamma_{j,k})^{-2}d_{l+1,k}^2]\\&\leq 32(1+c_k)^2\sum_{l=1}^{n-1}r_k^2\exp(\frac{2u_k|b|}{c}\sum_{j=1}^{l}\frac{t+1}{j^{\frac{5}{4}}})l^{2u_k}.\end{aligned} \tag{7}$$

So letting $\tilde{C}_{t,k}=32(1+c_k)^2r_k^2\exp(\frac{2u_k|b|}{c}\sum_{j=1}^{\infty}\frac{t+1}{j^{\frac{5}{4}}})$, we have the assertion.

$\square$

**Proposition 8** *There exist some constants $\hat{C}_{t,k}$, for each $k\in\mathbb{N}$ and $t>0$,*

*such that $E[|A_{n\wedge\tau,k}|^2]\leq\hat{C}_{t,k}n^{2u_k+1}$, $n>1$.*

*Proof*. We prove this proposition by induction in $k$.

Step 1. We consider the case that $k=1$. Notice that

$$\begin{aligned}|\varepsilon_{n,1}|&=|\alpha-c_1-\frac{(a+b)(1-\alpha)}{a+b\frac{N_n}{n}}c_1|=|\frac{(a+b)(1-\alpha)}{a+b\alpha}c_1-\frac{(a+b)(1-\alpha)}{a+b\frac{N_n}{n}}c_1|\\&\leq\frac{(a+b)(1-\alpha)|b|c_1}{c^2}|\frac{N_n-n\alpha}{n}|.\end{aligned} \tag{8}$$

By the definition of $A_{n,k}$ and Lemma 5 we have

$$
\begin{aligned}
&E[|A_{n\wedge\tau,1}|^2] \le nE\left[\sum_{l=1}^{(n-1)\wedge\tau}(\prod_{j=1}^{l}\gamma_{j,1})^{-2}|\varepsilon_{l,1}|^2\right] \\
&\le n\sum_{l=1}^{n-1}\left\{r_1^2\exp(\frac{2u_1|b|}{c}\sum_{j=1}^{l}\frac{t}{j^{\frac{5}{4}}})l^{2u_1}\frac{(u_1|b|c_1)^2}{c^2}E[\frac{(N_l-l\alpha)^2}{l^2}]\right\} \\
&\le 4r_1^2\exp(\frac{2u_1|b|}{c}\sum_{j=1}^{\infty}\frac{t+1}{j^{\frac{5}{4}}})\frac{(u_1|b|c_1)^2}{c^2}n\sum_{l=1}^{n-1}\{(\alpha-\alpha^2)l^{2u_1-1}+(2\alpha^2-3\alpha+1)l^{2u_1-2}\}.
\end{aligned}
$$

Letting $\hat{C}_{t,1}=8r_1^2\frac{(u_1|b|c_1)^2}{c^2}\exp(\frac{2u_1|b|}{c}\sum_{j=1}^{\infty}\frac{t+1}{j^{\frac{5}{4}}})$, then we have our assertion for $k=1$.

Step 2. Suppose that our assertion is valid for $k$. Because $(\prod_{j=1}^{n-1}\gamma_{j,k})^{-1}X_{n,k}=M_{n,k}+A_{n,k}$, by the assumption for $k$ and Proposition 7, we have $E[\max_{1\le m\le n}(\prod_{j=1}^{(m-1)\wedge\tau}\gamma_{j,k})^{-2}X_{m\wedge\tau,k}^2]\le 2(\tilde{C}_{t,k}+\hat{C}_{t,k})n^{2u_k+1}$, $n\ge 1$. Note that

$$
\begin{aligned}
&|\varepsilon_{n,k+1}| = |C_{n,k}f_{n,k}-c_k+1-C_{n,k+1}nc_{k+1}| \\
&\le |C_{n,k}(f_{n,k}-nc_k)|+|nC_{n,k}-u_k|c_k+|nC_{n,k+1}-u_{k+1}|c_{k+1} \\
&\le \frac{(ak+b)(1-\alpha)}{c}|\frac{X_{n,k}}{n}|+\frac{b}{c}(u_kc_k+u_{k+1}c_{k+1})|\alpha-\frac{N_n}{n}|. \qquad (9)
\end{aligned}
$$

If $n\le\tau$,

$$
\begin{aligned}
&(\prod_{j=1}^{n}\gamma_{j,k+1})^{-1}X_{n,k}=(\gamma_{n,k+1})^{-1}(\prod_{j=1}^{n-1}\gamma_{j,k+1})^{-1}X_{n,k} \\
&=\gamma_{n,k+1}^{-1}\frac{(\prod_{j=1}^{n-1}\gamma_{j,k+1})^{-1}}{(\prod_{j=1}^{n-1}\gamma_{j,k})^{-1}}\max_{1\le m\le n}(\prod_{j=1}^{m-1}\gamma_{j,k})^{-1}X_{m,k} \\
&\le \alpha^{-1}\frac{r_{k+1}}{s_k}\exp(\frac{|b|(u_{k+1}+u_k)}{c}\sum_{j=1}^{n-1}\frac{t+1}{j^{\frac{5}{4}}})(n-1)^{u_{k+1}-u_k}\max_{1\le m\le n}(\prod_{j=1}^{m-1}\gamma_{j,k})^{-1}X_{m,k}.
\end{aligned}
$$

By Lemma 6, we have

$$
\begin{aligned}
&E[(A_{n\wedge\tau,k+1})^2] \le E[n\sum_{h=1}^{(n-1)\wedge\tau}(\prod_{j=1}^{h}\gamma_{j,k+1})^{-2}\varepsilon_{h,k+1}^2] = n\sum_{h=1}^{n-1}E[(\prod_{j=1}^{h}\gamma_{j,k+1})^{-2}\varepsilon_{h,k+1}^2,\ h\le\tau]\\
&\le 2n\frac{(ak+b)^2(1-\alpha)^2}{c^2}\sum_{h=1}^{n-1}E[(\prod_{j=1}^{h}\gamma_{j,k+1})^{-2}\frac{X_{h,k}^2}{h^2},\ h\le\tau]\\
&\quad+2n(\frac{b}{c}(u_kc_k+u_{k+1}c_{k+1}))^2\sum_{h=1}^{n-1}E[(\prod_{j=1}^{h}\gamma_{j,k+1})^{-2}\frac{(N_h-h\alpha)^2}{h^2},h\le\tau]\\
&\le 2n\frac{(ak+b)^2(1-\alpha)^2}{c^2}\sum_{h=1}^{n-1}\alpha^{-2}\frac{r_{k+1}^2}{s_k^2}\exp(\frac{2|b|(u_{k+1}+u_k)}{c}\sum_{j=1}^{n-1}\frac{t+1}{j^{\frac{5}{4}}})h^{2u_{k+1}-2u_k-2}\\
&\quad E[\max_{1\le m\le h}(\prod_{j=1}^{m-1}\gamma_{j,k})^{-1}X_{m,k},h\le\tau]\\
&\quad+4n(\frac{b}{c}(u_kc_k+u_{k+1}c_{k+1}))^2r_{k+1}^2\sum_{h=1}^{n-1}\exp(\frac{2u_{k+1}|b|}{c}\sum_{j=1}^{h}\frac{t+1}{j^{\frac{5}{4}}})h^{2u_{k+1}-1}\\
&\le 2n\frac{(ak+b)^2(1-\alpha)^2}{c^2}\sum_{h=1}^{n-1}2(\tilde{C}_{t,k}+\hat{C}_{t,k})h^{2u_{k+1}-1}\frac{r_{k+1}^2}{s_k^2}\exp(\frac{2|b|(u_{k+1}+u_k)}{c}\sum_{j=1}^{h}\frac{t+1}{j^{\frac{5}{4}}})\\
&\quad+4n(\frac{b}{c}(u_kc_k+u_{k+1}c_{k+1}))^2r_{k+1}^2\exp(\frac{2u_{k+1}|b|}{c}\sum_{j=1}^{\infty}\frac{t+1}{j^{\frac{5}{4}}})h^{2u_{k+1}}.
\end{aligned}
$$

So there exists a constant $\hat{C}_{t,k+1}$ such that $E[(A_{n\wedge\tau,k+1})^2]\le\hat{C}_{t,k+1}n^{2u_{k+1}+1}$. This completes our assertion.

$\square$

Now let us prove Theorem 1. By Propositions 7 and 8 we have $E[\max_{1\le m\le n}(\prod_{j=1}^{m\wedge\tau-1}\gamma_{j,k})^{-2}X_{m\wedge\tau,k}^2]\le 2(\tilde{C}_{t,k}+\hat{C}_{t,k})n^{2u_k+1}$, $n\ge1$. By assertion (2) in *Lemma 6*, we have for every $\omega\in\{\tau=\infty\}$ $(\prod_{j=1}^{m}\gamma_{j,k})^{-2}\ge s_k^2\exp(-\frac{2u_k|b|}{c}\sum_{j=1}^{m}\frac{t}{j^{\frac{5}{4}}})m^{2u_k}$. Let $v=s_k^2\exp(-\frac{2u_k|b|}{c}\sum_{j=1}^{\infty}\frac{t}{j^{\frac{5}{4}}})$. So we see that

$$
\begin{aligned}
&E[\max_{1\le m\le n}vm^{2u_k}1_{\{\tau=\infty\}}(X_{m\wedge\tau,k})^2]\le E[\max_{1\le m\le n}(\prod_{j=1}^{m}\gamma_{j,k})^{-2}X_{m,k}^2,\tau=\infty]\\
&\le 2(\tilde{C}_{t,k}+\hat{C}_{t,k})(1-\frac{(ak+b)(1-\alpha)}{c})^{-2}n^{2u_k+1}.
\end{aligned}
$$

According to the Proposition 4 we get that $\frac{X_{n,k}1_{\{\tau=\infty\}}}{n^{\frac{1}{2}+\varepsilon}}$converges to 0 almost surely. Notice that $P(\tau_t = \infty) \to 1$, as $t \to \infty$. So $\frac{X_{n,k}}{n^{\frac{1}{2}+\varepsilon}}$ converges to 0 almost surely.

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