UTMS 2004–32 October 25, 2004

Discretized ingham type theorems

by

Vilmos KOMORNIK and Paola LORETI

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

DISCRETIZED INGHAM TYPE THEOREMS

VILMOS KOMORNIK AND PAOLA LORETI

Abstract. In [2] we developed, in collaboration with C. Baiocchi, a new approach, based on the systematic use of eigenfunctions of the Laplacian operator, in order to obtain various Ingham type generalizations of Parseval's equality. Recently in [14] Ingham's original theorem was adapted to the case where the integrals are replaced by Riemann sums. The purpose of this paper is to establish discrete versions of various discrete Ingham type theorems by using the approach of [2]. This leads to more precise results and to simpler proofs.

1. Introduction

In a classical paper devoted to the study of Dirichlet series, Ingham [5] proved an estimate for nonharmonic Fourier series, which extended Parseval's equality to cases where the exponents do not form an arithmetical sequence. His theorem was an improvement of earlier results of Paley and Wiener, and was subsequently generalized in many different directions. Ingham's original theorem turned out to be quite useful in control theory, too, where it became an essential tool in proving observability estimates of linear evolutionary systems. An overview of this subject was given by Russell [15]; see also [9].

In connection with some problems of numerical approximation, Negreanu and Zuazua [14] recently proved an Ingham type theorem where the integrals were replaced by Riemann sums. In the present note we propose a different approach, based on another method of Ingham. This easily leads to an improved version of the just mentioned result: first, we provide a weaker assumption on the exponents which appear in the estimates. Secondly, we give an explicit sufficient condition on the length of the interval on which the desired estimates hold. Both our assumption and condition are simple and natural.

Moreover, we also establish discrete versions of several multidimensional and vector-valued Ingham type theorems, obtained in [1], [2], [3], [4], [7] and [12]. As a matter of fact, the constants of our estimates below will allow us to pass to the limit and to recover the former theorems concerning the continuous case. We note that, on the other hand, the discrete versions of these theorems cannot be deduced from the continuous ones.

In the following section we give a discrete version of Ingham's theorem which improves an analogous theorem of Negreanu and Zuazua [14]. In the next one we illustrate the usefulness of this theorem by establishing a discrete observability result for a simple model problem. In the last two sections we generalize our discrete Ingham type theorem in two different directions: to functions of several variables and to vector-valued functions. Both generalizations play an important

Key words and phrases. observability, Fourier series, vibrating strings.

role in proving various observability and exact controllability theorems for systems of partial differential equations: we refer to [9] for various applications of this type.

2. A discrete version of Ingham's theorem

Let $(\omega_k)_{k\in K}$ be a family of real numbers, satisfying for some $\gamma > 0$ the *uniform gap condition*

$$|\omega_k - \omega_n| > \gamma \quad \text{for all} \quad k \neq n. \tag{2.1}$$

We consider functions of the form

$$x(t) = \sum_{k\in K} x_k e^{i\omega_k t} \tag{2.2}$$

with complex coefficients x_k, and we are going to establish estimates of the form

$$\delta \sum_{j=-J}^{J} |x(t' + j\delta)|^2 \asymp \sum_{k\in K} |x_k|^2, \tag{2.3}$$

where δ is a given positive number, J a positive integer and t' a real number.

Remarks.

- In the sequel we often write $A \asymp B$ instead of $c_1 A \leq B \leq c_2 A$ for brevity if we do not need to use explicitly the positive constants c_1, c_2. In all such estimates, the constants c_1, c_2 will be assumed to be independent of the particular choice of the coefficients x_k.
- The expression on the left-hand side is an approximation of the integral
$$\int_{t'-J\delta}^{t'+J\delta} |x(t)|^2 \ dt.$$

Theorem 1. *Assume* (2.1)*. Given $0 < \delta \leq \pi/\gamma$ arbitrarily, fix an integer J such that $J\delta > \pi/\gamma$. Then the estimates* (2.3) *hold true for every $t' \in \mathbb{R}$, for all functions* (2.2) *whose coefficients satisfy the condition*

$$x_k = 0 \quad \textit{whenever} \quad |\omega_k| > \frac{\pi}{\delta} - \frac{\gamma}{2}. \tag{2.4}$$

Moreover, the constants of the estimates (2.3) *depend only on γ and $J\delta$.*

Remarks.

- Since the constants of the estimates (2.3) depend only on γ and $J\delta$, letting $\delta \to 0$ in (2.3) we recover Ingham's theorem [5] :

  Assume (2.1) *and fix a bounded interval I of length $|I| > 2\pi/\gamma$. Then all functions* (2.2) *with square summable complex coefficients satisfy the estimates*
$$\int_I |x(t)|^2 \ dt \asymp \sum_{k\in K} |x_k|^2.$$
  Moreover, the constants of the estimates depend only on γ and $|I|$.
- As a matter of fact, the estimates (2.3) follow from the linear independence of the vectors
$$(e^{i\omega_k j\delta})_{j=-J}^{J},$$
  where k runs over the elements of K for which
$$|\omega_k| < \frac{\pi}{\delta} - \frac{\gamma}{2}.$$

Thanks to assumption (2.1) the number of such indices k is smaller than $2J+1$, the length of these vectors. Furthermore, none of the nonzero differences $(\omega_k-\omega_n)\delta$ is a multiple of 2π because $0<|\omega_k-\omega_n|\delta<2\pi$ if $k\neq n$. The linear independence hence follows because the corresponding Vandermonde determinants are different from zero.

However, this simple argument does not show the dependence of the constants on δ, and so it does not enable us to let $\delta\to 0$.

- Our theorem remains valid with the same proof if we assume instead of (2.4) that the exponents ω_k for which $x_k\neq 0$, belong to some fixed interval of length $\frac{2\pi}{\delta}-\gamma$, because we only use the estimates of the differences $\omega_k-\omega_n$. (Alternatively, we can also use a translation argument for all exponents.) In their theorem, Negreanu and Zuazua used the assumption

$$|\omega_k-\omega_n|\le\frac{2\pi}{\delta}-\delta^{p-1} \tag{2.5}$$

whenever $x_k\neq 0$ and $x_n\neq 0$, with some given number $0\le p<1/2$. It is easy to see that their proof requires $\delta=\Delta t$ be sufficiently close to 0, for otherwise the positivity of their constant $C_1(\Delta t,T,\gamma)$ is not ensured. (Note that the expression of $C_1(\Delta t,T,\gamma)$ contains an implicit constant C, which can only been neglected if Δt is sufficiently small.)
- Another feature of our theorem is that we give an explicit condition $2J\delta>2\pi/\gamma$ on the length of the the observation interval, instead of their corresponding condition $T>2\pi/\gamma+\varepsilon(\Delta t)$ with an implicit function ε whose limit is zero in zero.

Proof. We proceed in three steps.

First step. If G is a function belonging to $H_0^1(-\gamma,\gamma)$, then its Fourier transform is given by the formula

$$g(t)=\int_{-\infty}^{\infty}G(x)e^{-itx}\ dx$$

for all real t. Since $\pi/\delta\ge\gamma$, G vanishes outside the interval

$$I:=\Big(-\frac{\pi}{\delta},\frac{\pi}{\delta}\Big),$$

so that

$$g(j\delta)=\int_I G(x)e^{-ij\delta x}\ dx$$

for all integers j. Since the functions

$$\sqrt{\frac{\delta}{2\pi}}e^{ij\delta x},\quad j\in\mathbb{Z}$$

form an orthonormal basis in $L^2(I)$, applying the Dini–Lipschitz theorem (see, e.g., [17]) we conclude that

$$\delta\sum_{j=-\infty}^{\infty}g(j\delta)e^{ij\delta x}=2\pi G_\delta(x)$$

for all real x where G_δ is differentiable, where G_δ denotes the $2\pi/\delta$-periodic function which is equal to G in the interval I. Observe for later use that

$$G_\delta(x)=0\quad\text{if}\quad\gamma\le|x|\le\frac{2\pi}{\delta}-\gamma.$$

Assuming that G is differentiable in 0, it follows that the functions (2.2) satisfy the equalities

$$\begin{aligned}\delta\sum_{j=-\infty}^{\infty} g(j\delta)|x(j\delta)|^2 &= \delta\sum_{k,n\in K} x_k\overline{x_n}\sum_{j=-\infty}^{\infty} g(j\delta)e^{i(\omega_k-\omega_n)j\delta}\\ &= 2\pi\sum_{k,n\in K} x_k\overline{x_n}G_\delta(\omega_k-\omega_n)\\ &= 2\pi G(0)\sum_{k\in K}|x_k|^2.\end{aligned}$$

In the last two equalities we have also used the fact that

$$|\omega_k-\omega_n|\le\frac{2\pi}{\delta}-\gamma$$

whenever $x_k\ne 0$ and $x_n\ne 0$ by condition (2.4), and that $|\omega_k-\omega_n|\ge\gamma$ by condition (2.1) if $k\ne n$, so that $G_\delta(\omega_k-\omega_n)=0$.

More generally, we also have

$$\delta\sum_{j=-\infty}^{\infty} g(j\delta)|x(t'+j\delta)|^2 = 2\pi G(0)\sum_{k\in K}|x_k|^2 \tag{2.6}$$

for any fixed real number t'. This follows by applying the preceding equality to the function

$$y(t):=x(t'+t)=\sum_{k\in K}x_k e^{i\omega_k t'}e^{i\omega_k t}=:\sum_{k\in K}y_k e^{i\omega_k t}$$

and by observing that $|y_k|=|x_k|$ for every k.

Second step: proof of the direct inequality. Let us introduce the functions $H,G:\mathbb{R}\to\mathbb{R}$ by the formulae

$$H(x):=\begin{cases}\cos\frac{\pi x}{\gamma} & \text{if } |x|<\gamma/2,\\ 0 & \text{otherwise,}\end{cases}\qquad G:=H*H,$$

and let us denote by h and g their Fourier transforms. One can readily verify that

- G satisfies the conditions of the previous step;
- g has a positive minimum β in the interval $[-\pi/\gamma,\pi/\gamma]$.

Furthermore, one can readily verify that G is differentiable everywhere. Using these properties and denoting by J' the integer part of $\pi/(\gamma\delta)$, we deduce from (2.6) the inequality

$$\beta\delta\sum_{j=-J'}^{J'}|x(t'+j\delta)|^2\le 2\pi G(0)\sum_{k\in K}|x_k|^2.$$

Now given any positive integer J, the upper integer part m of $3\gamma\delta J/\pi$ satisfies the inequality $m(2J'+1)\ge 2J+1$ because (note that $J\ge 1$ and $J'\ge 0$)

$$\frac{2J+1}{2J'+1}\le\frac{3J}{J'+1}<\frac{3J}{\pi/(\gamma\delta)}=\frac{3\gamma\delta J}{\pi}.$$

Hence, applying the above estimate for m suitably chosen values of t' and summing these inequalities we obtain that

$$\delta\sum_{j=-J}^{J}|x(t'+j\delta)|^2\le\frac{2\pi G(0)m}{\beta}\sum_{k\in K}|x_k|^2$$

for any fixed t'.

Third step: proof of the inverse inequality. Let H and h be as in the preceding step. Set $R := J\delta$ and now define G and g by

$$G := R^2\ H * H + H' * H' \quad \text{and} \quad g(t) := \int_{-\infty}^{\infty} e^{-itx} G(x)\ dx.$$

It is easy to see that

- G satisfies the conditions of the first step;
- G is differentiable everywhere except in $x = \pm\gamma$;
- $G(0) > 0$;
- g is bounded from above by some constant α, and $g \le 0$ outside $(-R, R)$.

Using these properties we deduce from (2.6) the inequality

$$\sum_{k\in K} |x_k|^2 \le \frac{\alpha\delta}{2\pi G(0)} \sum_{j=-J}^{J} |x(t' + j\delta)|^2. \qquad \square$$

3. A discrete observability theorem

There is an extensive literature on the observability of distributed systems: see, e.g., Lions [10] , [11], Russell [15] and their references. In these works one usually considers problems with continuous observations, i.e., the observation of the solution or some derivatives of the solution at all points of some given time interval. However, sometimes this type of observation may be too expensive or even impossible. In many cases, it seems to be more realistic to make observations at a finite number of points of the given time interval. Of course, we cannot hope in this way to determine completely the unknown initial data, but we may hope to get better and better results as the number of observation points increases. As we will show, these results yield in the limit case the formerly established continuous observation theorems.

As an illustration, let us consider the following simple model system where ℓ is a given positive number:

$$\begin{cases} u_{tt} - u_{xx} + au = 0 & \text{in} \quad \mathbb{R} \times (0, \ell), \\ u(t, 0) = u(t, \ell) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u(0, x) = u_0(x) & \text{for} \quad x \in (0, \ell), \\ u_t(0, x) = u_1(x) & \text{for} \quad x \in (0, \ell). \end{cases} \tag{3.1}$$

For $a = 0$ this is a model of a vibrating string of length ℓ with fixed endpoints. Let us define the energy of the solutions by the usual formula

$$E := \frac{1}{2} \int_0^{\ell} u_t^2 + u_x^2\ dx.$$

We are going to establish the following discrete variant of Proposition 3.8 in [9]:

Proposition 1. *Assume that*

$$-\frac{3\pi^2}{4\ell^2} \le a \le 0. \tag{3.2}$$

Given $0 < \delta \le \ell$ arbitrarily, fix an integer J such that $J\delta > \ell$. Then we have

$$\delta \sum_{j=-J}^{J} |u_x(t' + j\delta, 0)|^2 \asymp E(0)$$

and

$$\delta \sum_{j=-J}^{J} |u_x(t' + j\delta, \ell)|^2 \asymp E(0)$$

for all real numbers t' and for all solutions of (3.1) whose initial data are finite linear combinations of the eigenfunctions $\sin \mu_k x$ for which

$$|k| < \frac{\ell}{\delta} - \frac{1}{2}.$$

Moreover, the constants in the estimates only depend on ℓ and on $J\delta$.

Proof. Setting

$$\mu_k = \frac{k\pi}{\ell} \quad \text{and} \quad \omega_k = \sqrt{\Big(\frac{k\pi}{\ell}\Big)^2 + a}$$

for brevity, it is well-known that the solutions of (3.1) are given by the sum

$$u(t,x) = \sum_{k=1}^{\infty} \big(a_k e^{i\omega_k t} + a_{-k} e^{-i\omega_k t}\big) \sin \mu_k x$$

with suitable complex coefficients a_k, a_{-k} depending on the initial data. (We only have a finite number of nonzero coefficients.)

Using this representation and the relation $|\mu_k|^2 \asymp k^2$, we obtain by an easy computation that

$$\begin{aligned} E(0) &= \frac{1}{2} \int_0^{\ell} \Big| \sum_{k=1}^{\infty} \big(i\omega_k a_k - i\omega_k a_{-k}\big) \sin \mu_k x \Big|^2 \, dx \\ &\quad + \frac{1}{2} \int_0^{\ell} \Big| \sum_{k=1}^{\infty} \big(i\mu_k a_k + i\mu_k a_{-k}\big) \cos \mu_k x \Big|^2 \, dx \\ &= \frac{\ell}{4} \sum_{k=1}^{\infty} \big(|\omega_k|^2 |a_k - a_{-k}|^2 + |\mu_k|^2 |a_k + a_{-k}|^2\big) \\ &\asymp \sum_{k=1}^{\infty} k^2 \big(|a_k|^2 + |a_{-k}|^2\big). \end{aligned}$$

On the other hand, it also follows from the representation of the solutions that

$$u_x(t,0) = \sum_{k=1}^{\infty} a_k \mu_k e^{i\omega_k t} + a_{-k} \mu_k e^{-i\omega_k t}$$

and

$$u_x(t,\ell) = \sum_{k=1}^{\infty} a_k \mu_k e^{i\omega_k t} - a_{-k} \mu_k e^{-i\omega_k t}.$$

By assumption (3.2) the family of the exponents $\{\pm\omega_k\}$ satisfies the condition (2.1) with $\gamma = \pi/\ell$. Since $|\mu_k|^2 \asymp k^2$, applying Theorem 1 we obtain that

$$\delta \sum_{j=-J}^{J} |u_x(t' + j\delta, 0)|^2 \asymp \sum_{k=1}^{\infty} k^2 \big(|a_k|^2 + |a_{-k}|^2\big)$$

and

$$\delta \sum_{j=-J}^{J} |u_x(t' + j\delta, \ell)|^2 \asymp \sum_{k=1}^{\infty} k^2 \big(|a_k|^2 + |a_{-k}|^2\big).$$

Taking into account the above estimate of $E(0)$, the proposition follows by . ☐

4. A multidimensional theorem

The proof of the preceding section can easily be adapted to the case of several variables. Let $(\omega_k)_{k\in K}$ be a family of vectors in $\mathbb{R}^N$, satisfying for some $1 \le p \le \infty$ and $\gamma > 0$ the *uniform gap condition*

$$\|\omega_k - \omega_n\|_p > \gamma \quad \text{for all} \quad k \ne n. \tag{4.1}$$

We consider functions of the form

$$x(t) = \sum_{k\in K} x_k e^{i\omega_k \cdot t}, \quad t \in \mathbb{R}^N \tag{4.2}$$

with complex coefficients x_k (the dot in the exponent denotes the usual scalar product of $\mathbb{R}^N$), and we are going to establish estimates of the form

$$\delta^N \sum_{j\in\mathbb{Z}_J^N} |x(t' + j\delta)|^2 \asymp \sum_{k\in K} |x_k|^2, \tag{4.3}$$

where δ is a given positive number, J a positive number, $t' \in \mathbb{R}^N$, and $\mathbb{Z}_J^N$ denotes the set of N-tuples $j = (j_1, \dots, j_N)$ satisfying $= \|j\|_2 \le J$.

Remark. The expression on the left-hand side is an approximation of the integral

$$\int_{B_{J\delta}} |x(t' + t)|^2 \; dt = \int_{t'+B_{J\delta}} |x(t)|^2 \; dt,$$

where B_R denotes the open ball of center 0 and radius R.

In the following theorem we denote by B_r^p the open ball of radius r with respect to the p-norm of $\mathbb{R}^N$, center ed at the origin, i.e.,

$$B_r^p = \{\omega \in \mathbb{R}^N \;:\; \|\omega\|_p < r\}.$$

We shall establish the following

Theorem 2. *Assume* (4.1) *and let us denote by μ_p the first eigenvalue of $-\Delta$ in the Sobolev space $H_0^1(B_{\gamma/2}^p)$. Given $0 < \delta \le \pi/\gamma$ arbitrarily, fix an integer J such that $J\delta > \sqrt{\mu_p}$. Then the estimates* (4.3) *hold true for all $t' \in \mathbb{R}$ and for all functions* (4.2) *whose coefficients satisfy the conditions*

$$x_k = 0 \quad \textit{whenever} \quad \|\omega_k\|_p > \frac{\pi}{\delta} - \frac{\gamma}{2}. \tag{4.4}$$

Remark. As in the preceding section, the constants of the estimates (4.3) depend only on γ and $J\delta$. This allows us to let $\delta \to 0$; this yields the following theorem, obtained earlier in [1], [2] and [12]:

Theorem 3. *Assume* (4.1) *and let us denote by* μ_p *the first eigenvalue of* $-\Delta$ *in the Sobolev space* $H_0^1(B_{\gamma/2}^p)$. *If* $R > \sqrt{\mu_p}$, *then all sums* (4.2) *with square summable complex coefficients* x_k *satisfy the following estimates :*

$$\int_{B_R} |x(t)|^2 \ dt \asymp \sum_{k\in K} |x_k|^2.$$

Proof. We proceed in three steps:

First step. If $G \in H_0^1(B_\gamma^p)$, then its Fourier transform is given by the formula

$$g(t) := \int_{B_\gamma} G(x)e^{-it\cdot x} \ dx, \quad t \in \mathbb{R}^N.$$

Since $\pi/\delta \ge \gamma$, G vanishes outside the N-dimensional interval

$$I := \Big(-\frac{\pi}{\delta}, \frac{\pi}{\delta}\Big)^N,$$

so that

$$g(j\delta) = \int_I G(x)e^{-ij\delta\cdot x} \ dx$$

for all $j \in \mathbb{Z}^N$. Since the functions

$$\sqrt[2N]{\frac{\delta}{2\pi}}e^{ij\delta x}, \quad j \in \mathbb{Z}^N$$

form an orthonormal basis in $L^2(I)$, we conclude that

$$\delta^N \sum_{j\in\mathbb{Z}^N} g(j\delta)e^{ij\delta x} = (2\pi)^N G_\delta(x)$$

for every $x \in \mathbb{R}^N$ where G_δ is infinitely many times differentiable. Here G_δ denotes the function which coincides with G in I and which is $2\pi/\delta$-periodic in each variable. Furthermore, we consider the pointwise convergence of the square partial sums: see [16].

Assuming that G is infinitely many times differentiable in $x = 0$, similarly to the preceding section it follows, thanks to (4.1) and (4.4), that the functions (4.2) satisfy the equalities

$$\delta^N \sum_{j\in\mathbb{Z}^N} g(j\delta)|x(t'+j\delta)|^2 = (2\pi)^N G(0)\sum_{k\in K}|x_k|^2 \tag{4.5}$$

for any fixed $t' \in \mathbb{R}^N$.

Second step: proof of the direct inequality. Let us denote by H the eigenfunction of $-\Delta$ in $H_0^1(B_{\gamma/2}^p)$, corresponding to the first eigenvalue μ. Multiplying by -1 if necessary, we may assume that $H > 0$ in $B_{\gamma/2}^p$. Extending by zero outside this ball, we obtain a continuous function on $\mathbb{R}^N$, still denoted by H. We obtain similarly to the preceding section that

- $G := H * H$ satisfies the conditions of the previous step;
- its Fourier transform g has a positive lower bound β in the ball $B_{\pi/\gamma}^q$, where q denotes the conjugate exponent of p;
- G is of class C^∞ outside the boundary of B_γ^p.

Denoting by J' the integer part of $\pi/(\gamma\delta)$, using these properties we deduce from (4.5) the inequality

$$\beta\delta^N \sum_{j\in\mathbb{Z}^N_{J'}} |x(t'+j\delta)|^2 \le (2\pi)^N G(0) \sum_{k\in K} |x_k|^2.$$

Now for any given positive integer J the set $\mathbb{Z}^N_J$ may be covered by a finite number, say m translates of $\mathbb{Z}^N_{J'}$. Therefore we deduce from the preceding inequality the more general estimate

$$\delta^N \sum_{j\in\mathbb{Z}^N_J} |x(t'+j\delta)|^2 \le \frac{m(2\pi)^N G(0)}{\beta} \sum_{k\in K} |x_k|^2$$

for any fixed $t'\in\mathbb{R}^N$.

Third step: proof of the inverse inequality. Define H and h as above. Choose a positive integer J satisfying $R := J\delta > \sqrt{\mu_p}$ and set $G := (R^2+\Delta)(H*H)$. A simple adaptation of the arguments of the preceding section shows that

- G satisfies the conditions of the first step;
- G is of class C^∞ outside the boundary of B^p_γ;
- $G(0) > 0$;
- g is bounded from above by some constant α and $g \le 0$ outside the ball B_R.

Therefore, applying (4.5) we obtain that

$$(2\pi)^N G(0) \sum_{k\in K} |x_k|^2 \le \alpha\delta^N \sum_{j\in\mathbb{Z}^N_J} |x(t'+j\delta)|^2;$$

since $G(0) > 0$, the inverse inequality follows. □

5. A theorem for vector-valued functions

In this section we establish a discrete version of a theorem obtained in [7], which played an important role in proving various observability and exact controllability theorems for systems of partial differential equations.

Let $(\omega_k)_{k\in K}$ be a family of real numbers, satisfying for some positive integer M and for some positive number γ the following *weakened gap condition*:

$$\begin{cases} \text{no interval } (\omega_k-\gamma,\omega_k+\gamma) \text{ contains more than} \\ M \text{ members of the family } (\omega_k). \end{cases} \tag{5.1}$$

Let $(E_k)_{k\in K}$ be a family of vectors in a complex Hilbert space $\mathcal{H}$, and denote by $\mathcal{Z}$ the linear hull of these vectors. Let $p(\cdot,\cdot)$ be a given semiscalar product on $\mathcal{Z}$, and denote by $p(\cdot)$ the corresponding seminorm.

Theorem 4. *Assume* (5.1). *Given* $0<\delta\le\pi/\gamma$ *arbitrarily, fix an integer* J *such that* $J\delta > \pi/\gamma$. *There exists a number* $\eta > 0$, *depending only on* γ *and* $J\delta$ *such that if*

$$|p(E_k,E_n)| \le \eta p(E_k)p(E_n) \quad \textit{whenever} \quad |\omega_k-\omega_n| < \gamma \quad \textit{but} \quad k\ne n, \tag{5.2}$$

then the estimates

$$\delta \sum_{j=-J}^{J} p(U(t'+j\delta))^2 \asymp \sum_{k\in K} |U_k|^2 p(E_k)^2 \tag{5.3}$$

hold true for all $t' \in \mathbb{R}$ and for all functions

$$U(t)=\sum_{k\in K} U_k e^{i\omega_k t}E_k,\quad U_k\in\mathbb{C}$$

whose coefficients satisfy the conditions

$$U_k=0\quad \textit{whenever}\quad |\omega_k|>\frac{\pi}{\delta}-\frac{\gamma}{2}.$$

Remark. Letting $\delta\to 0$ we recover the following theorem from [7]:

Theorem 5. *Assume* (5.1). *For every bounded interval I of length $|I|>2\pi/\gamma$, there exists a number $\eta>0$ such that if* (5.2) *is satisfied, then all* finite *sums*

$$U(t)=\sum_{k\in K} U_k e^{i\omega_k t}E_k,\quad U_k\in\mathbb{C}$$

satisfy the estimates

$$\int_I p(U(t))^2\ dt \asymp \sum_{k\in K}|U_k|^2 p(E_k)^2.$$

Proof. We may assume without loss of generality that $p(E_k)=1$ for all k. Indeed, terms with $p(E_k)=0$ do not contribute to either side of (5.3), while the other terms can be normalized. Then we have

$$|p(E_k,E_n)|\le 1\quad \text{for all}\quad k,n$$

and (5.2) takes the form

$$|p(E_k,E_n)|\le\eta\quad\text{whenever}\quad|\omega_k-\omega_n|<\gamma\quad\text{but}\quad k\ne n.$$

Let us choose the same functions H and h as in the proof of Theorem 1. Observe for further use that H is differentiable (it is even of class C^∞) in $(-\gamma/2,\gamma/2)$.

First step: If $G\in H_0^1(B_\gamma^\infty)$ is differentiable in $(-\gamma,\gamma)$, then repeating the first step of the proof of Theorem 1 we now obtain the following identity:

$$\delta\sum_{j=-\infty}^{\infty} g(j\delta)p(U(t'+j\delta))^2=2\pi\sum_{\substack{k,n\in K:\\ |\omega_k-\omega_n|<\gamma}} U_k\overline{U_n}G(\omega_k-\omega_n)p(E_k,E_n).$$

Second step: proof of the direct inequality. Let us introduce the same function G and integer J' as in the proof of the direct inequality in Section 2. Since G clearly attains its maximum in 0, now we have the following estimates:

$$
\begin{aligned}
\beta\delta \sum_{j=-J'}^{J'} p(U(t'+j\delta))^2 &\le \delta \sum_{j=-\infty}^{\infty} g(j\delta)p(U(t'+j\delta))^2 \\
&= 2\pi \sum_{\substack{k,n\in K:\\ |\omega_k-\omega_n|<\gamma}} U_k\overline{U_n}G(\omega_k-\omega_n)p(E_k,E_n) \\
&\le 2\pi G(0) \sum_{\substack{k,n\in K:\\ |\omega_k-\omega_n|<\gamma}} \frac{|U_k|^2+|U_n|^2}{2} \\
&\le 2\pi G(0) M \sum_{k\in K} |U_k|^2.
\end{aligned}
$$

In the last step we used that, thanks to assumption (5.1), no term $|U_k|^2$ appears more than M times in the sum.

Third step: proof of the inverse inequality. Let us define $R := J\delta$ and $G := R^2 H * H + H' * H'$ as in the proof of the inverse inequality in Section 2 and let β be the maximum of G. We have the following estimates:

$$
\begin{aligned}
\alpha\delta \sum_{j=-J}^{J} g(j\delta)p(U(t'+j\delta))^2 &\ge \delta \sum_{j=-\infty}^{\infty} g(j\delta)p(U(t'+j\delta))^2 \\
&= 2\pi \sum_{\substack{k,n\in K:\\ |\omega_k-\omega_n|<\gamma}} U_k\overline{U_n}G(\omega_k-\omega_n)p(E_k,E_n) \\
&= 2\pi G(0)\Big(\sum_{k\in K}|U_k|^2\Big) + 2\pi \sum_{\substack{k,n\in K:\\ |\omega_k-\omega_n|<\gamma,\\ k\ne n}} U_k\overline{U_n}G(\omega_k-\omega_n)p(E_k,E_n).
\end{aligned}
$$

The last sum can be majorized as follows:

$$
\begin{aligned}
\Big|\sum_{\substack{k,n\in K:\\ |\omega_k-\omega_n|<\gamma,\\ k\ne n}} U_k\overline{U_n}G(\omega_k-\omega_n)p(E_k,E_n)\Big| &\le \beta\eta \sum_{\substack{k,n\in K:\\ |\omega_k-\omega_n|<\gamma,\\ k\ne n}} |U_k\overline{U_n}| \\
&\le \beta\eta \sum_{\substack{k,n\in K:\\ |\omega_k-\omega_n|<\gamma,\\ k\ne n}} \frac{|U_k|^2+|U_n|^2}{2} \\
&\le (M-1)\beta\eta\Big(\sum_{k\in K}|U_k|^2\Big).
\end{aligned}
$$

It follows that

$$
\alpha\delta \sum_{j=-J}^{J} g(j\delta)p(U(t'+j\delta))^2 \ge 2\pi\big(G(0)-(M-1)\beta\eta\big)\sum_{k\in K}|U_k|^2.
$$

Choosing a sufficiently small $\eta > 0$ such that $(M-1)\beta\eta < G(0)$, the desired inequality follows. Observe that η depends only on γ and on the product $R = J\delta$. $\square$

References

[1] C. Baiocchi, V. Komornik and P. Loreti, *Théorèmes du type Ingham et application, à la théorie du contrôle*, C. R. Acad. Sci. Paris Sér. I Math. 326 (1998), 453–458.
[2] C. Baiocchi, V. Komornik and P. Loreti, *Ingham type theorems and applications to control theory*, Bol. Un. Mat. Ital. B (8) 2 (1999), no. 1, 33–63.
[3] C. Baiocchi, V. Komornik and P. Loreti, *Généralisation d'un théorème de Beurling et application à la théorie du contrôle*, C. R. Acad. Sci. Paris Sér. I Math. 330 (4) (2000) 281–286.
[4] C. Baiocchi, V. Komornik and P. Loreti, *Ingham-Beurling type theorems with weakened gap conditions*, Acta Math. Hungar. 97 (1-2) (2002), 55–95.
[5] A. E. Ingham, *Some trigonometrical inequalities with applications in the theory of series*, Math. Z. 41 (1936), 367–379.
[6] J.-P. Kahane, *Pseudo-périodicité et séries de Fourier lacunaires*, Ann. Sci. de l'E.N.S. 79 (1962), 93–150.
[7] V. Komornik and P. Loreti, *Ingham type theorems for vector-valued functions and observability of coupled linear systems*, SIAM J. Control Optim. 37 (1998), 461–485.
[8] V. Komornik and P. Loreti, *A further note on a theorem of Ingham and simultaneous observability in critical time*, Inverse Problems 20 (2004) 1649–1661.
[9] V. Komornik and P. Loreti, *Fourier Series in Control Theory*, Springer, New York, to appear.
[10] J.-L. Lions, *Exact controllability, stabilizability, and perturbations for distributed systems*, Siam Rev. 30 (1988), 1–68.
[11] J.-L. Lions, *Contrôlabilité exacte et stabilisation de systèmes distribués I-II*, Masson, Paris, 1988.
[12] P. Loreti, *On some gap theorems*, submitted to the Proceedings of the 11th General Meeting of the European Women in Mathematics (EWM), CIRM (Centre International de Rencontres Mathématiques), November 3–7, 2003.
[13] M. Mehrenberger, *Critical length for a Beurling type theorem*, Bol. Un. Mat. Ital. B (8), to appear.
[14] M. Negreanu and E. Zuazua, *Discrete Ingham inequalities and applications*, C. R. Acad. Sci. Paris Sér. I Math 338 (2004), No. 4, 281–286.
[15] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions*, SIAM Rev. 20 (1978), 639–739.
[16] L. Zhizhiashvili, *Uniform convergence and summability of Fourier series*, Bull. Georgian Acad. Sci. 160 (1999), no. 1, 5–7.
[17] A. Zygmund, *Trigonometric Series I-II*, Cambridge University Press, 1959.

Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7, rue René Descartes, 67084 Strasbourg Cedex, France
E-mail address: komornik@math.u-strasbg.fr

Dipartimento di Metodi e Modelli, Matematici per le Scienze Applicate, Università degli Studi di Roma "La Sapienza", Via A. Scarpa, 16, 00161 Roma, Italy
E-mail address: loreti@dmmm.uniroma1.it

UTMS

2004–21 Takaki Hayashi and Shigeo Kusuoka: *Nonsynchronous covariation measurement for continuous semimartingales.*

2004–22 Oleg Yu. Imanuvilov and Masahiro Yamamoto: *Carleman estimates for the three-dimensional non-stationary Lamé system and the application to an inverse problem.*

2004–23 Wuqing Ning and Masahiro Yamamoto: *An inverse spectral problem for a non-symmetric differential operator: Uniqueness and reconstruction formula.*

2004–24 Li Shumin: *An inverse problem for Maxwell's equations in biisotropic media.*

2004–25 Taro Asuke: *The Godbillon-Vey class of transversely holomorphic foliations.*

2004–26 Vilmos Komornik and Masahiro Yamamoto: *Estimation of point sources and the applications to inverse problems.*

2004–27 Oleg Yu. Imanuvilov and Masahiro Yamamoto: *Stability estimate in a Cauchy problem for a hyperbolic equation with variable coefficients.*

2004–28 Naoki Heya: *The absolute continuity of a measure induced by infinite dimensional stochastic differential equations.*

2004–29 Hidetaka Sakai: *Hypergeometric solution of* $2 \times 2q$*-Schlesinger system.*

2004–30 Yasuharu Nakae: *Taut foliations of torus knot complements.*

2004–31 Teruhisa Tsuda: *Universal characters and q-Painlevé systems.*

2004–32 Vilmos Komornik and Paola Loreti: *Discretized ingham type theorems.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012