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Taut foliations of torus knot complements

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TAUT FOLIATIONS OF TORUS KNOT COMPLEMENTS

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Abstract. We showed that for any torus knot K there is a family of taut foliations of the complement of K which realize all boundary slopes in $(-\infty, 1)$. This theorem is proved by a construction of branched surfaces and laminations carried by these branched surfaces which are used in the Roberts paper [6]. Applying this construction to a fibered knot K', we also showed that there exists a family of taut foliations of the complement of the cable knot K of K' which realize all boundary slopes in $(-\infty, 1)$. And more, we partially extend the theorem of Roberts to a link case.

1. Introduction

In this paper, we discuss taut foliations of the complement of a torus knot. A taut foliation of a 3-manifold is a codimension one foliation such that there is a circle which intersects every leaf transversely. There are a lot of studies on foliations of a 3-manifold, many of these indicate that the structure of foliations reflects well the topology of a manifold. Novikov [3] showed that if a 3-manifold other than $S^2 \times S^1$ possesses a foliation without Reeb components, it has topological properties that its fundamental group is infinite, the second homotopy group is trivial and its leaves are all π_1-injective. Rosenberg [8] showed that if a 3-manifold possesses a foliation without Reeb components, then the manifold is irreducible, where a 3-manifold is irreducible if all embedded 2-spheres bound 3-balls. Combining theorems of Novikov and Rosenberg with that of Palmeira [4], one can see that if a 3-manifold possesses a foliation without Reeb components its universal cover is homeomorphic to $\mathbb{R}^3$. An infinite fundamental group avoids a possibility that a 3-manifold is a lens space, the fact π_2 is trivial and moreover that a 3-manifold is irreducible imply that the universal cover is contractible. Therefore the existence of "Reebless" foliations plays an important role in studies of a 3-manifold. In fact, a Reeb component has no transverse circle which intersects all leaves, and hence a taut foliation has no Reeb component. Thus a taut foliation takes over the fruits of "Reebless" foliations with respect to the topological properties.

Rachel Roberts showed the following theorem.

Theorem 1.1. (**Roberts** [6]) *Let M be an orientable, fibered compact 3-manifold with single boundary component, whose fiber is a surface of negative Euler characteristic with one puncture. Then there is an interval $(-a, b)$ for some $a, b > 0$ such that for any rational number $\rho \in (-a, b)$ there is a taut foliation which realizes a boundary slope ρ.*

The boundary of such manifold M is a torus, and the boundaries of leaves of these taut foliations are parallel simple closed curves on the torus. Since a torus is homeomorphic to the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, a simple closed curve on a torus is regarded as a straight line on the quotient space. Then the boundary slope of a taut foliation means a slope of the simple closed curve which is a boundary of a leaf of the foliation, and it can be regarded as a fraction $\frac{q}{p}$ with two relatively prime integers p and q. If one performs the Dehn filling to the manifold in Theorem 1.1 with the slope ρ belonging to the interval $(-a, b)$, a taut foliation of a closed manifold is obtained, because a solid torus is a product of a disk and a circle, and then each disk is attached to each leaf of the taut foliation along its boundary. Hence one of the advantages of Theorem 1.1 is that one can estimate a range of slopes in which a taut foliation survives after doing the Dehn filling.

The Dehn surgery along a knot embedded in S^3 consists of two operations, drilling out a neighbourhood of a knot from S^3 and doing the Dehn filling. Lickorish [1] showed that all closed 3-manifold is obtained from S^3 by doing the Dehn surgery along some link embedded in S^3. Then how the Dehn surgery along knots and links yields a 3-manifold is one of important subjects in the study of 3-manifolds. A knot or link embedded in a 3-sphere is called fibered if the complementary space of the knot is a fiber bundle. The fiber is a surface with some number of holes, then the complementary space of a fibered knot is suitable for the application of Theorem 1.1. A torus knot embedded in S^3 is a simple closed curve on the boundary of the standardly embedded solid torus in S^3. It is well known that a torus knot is fibered. So we focus on the complement of a torus knot, then we prove the following theorem.

Theorem 1.2. (**Main Theorem**) *For any torus knot K embedded in S^3, there is a family of taut foliations in the complement of K which realize all boundary slopes in $(-\infty, 1)$.*

Theorem 1.2 leads one to the conclusion that all the Dehn surgeries along any torus knot by the slope belonging to the interval $(-\infty, 1)$ yield closed 3-manifolds with a taut foliation. As seen before, these manifolds with a taut foliation have properties that its fundamental group is infinite, its second homotopy group is trivial and its universal cover is homeomorphic to $\mathbb{R}^3$.

Theorem 1.2 is proved in Section 3 in the following way. First we give an explicit construction of the fibration of the complement of any torus knot. This construction is an analogy of the construction of the fibration on the complement of trefoil knot, which is one of the torus knots, written in Rolfsen's book [7]. Next, using this construction of the fibration, we construct a branched surface which carries a family of laminations. Finally we extend these laminations to taut foliations and prove these taut foliations satisfy the condition of the conclusion of Theorem 1.2.

In section 4, by using the construction of the fibration proved in section 3, we obtain the following result.

Corollary 4.6 *Let K be a fibered knot embedded in S^3. For this K, let $\hat{K}$ be a simple closed curve on the boundary of the regular neighbourhood of K, namely $\hat{K}$ is a cable knot of K. Then $\hat{K}$ is fibered, and moreover there is a family of taut foliations in the complement of $\hat{K}$ which realizes all boundary slopes in $(-\infty, 1)$.*

For a torus knot K_0 embedded in S^3, we can obtain a new knot K_1 as a simple closed curve on the boundary of the regular neighbourhood of K_0. By iterating this construction, there is a sequence of knots $\{K_i\}_{i=0,1,\cdots}$, and we call each of it a iterated torus knot. We obtain also in section 4 the following theorem.

Theorem 4.1 *Each iterated torus knot K_i is fibered, and moreover there is a family of taut foliations in the complement of K_i which realizes all boundary slopes in $(-\infty, 1)$.*

By the theorem of Lickorish stated before, in order to consider a topology of a 3-manifold in terms of the Dehn surgery it is needed to consider a link case. Therefore we partially extend the theorem of Roberts to a link case as follows.

Theorem 5.1 *Let M be an orientable, fibered compact 3-manifold with two boundary components, whose fiber is a surface with two punctures and its genus is more* *than two. If the monodromy of the fibration satisfy the condition* (1) *of Lemma 5.5, then there are intervals* $(-a_i, b_i)$ *for some* $a_i, b_i > 0$ *and* $i = 1, 2$ *such that there is a family of taut foliations which realizes all boundary slopes in each intervals, where* i *corresponds to each torus boundary component of* M.

2. Preliminaries

In this section, we review some definitions and explain backgrounds which are necessary to understand the main theorem of this paper. Throughout this paper, all manifolds and knots or links are oriented unless otherwise specified. For a manifold M and a submanifold B of M, $N(B)$ denotes the regular neighborhood of B in M.

Let $\mathcal{F}$ be a codimension one foliation on a 3-manifold M. For every leaf L of $\mathcal{F}$ if L has a closed transverse curve γ i.e. it is transverse to $\mathcal{F}$ and passes through L, we call that $\mathcal{F}$ is a *taut foliation.*

A *branched surface* B is a compact space modelled locally on the object of Figure 1.

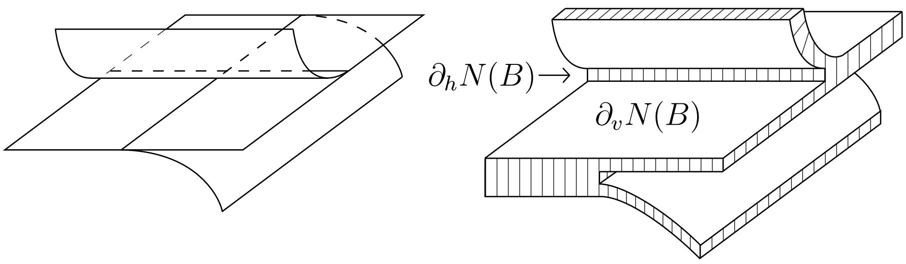

Figure 1

If B lies in a 3-manifold M, we denote a fibered regular neighbourhood of B in M by $N(B)$, locally modelled on Figure 1. When we regard that the branched surface B is embedded in $N(B)$, we consider that $N(B)$ is fibered by I-fibers normal to the branched surface B.

For such a fibered regular neighbourhood $N(B)$, we denote the part of $\partial N(B)$ which lies in the set of end points of the I-fibers of $N(B)$ by $\partial_h N(B)$, and the part of $\partial N(B)$ which contains sub arcs of the I-fibers by $\partial_v N(B)$ as in Figure 1. We call that $\partial_h N(B)$ is a *horizontal boundary*, and $\partial_v N(B)$ is a *vertical boundary.* If M has boundaries and the branched surface embedded in M intersects ∂M transversely, $\partial M \cap B$ is a *train track* τ, a space modelled locally on Figure 2. The train track on ∂M has also fibered regular neighbourhood $N(\tau)$ locally modelled on Figure 2 with I-fiber, and then we denote similarly the part which intersects the endpoints of I-fibers by $\partial_h N(\tau)$ and the part which contains sub arcs of the I-fibers by $\partial_v N(\tau)$.

If we denote the map which collapses all I-fibers by $\pi : N(B) \to B$, a *branch locus* is an arc on B which contains the image of the vertical boundary $\partial_v N(B)$ under the collapsing map π.

The *sectors* $\{S_i\}$ of B are the closures of the components of $B \setminus \{\text{branch locus}\}$. Now we put a weight $\{w_i \geqq 0\}$ on each sector $\{S_i\}$ of B, and we denote the correction of these weights by the vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)$. The *branch equation* is the equation among the sectors which intersect at the branch loci locally modelled in Figure 3. If we assign weights to sectors as in Figure 3, then the branch equations are $d = e + f$, $b = a + d$ and $c = a + e$. If the vector $\mathbf{w}$ satisfies the branch equations for all branches, we call the vector $\mathbf{w}$ an *invariant measure* of B. The branched surface B is called a *measured branched surface* if there is an invariant measure on B. The measures assigned on the sectors induce the measures on the train track τ

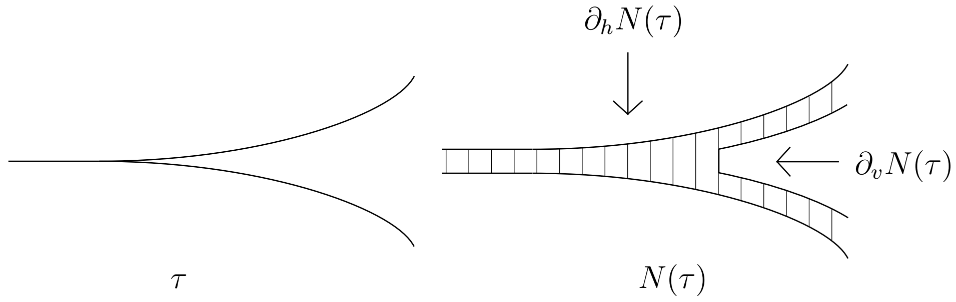

FIGURE 2

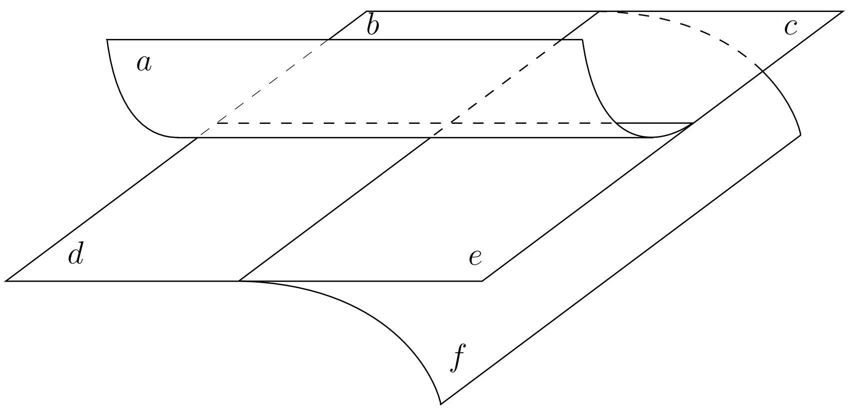

FIGURE 3

on the boundary ∂M. Therefore, if B is measured then the train track τ has also an invariant measure. In this case we call that the train track is a *measured train track*.

For a 3-manifold M we say λ is *a lamination* of M if λ is a foliation on a closed subset of M. We see that the measured branched surface B with positive integer weight carries a compact surface, then if we extend these weights to real numbers there is a non-compact surface on $N(B)$. These non-compact surface is a source of a measured lamination on $N(B)$.

We define that a lamination λ is *carried* by a branched surface B if it can be isotoped into $N(B)$ everywhere transverse to the fiber of the I-bundle, λ is *fully carried* by B if it also intersects every fiber of the I-bundle.

Related to the main theorem of this paper, we introduce the definition of affinely measured branched surface.

Definition 2.1. Let M be a compact 3-manifold and B be a branched surface embedded in M. If there is a family of the simple curves or simple properly embedded arcs $\{\gamma_i\}_{i=1,\cdots,n}$ such that $B \setminus \bigcup_{i=1}^n \gamma_i$ has an invariant measure $\mathbf{w}$, then we call that B is *affinely measured with respect to* $\bigcup_{i=1}^n \gamma_i$.

Let M_h be a surface bundle with monodromy h whose fiber is a once punctured oriented surface F of genus g. In fact we see that

$$M_h = F \times [0,1]/(x,1) \sim (h(x),0).$$

We take a family of properly embedded arcs $\{\alpha_i\}_{i=1,\cdots,n}$ in a fiber F and n copies of the fiber,

$$F_0 = F \times \{0\}\,,\ F_1 = F \times \left\{\frac{1}{n}\right\},\ \cdots,\ F_{n-1} = F \times \left\{\frac{n-1}{n}\right\}.$$

For the family of arcs $\{\alpha_i\}_{i=1,\cdots,n}$, we define the family of disks

$$D_1 = \alpha_1 \times \left[0, \frac{1}{n}\right],\ D_2 = \alpha_2 \times \left[\frac{1}{n}, \frac{2}{n}\right],\ \cdots,\ D_n = \alpha_n \times \left[\frac{n-1}{n}, 1\right].$$

Then we construct a branched surface embedded in M_h by combining these copies of fibers and disks whose branch loci are the arcs $\{\alpha_i\}_{i=1,\cdots,n}$, we denote the branched surface B by

$$B = \langle F_0,\ F_1,\ \cdots,\ F_{n-1}\ ;\ D_1,\ D_2,\ \cdots,\ D_n\rangle.$$

The operation that one removes a solid torus from a 3-manifold and reattach it with some identification map is called *Dehn surgery.* In the main theorem of this paper, we construct a family of taut foliation in torus knot complements. Therefore if we attach a solid torus to it in an appropriate way then we obtain a closed manifold with a taut foliation.

For a knot or link K embedded in a compact, oriented 3-manifold M we denote the *exterior* $M \setminus \overset{\circ}{N}(K)$ by M_K.

The Dehn surgery consists of two operations, *drilling* that one remove the solid torus from M, and *filling* that one reattaches a solid torus by an identification map f. Let T be a torus boundary component of a compact orientable 3-manifold M. For a homeomorphism $f : \partial(D^2 \times S^1) \to T$, the identification space $M(T; f) = (D^2 \times S^1) \bigcup_f M$ is obtained by identifying the points of $\partial(D^2 \times S^1)$ with their images of f, then we say that $M(T; f)$ is a *(Dehn) filling* of M along T. By this construction, $M(T; f)$ depends only on the isotopy class of f, and moreover depends only on the curve $f(m) \subset T$ where $m = \partial D^2 \times \{pt\} \subset \partial(D^2 \times S^1)$ is a simple closed curve. By this fact the isotopy class of a curve on a torus T plays an important role, then we define a *slope* r on a torus T to be the isotopy class of an essential, unoriented, simple closed curve on T. We denote the Dehn filling on M along T with an identification map f such that $f(m)$ represents the slope r by $M(T; r)$. If M has only one boundary component T, we write the abbreviation of $M(T; r)$ by $M(r)$.

There is a distinguished slope defined on any knot. The *meridian* m for a knot K embedded in a manifold M is any essential, simple closed curve on $\partial N(K)$ such that m is homologically trivial in $N(K)$. The slope represented by a meridian m is called *meridional slope* of K, and we denote it by μ_K. For the meridional slope m, if we glue a solid torus $D^2 \times S^1$ to ∂M_K with identification map $f : \partial(D^2 \times S^1) \to T$ such that $f(\partial D^2 \times \{pt\})$ is a representative of the meridional slope m, we say this operation is the *trivial* Dehn surgery since in this case $M_k(\mu_k) \cong M$.

For any knot K in the 3-sphere, there is a *Seifert surface* S of K such that the boundary of S is equivalent to K, more precisely S intersects $N(K)$ in an annulus whose boundary consists of K and an essential, simple closed curve on $\partial N(K)$. We call the latter curve $S \cap \partial N(K)$ *longitude* of K. The longitude is characterized up to isotopy, then we define the *longitudinal slope* of K denoted by λ_K such that λ_K is represented by any longitude of K.

By the fact that two oriented essential simple closed curve on a torus T are isotopic if and only if the 1-cycles which they define are homologous, the set of slopes on T corresponds bijectively to the set of $\pm$ pairs of primitive classes in $H_1(T)$. Then the slope r corresponds to a class α in $H_1(T)$. Since $H_1(T)$ has an ordered basis $\{\alpha, \beta\}$, for this basis the slope r corresponds to the element $p\alpha + q\beta \in H_1(T)$.

Then we obtain a bijection between the space of all slopes on T and $\mathbb{Q} \cup \{\frac{1}{0}\}$ by the identification $p\alpha + q\beta \leftrightarrow \frac{p}{q}$. Let K be a knot embedded in S^3. For a slope r of the boundary of M_K which corresponds to the fraction $\frac{p}{q}$, we denote the surgered manifold $M_K(r)$ also by $M_K(\frac{p}{q})$.

Now we prepare the convention for the paper. For given two oriented simple closed curves α and β properly embedded in a surface F, we denote the homological intersection number by $\langle \alpha, \beta \rangle$ with the sign convention for orientation such that if the positive vector of the first curve overlaps to the next one by rotating clockwise by angle $\frac{\pi}{2}$ then $\langle \alpha, \beta \rangle = 1$ (see Figure 4).

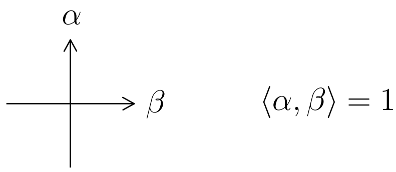

FIGURE 4

For a torus boundary T of a 3-manifold M, we take distinguished two simple closed curves μ and λ on T which satisfy $\langle \mu, \lambda \rangle = 1$. They are called meridian and longitude as defined above when M is an exterior of some knots. The pair μ and λ represents a basis of $H_1(T)$, then we also write this basis by μ and λ. Corresponding to the basis (μ, λ), for any given essential simple closed curve γ in T we define the corresponding fraction of the slope which represented by γ by the formula

$$\text{slope } \gamma = \frac{\langle \gamma, \lambda \rangle}{\langle \mu, \gamma \rangle}.$$

Note that by the above definition the slope of λ corresponds to $\frac{0}{1}$, and the slope of μ corresponds to $\frac{1}{0}$.

Let M be a compact 3-manifold with single boundary component which is homeomorphic to a torus, and let $\mathcal{F}$ be a taut foliation of M whose leaves intersect ∂M in parallel simple closed curves. Then the boundary of every leaf of $\mathcal{F}$ is a union of closed curves. We take an appropriate coordinate (μ, λ) on ∂M, that is, a basis for $H_1(\partial M)$, and suppose that the parallel simple closed curves which are boundary leaves of $\mathcal{F}$ represent a slope r. If we do the Dehn filling along ∂M with slope r, we obtain the closed manifold $M(r)$. Since the solid torus $D^2 \times S^1$ is trivially foliated by the disks $\{D^2 \times \{x\}\}_{x \in S^1}$, the disk $D^2 \times \{x\}$ is attached to the leaf of $\mathcal{F}$ along each boundary, then we simultaneously obtain the foliation $\widehat{\mathcal{F}}$ in the closed manifold $\hat{M}(r)$. By this construction, resultant foliation $\widehat{\mathcal{F}}$ remains taut, thus we obtain a taut foliation of a closed 3-manifold by this procedure.

In this paper we mainly deal with a torus knot embedded in S^3. Then we introduce the definition of torus knot.

Definition 2.2. Let T be a solid torus standardly embedded in S^3. For an appropriate basis (α, β) of $H_1(\partial T)$, we take simple closed curves γ on ∂T which has a representation $r\alpha + s\beta \in H_1(\partial T)$ where r and s are integers. Then we call γ is *torus knot or link of type* (r, s), and denote it $K(r, s)$.

Note that if r and s are relatively prime, then γ is one simple closed curve, thus $K(r, s)$ is a knot embedded in S^3. Otherwise $K(r, s)$ is a link embedded in S^3, whose number of components is equal to the greatest common divisor between r and s.

For a knot K embedded in S^3, K is called *fibered knot* if the exterior of K is a surface bundle over a circle. A torus knot has following property.

Lemma 2.3. *$K(r,s)$ is a fibered knot.*

We shall prove this Lemma by constructing a fiber bundle directly in the exterior of the torus knot $K(r,s)$ in Section 3, but usually it is a well known fact by the theory of singularity of complex functions (see Milnor's book [2]).

3. Main theorem

Theorem 3.1. (Main Theorem)
Let $K(r,s)$ be the torus knot of type (r,s), where (r,s) is a pair of relatively prime integers. Then there is a family of taut foliations $\{\mathcal{F}_x\}$ of the exterior of $K(r,s)$ which realizes any boundary slope in the open interval $(-\infty,1)$.

This theorem is proved as follows. All types of torus knots are fiber knots (see Lemma 2.3). First we construct explicitly a fiber bundle structure of the exterior of the (r,s)-type torus knot $K(r,s)$. Next we choose an arc properly embedded in a fiber surface and then, by the explicit construction of the fibration, we can see the image of this arc under the action of the monodromy of this fibration. Finally, we shall prove that this properly embedded arc and its image is a "good pair" in the sense of the theorem of Roberts and we obtain the desired family of taut foliations $\{\mathcal{F}_x\}$ with parameter x. Then we shall prove that the family $\{\mathcal{F}_x\}$ realizes all boundary slopes in the open interval $(-\infty,1)$.

3.1. **Constructing fibrations of the exterior of tours knots.** Let V be a solid torus standardly embedded in the 3-sphere S^3. We consider that the (r,s)-type torus knot $K(r,s)$ is a simple closed curve on the boundary ∂V of V. Cutting V by a meridian disk D and joining infinitely many copies of this piece, we get the universal cover $\tilde{V}$ of V and the covering $\tilde{K}$ of K on $\partial\tilde{V}$. $\tilde{V}$ becomes a cylinder of infinite length, so we put $\tilde{V}$ into $\mathbb{R}^3$ such that the x-axis is the core of this cylinder. Notice that the number of components of $\tilde{K}$ is s, and then let $k_1(x)$, $k_2(x)$, $\cdots$, $k_s(x)$ be components of $\tilde{K}$.

These components $k_1(x), k_2(x), \cdots, k_s(x)$ are the curves represented by following formulae;

$$k_i(x) = (x,\ \cos\frac{r}{s}(x+\frac{2(i-1)\pi}{r}),\ \sin\frac{r}{s}(x+\frac{2(i-1)\pi}{r}))\ \ (i=1,\cdots,s).$$

Now we construct a surface in the cylinder $\tilde{V}$ as follows. Let ${G_B}^i$ be the twisted band embedded in the part of the cylinder $\tilde{V}$ where $x\in[0,\frac{2\pi}{r}]$ represented by following formulae;

$$\begin{aligned}{G_B}^i = \Bigg\{ & r_i k_i(x) + (1-r_i)k_{i-1}\left(\frac{2\pi}{r}-x\right) + \left(\frac{2\pi}{r}n,0,0\right) \\ & \Bigg|\ 0 \leqq x \leqq \frac{\pi}{r},\ 0<r_i<1,\ n=0,\pm1,\pm2,\cdots \Bigg\} \\ & (i=1,\cdots,s,\ \ k_0=k_s).\end{aligned}$$

For the parameter value $x=\frac{\pi}{r}$, there is a disk with s points removed from the boundary. It is the regular polygon with s edges which are parts of boundaries of these bands. Then let G_P be the regular polygonal disks embedded into the disks $\{(\frac{\pi}{r}+\frac{2\pi}{r}k,\ y,\ z)\,|\, y^2+z^2\leqq 1,\ k=0,\pm1,\pm2,\cdots\}$ such that the boundary edges of one of these disks P_k are the arcs represented by the following formulae;

$$\partial P_k = \bigcup_{i=1}^{s} \left\{ r_i k_i \left(\frac{\pi}{r} + \frac{2\pi}{r} k \right) + (1 - r_i) k_{i-1} \left(\frac{\pi}{r} + \frac{2\pi}{r} k \right) \mid 0 < r_i < 1 \right\}.$$

The regular polygonal disk P_k is bounded by the above arcs ∂P_k and embedded in the disk $\left\{ (\frac{\pi}{r} + \frac{2\pi}{r} k \ , y \ , z) \mid y^2 + z^2 \leqq 1 \right\}$, therefore $G_P = \bigcup_{k \in \mathbb{Z}} P_k$. Then the surface G which we want to construct in $\tilde{V}$ is defined as the union of G_B and G_P.

Next we define the map $R_\theta : \tilde{V} \longrightarrow \tilde{V}$ given by

$$R_\theta(x, y, z) = \left(x + \frac{\theta}{r} \ , \ y \cos \frac{\theta}{s} - z \sin \frac{\theta}{s} \ , \ y \sin \frac{\theta}{s} + z \cos \frac{\theta}{s} \right).$$

Lemma 3.2. *R_θ turns $\tilde{V}$ by the angle $\dfrac{\theta}{s}$ keeping components $k_1(x),\ k_2(x),\ \cdots,\ k_s(x)$ of $\tilde{K}$ invariant.*

Proof. Let $k_i(t) = \left(t,\ \cos \frac{r}{s}(t + \frac{2(i-1)\pi}{r}),\ \sin \frac{r}{s}(t + \frac{2(i-1)\pi}{r}) \right)$ be a component of $\tilde{K}$. Then

$$\begin{aligned}
R_\theta(k_i(t)) &= R_\theta \left(t,\ \cos \frac{r}{s}(t + \frac{2(i-1)\pi}{r}),\ \sin \frac{r}{s}(t + \frac{2(i-1)\pi}{r}) \right) \\
&= \Bigg(t + \frac{\theta}{r},\ \cos \frac{r}{s}(t + \frac{2(i-1)\pi}{r}) \cos \frac{\theta}{s} - \sin \frac{r}{s}(t + \frac{2(i-1)\pi}{r}) \sin \frac{\theta}{s}, \\
&\qquad \cos \frac{r}{s}(t + \frac{2(i-1)\pi}{r}) \sin \frac{\theta}{s} + \sin \frac{r}{s}(t + \frac{2(i-1)\pi}{r}) \cos \frac{\theta}{s} \Bigg) \\
&= \left(t + \frac{\theta}{r},\ \cos (\frac{r}{s}(t + \frac{2(i-1)\pi}{r}) + \frac{\theta}{s}),\ \sin (\frac{r}{s}(t + \frac{2(i-1)\pi}{r}) + \frac{\theta}{s}) \right) \\
&= \left(t + \frac{\theta}{r},\ \cos \frac{r}{s}((t + \frac{\theta}{r}) + \frac{2(i-1)\pi}{r}),\ \sin \frac{r}{s}((t + \frac{\theta}{r}) + \frac{2(i-1)\pi}{r}) \right) \\
&= k_i(t + \frac{\theta}{r})
\end{aligned}$$

□

We define $G_\theta = R_\theta(G),\ 0 \leqq \theta \leqq 2\pi$.

Lemma 3.3. *The family of surfaces $\{G_\theta | 0 \leqq \theta \leqq 2\pi\}$ fills up $\tilde{V} \setminus \bigcup k_i$. If $\theta_i,\ \theta_j \in [0, 2\pi]$ and $\theta_i \neq \theta_j$, then $G_{\theta_i} \cap G_{\theta_j} = \emptyset$ except for $\theta_i,\ \theta_j \in \{0, 2\pi\}$.*

Proof. Let $p = (t,\ u,\ v) \in \tilde{V} \subset \mathbb{R}^3$ be a point in $\tilde{V}$. It is sufficient to prove when $0 \leqq t \leqq \dfrac{2\pi}{r}$. Let D_t and $D_t{}'$ be the disks in $\tilde{V}$ given by

$$D_t = \{(t,\ y,\ z) | y^2 + z^2 \leqq 1\},\ \ D_t{}' = \{(t,\ y,\ z) | y^2 + z^2 < 1\}.$$

Now we define a flow ψ on $\tilde{V}$ by $\psi = \{R_\theta(w) | \theta \in \mathbb{R}\}_{w \in D_0}$. We denote ψ' the flow ψ restricted to $\tilde{V}' = \{(x,\ y,\ z) | x \in \mathbb{R},\ y^2 + z^2 < 1\}$, G' the surface G restricted to $\tilde{V}'$.

Claim. The intersection of a flow line l of $\psi'|_{0 \leqq \theta \leqq 2\pi}$ and a surface $G'|_{0 \leqq x \leqq \frac{2\pi}{r}}$ is one point.

Proof of Claim. Let proj : $\tilde{V}|_{0 \leqq x \leqq \frac{2\pi}{r}} \longrightarrow D_0{}'$ be the projection map given by $\mathrm{proj}(x,\ y,\ z) = (0,\ y,\ z)$. Then this map is shown to be one to one and onto when it is restricted to G' as follows. For a point $p \in G'$, if $p \in G_P$, the point p is written as $p = \left(\frac{\pi}{r},\ u,\ v \right)$ and then $\mathrm{proj}(p) = (u, v)$ which belongs to the regular polygonal disk P on D_0. If $p \notin G_P$, let ${G^i_B}'$ be the surface G^i_B restricted to $\tilde{V}'$ and we set $p \in {G^i_B}'$. We can write ${G^i_B}'$ and $k_i(x)$ as follows;

$$
\begin{aligned}
{G^i_B}' &= \left\{ r_i k_i(x) + (1-r_i)k_{i-1}(\frac{2\pi}{r} - x) \mid 0 < x < \frac{\pi}{r},\ 0 < r_i < 1 \right\} \\
k_i(x) &= \left(x,\ \cos\frac{r}{s}(x + \frac{2(i-1)\pi}{r}),\ \sin\frac{r}{s}(x + \frac{2(i-1)\pi}{r}) \right).
\end{aligned}
$$

Then there are real numbers r_p and t_p such that $0 < r_p < 1$ and $0 < t_p < \frac{\pi}{r}$, and we can write $\mathrm{proj}(p)$ as follows;

$$
\begin{aligned}
\mathrm{proj}(p) &= \mathrm{proj}\left(r_p k_i(t_p) + (1-r_p)k_{i-1}(\frac{2\pi}{r} - t_p) \right) \\
&= \left(r_p \cos\frac{r}{s}(t_p + \frac{2(i-1)\pi}{r}) + (1-r_p)\cos\frac{r}{s}(\frac{2\pi}{r} - t_p + \frac{2(i-2)\pi}{r}), \right. \\
&\qquad \left. r_p \sin\frac{r}{s}(t_p + \frac{2(i-1)\pi}{r}) + (1-r_p)\sin\frac{r}{s}(\frac{2\pi}{r} - t_p + \frac{2(i-2)\pi}{r}) \right) \\
&= \left(r_p \cos\frac{r}{s}(t_p + \frac{2(i-1)\pi}{r}) + (1-r_p)\cos\frac{r}{s}(\frac{2(i-1)\pi}{r} - t_p), \right. \\
&\qquad \left. r_p \sin\frac{r}{s}(t_p + \frac{2(i-1)\pi}{r}) + (1-r_p)\sin\frac{r}{s}(\frac{2(i-1)\pi}{r} - t_p) \right).
\end{aligned}
$$

By putting $\gamma = r_p$ and $\xi = \frac{2(i-1)\pi}{r}$,

$$
\begin{aligned}
\mathrm{proj}(p) &= \left(\gamma\cos\frac{r}{s}(t_p + \xi) + (1-\gamma)\cos\frac{r}{s}(\xi - t_p),\ \gamma\sin\frac{r}{s}(t_p + \xi) + (1-\gamma)\sin\frac{r}{s}(\xi - t_p) \right) \\
&= \gamma\left(\cos\frac{r}{s}(t_p + \xi),\ \sin\frac{r}{s}(t_p + \xi) \right) + (1-\gamma)\left(\cos\frac{r}{s}(\xi - t_p),\ \sin\frac{r}{s}(\xi - t_p) \right).
\end{aligned}
$$

Let $\alpha(t)$ and $\beta(t)$ be the points on the boundary $\partial D_0 = \{(y,\ z) \in D_0 \mid y^2 + z^2 = 1\}$ such that

$$
\begin{aligned}
\alpha(t) &= \left(\cos\frac{r}{s}(t + \xi), \sin\frac{r}{s}(t + \xi) \right) \\
\beta(t) &= \left(\cos\frac{r}{s}(\xi - t), \sin\frac{r}{s}(\xi - t) \right).
\end{aligned}
$$

Then we can write the image of ${G^i_B}'$ under the map proj as follows;

$$
\mathrm{proj}({G^i_B}') = \left\{ \gamma\alpha(t) + (1-\gamma)\beta(t) \in D_0 \mid 0 < t < \frac{\pi}{r},\ 0 < \gamma < 1 \right\}.
$$

Gathering this image for all $i = 1, 2, \cdots, s$, these fill the complement of P in ${D_0}'$. Thus we proved that the map proj is one to one and onto.

For a point $p = (t,\ u,\ v) \in \tilde{V}'|_{0 \leqq x \leqq \frac{2\pi}{r}}$, let l be the flow line of ψ' which contains the point p. Since the perpendicular projection map is one to one and onto by the above argument, the flow line l intersects G' at one point $p' = (t',\ u',\ v')$. By the definition of the flow ψ', $R_{\frac{t-t'}{r}}(p') = p$. Therefore this point p exists on $R_{\frac{t-t'}{r}}(G') = {G_{\frac{t-t'}{t}}}'$, so there is an unique θ such that $p \in G_\theta$.

For a point $p = (t, u, v) = (t,\ \cos\tau,\ \sin\tau) \in (\tilde{V} \setminus \tilde{V}'|_{0 \leqq x \leqq \frac{2\pi}{r}}) \setminus \bigcup k_i$, there exists some i and a path l on the boundary $\partial\tilde{V}$ such that

$$
\begin{aligned}
p \in l &= \left\{ r_i k_i(\tau - \frac{2(i-1)\pi}{r}) + (1-r_i)k_{i-1}(\frac{2\pi}{r} - (\tau - \frac{2(i-1)\pi}{r})) \mid 0 < r_i < 1 \right\} \\
&= R_{r(\tau - \frac{2(i-1)\pi}{r})}\left(\{ r_i k_i(0) + (1-r_i)k_{i-1}(\frac{2\pi}{r}) \mid 0 < r_i < 1 \} \right).
\end{aligned}
$$

Since the path l is contained in $R_{r\left(\tau-\frac{2(i-1)\pi}{r}\right)}(G_B^i)$, the point p is contained in this image. Therefore there exists an unique θ such that $p \in G_\theta$. This completes the proof of Lemma 3.3.

□

Adding some points to G_θ and modifying G_θ in a neighbourhood of $\partial\tilde{V}$, we see that the boundary of G_θ consists of $k_1(x),\ \cdots,\ k_s(x)$ and s lines $\{l_i\}_{i=1}^s$, where $l_i = \left\{\left(x,\ \cos\frac{2(i-1)\pi}{r},\ \sin\frac{2(i-1)\pi}{r}\right) \mid x \in \mathbb{R}\right\}$. By the above explicit construction of G_θ on $\tilde{V}$, every G_θ is invariant under the covering transformation of $\tilde{V}$. Then we can project G_θ to the surface ${F_\theta}' = q(G_\theta)$ on V. The family of surfaces $\{{F_\theta}' = q(G_\theta) \mid 0 \leqq \theta < 2\pi\}$ fills up V since all G_θ are disjoint by Lemma 3.3, and each surfaces satisfy $\partial {F_\theta}' \supset K(r,s)$.

Next we define the s lines $\tilde{C^1},\ \tilde{C^2},\ \cdots,\ \tilde{C^i},\ \cdots,\ \tilde{C^s}$ on $\partial\tilde{V}$ as follows;

$$\tilde{C^i} = \left\{(x,\ \cos\frac{2(i-1)\pi}{s},\ \sin\frac{2(i-1)\pi}{s}) \mid x \in \mathbb{R}\right\},\quad (i = 1,\cdots,s).$$

We define the family of lines $\{\tilde{C^i_\theta} \mid 0 \leqq \theta < 2\pi\}$ on $\partial\tilde{V}$ by $\tilde{C^i_\theta} = R_\theta(\tilde{C^i})$. Now we project this family to V by the covering map q, and get the family of curves $\{C^i_\theta = q(\tilde{C^i_\theta}) \mid 0 \leqq \theta < 2\pi\}$ on ∂V. The boundary of ${F_\theta}'$ on ∂V consists of the union of our torus knot $K(r,s)$ and this family of curves, that is,

$$\partial {F_\theta}' = K(r,s) \cup \left(\bigcup_{i=1}^{s} C^i_\theta\right).$$

By definition, V is a solid torus standardly embedded into S^3. So let W be the complement of V in S^3, then W also is a solid torus standardly embedded into S^3. By the above construction, a curve of this family $\{\tilde{C^i_\theta} \mid 0 \leqq \theta < 2\pi\}$ is a longitude curve on ∂V, then it is a meridian curve on ∂W and it bounds a meridian disk K in W. We define the meridian disks D^i_θ such that $\partial D^i_\theta = C^i_\theta$.

Finally we define the surface

$$F_\theta = \left({F_\theta}' \cup \left(\bigcup_{i=1}^{s} D^i_\theta\right)\right) \setminus K(r,s),$$

and the map $p : S^3 \setminus K(r,s) \longrightarrow S^1$ such that if $x \in F_\theta \subset S^3 \setminus K(r,s)$, $p(x) = e^{i\theta} \in S^1$.

Lemma 3.4. *This map $p : S^3 \setminus K(r,s) \longrightarrow S^1$ defines a fibration on $S^3 \setminus K(r,s)$ whose fiber is F_θ.*

Proof. Since the surfaces G_θ are disjoint in $\tilde{V}$ by Lemma 3.3 and q is a covering map, the surfaces F_θ are disjoint in $S^3 \setminus K(r,s)$. Let $I \subset S^1$ be an open interval on S^1. By the definition of p, for $x \in S^1$, $p^{-1}(x) = F_x$, and then $F_x \cap F_y = \emptyset$ when $x \neq y$. Thus $p^{-1}(I) = \coprod_{x\in I} F_x$ which is a disjoint union of fibers. For any $x \in S^1$, F_x is a open set in $S^3 \setminus K(r,s)$, so $p^{-1}(I)$ is open and the map p is continuous. As seen before there is the flow ψ on $\tilde{V}$. If we project it to V and denote this flow by $\hat{\psi}$, the flow $\hat{\psi}$ is transverse to ${F_\theta}'$ for any θ in V. The solid torus W is trivially foliated by disks D^i_θ, and there is a flow ϕ transverse to any disk D^i_θ which coincide with $\hat{\psi}$ on the boundary ∂W. By gathering these transverse flows $\hat{\psi}$ and ϕ, we obtain the flow φ on $S^3 \setminus K(r,s)$ transverse to F_θ for any θ. For any point $x \in S^1$ and any interval $x \in I \subset S^1$ we define a map $\eta : p^{-1}(I) \to F_x \times I$ such that $\eta(q) = (\varphi_\tau(q), t)$ where the point $\varphi_\tau(q)$ is the point on which the flow line of φ through q intersects $F_x = p^{-1}(x)$, and $t = p(q)$. Since φ is a transverse flow, the map η becomes a trivialization map of this fibration.

□

Thus we complete an explicit construction of fibration of the complement of our torus knot $K(r, s)$.

3.2. **Proof of main theorem.** Next we choose two properly embedded arcs α and β on the fiber F_0. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the arcs on $\partial\tilde{V}$ such that

$$\tilde{\alpha} = \left\{(t,\ 1,\ 0) \in \partial\tilde{V} \mid 0 \leqq t \leqq \frac{2\pi}{r}\right\}$$
$$\tilde{\beta} = \left\{(t,\ \cos\frac{2\pi}{s},\ \sin\frac{2\pi}{s}) \in \partial\tilde{V} \mid \frac{2\pi}{r} \leqq t \leqq \frac{4\pi}{r}\right\}.$$

We define the two arcs α and β on the fiber F_0 so that $\alpha = q(\tilde{\alpha})$ and $\beta = q(\tilde{\beta})$. The fiber F_0 is an open surface, but attaching a copy of our torus knot $K(r, s)$ to it we regard it as a closed surface whose boundary is on $\partial N(K(r, s))$. If we regard the fiber as a closed surface, these two arcs α and β are properly embedded arcs whose each end points $\partial\alpha$ and $\partial\beta$ sit on $\partial N(K(r, s))$. In the later argument, we always regard a fiber surface as a closed surface whose boundary is on $\partial N(K(r, s))$.

Let $h : F_0 \longrightarrow F_0$ be the monodromy map of this fibration. This map is defined as the composition of the rotation map R_θ and the covering map q ; $h = q \circ R_{2\pi}$. By the construction of this fibration, the monodromy h maps α to β, i.e. $h(\alpha) = \beta$.

Now we consider the complement $M = \overline{S^3 \setminus N(K(r, s))}$ of the torus knot $K(r, s)$ as the quotient space of the product of the fiber F_0 and the unit interval $I = [0, 1]$;

$$M = F_0 \times [0, 1]/(x,\ 1) \sim (h(x),\ 0).$$

Note that the boundary ∂M is homeomorphic to a torus since the boundary of F_0 is a circle and h maps this circle to itself. The positive side of F_0 is defined by a positive direction of the unit interval $[0, 1]$.

Using this interval, we define a disk D in M such that $D = \alpha \times [0, 1]$. Note that the boundary ∂D of the disk consists of four arcs, $\alpha,\ \beta$ on F_0 and $\partial\alpha \times [0, 1]$ on ∂M.

We define the coordinate system on the torus boundary $\partial M = \partial N(K(r, s))$ by choosing two specific oriented simple closed curves λ and μ as follows. Let λ be a curve such that $\lambda = \partial F_0$, and we call it a longitude. The orientation of λ is induced from the orientation of F_0. Let μ be a curve on ∂M such that it satisfies $\langle\lambda, \mu\rangle = 1$ and bounds an essential disk in $N(K(r, s))$.

Now we define a branched surface B_- such that $B_- = \langle F_0; D\rangle$. We shall prove that this branched surface B_- carries laminations λ_x which realize all boundary slopes in $(-\infty, 0]$, and then these laminations λ_x extend to taut foliations $\mathcal{F}_x$ by filling up complementary regions. To prove this, we need some definitions and lemmas.

Definition 3.5. Let F be a compact surface with a single circle boundary component and negative Euler characteristic, and δ and δ' be simple arcs properly embedded in F. The pair (δ, δ') is called *good* if δ and δ' are disjoint on F, and their endpoints alternate along ∂F as are shown in Figure 5.

Note that for a good pair (δ, δ'), each simple arc is non-separating on F.

Lemma 3.6. *Let α and β be the simple arcs on F_0 defined above, then (α, β) is a good pair.*

Proof. The arc α is a part of the boundary of D_0^1 and β is a part of the boundary of D_0^2, $D_0^i \cap D_0^j = \emptyset$ if $i \neq j$, and each disk D_0^i is a properly embedded meridian disk in the solid torus W. Then clearly α and β have no self intersection and are disjoint each other.

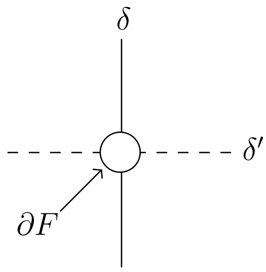

FIGURE 5

Let $\partial^1\tilde{\alpha}$ be the point on $\partial\tilde{V}$ such that $\partial^1\tilde{\alpha} = (0,\ 1,\ 0) \in \tilde{V} \subset \mathbb{R}^3$. It is one of the end points of $\tilde{\alpha}$ and is on the component $k_1(x)$ of $\tilde{K}(r,s)$. When we consider G_0 as a closed surface with boundary $\tilde{K}(r,s)$, $\partial^1\tilde{\alpha}$ can move along $k_1(x)$ in the direction induced by the rotation map R_θ, then $\partial^1\tilde{\alpha}$ meets one end point of $\tilde{\beta}$. We denote this point by $\partial^1\tilde{\beta}$.

Let $\overline{k_i(x)}$ be the sub arc of $k_i(x)$ restricted to $\tilde{V}|_{0\leqq x\leqq 2\pi}$, then our torus knot $K(r,s)$ consists of the union of arcs $q(\overline{k_i(x)})$, $i = 1, 2, \cdots, s$ and these arcs are disjoint except at each end points. We define an orientation on $\overline{k_i(x)}$ by the orientation induced by the rotation map R_θ, and then we can specify a starting point and an ending point of each $\overline{k_i(x)}$. One end point of $\overline{k_1(x)}$ which is not $\partial^1\tilde{\alpha}$ is the ending point of $\overline{k_1(x)}$ and is connected to the starting point of $\overline{k_2(x)}$ when they are projected to V. Similarly, the ending point of $\overline{k_i(x)}$ is connected to the starting point of $\overline{k_{i+1}(x)}$. By this consideration, after passing $\partial^1\tilde{\beta}$, $\partial^1\tilde{\alpha}$ will meet the remaining end point $\partial^2\tilde{\alpha}$ of $\partial\tilde{\alpha}$ on $\overline{k_s(x)}$ when they are projected to V, and finally meets the remaining end point $\partial^2\tilde{\beta}$ on $\overline{k_s(x)}$.

This means that these points $\partial^1\tilde{\alpha}$, $\partial^1\tilde{\beta}$, $\partial^2\tilde{\alpha}$ and $\partial^2\tilde{\beta}$ project to the end points of α and β on ∂F_0, and these points are in the order along ∂F_0 such that $\partial^1\alpha = q(\partial^1\tilde{\alpha})$, $\partial^1\beta = q(\partial^1\tilde{\beta})$, $\partial^2\alpha = q(\partial^2\tilde{\alpha})$ and $\partial^2\beta = q(\partial^2\tilde{\beta})$. Therefore end points of α and β alternate along ∂F_0, thus (α, β) is a good pair.

□

Next we choose a properly embedded curve on F_0 which cuts the branched surface B_- nicely.

Lemma 3.7. *There is a simple arc γ_- properly embedded in F_0 such that B_- is affinely measured with respect to γ_-.*

Proof. Let T be a regular neighbourhood of $\alpha \cup \beta \cup \partial F_0$. Since α and β form a good pair, T is homeomorphic to a twice punctured torus. ∂T has two boundary components. One of these is ∂F_0, and we denote the other component by C. To define a desired simple arc, we put an orientation on α and β. We choose an orientation of $\tilde{\alpha}$ so that the direction in which the x-coordinate is increasing is positive and choose one on $\tilde{\beta}$ to be the same direction as $\tilde{V}$.

Then we define a simple arc γ_- properly embedded in T so that it satisfies

$$[\gamma_-] = -[\alpha] + [\beta] \in H_1(F_0, \partial F_0).$$

The orientations on α and β are induced from those on $\tilde{\alpha}$ and $\tilde{\beta}$ by the covering projection q on V.

There are two choices of the simple arc γ_- up to isotopy as are shown in Figure 6.

Each choice of the simple arc γ_- separates $T \setminus (\alpha \cup \beta)$ into two regions R and R_C; R does not intersect C and R_C contains C. To prove that $B_- \setminus \gamma_-$ has an

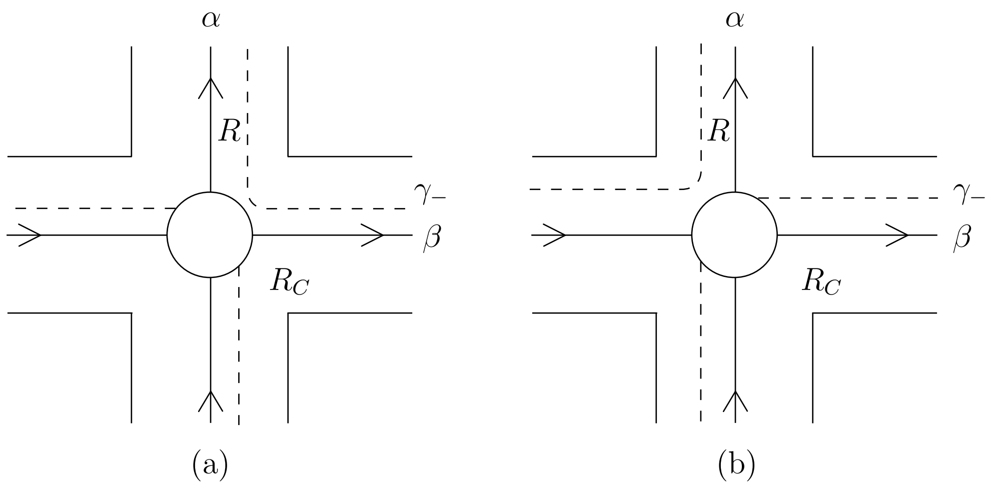

FIGURE 6

affine measure, we assign weight 1 to R_C and $1 + x$ to R as in Figure 7, where R_C and R are parts of sectors of $B_- \cap T$.

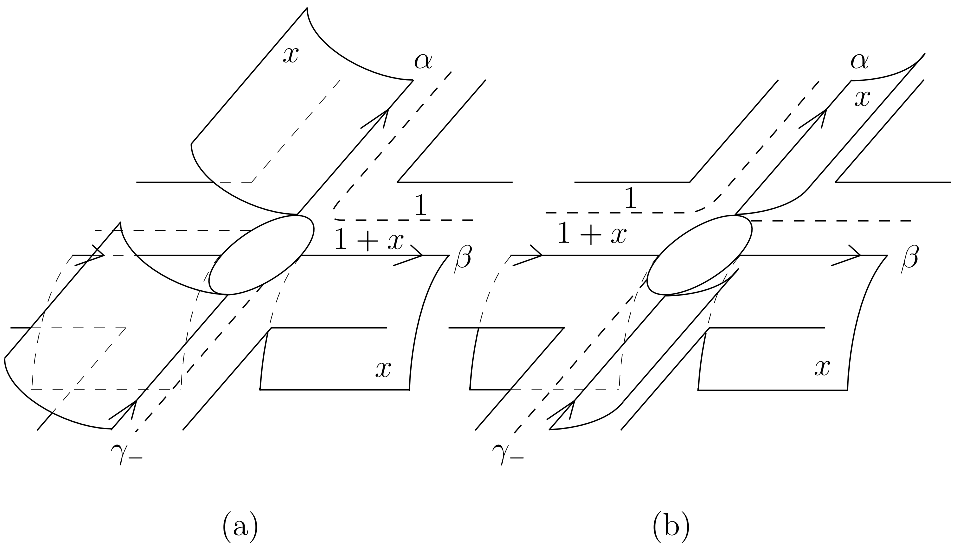

FIGURE 7

Since the simple closed curve C is contained only in R_C and the weight of R_C is 1, we can put weight 1 on the region which is bounded by C on F_0, then B_- is affinely measured with respect to γ_-.

□

Let us take the choice of γ_- indicated in Figure 7 (a). Let ${\lambda_x}'$ be an affinely measured lamination which is carried by $B_- \setminus \gamma_-$. $\partial_v(N(B_- \setminus \gamma_-)) \setminus \partial_v N(B_-)$ is two copies of annuli $\gamma_- \times I$. Let f be a scaling map on $\gamma_- \times I$ such that

$$f : \gamma_- \times [0, 1] \to \gamma_- \times [0, 1] : (p, t) \mapsto (p, (1 + x)t)$$

where p is a point of γ_-, x is the weight parameter defined before. We glue two copies of $\gamma_- \times I$ on $\partial_v(N(B_- \setminus \gamma_-)) \setminus \partial_v N(B_-)$ by the scaling map f, we get the lamination λ_x which is carried by B_-.

Lemma 3.8. *λ_x realizes the boundary slope* $\dfrac{-x^2}{x+1}$.

Proof. Let $\tau = B_- \cap \partial M$ be the train track on ∂M. By the definition of γ_-, $\partial\gamma_-$ intersects τ at two points $\partial^1\gamma_-$ and $\partial^2\gamma_-$ on $\lambda = F_0 \cap \partial M$. Let τ' be the branched arc system which is made from τ by cutting at $\partial^1\gamma_-$ and $\partial^2\gamma_-$. We assign τ' the affine measure induced from the affine measure already defined on $B_- \setminus \gamma_-$.

We defined the disk $D = \alpha \times [0,1]$ which is a sector of B_-. The boundary of D consists of four arcs; two of them are α and β, the other two arcs are bounded by $\partial^1\alpha$ and $\partial^1\beta$, $\partial^2\alpha$ and $\partial^2\beta$, respectively. We denote by κ_1 the arc which is bounded by $\partial^1\alpha$ and $\partial^1\beta$, and denote the other by κ_2. By construction of B_-, sectors of τ which is not contained in λ are κ_1 and κ_2. Therefore the point $\partial^1\alpha$ is connected with $\partial^1\beta$ by κ_1, and $\partial^2\alpha$ is connected with $\partial^2\alpha$ by κ_2 on ∂M. Then we see that the train track τ is divided into exactly two components by cutting at $\partial^1\gamma_-$ and $\partial^2\gamma_-$, that is, τ' has two components.

Let ${\tau_1}'$ be one of the components of τ' which contains $\partial^1\alpha$ and $\partial^1\beta$, ${\tau_2}'$ be the other one. τ' has the affine measure assigned as before, we denote this measure by w'. Then ${\tau_1}'$ and ${\tau_2}'$ also have the affine measure ${w_1}'$ and ${w_2}'$. Since we construct the measured branched surface B_- from $B_- \setminus \gamma_-$ by using the scale map f, the measure ${w_2}'$ changes into ${w_2}' \times \frac{1}{1+x}$. Then we obtain the measure w on the train track τ induced from the measure on B_- as pictured in Figure 8.

Let $\tau(w)$ be a lamination on $N(\tau) \subset \partial M$ carried by the measured train track (τ, w), then $\tau(w)$ is the restriction of the lamination λ_x to ∂M. It means that the boundary slope of λ_x is the slope of a simple closed curve which is a leaf of $\tau(w)$.

We assign an orientation to the measured train track (τ, w), and the meridian μ and the longitude λ as Figure 8.

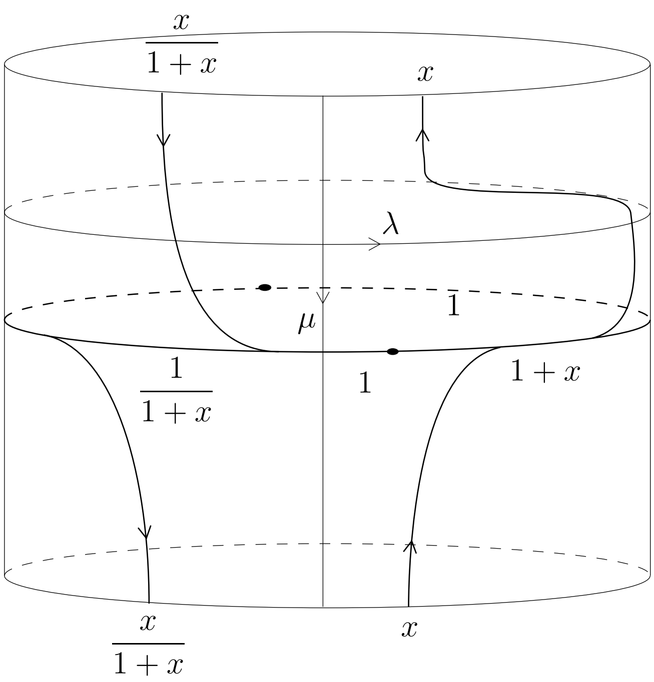

FIGURE 8

Notice that in Figure 8 we look at ∂M putting our viewpoint in the interior of M, but for the purpose of estimating a homological intersection number of simple closed curves on ∂M we must observe them from the viewpoint which is in the interior of $N(K(r,s))$.

By this observation on Figure 8, we can calculate the homological intersection number between $\tau(w)$ and the coordinate system (μ, λ) as follows;

$$\langle \tau(w), \lambda \rangle = \frac{x}{1+x} - x$$
$$\langle \mu, \tau(w) \rangle = 1.$$

Then the slope of a simple closed curve of $\tau(w)$ is

$$\text{slope } \tau(w) = \frac{\langle \tau(w), \lambda \rangle}{\langle \mu, \tau(w) \rangle} = \frac{x}{1+x} - x = \frac{-x^2}{x+1}.$$

Therefore, we see that the lamination λ_x realizes the boundary slope $\frac{-x^2}{x+1}$.

□

By this Lemma 3.8 and letting x range over $[0, \infty)$, we conclude the following Proposition.

Proposition 3.9. *The family of laminations $\{\lambda_x\}$ realizes all boundary slopes in $(-\infty, 0]$.*

To prove the main theorem we shall prove the following Proposition.

Proposition 3.10. *The lamination λ_x extends to a taut foliation $\mathcal{F}_x$ of M which realizes any boundary slope in $(-\infty, 0]$.*

Proof. Let λ_x be the lamination on $N(B_-)$ defined above. Since the disk D is defined by $D = \alpha \times [0,1]$ in M, we can consider that D is properly embedded in $F_0 \times [0,1]$. By definition, (α, β) is a good pair and then α is non-separating. Therefore D does not separate $F_0 \times [0,1]$.

Let M_D be a complementary region of D in $F_0 \times [0,1]$, F_α be a complementary region of α in F_0, i.e. if we set

$$M_D{}' = (F_0 \times [0,1]) \setminus D, \quad F_\alpha{}' = F_0 \setminus \alpha,$$

M_D and F_α are the metric completion of $M_D{}'$ and $F_\alpha{}'$, respectively. Then M_D is the product space of F_α and $[0,1]$, and M_D has corners. The boundary of M_D consists of four parts, two copies of F_α and two copies of $\alpha \times [0,1]$.

We denote these parts as follows: $F_{\alpha,0}$ is the copy of F_α embedded in $F_0 \times \{0\}$, $F_{\alpha,1}$ is the copy in $F_0 \times \{1\}$, D_+ and D_- are two copies of $\alpha \times [0,1]$, where D_+ is the positive side of the sector D in B_- with respect to the orientation of B_-, D_- is a negative side. By shrinking M_D and sliding up the curve $F_{\alpha,0} \cap D_+$ to near the curve $F_{\alpha,1} \cap D_+$ along D_+, and sliding down the curve $F_{\alpha,1} \cap D_-$ to $F_{\alpha,0} \cap D_-$ along D_-, we can consider that M_D is embedded into the complementary region $M_{B_-} = \overline{M \setminus N(B_-)}$.

The embedded M_D satisfies that the boundaries D_+ and D_- coincide with the two components of $\partial_v N(B_-)$ and M_D is homeomorphic to the product of one component of $\partial_h N(B_-)$ which is homeomorphic to $F_0 \setminus \alpha$ and an interval $[0,1]$. Therefore if we foliate the complementary region M_{B_-} by the product foliation and fill the complementary region of λ_x in $N(B_-)$ with parallel leaves, we can extend λ_x to a foliation $\mathcal{F}_x$.

When $x = 0$ we consider that $\mathcal{F}_x$ is the original fibration of M. Otherwise, at the boundary ∂M the meridian curve meets each leaf of $\mathcal{F}_x$ transversely. In both cases, $\mathcal{F}_x$ is a taut foliation.

□

Now we have shown the existence of taut foliations which realizes all boundary slopes in $(-\infty, 0]$. To complete the proof of main theorem, we must fill up the left part of the interval, $[0, 1)$.

We take a fiber F_0, and the two simple arcs α and β properly embedded in F_0 as before. Let F_1 and F_2 be the fibers given by

$$F_1 = F_0 \times \left\{\frac{1}{3}\right\}, \quad F_2 = F_0 \times \left\{\frac{2}{3}\right\}.$$

Since F_1 and F_2 are copies of F_0, there are copies of the pair of arcs α and β on each F_i, $i = 0, 1, 2$. We denote these arcs on F_i by α_i and β_i. The orientation of the pair of curves (α_i, β_i) is induced from the pair (α, β).

In the proof of Lemma 3.7, we defined a simple arc γ_- properly embedded in F_0, and there are two choices of γ_- up to isotopy. Now we temporary denote one choice of γ_- which is the type shown in the left figure of Figure 6 by γ_a, the other by γ_b. Then we define three simple arcs γ_+^0, γ_+^1 and γ_+^2 properly embedded in F_0, F_1 and F_2, respectively, such that γ_+^0 has the type of γ_b, γ_+^1 has the type of γ_a and γ_+^2 has the type of γ_b.

We define three disks D_0, D_1, D_2 in M which will be sectors of our desired branched surface such that

$$D_0 = \alpha_0 \times \left[0, \frac{1}{3}\right], \quad D_1 = \beta_1 \times \left[\frac{1}{3}, \frac{2}{3}\right], \quad D_2 = \alpha_2 \times \left[\frac{2}{3}, 1\right].$$

Then we obtain the branched surface B_+ such that

$$B_+ = \langle F_0, F_1, F_2 \; ; \; D_0, D_1, D_2 \rangle$$

with the orientations given in Figure 9.

Lemma 3.11. *B_+ is affinely measured with respect to the simple arcs γ_+^0, γ_+^1 and γ_+^2.*

Proof. Similar to the proof of Lemma 3.7, for $i = 0, 1, 2$, each $\alpha_i \cap \beta_i \cap \partial F_i$ has a regular neighbourhood T_i which is homeomorphic to a twice punctured torus. We can consider that each γ_i is properly embedded in T_i. We assign measures on D_i and T_i as shown in Figure 7.

Then each region $F_i \setminus T_i$ is bounded by the boundaries of T_i which are assigned the measure 1. We put the measure 1 on each region $F_i \setminus T_i$, then B_+ is affinely measured with respect to γ_+^0, γ_+^1 and γ_+^2.

□

Let λ_x be the lamination on $N(B_+)$ which is obtained by gluing via the scaling map f on each region $\gamma_+^i \times [0, 1]$.

Lemma 3.12. *λ_x realizes the boundary slope* $\dfrac{x^2}{x^2 + 3x + 3}$.

Proof. Let $\tau = B_+ \cap \partial M$ be the train track on ∂M. Each γ_+^i has two end points on $\lambda_i = F_i \cap \partial M$. We denote these end points of γ_+^i by $\partial^1 \gamma_+^i$ and $\partial^2 \gamma_+^i$. Similar to the proof of Lemma 3.8, these six end points $\partial^1 \gamma_+^i$ and $\partial^2 \gamma_+^i$, $i = 0, 1, 2$ cuts τ into two parts. We denote these parts by $\tau_1{}'$ and $\tau_2{}'$ such that $\tau_1{}'$ contains a point $\partial^2 \beta_0$. We put the affine measure on $B_+ \setminus (\gamma_+^0 \cup \gamma_+^1 \cup \gamma_+^2)$ as pictured in Figure 9. Then $\tau_1{}'$ and $\tau_2{}'$ have the affine measures induced from the affine measure on $B_+ \setminus (\gamma_+^0 \cup \gamma_+^1 \cup \gamma_+^2)$, let $w_1{}'$ and $w_2{}'$ be affine measures on $\tau_1{}'$ and $\tau_2{}'$ respectively. When we construct the measured branched surface B_+ from $B_+ \setminus (\gamma_+^0 \cup \gamma_+^1 \cup \gamma_+^2)$ by gluing with the scale map f, the measure $w_2{}'$ changes into $w_2{}' \times \frac{1}{1+x}$. Then we obtain the affine measure w on τ (See Figure 10).

Let $\tau(w)$ be a lamination on $N(\tau) \subset \partial M$ carried by the measured train track (τ, w). By construction, the leaves of $\tau(w)$ are the boundary of leaves of λ_x, that is, the boundary slope of λ_x is calculated by computing the slope of a leaf of $\tau(w)$. To do this, we put an orientation on τ as in Figure 10. By the same consideration

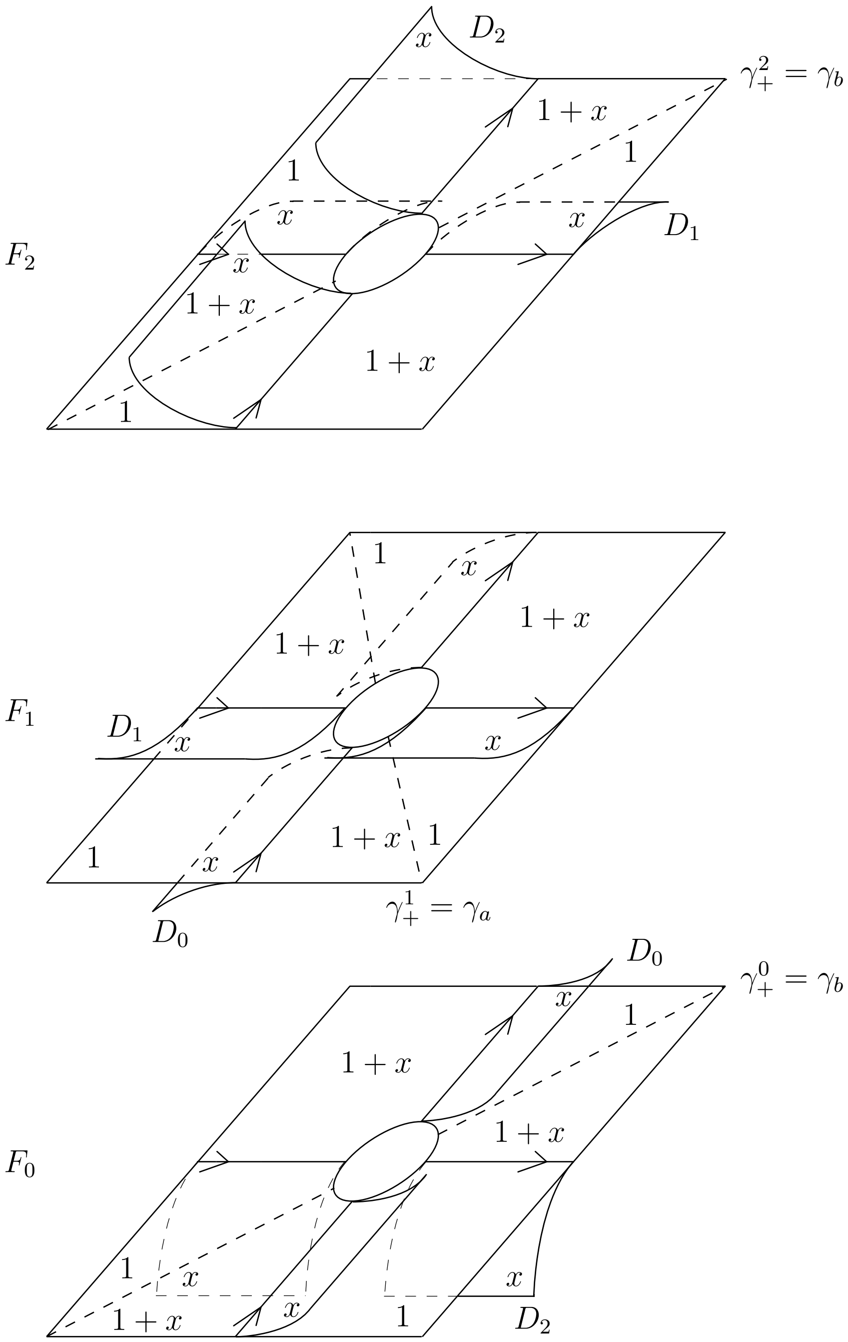

FIGURE 9

as in the proof of Lemma 3.8, we obtain the homological intersection numbers of $\tau(w)$ with respect to the coordinate system (μ, λ) as follows;

$$\langle \tau(w), \lambda \rangle = -\frac{x}{1+x} + x = \frac{x^2}{1+x}$$
$$\langle \mu, \tau(w) \rangle = (1+x) + 1 + \frac{1}{1+x} = \frac{x^2+3x+3}{1+x},$$

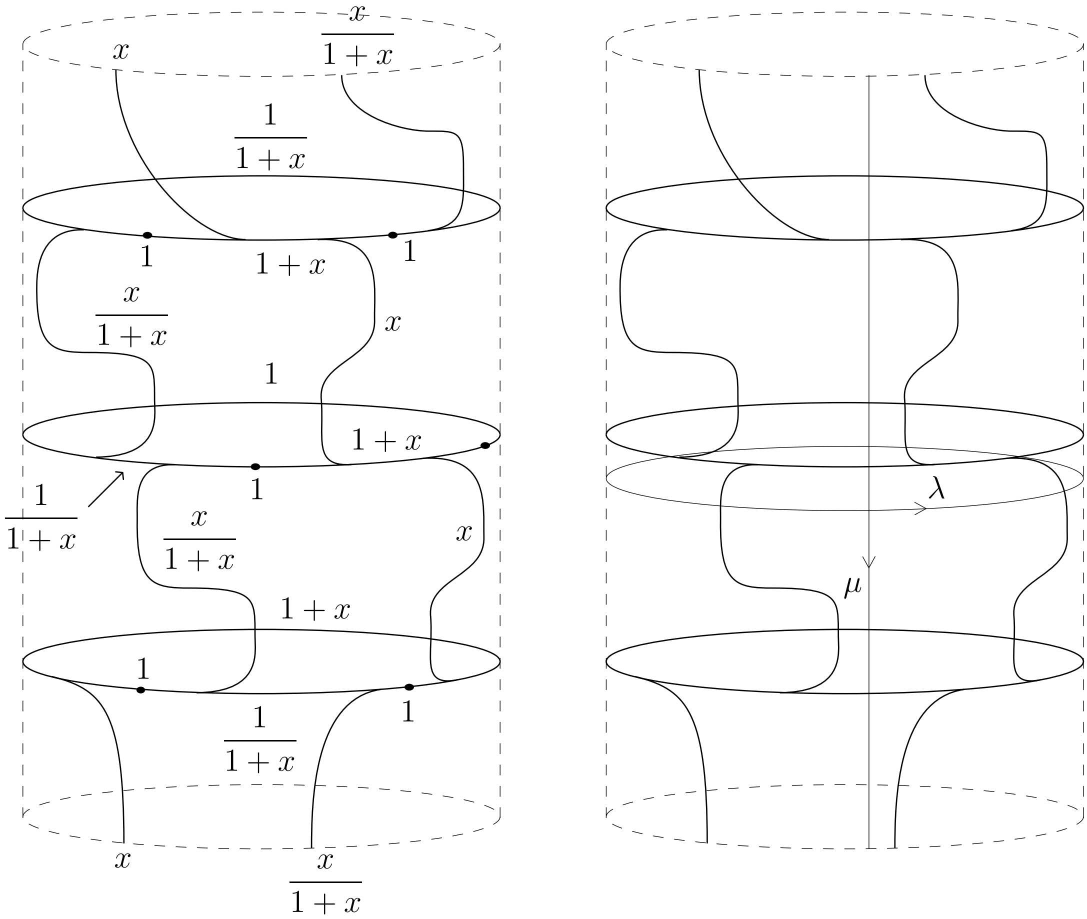

FIGURE 10

where the intersections between τ and λ are two points, the intersections between μ and τ are three points. Then the slope of a simple closed curve of $\tau(w)$ is

$$\text{slope}\ \tau(w) = \frac{\langle \tau(w), \lambda \rangle}{\langle \mu, \tau(w) \rangle} = \frac{x^2}{x^2 + 3x + 3}.$$

□

By the Lemma 3.12 and letting x range over $[0, \infty)$, we conclude as follows.

Proposition 3.13. *The family of laminations $\{\lambda_x\}$ realizes all boundary slopes in $[0, 1)$.*

To finish the proof of main theorem, we shall prove following Proposition.

Proposition 3.14. *The above family of laminations $\{\lambda_x\}$ extends to the family of taut foliations $\{\mathcal{F}_x\}$ on M which realizes all boundary slopes in $[0, 1)$.*

Proof. Similar to the proof of Proposition 3.10, we see that there are three components of complementary regions of B_+ in $F_0 \times [0, 1]$. Each component is homeomorphic to a region M_{D_i} which is a product of a subsurface of F_0 and an interval. Therefore each complementary region of $N(B_+)$ in M is filled by a product foliation whose leaves intersect $\partial_v N(B)$ transversely. Then λ_x extends to the whole foliation $\mathcal{F}_x$ on M. Considering the leaves of $\mathcal{F}_x$ on ∂M, we can see $\mathcal{F}_x$ is a taut foliation.

□

By Proposition 3.10 and Proposition 3.14, the proof of main theorem is completed.

4. Iterated torus knot case

In this section, we extend the result of section 3 to an iterated torus knot. To define an iterated torus knot, at first we define a sequence of solid tori $\{T_i\}$ and knots $\{K_i\}$ embedded in S^3 as follows. The first solid torus T_0 is standardly embedded in S^3 and let K_0 be a simple closed curve on the boundary ∂T_0. We called K_0 a torus knot before, now we will call it a standard torus knot. A regular neighbourhood of K_0 is also a solid torus, and we denote this solid torus by T_1 which is embedded in S^3. Then we define a new knot K_1 which is a simple closed curve on the boundary ∂T_1. By iterating this construction, the knot K_{i-1} has a regular neighbourhood T_i homeomorphic to a solid torus and there is a new knot K_i which is a simple closed curve on the boundary ∂T_i. To avoid complicated arguments, we assume that each K_i is not homotopic to a meridian curve or a longitude curve on ∂T_i.

To construct a taut foliation made as a modification of fibration, we must define these $\{K_i\}$ precisely.

Let T_0 be a solid torus standardly embedded in S^3 and $K_0(r_0, s_0)$ be a simple closed curve on ∂T_0 which has a homological representation $r_0 m_0 + s_0 l_0 \in H_1(\partial T_0)$, where m_0 is the standard meridian and l_0 is the standard longitude of ∂T_0. Let T_1 be a regular neighbourhood of K_0 which is homeomorphic to a solid torus. The complement $M_0 = \overline{S^3 \setminus N(K_0)}$ has the fibration ξ_0 as seen before, then we define that the longitude l_1 of ∂T_1 is a simple closed curve which coincides with the boundary of a fiber of the fibration ξ_0, and we define the meridian m_1 such that m_1 intersects l_1 transversely at one point and m_1 bounds a disk in T_1. For this meridian-longitude pair we define a new knot $K_1(r_1, s_1)$ which is a simple closed curve on ∂T_1 and has a homological representation $r_1 m_1 + s_1 l_1 \in H_1(\partial T_1)$.

In section 3.1, we construct a sub surface ${F_\theta}'$ in the solid torus V and prove that the family of surfaces $\{{F_\theta}' | 0 \leqq \theta < 2\pi\}$ fills up V. The boundaries of ${F_\theta}'$ consist of circles $\{C_\theta^i\}_{i=1,\cdots,s}$ and the torus knot $K(r, s)$ on ∂V. By construction, the circles $\{C_\theta^i\}_{i=1,\cdots,s}$ are parallel on ∂V. Then we replace T_1 by this solid torus V so that the circles $\{C_\theta^i\}_{i=1,\cdots,s}$ coincide with the curves parallel to the longitude l_1 and the torus knot $K(r, s)$ on ∂V coincides with $K_1(r_1, s_1)$, that is, $r = r_1$ and $s = s_1$. Since any boundary of fibers of ξ_0 is a curve on ∂T_1 which parallel to a longitude, any surface of the family $\{{F_\theta}' | 0 \leqq \theta < 2\pi\}$ is connected to a fiber of ξ_0 via the boundary circles $\{C_\theta^i | i = 1, \cdots, s_1,\ 0 \leqq \theta < 2\pi\}$. Let F_θ^1 be one of the surfaces made by this construction. F_θ^1 consists of one sub surface ${F_\theta}'$ and s_1 copies of a fiber of ξ_0 which are connected to ${F_\theta}'$ on the circles $\{C_\theta^i\}_{i=1,\cdots,s_1}$. Since the family $\{{F_\theta}'\}$ fills up the solid torus T_1 and M_0 is fibered, the family of surfaces $\{F_\theta^1 | 0 \leqq \theta < 2\pi\}$ fills up the complement $M_1 = \overline{S^3 \setminus N(K_1(r_1, s_1))}$. Similar to the proof of Lemma 3.3, we can see that surfaces of the family $\{F_\theta^1\}$ are disjoint. Then the map $p : M_1 \to S^1 : x \in F_\theta^1 \mapsto \theta$ defines the fibration ξ_1.

Therefore, $K_1(r_1, s_1)$ is a fibered knot embedded in S^3. Let T_2 be a regular neighbourhood of $K_1(r_1, s_1)$, T_2 is also solid torus. We define the longitude l_2 on ∂T_2 so that its homology class coincides with the homology class of a curve which is the boundary of a fiber of the fibration ξ_1, and define the meridian m_2 so that it intersects l_2 at one point and bounds a disk in T_2. Then we define a new knot $K_2(r_2, s_2)$ which is a simple closed arc on ∂T_2 whose homology class is represented by $r_2 m_2 + s_2 l_2 \in H_1(\partial T_2)$. Replacing the solid torus T_2 by same V, we can construct the fibration ξ_2 on $M_2 = \overline{S^3 \setminus N(K_2(r_2, s_2))}$.

Iterating this construction, we can get the sequence of knots $\{K_i(r_i, s_i)\}$, and then we call it an iterated torus knot sequence. Simply, we call $K_i(r_i, s_i)$ an iterated torus knot. By this construction, the complement M_i of every iterated torus knot is fibered with the fibration ξ_i.

For every iterated torus knot, we can extend the result of Theorem 3.1.

Theorem 4.1. *Let $K_i(r_i, s_i)$ be the iterated torus knot defined as above. Then there is a family of taut foliations $\{\mathcal{F}_x\}$ of the exterior of $K_i(r_i, s_i)$ which realizes any boundary slope in the open interval $(-\infty, 1)$.*

We shall prove this theorem by the same steps as in the proof of the main theorem. First we define two arcs on the fiber of fibration ξ_i. We denote the infinite cover of the solid torus V by $\tilde{V} \subset \mathbb{R}^3$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the arcs on $\partial\tilde{V}$ such that

$$\tilde{\alpha} = \left\{(t,\ 1,\ 0) \in \partial\tilde{V} \mid 0 \leqq t \leqq \frac{2\pi}{r_i}\right\}$$
$$\tilde{\beta} = \left\{(t,\ \cos\frac{2\pi}{s_i},\ \sin\frac{2\pi}{s_i}) \in \partial\tilde{V} \mid \frac{2\pi}{r_i} \leqq t \leqq \frac{4\pi}{r_i}\right\}.$$

We define the two arcs α and β on the boundary of ∂V so that $\alpha = q(\tilde{\alpha})$ and $\beta = q(\tilde{\beta})$ where the map $q : \tilde{V} \to V$ is the covering map. Since α and β are the arcs on C_0^1 and C_0^2 respectively, α and β are properly embedded in the fiber F_0 of the fibration ξ_i, and these end points are on the boundary of a regular neighbourhood of the iterated torus knot $K_i(r_i, s_i)$.

Let $h : F_0 \to F_0$ be the monodromy map of the fibration ξ_i. The rotation map $R_\theta : \tilde{V} \to \tilde{V}$ defined in previous section induces the map $\widehat{R_\theta} : V \to V$ by composition with the covering map q, $\widehat{R_\theta} = q \circ R_\theta$. We define a map $h' : V \to V$ by $h' = \widehat{R_{2\pi}}$. By construction of the fibration ξ_i, the map h' maps the subsurface $F_0 \cap V$ to $F_0 \cap V$. If we define the map h' on complementary region of V such that for $k = 1, \cdots, s_i$, h' maps the fiber F_k^{i-1} of the fibration ξ_{i-1} connected with $F_0{}'$ via C_0^k to the fiber F_{k+1}^{i-1} of ξ_{i-1} connected with $F_0{}'$ via C_0^{k+1}, we can extend h' to the monodromy h. Because of this extension, we can see that $h(\alpha) = \beta$.

Now we consider the complement $M_i = \overline{S^3 \setminus N(K_i(r_i, s_i))}$ of the torus knot $K_i(r_i, s_i)$ as the quotient space of the product of the fiber F_0 and a unit interval $I = [0, 1]$;

$$M_i = F_0 \times [0,1] \big/ (x, 1) \sim (h(x), 0).$$

We define the orientation of F_0 such that the positive side is the positive direction of the unit interval $[0, 1]$.

We define a coordinate system on the torus boundary $\partial M_i = \partial N(K_i(r_i, s_i))$ by choosing two specific oriented simple closed curves λ and μ which are called a longitude curve and a meridian curve so that λ is the boundary of ∂F_0 and μ satisfies that $\langle \lambda, \mu \rangle = 1$ and μ bounds an essential disk in $N(K_i(r_i, s_i))$. In this definition, the orientation of λ is induced from the orientation of F_0.

Lemma 4.2. *The pair of properly embedded arcs (α, β) defined above is a good pair.*

Proof. By the construction of ξ_i, for some parameter $k, k' \in [0, 2\pi)$, α is a part of the boundary of the fiber F_k^{i-1} of the fibration ξ_{i-1} and β is a part of the boundary of the fiber $F_{k'}^{i-1}$. Since F_k^{i-1} and $F_{k'}^{i-1}$ are disjoint, α and β have no self intersection and are disjoint.

We define a temporary orientation on $K_i(r_i, s_i)$ which is induced from the orientation of the rotation map R_θ, and denote the end points of α and β by $\partial^1\alpha$ and $\partial^2\alpha$, $\partial^1\beta$ and $\partial^2\beta$. The distinction between ∂^1 and ∂^2 is defined by the direction induced from the direction of x-axis of $\tilde{V}$. Similar to the proof of Lemma 3.6, tracing these four end points along $K_i(r_i, s_i)$ with the orientation, we can see that these points are placed along ∂F_0 in the order $\partial^1\alpha$, $\partial^1\beta$, $\partial^2\alpha$, $\partial^2\beta$.

□

Now we define the branched surface B_- such that $B_- = \langle F_0 \ ; \ D\rangle$.

Lemma 4.3. *There is a simple arc γ_- properly embedded in F_0 such that B_- is affinely measured with respect to γ_-.*

Proof. Let T be a regular neighbourhood of $\alpha \cap \beta \cup \partial F_0$. By the argument of the proof of Lemma 3.7, we can take a simple arc γ_- properly embedded in T so that it satisfies

$$[\gamma_-] = -[\alpha] + [\beta] \in H_1(F_0, \partial F_0).$$

Although there are two choices of simple arc γ_- up to isotopy, we can see that $B_- \setminus \gamma_-$ consists of two branched surfaces and they have affine measure for each choice of γ_-. □

In order to define the branched surfaces B_-, we take the left type of the two choices of γ_- in Figure 6. Let ${\lambda_x}'$ be an affinely measured lamination carried by $B_- \setminus \gamma_-$ with the weight as shown in Figure 7, and f be the scaling map on $\gamma_- \times I$ such that

$$f : \gamma_- \times [0,1] \to \gamma_- \times [0,1] : (p,t) \mapsto (p, (1+x)t).$$

We glue two copies of $\gamma_- \times I$ on $\partial_v N(B_- \setminus \gamma_-) \setminus \partial_v N(B_-)$ by f, and get the lamination λ_x in $N(B_-)$ carried by the branched surface B_-.

Lemma 4.4. *λ_x realizes the boundary slope $\dfrac{-x^2}{x+1}$.*

Proof. Let $\tau = B_- \cap \partial M_i$ be the train track on ∂M_i. By the definition of γ_- the end points $\partial\gamma_-$ intersects τ at two points. As the proof of Lemma 3.8, we can see that these two points separate τ into two parts τ_1' and τ_2'. We assign the affine measure on τ_1' and τ_2' induced from an affine measure on $B_- \setminus \gamma_-$. Gluing by the scaling map f implies that the affine measure w_2' on τ_2' changes into $w_2' \times \frac{1}{1+x}$ and in consequence τ has the affine measure w.

Let $\tau(w)$ be a lamination on $N(\tau) \subset \partial M$ carried by the measured train track τ with the affine measure w. By construction $\tau(w)$ is the restriction of the lamination λ_x to ∂M_i. It means that the boundary slope of λ_x is calculated by the slope of a leaf of $\tau(w)$. We calculate the slope of a leaf of $\tau(w)$ as in Figure 8, then we obtain that

$$\text{slope } \tau(w) = \frac{\langle \tau(w), \lambda\rangle}{\langle \mu, \tau(w)\rangle} = \frac{x}{1+x} - x = \frac{-x^2}{x+1}.$$

Therefore, we see that the lamination λ_x realizes boundary slope $\frac{-x^2}{x+1}$.

□

Letting the parameter x range over $[0, \infty)$ in the above formula of the slope, we conclude that the family of laminations $\{\lambda_x\}$ realizes all boundary slopes in $(-\infty, 0]$. To prove Theorem 4.1, we extend the lamination λ_x to the taut foliation $\mathcal{F}_x$.

Proposition 4.5. *The lamination λ_x extends to a taut foliation $\mathcal{F}_x$ on M_i which realizes any boundary slope in $(-\infty, 0]$.*

Proof. Let λ_x be the lamination on $N(B_-)$ defined above. Similar to the proof of Proposition 3.10, the complementary region $M_{B_-} = \overline{M_i \setminus N(B_-)}$ is homeomorphic to the product space of the subsurface $F_0 \setminus \alpha$ and the interval $[0,1]$. We foliate this complementary region by the product foliation whose vertical boundary coincides with the $\partial_v N(B_-)$ and fill the complementary region of λ_x in $N(B_-)$ by parallel leaves. Then we can extend λ_x to a foliation $\mathcal{F}_x$.

If $x = 0$ we consider that $\mathcal{F}_x$ is the original fibration ξ_i on M_i. Otherwise, at the boundary ∂M_i the meridian curve meets each leaf of $\mathcal{F}_x$ transversely. In both cases, $\mathcal{F}_x$ is a taut foliation.

□

The existence of taut foliations which realizes all boundary slope in the interval $[0, 1)$ is shown as same as the latter part of Section 3. By joining the results of Proposition 4.5, the proof of Theorem 4.1 is completed.

Using the method of the proof of Theorem 4.1, we obtain the following Corollary.

Corollary 4.6. *Let K be a fibered knot embedded in S^3. The boundary ∂T of the regular neighbourhood T of K is a torus. Let $\hat{K}(r, s)$ be a simple closed curve on ∂T whose homology class is represented by $rm + sl \in H_1(\partial T)$ where m and l are the meridian and the longitude respectively. Then there is a family of taut foliations $\{\mathcal{F}_x\}$ of the exterior of $\hat{K}(r, s)$ which realizes any boundary slope in the open interval $(-\infty, 1)$.*

Proof. Since K is a fibered knot embedded in S^3, there is a fibration ξ in $M' = \overline{S^3 \setminus N(K)}$. We define a longitude curve l on ∂T as the boundary of a fiber of ξ and a meridian curve m such that m intersects l at one point and bounds a disk in $N(K)$. We replace the solid torus T by the solid torus V defined before such that the circles $\{C^i_\theta\}_{i=1,\cdots,s}$ coincide with the parallel curves of the longitude l. By joining the internal surfaces in V and original fiber surfaces of ξ along the family of circles $\{C^i_\theta | i = 1, \cdots, s,\ 0 \leqq \theta < 2\pi\}$, we obtain a fibration $\hat{\xi}$ on $M = \overline{S^3 \setminus \hat{K}(r, s)}$.

By above construction, if we take two arcs α and β properly embedded in a fiber of $\hat{\xi}$ as defined in this section, we can construct the branched surfaces B_- and B_+ by using the pair (α, β) which carry the laminations λ_x with the property that they realize all boundary slopes in $(-\infty, 0]$ and $[0, 1)$ respectively. We extend these laminations to taut foliations by filling the complementary region, then we obtain a family of taut foliations which realizes all boundary slopes in $(-\infty, 1)$.

□

5. Extension to a link case

In this section, we partially extend theorem of Rachel Roberts (**Theorem 4.1** in [6]) to a fibered link case.

We denote a surface whose genus is i and has j boundaries by $\Sigma_{i,j}$. Let M be an oriented, compact, fibered 3-manifold with a monodromy h and an orientable fiber $\Sigma_{i,j}$. Any boundary component of M is homeomorphic to a torus. We suppose that $j = 2$ and i is more than two, for simplicity we write $\Sigma_{i,2}$ by F, and the monodromy h maps each boundary to itself. We consider M as a quotient space; $M = F \times [0, 1]/(h(x), 0) \sim (x, 1)$ where $F = \Sigma_{i,2}$ and $x \in F$. The orientation of F is defined by the increasing direction of this interval $[0, 1]$. For this orientation of F we define a coordinate system (μ, λ) for each component of ∂M such that λ is a component of ∂F with the orientation induced from F and μ satisfies that $\langle \mu, \lambda \rangle = 1$.

Let α be a simple non-separating arc properly embedded in $F \times \{0\}$. Setting $D = \alpha \times [0, 1]$ we consider that D is properly embedded in $F \times [0, 1]$ such that ∂D consists of four arcs, $\partial\alpha \times [0, 1]$ on $\partial F \times [0, 1]$, α_+ on $F \times \{0\}$ and α_- on $F \times \{1\}$. In order to prepare for the later section where we construct a branched surface by these fibers and disks, we take the convention for the orientation of D such that if we fix the orientation of α the positive orientation of D is coherent with the positive turn direction of α with the right screw rule (see Figure 11).

For this settings, we state the theorem extended to a link case.

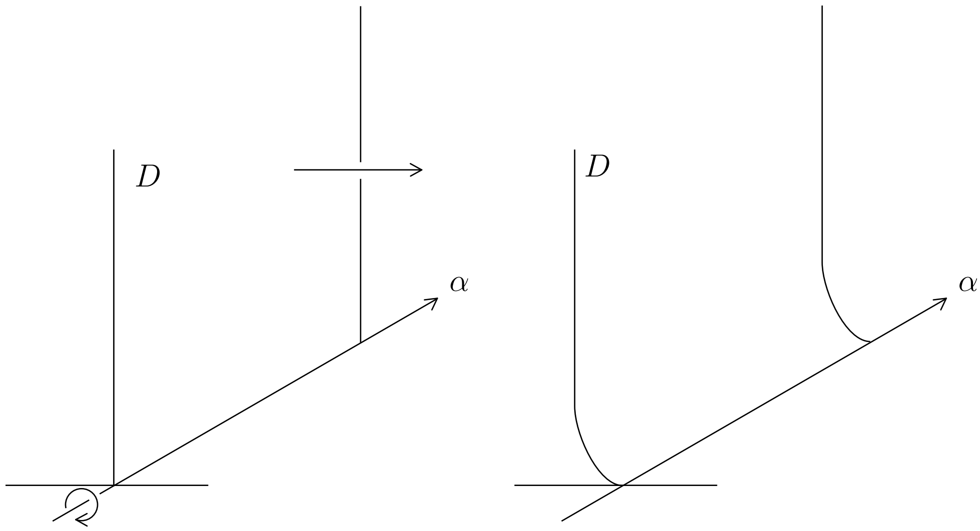

FIGURE 11

Theorem 5.1. *For $i = 1, 2$, let α^i be simple non-separating arcs properly embedded in F such that the end points of α^i are on one component of ∂F and α^i are disjoint each other. Let D^i be disks in M such that $D^i = \alpha^i \times [0, 1]$. If the arcs α^i and the monodromy h satisfy the condition (1) of Lemma 5.5, there is a branched surface which is made from a splitting of $B = \langle F; D_1, D_2\rangle$ such that it carries a family of laminations λ_x realizing all boundary slopes in $(-a_i, b_i)$ for some $a_i, b_i > 0$, $i = 1, 2$ where i corresponds to the component of the torus boundaries of M. Moreover, these laminations λ_x extend to taut foliations $\mathcal{F}_x$ with same property of slopes.*

We shall prove Theorem 5.1 by the following steps. In Lemma 5.2, we define a branched surface which is made from a sequence of arcs properly embedded in each fiber, and prove the existence of arcs $\{\gamma^i_k\}$ such that the branched surface is affinely measured with respect to $\{\gamma^i_k\}$. In Lemma 5.3, we prove that there are sequences of arcs which are the source of above branch surface. In Lemma 5.4, depending on the property of orientations of a sequence of arcs we can see that the branched surface carries laminations which realize all boundary slopes in a positive or negative part of above intervals. In Lemma 5.5, for sequences of arcs which are given by the arcs α^i and the monodromy h, this sequences can be modified to sequences with a certain property. There are two cases, one case is suitable for the assumption of Lemma 5.3 and then there is a desired branched surface, the other case is not suitable for the assumption of Lemma 5.3.

Recall that a pair of arcs δ and δ' properly embedded in a surface F is good if they are disjoint and their end points alternate along the boundary of F. For the sequence $\sigma = (\alpha_0, \alpha_1, \cdots, \alpha_n)$ of arcs properly embedded in F, if each pair (α_k, α_{k+1}) for $0 \leqq k < n$ is good we call the sequence σ is a good sequence.

Lemma 5.2. *Let $\sigma^i = (h(\alpha^i_n) = \alpha^i_0, \alpha^i_1, \cdots, \alpha^i_n)$, $i = 1, 2$ be good sequences and we suppose that four arcs $(\alpha^1_k, \alpha^1_{k+1})$ and $(\alpha^2_k, \alpha^2_{k+1})$ are disjoint $(k = 0, \cdots, n-1)$. For $1 \leqq k \leqq n$, we take disks $\{D^i_k\}$ in M such that*

$$D^i_k = \alpha^i_k \times \left[\frac{k-1}{n}, \frac{k}{n}\right].$$

We fix an orientation for each α_k^i and define orientations on D_k^i by our convention. We define the branched surface

$$B = \langle F_0, F_1, \cdots, F_{n-1}; D_1^1, D_1^2, D_2^1, D_2^2, \cdots, D_n^1, D_n^2 \rangle.$$

Then there is a family of simple arcs $\{\gamma_k^i\}$ properly embedded in each F_k for $0 \leqq k \leqq n-1, i = 1, 2$, such that B is affinely measured with respect to $\bigcup_{i=1,2} \bigcup_{k=0}^{n-1} \gamma_k^i$.

Proof. We denote each boundary of the fiber F_k by $\partial^i F_k$ for $i = 1, 2$. For $0 \leqq k \leqq n-1$ and $i = 1, 2$, there is a regular neighbourhood T_k^i of $\alpha_k^i \cup \alpha_{k+1}^i \cup \partial^i F_k$ which is homeomorphic to a torus with two boundary components in each F_k. One of these boundary components is $\partial^i F_k$, we denote the other component by C_k^i. Since the four arcs $(\alpha_k^1, \alpha_{k+1}^1)$ and $(\alpha_k^2, \alpha_{k+1}^2)$ are disjoint by assumption, we can take these T_k^1 and T_k^2 so that they are disjoint. Then two simple closed curves C_k^1 and C_k^2 are disjoint. Similar to the proof of Lemma 3.7, we define a simple arc γ_k^i properly embedded in T_k^i so that it satisfies

$$[\gamma_k^i] = -[\alpha_k^i] + [\alpha_{k+1}^i] \in H_1(F_k, \partial F_k).$$

There are two choices of the simple arc γ_k^i up to isotopy as in Figure 6, but now we choose the candidate suitable for our convention.

The properly embedded arc γ_k^i separates $T_k^i \setminus (\alpha_k^i \cup \alpha_{k+1}^i)$ into exactly two regions R and $R_{C_k^i}$, R does not intersect C_k^i and $R_{C_k^i}$ contains C_k^i. To prove this Lemma, we assign the weight 1 to $R_{C_k^i}$ and $1 + x$ to R for all T_k^i. Since the simple closed curve C_k^i is contained only in $R_{C_k^i}$ and the weight of $R_{C_k^i}$ is 1, we can put the weight 1 on the region which is bounded by C_k^1 and C_k^2 on F_k. Putting the weight x on each D_k^i, we see that $B \setminus (\bigcup_{i=1,2} \bigcup_{k=0}^{n-1} \gamma_k^i)$ has an affine measure. Then B is affinely measured with respect to $\bigcup_{i=1,2} \bigcup_{k=0}^{n-1} \gamma_k^i$.

□

Lemma 5.3. *Let α^1 and α^2 be two disjoint non-separating simple arcs properly embedded in F such that the boundary points $\partial\alpha^1$ are on $\partial^1 F$ and $\partial\alpha^2$ are on $\partial^2 F$. Then there are good sequences*

$$\sigma^i = (h(\alpha^i) = \alpha_0^i, \alpha_1^i, \cdots, \alpha_n^i = \alpha^i), i = 1, 2,$$

such that for $0 \leqq k \leqq n-1$, four arcs $(\alpha_k^1, \alpha_{k+1}^1)$ and $(\alpha_k^2, \alpha_{k+1}^2)$ are disjoint.

Proof. We suppose the arcs α^1 and α^2 are in the configuration shown in Figure 12.

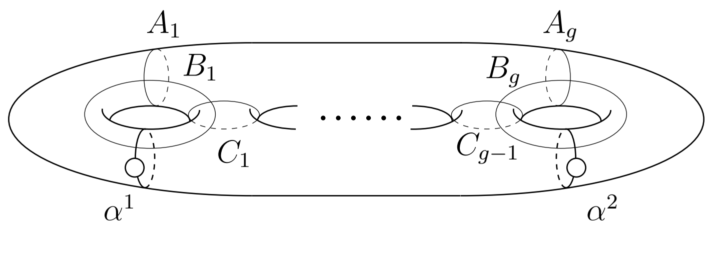

FIGURE 12

Lickorish proved in [1] that the group of orientation preserving automorphisms $Aut_+(F)$ of the surface of genus g is generated by the set $\mathcal{D}$ of Dehn twists with respect to the curves $A_1, \cdots, A_g, B_1, \cdots, B_g, C_1, \cdots, C_{g-1}$ in Figure 12. We denote the Dehn twist along these curves also by $A_1, \cdots, A_g, B_1, \cdots, B_g, C_1, \cdots, C_{g-1}$, then $\mathcal{D} = \{A_1, \cdots, A_g, B_1, \cdots, B_g, C_1, \cdots, C_{g-1}\}$. Let γ be a properly embedded arc in F whose endpoints are on one boundary component. If γ intersects only one

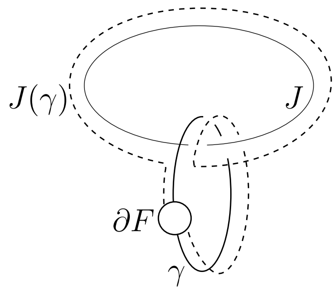

FIGURE 13

of the curves of $\mathcal{D}$ which we denote by J, then we easily see that $(\gamma, J(\gamma))$ is a good pair by isotoping $J(\gamma)$ slightly in the neighbourhood of $\gamma \cup J$ (see Figure 13).

Let J be any element of $\mathcal{D}$. If $J = B_1$, $(\alpha^1, A_1J(\alpha^1))$ is a good pair, and also $(A_1J(\alpha^1), (A_1{}^{-1})A_1J(\alpha^1) = J(\alpha^1))$ is a good pair. Then $(\alpha^1, A_1J(\alpha^1), J(\alpha^1))$ is a good sequence. By the same considerations, $(\alpha^2, B_g(\alpha^2), (JB_g{}^{-1})B_g(\alpha^2) = J(\alpha^2))$ is a good sequence. If $J = B_g$, by a symmetrical argument, $(\alpha^1, B_1(\alpha^1), J(\alpha^1))$ and $(\alpha^2, A_gJ(\alpha^2), J(\alpha^2))$ are good sequences. If $J \neq B_1$ and B_g, $(\alpha^1, B_1(\alpha^1), J(\alpha^1))$ and $(\alpha^2, B_g(\alpha^2), J(\alpha^2))$ are good sequences. In all cases, there are good sequences $\hat{\sigma^1} : \alpha^1 \to J(\alpha^1)$ and $\hat{\sigma^2} : \alpha^2 \to J(\alpha^2)$ with the same number of terms.

Now we decompose the monodromy h into compositions of elements of $\mathcal{D}$, $h = J_m \circ J_{m-1} \circ \cdots \circ J_1$ where $J_i \in \mathcal{D}$. By the above argument, there are good sequences

$$\hat{\sigma^i_k} : \alpha^i \to J_k(\alpha^i)$$

for each $i = 1, 2$ and $k = 1, \cdots, m$. For any good sequence $\sigma : \delta \to \delta'$ and $\psi \in Aut_+(F)$, $\psi(\sigma)$ is also a good sequence

$$\psi(\sigma) : \psi(\delta) \to \psi(\delta').$$

Hence if we concatenate these good sequences,

$$\begin{aligned}
\hat{\sigma}^i_m &: \alpha^i \to J_m(\alpha^i) \\
J_m(\hat{\sigma}^i_{m-1}) &: J_m(\alpha^i) \to J_m \circ J_{m-1}(\alpha^i) \\
J_m \circ J_{m-1}(\hat{\sigma}^i_{m-2}) &: J_m \circ J_{m-1}(\alpha^i) \to J_m \circ J_{m-1} \circ J_{m-2}(\alpha^i) \\
&\vdots \\
J_m \circ J_{m-1} \circ \cdots J_2(\hat{\sigma}^i_1) &: J_m \circ J_{m-1} \circ \cdots J_2(\alpha^i) \to J_m \circ J_{m-1} \circ \cdots J_2 \circ J_1(\alpha^i) \\
&\qquad\qquad = h(\alpha^i)
\end{aligned}$$

we obtain a good sequence $\hat{\sigma}^i : \alpha^i \to h(\alpha^i)$ for $i = 1, 2$ with same number of terms. If a pair (α_k, α_{k+1}) is good, the opposite pair (α_{k+1}, α_k) is also good. Therefore if we reverse the order of the sequence $\hat{\sigma}^i$, we can obtain the desired good sequence $\sigma^i = (h(\alpha^i) = \alpha^i_0, \alpha^i_1, \cdots, \alpha^i_n = \alpha^i)$, for $i = 1, 2$.

□

In Lemma 5.4, we shall construct two branched surfaces such that the one of them carries the family of laminations which realizes all boundary slopes in negative part of the interval of the conclusion of Theorem 5.1, the other carries positive part. In order to define these specific branched surfaces, we define some notation for orientations of simple arcs on the surface.

Let α and β be simple arcs properly embedded in F and we suppose the pair (α, β) is good. If we give the orientations for α and β, there are two cases for the orientation of the pair (α, β) in the neighbourhood of ∂F as in Figure 14.

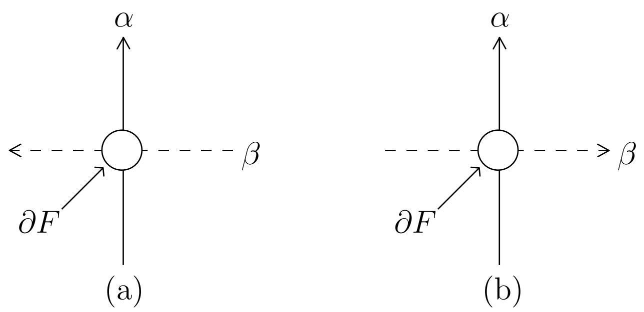

Figure 14

We call a good pair (α, β) is *a negatively oriented pair* if it is oriented as in Figure 14 (a), otherwise if it is oriented as in Figure 14 (b) we call it *a positively oriented pair*. For a good sequence $\sigma = (\alpha_0, \alpha_1, \cdots, \alpha_n)$, we call σ is *a negatively oriented good sequence* if each pair (α_{i-1}, α_i) is a negatively oriented pair for $i = 1, 2, \cdots, n$, and we call σ is *a positively oriented good sequence* if each pair (α_{i-1}, α_i) is a positively oriented pair.

For the pair of good sequences $\sigma = (\sigma^1, \sigma^2)$ defined in Lemma 5.2, we denote the branched surface defined in Lemma 5.2 by B_σ, and we consider that each sector of B_σ constructed from $\{D^i_k\}$ has the orientation induced from the arcs $\{\alpha^i_k\}$ with our convention defined before. We denote the two boundaries of M which are homeomorphic to a torus by $\partial^i M$ for $i = 1, 2$, which corresponds to $\partial^i F \times [0, 1]/(h(x), 0) \sim (x, 1)$ where $x \in \partial^i F$.

Lemma 5.4. *For α^1 and α^2 defined in the proof of Lemma 5.3, if $\sigma^1 = (h(\alpha^1) = \alpha^1_0, \alpha^1_1, \cdots, \alpha^1_n = \alpha^1)$ and $\sigma^2 = (h(\alpha^2) = \alpha^2_0, \alpha^2_1, \cdots, \alpha^2_n = \alpha_2)$ are both negatively oriented good sequences, then the branched surface B_σ carries the family of laminations $\{\lambda_x\}$ which realizes all boundary slopes of $\partial^i M$ in $(-a_i, 0]$ for some $a_i > 0$. If σ^1 and σ^2 are both positively oriented good sequences, B_σ carries the family of laminations $\{\lambda_x\}$ which realize all boundary slopes of $\partial^i M$ in $[0, b_i)$ for some $b_i > 0$.*

Proof. We assume that σ^1 and σ^2 are both positively oriented. Let τ_{σ^i} be the train track on $\partial^i M$ such that $\tau_{\sigma^i} = B_\sigma \cap \partial^i M$ for $i = 1, 2$. By Lemma 5.2 there is a family of properly embedded arcs $\{\gamma^i_k\}$ such that B_σ is affinely measured with respect to $\bigcup_{i=1,2} \bigcup_{k=0}^{n-1} \gamma^i_k$. Let λ_x be a lamination which is carried by B_σ with this measure. By assumption that σ is positively oriented, we can see that the boundary points of the family $\{\gamma^i_k\}$ cut each τ_{σ^i} on $\partial^i M$ into two parts as follows. For each $k = 0, 1, \cdots, n-1$, we define the orientation on $\partial^i F_k$ induced from the orientation of F_k. Since each $\partial^i F_k$ is parallel to the longitude curve on $\partial^i M$ and the orientation of ∂F_k is induced from the interval $[0, 1]$ used in the definition of quotient space M, these circles $\{\tau_{\sigma^i} \cap \partial F_k\}$ are parallel circles on $\partial^i M$ with coherent orientation. By applying this orientation to the train tracks $\{\tau_{\sigma^i}\}$, the two sub arcs $\partial D^i_k = D^i_k \cap \partial^i M$ have two types, one is oriented downwards, the other is oriented upwards (see Figure 15).

In each $\partial^i F_k$ the end points of γ^i_k cut the circle $\tau_{\sigma^i} \cap F_k$ into exactly two parts. We temporary denote these components by σ^i_k and ${\sigma^i_k}'$. If σ^i_k intersects the component of ∂D^i_{k+1} which is oriented downwards, σ^i_k intersects the component of ∂D^i_k which is also oriented downwards, and also ${\sigma^i_k}'$ intersects the components of ∂D^i_{k+1} and ∂D^i_k which are both oriented upwards. By these considerations, any components of ∂D^i_k for $k = 1, 2, \cdots, n$, oriented downwards are connected via σ^i_k for $k = 0, 1, \cdots, n-1$,

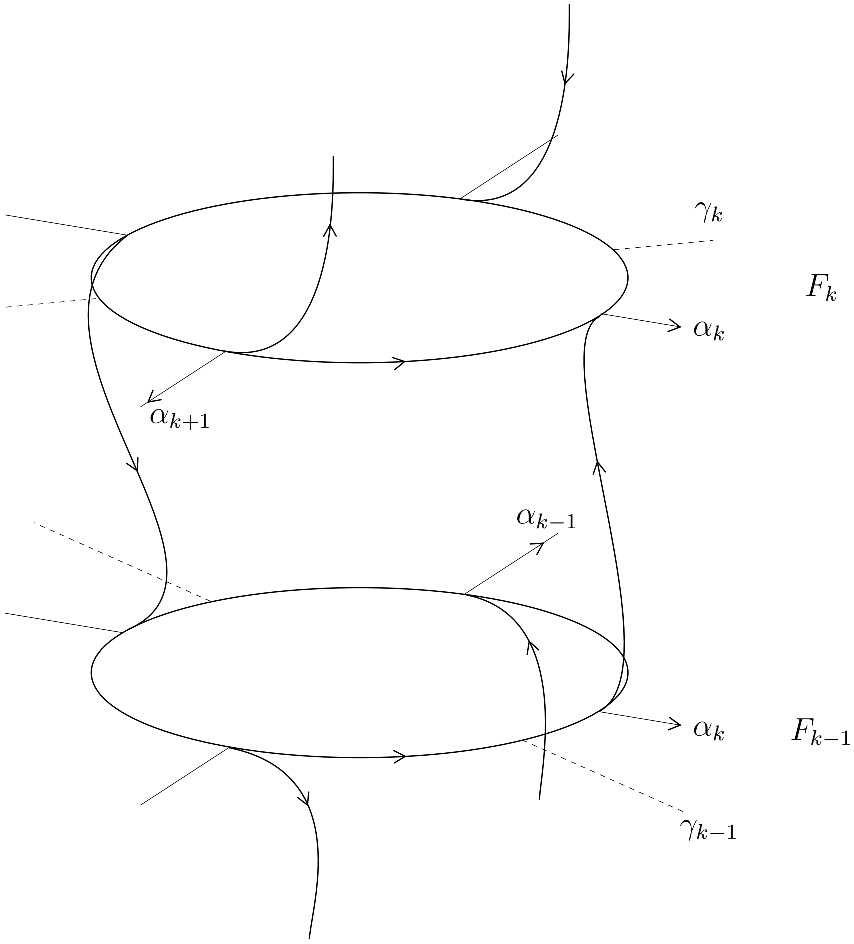

FIGURE 15

and also any components of ∂D_k^i oriented upwards are connected via ${\sigma_k^i}'$. Therefore τ_{σ^i} on $\partial^i M$ is separated into exactly two parts.

By this conclusion we can apply the scale map f as before, then τ_{σ^i} has the affine measure w_{σ^i} induced from the measure on the branched surface B_σ. Hence the lamination λ_x intersects $\partial^i M$ in the measured lamination $\tau_{\sigma^i}(w_{\sigma^i})$.

We calculate boundary slopes of $\tau_{\sigma^i}(w_{\sigma^i})$ by converting the measured train track $(\tau_{\sigma^i}, w_{\sigma^i})$ into a *combinatorially equivalent* (see [5] Chapter 2) measured train track $({\tau_{\sigma^i}}', {w_{\sigma^i}}')$.

We denote one of the branches of $\partial^i D_n^i \cap \partial^i M \subset \tau_{\sigma^i}$ which is weighted $\frac{x}{x+1}$ by ν_1^i and the other weighted x by ν_2^i. We remove ν_1^i from τ_{σ^i} and changes the weight of ν_2^i into

$$x - \frac{x}{x+1} = \frac{x^2}{x+1},$$

and except for the weight of sectors on $\partial^i F_0$ and $\partial^i F_{n-1}$ which are weighted $x+1$, fix the weight on all sectors of τ_{σ^i}. The excepted sectors get the weight

$$x - \frac{x}{x+1} + 1 = \frac{x^2}{x+1} + 1$$

(see Figure 16).

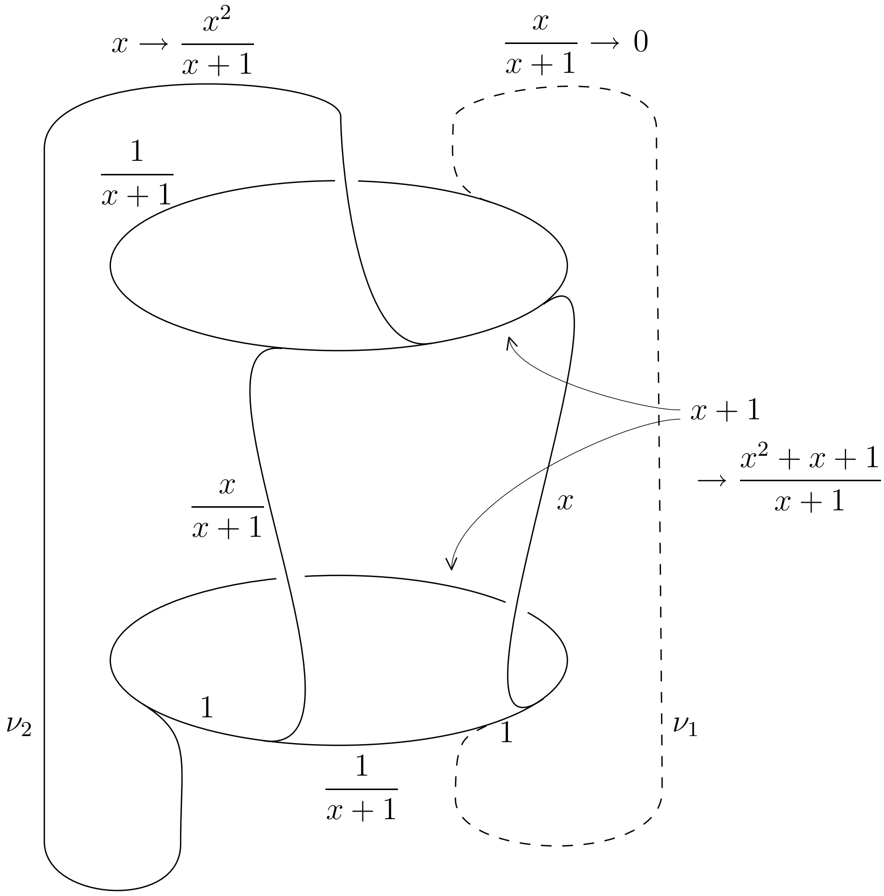

Figure 16

Next we collapse the resultant train track along $\partial^i F_0 \times [0, \frac{n-1}{n}]$ to get ${\tau_{\sigma^i}}'$. Let $z(x)$ be a linear combination of 1, x and $\frac{1}{x+1}$ such that

$$z(x) = c_2 x + c_1 + \frac{c_0}{x+1},$$

where c_0,c_1 and c_2 are non-negative constants. Since the sectors of τ_{σ^i} contained in $\partial^i F_0 \cup \partial^i F_1 \cup \cdots \partial^i F_{n-1}$ are positively weighted by non-negative linear combinations of 1, x and $\frac{1}{x+1}$, the three sectors of ${\tau_{\sigma^i}}'$ are weighted $\frac{x^2}{x+1}$, $z(x)$ and $z(x) + \frac{x^2}{x+1}$ (see Figure 17).

By taking the two candidates of the meridian μ there are two cases of the slope of ${\tau_{\sigma^i}}'$, and hence of τ_{σ^i} such that

$$\frac{\langle {\tau_{\sigma^i}}'({w_{\sigma^i}}'), \lambda \rangle}{\langle \mu, {\tau_{\sigma^i}}'({w_{\sigma^i}}') \rangle} = \frac{\frac{x^2}{x+1}}{z(x)},$$

or

$$\frac{\langle {\tau_{\sigma^i}}'({w_{\sigma^i}}'), \lambda \rangle}{\langle \mu, {\tau_{\sigma^i}}'({w_{\sigma^i}}') \rangle} = \frac{\frac{x^2}{x+1}}{z(x) + \frac{x^2}{x+1}}.$$

Therefore, letting x range over $[0, \infty)$ we obtain a family of laminations $\{\lambda_x\}$ which realizes all boundary slopes either in $[0, \frac{1}{c_2})$ or in $[0, \frac{1}{c_2+1})$ respectively. Then we see that B_σ carries the family of laminations which realize all boundary slopes of $\partial^i M$ in $[0, b_i)$ for some $b_i > 0$.

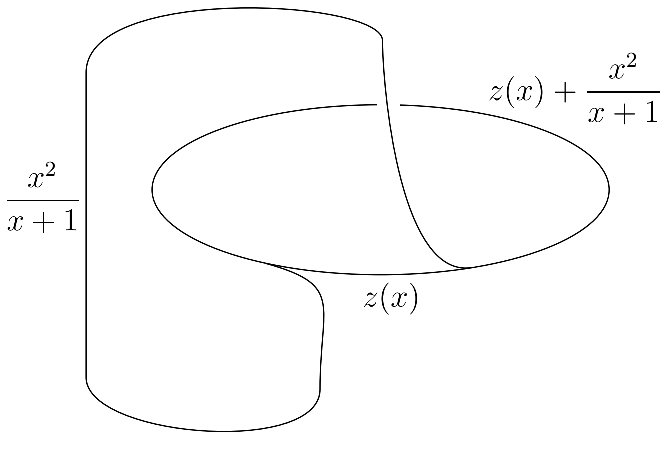

FIGURE 17

We can apply the symmetrical argument to obtain a similar conclusion for a negatively oriented good sequence σ', and then we see that $B_{\sigma'}$ carries the family of laminations which realize all boundary slopes of $\partial^i M$ in $(-a_i, 0]$ for some $a_i > 0$. By gathering these intervals, we complete the proof of this Lemma.

□

Lemma 5.5. *Let $\sigma^i = (h(\alpha^i_n) = \alpha^i_0, \alpha^i_1, \cdots, \alpha^i_n)$, $i = 1, 2$ be good sequences and we suppose that four arcs $(\alpha^1_k, \alpha^1_{k+1})$ and $(\alpha^2_k, \alpha^2_{k+1})$ are disjoint for $0 \leqq k \leqq n$. Then we can modify the sequence $\sigma = (\sigma^1, \sigma^2)$ into $\bar{\sigma} = (\bar{\sigma^1}, \bar{\sigma^2})$ with one of the following two properties,*

1. *both of $\bar{\sigma^1}$ and $\bar{\sigma^2}$ are either positively oriented good sequences or negatively oriented good sequences,*
2. *$\bar{\sigma} = (\bar{\sigma^1}, \bar{\sigma^2})$ has the property that $(\alpha^1_{n-1}, \alpha^1_n)$ is a positively oriented good pair and $(\alpha^2_{n-1}, \alpha^2_n)$ is a negatively oriented good pair, or $(\alpha^1_{n-1}, \alpha^1_n)$ is a negatively oriented good pair and $(\alpha^2_{n-1}, \alpha^2_n)$ is a positively oriented good pair. Other pairs $(\alpha^i_{k-1}, \alpha^i_k)$, $k = 1, \cdots, n-2$, $i = 1, 2$ are either all positive or all negative pair.*

Proof. For the original good sequences

$$\sigma^1 = (\alpha^1_0, \alpha^1_1, \cdots, \alpha^1_n)$$
$$\sigma^2 = (\alpha^2_0, \alpha^2_1, \cdots, \alpha^2_n),$$

there are the following eight cases:

$(NP)^P_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is positively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is negatively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is positively oriented.

$(NP)^N_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is negatively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is negatively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is positively oriented.

$(PN)^P_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is positively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is positively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is negatively oriented.

$(PN)^N_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is negatively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is positively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is negatively oriented.

$(NN)^P_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is positively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is negatively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is negatively oriented.

$(NN)^N_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is negatively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is negatively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is negatively oriented.

$(PP)^P_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is positively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is positively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is positively oriented.

$(PP)^N_k$: For $0 \leqq j < k$ and $i = 1, 2$, each pair $(\alpha^i_j, \alpha^i_{j+1})$ is negatively oriented; $(\alpha^1_k, \alpha^1_{k+1})$ is positively oriented and $(\alpha^2_k, \alpha^2_{k+1})$ is positively oriented.

For each of eight cases, we define operations as follows.

For the case $(NP)^P_k$, we replace the pair $(\alpha^1_k, \alpha^1_{k+1})$ by the sequence $(\alpha^1_k, -\alpha^1_{k+1}, -\alpha^1_k, \alpha^1_{k+1})$, and replace the pair $(\alpha^2_k, \alpha^2_{k+1})$ by the sequence $(\alpha^2_k, \alpha^2_{k+1}, -\alpha^2_k, -\alpha^2_{k+1})$ and rewrite $-\alpha^2_{k+1}$ to α^2_{k+1}, i.e. we reverse the orientation of α^2_{k+1}. Then we can see that $(\alpha^1_k, -\alpha^1_{k+1}, -\alpha^1_k, \alpha^1_{k+1})$ and $(\alpha^2_k, \alpha^2_{k+1}, -\alpha^2_k, \alpha^2_{k+1})$ are positively oriented good sequences. Then all pairs before α^i_{k+1} are a positive good pairs. It means that we modify the cases $(NP)^P_k$ into the cases $(NP)^P_{k+1}$, $(PN)^P_{k+1}$, $(PP)^P_{k+1}$, or $(NN)^P_{k+1}$.

For other cases, the operations are as follows.

$(NP)^N_k$: $(\alpha^1_k, \alpha^1_{k+1}) \to (\alpha^1_k, \alpha^1_{k+1}, -\alpha^1_k, -\alpha^1_{k+1})$ and rewrite the last term,
$(\alpha^2_k, \alpha^2_{k+1}) \to (\alpha^2_k, -\alpha^2_{k+1}, -\alpha^2_k, \alpha^2_{k+1})$

$(PN)^P_k$: $(\alpha^1_k, \alpha^1_{k+1}) \to (\alpha^1_k, \alpha^1_{k+1}, -\alpha^1_k, -\alpha^1_{k+1})$ and rewrite the last term,
$(\alpha^2_k, \alpha^2_{k+1}) \to (\alpha^2_k, -\alpha^2_{k+1}, -\alpha^2_k, \alpha^2_{k+1})$

$(PN)^N_k$: $(\alpha^1_k, \alpha^1_{k+1}) \to (\alpha^1_k, -\alpha^1_{k+1}, -\alpha^1_k, \alpha^1_{k+1})$,
$(\alpha^2_k, \alpha^2_{k+1}) \to (\alpha^2_k, \alpha^2_{k+1}, -\alpha^2_k, -\alpha^2_{k+1})$ and rewrite the last term

$(NN)^P_k$: $(\alpha^1_k, \alpha^1_{k+1}) \to (\alpha^1_k, -\alpha^1_{k+1}, -\alpha^1_k, \alpha^1_{k+1})$,
$(\alpha^2_k, \alpha^2_{k+1}) \to (\alpha^2_k, -\alpha^2_{k+1}, -\alpha^2_k, \alpha^2_{k+1})$

$(NN)^N_k$: no operations

$(PP)^P_k$: no operations

$(PP)^N_k$: $(\alpha^1_k, \alpha^1_{k+1}) \to (\alpha^1_k, -\alpha^1_{k+1}, -\alpha^1_k, \alpha^1_{k+1})$,
$(\alpha^2_k, \alpha^2_{k+1}) \to (\alpha^2_k, -\alpha^2_{k+1}, -\alpha^2_k, \alpha^2_{k+1})$

By doing these operations, in each case the resultant sequences satisfy the condition of one of the cases $(NP)^P_{k+1}$, $(NP)^N_{k+1}$, $(PN)^P_{k+1}$, $(PN)^N_{k+1}$, $(NN)^P_{k+1}$, $(NN)^N_{k+1}$, $(PP)^P_{k+1}$, $(PP)^N_{k+1}$. Therefore if we iterate these operations, finally we reach one of the following situations:

(1a) the resultant sequences $\bar{\sigma^1}$ and $\bar{\sigma^2}$ are both positively oriented.
(1b) the resultant sequences $\bar{\sigma^1}$ and $\bar{\sigma^2}$ are both negatively oriented.
(2a) For $0 \leqq k \leqq n-1$ and $i = 1, 2$ each pair $(\alpha^i_k, \alpha^i_{k+1})$ is positively oriented but $(\alpha^1_{n-1}, \alpha^1_n)$ is negatively oriented and $(\alpha^2_{n-1}, \alpha^2_n)$ is positively oriented, or $(\alpha^1_{n-1}, \alpha^1_n)$ is positively oriented and $(\alpha^2_{n-1}, \alpha^2_n)$ is negatively oriented.
(2b) For $0 \leqq k \leqq n-1$ and $i = 1, 2$ each pair $(\alpha^i_k, \alpha^i_{k+1})$ is negatively oriented but $(\alpha^1_{n-1}, \alpha^1_n)$ is negatively oriented and $(\alpha^2_{n-1}, \alpha^2_n)$ is positively oriented, or $(\alpha^1_{n-1}, \alpha^1_n)$ is positively oriented and $(\alpha^2_{n-1}, \alpha^2_n)$ is negatively oriented.

Hence the cases (1a) and (1b) is the case (1) of the conclusion of this Lemma, and the cases (2a) and (2b) is the case (2).

□

Lemma 5.6. *Let λ_x be the lamination obtained in Lemma 5.4. Then the lamination λ_x extends to a taut foliation $\mathcal{F}_x$ with the same boundary slope property.*

Proof. For any point x^1 on α^1_k and any point x^2 on α^2_k, let δ be a simple arc on F_{k-1} whose end points are $\partial^1\delta = x^1$ and $\partial^2\delta = x^2$ and such that δ does not intersect α^1_k and α^2_k. Since α^1_k and α^2_k are disjoint and both non-separating, there is such a simple arc δ. Let ${F_{k-1}}'$ be a sub surface which is a metrically completed surface of the open surface $F_{k-1} \setminus (\alpha^1_k \cup \alpha^2_k)$. Then we can regard that δ is properly embedded in ${F_{k-1}}'$. The boundaries of the sub surface ${F_{k-1}}'$ has four components, two of them are copies of α^1_k and the others are copies of α^2_k. We denote these boundaries by α_k^{i+} and α_k^{i-} for $i = 1, 2$, where the signs mean that the copy with $+$ sign is on the right side of the original arc with respect to the orientation of original arc, the sign $-$ means that it is on opposite side.

Since we construct the disks $\{D_k^i\}$ by using the sub interval $[\frac{k-1}{n}, \frac{k}{n}]$, we denote the image of $\partial^i\delta$ on F_k induced from this construction of disks by $\partial^i\bar{\delta}$ for $i = 1, 2$, and by the same construction we can consider the image of δ on F_k, we denote it by $\bar{\delta}$. The arcs α_k^1 and α_k^2 also separate the surface F_k into sub surface $F_k{}'$ with four boundary components α_k^{i+} and α_k^{i-} for $i = 1, 2$. In order to specify these arcs we denote them by $\bar{\alpha}_k^{i+}$ and $\bar{\alpha}_k^{i-}$. Because of our convention for the orientation of the disks $\{D_k^i\}$, we can see that α_k^{i-} correspond to the vertical boundary $\partial_v N(B_\sigma)$ near the surface F_{k-1} and $\bar{\alpha}_k^{i+}$ correspond to $\partial_v N(B_\sigma)$ near the surface F_k.

There are four cases related to the endpoints condition of δ. If $\partial^1\delta \in \alpha_k^{1+}$ and $\partial^2\delta \in \alpha_k^{2+}$, then $\partial^1\bar{\delta} \in \bar{\alpha}_k^{1+}$ and $\partial^2\bar{\delta} \in \bar{\alpha}_k^{2+}$. In this case, by the condition of the orientation of disks $\{D_k^i\}$, we can modify δ by sliding the end point $\partial^1\delta$ to the point $\partial^1\bar{\delta}$ along the disk D_k^1 and $\partial^2\delta$ to the point $\partial^2\bar{\delta}$ along the disk D_k^2. The resultant arc is smooth arc on B_σ with endpoints on $\bar{\alpha}_k^{1+}$ and $\bar{\alpha}_k^{2+}$. By the same argument, if $\partial^1\delta \in \alpha_k^{1+}$ and $\partial^2\delta \in \alpha_k^{2-}$, then there is a smooth arc on B_σ with endpoints on $\bar{\alpha}_k^{1+}$ and α_k^{2-}; if $\partial^1\delta \in \alpha_k^{1-}$ and $\partial^2\delta \in \alpha_k^{2+}$, then there is a smooth arc on B_σ with endpoints on α_k^{1-} and $\bar{\alpha}_k^{2+}$; and if $\partial^1\delta \in \alpha_k^{1-}$ and $\partial^2\delta \in \alpha_k^{2-}$, then there is a smooth arc on B_σ with endpoints on α_k^{1-} and α_k^{2-}. In all cases, each end points of the modified arc δ correspond to the points on $\partial_v N(B)$.

Therefore we can foliate the complementary region $F_{k-1} \times [\frac{k-1}{n}, \frac{k}{n}] \setminus N(\overset{\circ}{B_\sigma})$ by the product foliation $F_{k-1}{}' \times [0, 1]$ with the property that the vertical boundaries of $F_{k-1}{}' \times [0, 1]$ are connected to the vertical boundaries $\partial_v N(B_\sigma)$. Filling the complementary region of the lamination λ_x in $N(B_\sigma)$ with parallel leaves, we can extend λ_x to a foliation $\mathcal{F}_x$. In the boundary ∂M a meridian curve intersects all leaves of $\mathcal{F}_x$ transversely, thus $\mathcal{F}_x$ is a taut foliation.

□

In summary, we proved the existence of the good sequences $\sigma = (\sigma^1, \sigma^2)$ in Lemma 5.3 and we modify these sequences suitable for the assumption of Lemma 5.4. By Lemma 5.2 and Lemma 5.4, for these modified good sequences with good property there are two branched surfaces B_{σ_-} and B_{σ_+} which carry the families of laminations $\{\lambda_x\}$ which realize all boundary slope in $(-a_i, 0]$ and $[0, b_i)$ on $\partial^i M$ for some $a_i > 0$ and $b_i > 0$, $i = 1, 2$ respectively. Then the lamination λ_x is extended to the taut foliations $\mathcal{F}_x$ by Lemma 5.6, we complete the proof of Theorem 5.1.

Example 5.7. Now we calculate these intervals of slopes for the complement of $(6, 4)$-torus link. First we propose an explicit construction of the fibration on the complement of $(6, 4)$-torus knot as similar to the construction established in Section 3.1.

Let K be the $(6, 4)$-torus link which is a pair of simple closed curves on the solid torus V standardly embedded in S^3. We denote these components by K_1 and K_2. Taking the infinite cover $\tilde{V}$ of V with the covering map $q : \tilde{V} \to V$, we denote the cover of K on $\tilde{V}$ by $\tilde{K}$. Then $\tilde{V}$ is a cylinder of infinite length, and $\tilde{K}$ has four components. If we embed $\tilde{V}$ into $\mathbb{R}^3$ in the same way as in section 3.1, these components are the curves represented by the following formulae;

$$k_i^1(x) = (x,\ \cos\frac{3}{2}(x + \frac{2(i-1)\pi}{3}),\ \sin\frac{3}{2}(x + \frac{2(i-1)\pi}{3}))\ \ (i = 1, 2).$$

$$k_i^2(x) = (x,\ \cos\frac{3}{2}(x + \frac{2(i-1)\pi}{3} + \frac{\pi}{3}),\ \sin\frac{3}{2}(x + \frac{2(i-1)\pi}{3}) + \frac{\pi}{3})\ \ (i = 1, 2),$$

where each $k_i^j(x)$ projects to K_j by the covering map q.

Now we construct a surface in $\tilde{V}$ as follows. We define twisted bands $G_B^{i,j}$ by the following formulae;

$$G_B{}^{1,1} = \left\{ rk_1^1(x) + (1-r)k_2^2\left(\frac{2\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \right.$$
$$\left. \middle| \, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \cdots \right\}$$

$$G_B{}^{1,2} = \left\{ rk_2^2(x + \frac{2\pi}{6}) + (1-r)k_2^1\left(\frac{4\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \right.$$
$$\left. \middle| \, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \cdots \right\}$$

$$G_B{}^{2,1} = \left\{ rk_1^2(x) + (1-r)k_1^1\left(\frac{2\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \right.$$
$$\left. \middle| \, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \cdots \right\}$$

$$G_B{}^{2,2} = \left\{ rk_1^1(x + \frac{2\pi}{6}) + (1-r)k_2^2\left(\frac{4\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \right.$$
$$\left. \middle| \, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \cdots \right\}$$

$$G_B{}^{3,1} = \left\{ rk_2^1(x) + (1-r)k_1^2\left(\frac{2\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \right.$$
$$\left. \middle| \, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \cdots \right\}$$

$$G_B{}^{3,2} = \left\{ rk_1^2(x + \frac{2\pi}{6}) + (1-r)k_1^1\left(\frac{4\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \right.$$
$$\left. \middle| \, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \cdots \right\}$$

$$G_B{}^{4,1} = \left\{ rk_2^2(x) + (1-r)k_2^1\left(\frac{2\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \right.$$
$$\left. \middle| \, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \cdots \right\}$$

$$G_B{}^{4,2} = \left\{ rk_2^1(x + \frac{2\pi}{6}) + (1-r)k_1^2\left(\frac{4\pi}{6} - x\right) + \left(\frac{2\pi}{3}n, 0, 0\right) \right.$$
$$\left. \middle| \, 0 \leqq x \leqq \frac{\pi}{6},\ 0 < r < 1,\ n = 0, \pm 1, \pm 2, \cdots \right\}$$

The boundaries of these bands bound the squares $\{P_k\}$ on each disk $D_k = \{(x,y,z) | x = \frac{2k+1}{6}\pi, y^2 + z^2 \leqq 1\}$, $k \in \mathbb{Z}$. Then we define a surface G in $\tilde{V}$ as the union of all $G_B^{i,j}$ and P_k. Next we define the map $R_\theta : \tilde{V} \to \tilde{V}$ given by

$$R_\theta(x,y,z) = \left(x + \frac{\theta}{6}\ ,\ y\cos\frac{\theta}{4} - z\sin\frac{\theta}{4}\ ,\ y\sin\frac{\theta}{4} + z\cos\frac{\theta}{4} \right).$$

As seen in section 3.1, R_θ keeps the components $k_i^j(x)$ invariant and rotates $\tilde{V}$ by angle $\frac{\pi}{2}$, moreover if we set $G_\theta = R_\theta(G)$, $0 \leqq \theta \leqq 2\pi$, the family of surfaces $\{G_\theta | 0 \leqq \theta < 2\pi\}$ fills up $\tilde{V}$ and all G_θ are disjoint. G_θ has four line boundaries $\{\tilde{C_\theta^i}\}_{i=1,2,3,4}$ on $\partial\tilde{V}$. We set $F_\theta{}' = q(G_\theta)$ and then the family of surfaces $\{F_\theta{}' | 0 \leqq$

$\theta < 2\pi\}$ fills up V. The images of $\tilde{C}_\theta$ are four longitudinal circles $\{C^i_\theta\}_{i=1,2,3,4}$ on ∂V. Since the complement of V is also a solid torus, we connect meridian disks of the complement to each C_θ along its boundaries, then we obtain a surface F_θ in $S^3 \setminus K$. The family of surfaces $\{F_\theta | 0 \leqq \theta < 2\pi\}$ fills up $S^3 \setminus K$, and as seen in section 3.1, the map $p : S^3 \setminus K \to S^1$ defines a fibration.

Next we take two arcs. Let $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ be arcs on $\partial\tilde{V}$ such that

$$\tilde{\alpha}^1 = \left\{(t,\ 1,\ 0) \in \partial\tilde{V} \mid 0 \leqq t \leqq \frac{2\pi}{3}\right\}$$

$$\tilde{\alpha}^2 = \left\{(t,\ 1,\ 0) \in \partial\tilde{V} \mid \pi \leqq t \leqq \frac{5\pi}{3}\right\}.$$

We project each arc to V and denote their images by $\alpha^{1'}$ and $\alpha^{2'}$. These arcs are on the circle C_0, then we modify these arcs slightly in the neighbourhood of C_0 such that we fix each end points and shift each center of arcs forward to the direction of the center of the meridian disk whose boundary is C_0, so that each arc does not intersect the link K in its interior. We denote the resultant arcs by α^1 and α^2.

Let h be the monodromy of the fibration, and set $\beta^1 = h(\alpha^1)$ and $\beta^2 = h(\alpha^2)$. These arcs β^1 and β^2 are on the meridian disk whose boundary is $C_{2\pi}$, then four arcs are on the one fiber surface F_0. The four arcs are mutually disjoint, and each arc is non-separating on the fiber. The pair (α^1, β^1) is a good pair, and so is (α^2, β^2). Hence by tracing the method of section 3.2, we can construct the branched surfaces, then we obtain a family of taut foliations $\{\mathcal{F}_x\}$ such that $\mathcal{F}_x$ realizes all boundary slope in $(-\infty, 1)$ on each boundary components.

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