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Hypergeometric solution of 2×2 q-Schlesinger system

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1 Introduction

The sixth Painlevé equation is derived from monodromy preserving deformation of a 2nd order linear differential equation of Fuchsian type which has 4 regular singular points ([1]). The Garnier system is a natural extension of the sixth Painlevé equation after this idea. The system consists of nonlinear partial differential equations which arise as the conditions for preserving the monodromy of Fuchsian equation which has $N+3$ generic regular singularities (see [2, 4]).

It is natural to consider the same problem for $m \times m$ matrix system. The system obtained as the condition is called Schlesinger system ([7]). The 2×2 Schlesinger system is equivalent to the Garnier system. When we write the Fuchsian equation as

$$\frac{\mathrm{d}}{\mathrm{d}x} Y = \sum_{i=0}^{N+1} \frac{A_i}{x - t_i} Y,$$

then the Schlesinger system is expressed by

$$\frac{\partial A_l}{\partial t_k} = \frac{[A_l, A_k]}{t_l - t_k}, \quad (k \neq l), \tag{1.1a}$$

$$\frac{\partial A_l}{\partial t_l} = - \sum_{\substack{j=0 \\ (j \neq l)}}^{N+1} \frac{[A_j, A_l]}{t_j - t_l}. \tag{1.1b}$$

Besides, it is known that the Painlevé equations admit, under a certain restriction on the parameters, particular solutions which are expressed in terms of the Gauss' hypergeometric function and its confluents. As a generalization, Okamoto K and Kimura H constructed a particular solution of the Garnier system which is written by the Lauricella's hypergeometric functions ([5]).

Recall that the Garnier system is expressed by the following Hamiltonian system:

$$\frac{\partial q_k}{\partial s_j} = \frac{\partial H_j}{\partial p_k}, \quad \frac{\partial p_k}{\partial s_j} = -\frac{\partial H_j}{\partial q_k}, \qquad (j, k = 1, \dots, N), \tag{1.2}$$

with the Hamiltonian

$$H_i = \frac{1}{s_i(s_i-1)} \left[\sum_{j,k=1}^{N} E_{ijk}(s,q) p_j p_k - \sum_{j=1}^{N} F_{ij}(s,q) p_j + \kappa q_i \right]. \tag{1.3}$$

Here E_{ijk}, $F_{ij} \in \mathbb{C}(s)[q]$ are given by

$$E_{ijk} = E_{ikj} = \begin{cases} q_i q_j q_k, & \text{if } i, j, k \text{ are distinct}, \\ q_i q_j \left(q_j - \frac{s_j(s_i-1)}{s_i - s_j} \right), & \text{if } j = k \neq i, \\ q_i q_j \left(q_i - \frac{s_i(s_j-1)}{s_j - s_i} \right), & \text{if } k = i, j \neq i, \\ q_i(q_i-1)(q_i-s_i) - \sum_{l \neq i} \frac{s_i(s_i-1)}{s_i - s_l} q_i q_l, & \text{if } i = j = k, \end{cases}$$

$$F_{ij} = \begin{cases} \left(\kappa_0 + \kappa_1 + \sum_{l=1}^{N} \theta_l - 1 \right) q_i q_j - \theta_j \frac{s_j(s_i-1)}{s_i - s_j} q_i - \theta_i \frac{s_i(s_j-1)}{s_j - s_i} q_j, & \text{if } i \neq j, \\ (\kappa_0 - 1) q_i (q_i - 1) + \kappa_1 q_i (q_i - s_i) + \theta_i (q_i - 1)(q_i - s_i) & \\ \quad + \sum_{l \neq i} \left\{ \theta_l q_i \left(q_i - \frac{s_i(s_l-1)}{s_l - s_i} \right) - \theta_i \frac{s_i(s_i-1)}{s_i - s_l} q_l \right\}, & \text{if } i = j, \end{cases}$$

and

$$\kappa = \frac{1}{4} \left[\left(\kappa_0 + \kappa_1 + \sum_{j=1}^{N} \theta_j - 1 \right)^2 - \kappa_\infty^2 \right].$$

Notice that the Garnier system contains $N+3$ parameters

$$(\kappa_0, \kappa_1, \theta_1, \theta_2, \dots, \theta_N, \kappa_\infty) \in \mathbb{C}^{N+3}.$$

These are connected to the Riemannian scheme of an associated Fuchsian equation:

$$\begin{pmatrix} x=0 & x=1 & x=t_j & x=u_k & x=\infty \\ \frac{1-\kappa_0}{2} & \frac{1-\kappa_1}{2} & \frac{1-\theta_j}{2} & -\frac{1}{2} & \frac{-1-\kappa_\infty}{2} \\ \frac{1+\kappa_0}{2} & \frac{1+\kappa_1}{2} & \frac{1+\theta_j}{2} & \frac{3}{2} & \frac{-1+\kappa_\infty}{2} \end{pmatrix}, \qquad j, k = 1, \dots, N.$$

Theorem 1.1 ([5]). *Suppose that $\kappa = 0$, then Hamiltonian system* (1.2) *admits solutions* $(q_1, \dots, q_N, p_1 \dots, p_N)$*:*

$$q_l = A s_l (s_l - 1) \frac{\partial}{\partial s_l} \log \left((1 - s_l)^{\theta_l} u(s) \right), \qquad p_l = 0, \tag{1.4}$$

where $u(s)$ is the Lauricella's hypergeometric series in N variables:

$$u(s) = F_D \left(\begin{matrix} \alpha; \beta_1, \dots, \beta_N \\ \gamma \end{matrix} ; s \right) = \sum_{m \in (\mathbb{Z}_{\geq 0})^N} \frac{(\alpha)_{|m|} (\beta_1)_{m_1} \cdots (\beta_N)_{m_N}}{(\gamma)_{|m|} (1)_{m_1} \cdots (1)_{m_N}} s_1^{m_1} \cdots s_N^{m_N}, \tag{1.5}$$

with $\alpha = 1 - \kappa_1$, $\beta_j = \theta_j$, $\gamma = \kappa_0 + \sum_{i=1}^{N} \theta_i$ and $A^{-1} = \kappa_0 + \kappa_1 + \sum_{i=1}^{N} \theta_i - 1$.

In the present paper we will show a q-analog of this theorem. The text is organized as follows. In Section 2 we recall the setting of q-Schlesinger system and consider the reducible case especially. In Section 3 we give some formulas about a q-analog of Lauricella's hypergeometric series, which we use later. In final Section we state the main theorem, that is a q-analog of the above theorem, and we prove it.

2 2×2 q-Schlesinger system and its reducible case

Consider an $m \times m$ matrix system with a polynomial coefficient

$$Y(qx) = A(x)Y(x), \quad A(x) = A_0 + A_1 x + \cdots + A_{N+1}x^{N+1}. \tag{2.1}$$

More general case of a rational coefficient can be reduced to this case by solving a scaler q-difference equation.

A deformation theory is studied in the previous papers ([3, 6]). In the theory of monodromy preserving deformation of Fuchsian equations, extra parameter $t = (t_j)$ is introduced as the position of regular singular points. In the setting of q-difference equations we put the (discrete) deformation parameters in zeros of $\det A(x)$, $a_1, \ldots, a_{2N+2}$, and the eigen values of A_0, $\theta_1, \ldots, \theta_m$, and the eigen values of A_{N+1}, $\kappa_1, \ldots, \kappa_m$. Notice that

$$\left(\prod_{h=1}^{m} \kappa_h\right)\left(\prod_{i=1}^{2N+2} a_i\right) = \prod_{h=1}^{m} \theta_h.$$

We normalize the leading term to a diagonal form as $A_{N+1} = \mathrm{diag}(\kappa_1, \ldots, \kappa_m)$.

The connection preserving deformation of the linear q-difference equation, which is a discrete counter part of monodromy preserving deformation, is equivalent to the existence of linear deformation equation whose coefficients are rational in x. We express the deformation equation as

$$T(Y(x)) = B(x)Y(x). \tag{2.2}$$

Last of all, q-Schlesinger equation is written by the compatibility of the deformation equation and the original linear q-difference equation:

$$T(A(x))B(x) = B(qx)A(x). \tag{2.3}$$

Here T express a discrete time evolution.

We restrict the subject to 2×2 matrix system. We introduce a function w by the leading term of the $(1,2)$-element $A_{1,2}(x) = \kappa_2 w x^N + \cdots$. The function w is related to the gauge freedom.

We denote by $T = T_{r,s}$ the deformation:

$$f(a_1, \dots, a_{2N+2}, \theta_1, \theta_2, \kappa_1, \kappa_2) \mapsto T_{r,s}(f) = f(a_1, \dots, qa_r, \dots, qa_s, \dots, a_{2N+2}, q\theta_1, q\theta_2, \kappa_1, \kappa_2),$$

where f is a function of the variables $(a_i, \kappa_h, \theta_h)_{i=1,\dots,2N+2}^{h=1,2}$. In this case, if we parameterize

$$A(a_l) = y_l \begin{pmatrix} 1 \\ z_l/w \end{pmatrix} (w_l, w), \qquad (l = 1, \dots, 2N+2), \tag{2.4}$$

the coefficient of the deformation equation, $B(x) = B_{r,s}(x)$, can be expressed by the elements of $A(a_h)$ $(h = r, s)$:

$$B_{r,s}(x) = \frac{x}{(x - qa_r)(x - qa_s)} \left\{ x1 - \begin{pmatrix} \frac{qa_s z_r - qa_r z_s}{z_r - z_s} & qw\frac{a_r - a_s}{z_r - z_s} \\ \frac{z_s}{w} T_{r,s}\left(w_s \frac{a_r - a_s}{w_r - w_s} \right) & T_{r,s}\left(\frac{a_s w_r - a_r w_s}{w_r - w_s} \right) \end{pmatrix} \right\}. \tag{2.5}$$

This 2×2 q-Schlesinger system,

$$T_{r,s}(A(x)) B_{r,s}(x) = B_{r,s}(qx) A(x), \tag{2.6}$$

can be regarded as a q-analog of the Garnier system, and this tends to the Garnier system as a natural continuous limit (see [6]).

Now we consider the reducible case. In the reducible case, we can write $A(x)$ as

$$A(x) = \begin{pmatrix} \kappa_1 \prod_{i=0}^N (x - a_{2i+1}) & A_{12}(x) \\ 0 & \kappa_2 \prod_{i=0}^N (x - a_{2i+2}) \end{pmatrix}. \tag{2.7}$$

The parameters satisfy constraints

$$\theta_1 = \kappa_1 \prod_{i=0}^N (-a_{2i+1}), \quad \theta_2 = \kappa_2 \prod_{i=0}^N (-a_{2i+2}). \tag{2.8}$$

Immediately we have

$$\begin{aligned} &y_{2l+1} z_{2l+1} = \kappa_2 \prod_{i=0}^N (a_{2l+1} - a_{2i+2}), \quad y_{2l+2} w_{2l+2} = \kappa_1 \prod_{i=0}^N (a_{2l+2} - a_{2i+1}), \\ &w_{2l+1} = z_{2l+2} = 0, \quad (l = 0, 1, \dots, N). \end{aligned} \tag{2.9}$$

We consider the case that r is odd and s is even. Assume that initial data of $A(x)$ is expressed by upper triangular one, then $T_{r,s}^l(A(x))$ (r:odd, s:even, $l \in \mathbb{Z}$) remains still upper triangular because $B_{r,s}(x)$ is also upper triangular.

In this case, the q-Schlesinger system, (2.6), is expressed by

$$\begin{aligned} &(x - a_r) T_{r,s}(A_{12}(x)) - (x - qa_s) A_{12}(x) \\ &= \frac{(a_r - a_s)w}{z_r} \left[q\kappa_1 \prod_{i=0}^N (x - a_{2i+1}) - \kappa_2 \frac{x - qa_s}{x - a_s} \prod_{i=0}^N (x - a_{2i+2}) \right]. \end{aligned} \tag{2.10}$$

This equation is an identity of polynomials of degree $N+1$ in x. Hence it is sufficient to show that the equation is established at $N+2$ distinct points. We choose $x = a_{2l+1}$ $(l = 0, \dots, N)$ and $x = qa_s$. At $x = a_r$, it is trivial. For the rest we obtain $N+1$ equations:

$$\begin{aligned} T_{r,s}(wy_{2l+1}) = {} & \frac{a_{2l+1} - qa_s}{a_{2l+1} - a_r} wy_{2l+1} + \\ & + \frac{a_s - a_r}{a_{2l+1} - a_r} \frac{\prod_{i=0}^{N}(a_{2l+1} - a_{2i+2})}{\prod_{i=0}^{N}(a_r - a_{2i+2})} \frac{a_{2l+1} - qa_s}{a_{2l+1} - a_s} wy_r, \quad (2l+1 \neq r), \end{aligned} \tag{2.11a}$$

$$T_{r,s}(wy_s) = -\frac{q\kappa_1}{\kappa_2} \frac{\prod_{i=0}^{N}(qa_s - a_{2i+1})}{\prod_{i=0}^{N}(a_r - a_{2i+2})} \frac{a_s - a_r}{qa_s - a_r} wy_r. \tag{2.11b}$$

3 A q-analog of Lauricella's hypergeometric series

Here we consider the following q-analog of Lauricella's hypergeometric series:

$$\varphi_D\left(\begin{matrix} a; b_1, \dots, b_N \\ c \end{matrix}; q; t_1, \dots, t_N\right) = \sum_{m \in (\mathbb{Z}_{\geq 0})^N} \frac{(a;q)_{|m|}(b_1;q)_{m_1} \cdots (b_N;q)_{m_N}}{(c;q)_{|m|}(q;q)_{m_1} \cdots (q;q)_{m_N}} t_1^{m_1} \cdots t_N^{m_N}, \tag{3.1}$$

where $(x;q)_k = (1-x)(1-qx)\cdots(1-q^{k-1}x)$ and $|m| = m_1 + \cdots + m_N$. This q-hypergeometric series defines a holomorphic function around the origin $t_1 = t_2 = \cdots = t_N = 0$.

We know that the series $\varphi_D(t)$ satisfies the system of q-difference equations:

$$\mathcal{L}_k \varphi_D = \left\{(1 - cq^{-1}\widetilde{T})(1 - \widetilde{T}_k) - t_k(1 - a\widetilde{T})(1 - b_k\widetilde{T}_k)\right\} \varphi_D = 0, \tag{3.2a}$$

$$\mathcal{M}_{i,j} \varphi_D = \left\{t_i(1 - b_i\widetilde{T}_i)(1 - \widetilde{T}_j) - t_j(1 - \widetilde{T}_i)(1 - b_j\widetilde{T}_j)\right\} \varphi_D = 0. \tag{3.2b}$$

We here denote by $\widetilde{T}_j$ the q-shift operator in the variable t_j:

$$\widetilde{T}_j f(t) = f(t_1, \dots, t_{j-1}, qt_j, t_{j+1}, \dots, t_N),$$

and $\widetilde{T} = \widetilde{T}_1 \cdots \widetilde{T}_N$.

We give some formulas about q-Lauricella's. Setting

$$\varphi_l(t) = (1 - \widetilde{T}_l)\varphi_D(t), \tag{3.3}$$

we rewrite equation (3.2b) as

$$(b_k t_k - b_l t_l)\widetilde{T}_k \varphi_l = (t_k - b_l t_l)\varphi_l + (b_l - 1)t_l \varphi_k. \tag{3.4}$$

Moreover we set

$$\varphi_0(t) = (1 - a\widetilde{T})\varphi_D(t), \tag{3.5}$$

and rewrite equation (3.2a) to the following relations which are equivalent to each other:

$$(cq^{-1} - ab_lt_l)\widetilde{T}\varphi_l = (1-b_lt_l)\varphi_l + (b_l-1)t_l\varphi_0, \tag{3.6a}$$

$$(b_kt_k - 1)\widetilde{T}_k\varphi_0 = (t_k-1)\varphi_0 + (cq^{-1}-a)\varphi_k. \tag{3.6b}$$

Using this notation, we obtain the following formulas:

Proposition 3.1.

$$\widetilde{T}\varphi_l = \widetilde{T}_l\left((b_l-1)t_l\sum_{j=1}^{N}\frac{\prod_{i=1}^{N}(b_jt_j-t_i)}{\prod_{\substack{i=1\\(i\neq j)}}^{N}(b_jt_j-b_it_i)}\frac{\varphi_j}{(b_jt_j-t_l)(b_j-1)t_j}\right), \tag{3.7a}$$

$$(1-x\widetilde{T})\varphi_D = (1-x)\varphi_D + x\sum_{j=1}^{N}\frac{\prod_{i=1}^{N}(b_jt_j-t_i)}{\prod_{\substack{i=1\\(i\neq j)}}^{N}(b_jt_j-b_it_i)}\frac{\varphi_j}{(b_j-1)t_j}. \tag{3.7b}$$

Proof. Concerning the first relation, (3.7a), we have $\widetilde{T}\varphi_l = \widetilde{T}_l\widetilde{T}_1\cdots\widetilde{T}_{l-1}\widetilde{T}_{l+1}\cdots\widetilde{T}_N\varphi_l$. Iterating to use equation (3.4), we see that $\widetilde{T}\varphi_l$ can be expressed by a linear combination of $\widetilde{T}_l\varphi_j$'s. A straight calculus leads to the relation.

The second relation is obtained from equation

$$(1-x\widetilde{T})\varphi_D = x\left\{\frac{1}{x} - 1 + (1-\widetilde{T}_1) + \widetilde{T}_1(1-\widetilde{T}_2) + \cdots + \widetilde{T}_1\widetilde{T}_2\cdots\widetilde{T}_{N-1}(1-\widetilde{T}_N)\right\}\varphi_D.$$

The right hand side can be calculated by using formula (3.4) and the second equation is obtained. □

Now then we consider contiguity relations. We construct ladder operators; the following one is easily obtained by comparison between coefficients of series:

Proposition 3.2.

$$\frac{1-b_k\widetilde{T}_k}{1-b_k}\varphi_D\left(\begin{matrix}a;b_1,\ldots,b_N\\c\end{matrix};q;t\right) = \varphi_D\left(\begin{matrix}a;b_1,\ldots,qb_k,\ldots,b_N\\c\end{matrix};q;t\right). \tag{3.8a}$$

$$\frac{1-cq^{-1}\widetilde{T}}{1-cq^{-1}}\varphi_D\left(\begin{matrix}a;b_1,\ldots,b_N\\c\end{matrix};q;t\right) = \varphi_D\left(\begin{matrix}a;b_1,\ldots,b_N\\qc\end{matrix};q;t\right). \tag{3.8b}$$

The next one, which we need later, is about the following ladder operators:

Proposition 3.3.

$$\frac{(b_kt_k - q^{-1}b_lt_l)\widetilde{T}_k - t_k + q^{-1}b_lt_l}{(b_k-1)t_k}\varphi_D\left(\begin{matrix}a;b\\c\end{matrix};q;t\right)$$
$$= \varphi_D\left(\begin{matrix}a;b_1,\ldots,q^{-1}b_lt_l,\ldots,qb_k,\ldots,b_N\\c\end{matrix};q;t\right). \tag{3.9a}$$

$$\frac{c-1}{(c-a)(1-b_k)t_k}\left\{(c-ab_kt_k)\widetilde{T} - c + at_k\right\}\varphi_D\left(\begin{matrix}a;b\\c\end{matrix};q;t\right)$$
$$= \varphi_D\left(\begin{matrix}a;b_1,\ldots,qb_k,\ldots,b_N\\qc\end{matrix};q;t\right). \tag{3.9b}$$

Proof. We can write $\mathcal{M}_{i,j}$ in equation (3.2b) as

$$\mathcal{M}_{l,k} = \left\{ t_l(1-\widetilde{T}_k) - \frac{t_k}{b_l}(1-b_k\widetilde{T}_k) \right\}(1-b_l\widetilde{T}_l) + \frac{t_k}{b_l}(1-b_k)(1-b_k\widetilde{T}_k).$$

Therefore, using equation (3.8a), we get

$$\left\{ (b_lt_l - t_k) - (b_lt_l - b_kt_k)\widetilde{T}_k \right\} \varphi_D \left(\begin{matrix} a; b_1, \dots, qb_l, \dots b_N \\ c \end{matrix} ; q; t \right) +$$

$$+ (1-b_k)t_k\varphi_D \left(\begin{matrix} a; b_1, \dots, qb_k, \dots b_N \\ c \end{matrix} ; q; t \right) = 0,$$

and we obtain the first relation by replacing b_l to $q^{-1}b_l$.

We obtain the second relation similarly by using $\mathcal{L}_k\varphi_D = 0$. $\square$

4 Main result

We state the main theorem.

Theorem 4.1. *Suppose that*

$$\theta_1 = \kappa_1 \prod_{i=0}^{N}(-a_{2i+1}), \quad \theta_2 = \kappa_2 \prod_{i=0}^{N}(-a_{2i+2}),$$

then the 2×2 *q-Schlesinger system* (2.6) *(r:odd, s:even) admits solutions* $A(a_l) = y_l \begin{pmatrix} 1 \\ z_l/w \end{pmatrix} (w_l, w)$*:*

$$w_{2l+1} = z_{2l+2} = 0, \quad z_{2l+1} = \frac{\kappa_2\Psi(a_{2l+1})}{y_{2l+1}}, \quad w_{2l+2} = \frac{\kappa_1\Phi(a_{2l+2})}{y_{2l+2}}, \tag{4.1a}$$

$$w = \left(1 - \frac{q\kappa_1}{\kappa_2}\right)\frac{u}{\kappa_2}, \tag{4.1b}$$

$$y_{2l+1} = \frac{\Psi(a_{2l+1})}{(a_{2l+1} - a_{2l+2})w}(1-\widetilde{T}_l)u, \quad (l \neq 0), \tag{4.1c}$$

$$y_{2l+2} = -\frac{q\kappa_1}{\kappa_2}\frac{\Phi(a_{2l+2})}{(a_{2l+2} - a_{2l+1})w}(1-\widetilde{T}_l^{-1})u, \quad (l \neq 0), \tag{4.1d}$$

$$y_1 = \frac{\Psi(a_1)}{(a_1 - a_2)w}\left(1 - \frac{q\kappa_1}{\kappa_2}\widetilde{T}^{-1}\right)u, \tag{4.1e}$$

$$y_2 = -\frac{q\kappa_1}{\kappa_2}\frac{\Phi(a_2)}{(a_2 - a_1)w}\left(1 - \frac{\kappa_2}{q\kappa_1}\widetilde{T}\right)u, \tag{4.1f}$$

where $\Phi(x) = \prod_{i=0}^{N}(x - a_{2i+1})$, $\Psi(x) = \prod_{i=0}^{N}(x - a_{2i+2})$ $('=\frac{\mathrm{d}}{\mathrm{d}x})$ *and* u *is expressed by the q-analog of the Lauricella's hypergeometric series in* N *variables:*

$$u = u(t) = \left(\frac{q\kappa_1}{\kappa_2}\right)^{\frac{\log a_1}{\log q}} \frac{\left(\frac{qa_2}{a_1}; q\right)_\infty}{\left(\frac{\kappa_2 a_2}{\kappa_1 a_1}; q\right)_\infty} \varphi_D \left(\begin{matrix} a; b_1, \dots, b_N \\ c \end{matrix} ; q; t \right), \tag{4.2}$$

with $a = \frac{\kappa_2}{q\kappa_1}$, $b_j = \frac{a_{2j+1}}{a_{2j+2}}$, $c = \frac{\kappa_2 a_2}{\kappa_1 a_1}$ *and* $t_j = \frac{q a_{2j+2}}{a_1}$ $(j = 1, \dots, N)$.

Proof. We will prove that it is a solution of the reducible case.

Firstly we see that the solution is well-defined, that is to say, that the solution defines 2×2 coefficient matrix whose elements are polynomials of degree $N+1$. It is necessary to show the relations coming from Lagrange's interpolations:

$$A(a_{2l+2}) = \Phi(a_{2l+2}) \left(A_{N+1} + \sum_{j=0}^{N} \frac{A(a_{2j+1})}{\Phi'(a_{2j+1})(a_{2l+2} - a_{2j+1})} \right), \qquad (l = 0, 1, \dots, N). \quad (4.3)$$

Diagonal part of equation (4.3) is trivial. For $l \neq 0$, the $(1,2)$-element can be shown as follows:

$$\begin{aligned}
\widetilde{T}_l^{-1} \varphi_l &= (b_l - 1) t_l \sum_{j=1}^{N} \frac{\prod_{i=1}^{N} (b_j t_j - t_i)}{\prod_{\substack{i=1 \\ (i \neq j)}}^{N} (b_j t_j - b_i t_i)} \frac{\widetilde{T}^{-1}(\varphi_j)}{(b_j t_j - t_l)(b_j - 1) t_j} \\
&= (a_{2l+1} - a_{2l+2}) \sum_{j=1}^{N} \frac{\Psi(a_{2j+1})(a_{2j+1} - a_1)}{\Phi'(a_{2j+1})(a_{2j+1} - a_2)} \frac{\widetilde{T}^{-1}(\varphi_j)}{(a_{2j+1} - a_{2l+2})(a_{2j+1} - a_{2j+2})} \\
&= (a_{2l+1} - a_{2l+2}) \sum_{j=1}^{N} \frac{\Psi(a_{2j+1})}{\Phi'(a_{2j+1})} \frac{\frac{\kappa_2}{q\kappa_1} \varphi_j + \frac{a_{2j+1} - a_{2j+2}}{a_{2j+1} - a_2} \widetilde{T}^{-1}(\varphi_0)}{(a_{2j+1} - a_{2l+2})(a_{2j+1} - a_{2j+2})} \\
&= \frac{\kappa_2}{q\kappa_1} (a_{2l+1} - a_{2l+2}) \left(\sum_{j=1}^{N} \frac{\Psi(a_{2j+1})}{\Phi'(a_{2j+1})} \frac{\varphi_j}{(a_{2j+1} - a_{2l+2})(a_{2j+1} - a_{2j+2})} + \right. \\
&\qquad \left. - \frac{\Psi(a_1)}{\Phi'(a_1)} \frac{(1 - a^{-1} \widetilde{T}^{-1}) \varphi_D}{(a_1 - a_2)(a_1 - a_{2l+2})} \right).
\end{aligned}$$

In this calculation we have used equations (3.7a) and (3.6a).

In the case $l = 0$ we have

$$\begin{aligned}
(1 - a\widetilde{T}) \varphi_D &= (1 - a\widetilde{T}) \varphi_D + (1 - \widetilde{T}^{-1}) \varphi_D - (a - \widetilde{T}^{-1}) \varphi_D \\
&= a \sum_{j=1}^{N} \frac{\prod_{i=1}^{N} (b_j t_j - t_i)}{\prod_{\substack{i=1 \\ (i \neq j)}}^{N} (b_j t_j - b_i t_i)} \frac{(a - \widetilde{T}^{-1}) \varphi_j}{(b_j - 1) t_j} - (a - \widetilde{T}^{-1}) \varphi_D \\
&= \frac{\kappa_2}{q\kappa_1} \sum_{j=1}^{N} \frac{\Psi(a_{2j+1})}{\Phi'(a_{2j+1})} \frac{(a_2 - a_1) \varphi_j - (a_{2j+1} - a_{2j+2})(1 - a^{-1} \widetilde{T}) \varphi_D}{(a_{2j+1} - a_2)(a_{2j+1} - a_{2j+2})} - (a - \widetilde{T}^{-1}) \varphi_D \\
&= \frac{\kappa_2}{q\kappa_1} \sum_{j=1}^{N} \frac{\Psi(a_{2j+1})}{\Phi'(a_{2j+1})} \frac{(a_2 - a_1) \varphi_j}{(a_{2j+1} - a_2)(a_{2j+1} - a_{2j+2})} - \frac{\kappa_2}{q\kappa_1} \frac{\Psi(a_1)}{\Phi'(a_1)} \frac{(1 - a^{-1} \widetilde{T}^{-1}) \varphi_D}{(a_1 - a_2)}.
\end{aligned}$$

In this calculation we used equations (3.7b) and (3.6a).

The coefficient of the leading term of $A_{12}(x)$ is $\kappa_2 w$ and we have

$$\begin{aligned}
\kappa_2 w &= \sum_{j=1}^{N} \frac{(a_{2j+1}-a_1)y_{2j+1}w}{\Phi'(a_{2j+1})(a_{2j+1}-a_2)} + \frac{(a_2-a_1)y_2 w}{\Phi(a_2)} \\
&= \sum_{j=1}^{N} \frac{\prod_{\substack{i=1\\(i\neq j)}}^{N}(b_j t_j - t_i)(1-\widetilde{T}_j)u}{\prod_{\substack{i=1\\(i\neq j)}}^{N}(b_j t_j - b_i t_i)} - \left(\frac{q\kappa_1}{\kappa_2} - \widetilde{T}\right)u \\
&= \left(1 - \frac{q\kappa_1}{\kappa_2}\right)u.
\end{aligned}$$

We used equation (3.7b) above.

Secondly we show equation (2.11a) in the case that $T_{r,s} = T_{2k+1,2k+2}$. The time evolution $T_{2k+1,2k+2}$ acts on u as $\widetilde{T}_k$ for $k \neq 0$. Equation (2.11a) is equivalent to (3.4) for $k, l \neq 0$ and is equivalent to (3.6b) for $l = 0$. Furthermore $T_{1,2}$ acts on φ_D as $\widetilde{T}^{-1}$ and equation (2.11a) for $k = 0$ is equivalent to relation (3.6a). Equation (2.11b) is trivial in that case.

Furthermore we consider the other case of $T_{r,s}$ (r: odd, s: even). Putting $T_{r,s} = T_{2n+1,2m+2}$ $(n, m \neq 0)$, we find that $T_{r,s}$ acts on u as

$$(a; b; c; t) \mapsto (a; b_1, \ldots, qb_n, \ldots, q^{-1}b_m, \ldots, b_N; c; t_1, \ldots, qt_m, \ldots, t_N).$$

By using ladder operators (3.9a), this action is written by q-shift operators with respect to t:

$$\frac{1}{(1-b_n)t_n}\left\{(b_m t_m - b_n t_n)\widetilde{T}_n - b_m t_m + t_n\right\}\widetilde{T}_m \varphi_D = \frac{1}{(1-b_n)t_n}\left\{(t_m - b_n t_n)\widetilde{T}_n - t_m + t_n\right\}\varphi_D.$$

Using equation (3.4), we obtain equation (2.11a) $(l \neq 0)$. In the case that $l = 0$, we get equation (2.11a) similarly by using relation (3.6b). Furthermore equation (2.11b) are obtained from the following calculation:

$$\begin{aligned}
&(1-\widetilde{T}_m^{-1})\left\{(t_m - b_n t_n)\widetilde{T}_n - t_m + t_n\right\}\varphi_D \\
&= -(1-\widetilde{T}_m)\widetilde{T}_m^{-1}\left\{(b_m t_m - b_n t_n)\widetilde{T}_n - b_m t_m + t_n\right\}\widetilde{T}_m \varphi_D \\
&= -t_n(1-b_n\widetilde{T}_n)(1-\widetilde{T}_m)\varphi_D + q^{-1}b_m t_m(1-q\widetilde{T}_m)(1-\widetilde{T}_n)\varphi_D \\
&= (q^{-1}b_m - 1)t_m(1-\widetilde{T}_n)\varphi_D.
\end{aligned}$$

As concerns the case $T_{r,s} = T_{2n+1,2}$ we find that $T_{2n+1,2}$ acts on u as

$$(a; b; c; t) \mapsto (a; b_1, \ldots, qb_n, \ldots, b_N; qc; t).$$

By using ladder operator (3.9b), this action is written by the q-shift operator

$$\frac{c-1}{(c-a)(1-b_n)t_n}\left\{(c - ab_n t_n)\widetilde{T}_n - c + at_n\right\}.$$

Equations (2.11a) and (2.11a) can be obtained similarly as above by using equations (3.4) and (3.2a). For $T_{r,s} = T_{1,2m+2}$, equation (2.11a) and (2.11b) are derived from $T_{1,2m+2} = T_{1,2} \circ T^{-1}_{2n+1,2} \circ T_{2n+1,2m+2}$ (see [6]). $\square$

***Remark* 4.1.** The coefficient, $A(x)$, is written by using this solution as follows:

$$A(x) = \begin{pmatrix} \kappa_1 \prod_{i=0}^{N}(x - a_{2i+1}) & A_{12}(x) \\ 0 & \kappa_2 \prod_{i=0}^{N}(x - a_{2i+2}) \end{pmatrix}, \tag{4.4}$$

where

$$A_{12}(x) = \Phi(x) \left[\frac{\Psi(a_1)}{\Phi'(a_1)(a_1 - a_2)} \frac{1 - \frac{q\kappa_1}{\kappa_2}\widetilde{T}^{-1}}{x - a_1} + \sum_{l=1}^{N} \frac{\Psi(a_{2l+1})}{\Phi'(a_{2l+1})(a_{2l+1} - a_{2l+2})} \frac{1 - \widetilde{T}_l}{x - a_{2l+1}} \right] u. \tag{4.5}$$

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