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STABILITY ESTIMATE IN A CAUCHY PROBLEM FOR A HYPERBOLIC EQUATION WITH VARIABLE COEFFICIENTS

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Abstract. In a bounded domain $\Omega \subset \mathbb{R}^n$, we consider a hyperbolic operator P with the principal term $\partial_t^2 - p(x,t)\Delta$. Under the assumption that the outer normal derivative of p is non-positive, we will estimate u in $\mathcal{U} \times (-t_0, t_0)$ by the Cauchy data on an open subset of $\partial\Omega \times (-T, T)$, where $t_0 < T$ is some constant and $\mathcal{U}$ is a neighbourhood of $\partial\Omega$. The condition on the normal derivative is physically understood and means that the wave speed does not decrease inward on $\partial\Omega$.

§1. Introduction and main result.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary $\partial\Omega$ is of class C^2. We consider a hyperbolic operator:

$$(Pu)(x,t) \equiv \partial_t^2 u(x,t) - p(x,t)\Delta u(x,t)$$
$$-\sum_{k=1}^n q_k(x,t)\partial_k u(x,t) - q_{n+1}(x,t)\partial_t u(x,t) - q_0(x,t)u(x,t), \quad x \in \Omega,\ 0 < t < T. \tag{1.1}$$

Here we set $x = (x_1, ..., x_n) \in \mathbb{R}^n$,

$$\partial_j = \frac{\partial}{\partial x_j}, \quad 1 \le j \le n, \quad \partial_t = \frac{\partial}{\partial t}.$$

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Henceforth we assume that

$$p \in C^1(\overline{\Omega} \times \mathbb{R}), \quad q_j \in L^\infty(\Omega \times \mathbb{R}), \quad 0 \le j \le n+1 \tag{1.2}$$

and

$$p(x,t) > 0, \qquad (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{1.3}$$

Let $\Gamma \subset \partial\Omega$ be a relatively open subset.

Stability for the Cauchy problem. Let $Pu = 0$ in $\Omega \times (-T,T)$. Then estimate $\|u\|_{H^1(\mathcal{D})}$ by $u, \nabla u$ on $\Gamma \times (-T,T)$. Here $\mathcal{D} \subset \Omega \times (-T,T)$ is some open subset.

In some cases, we can take $\mathcal{D} = \Omega \times (-T,T)$ for suitable $T > 0$ and $\Gamma = \partial\Omega$. Then the resulting estimate is closely related with the observability inequality. For stability estimates including observability inequalities for Cauchy problems, we refer to: Amirov and Yamamoto [1], Bardos, Lebeau and Rauch [2], Cheng, Isakov, Yamamoto and Zhou [3], Ho [4], Hörmander [5], Isakov [7], [8], Kazemi and Klibanov [9], Khaĭdarov [10], Klibanov and Malinsky [11], Klibanov and Timonov [12], Komornik [13], [14], Lasiecka and Triggiani [15], Lasiecka, Triggiani and Yao [16], Lasiecka, Triggiani and Zhang [17], Lions [18], Yao [22], and the references therein.

For establishing stability estimates, we can refer to the three methods:

(1) the multiplier method: [4], [13], [14].[18].

(2) the Carleman estimate and the related estimates: [1], [3], [5], [7]-[12], [15]-[17], [22].

(3) the micrlocal analysis: [2].

Our main interest is to establish the stability estimate for general variable coefficient $p(x)$. The multiplier method is widely applicable but is not feasible for treating the first order terms $\sum_{j=1}^n q_j\partial_j u$ in (1.1). On the other hand, the microlocal analytical technique gives a sharp condition for the observability inequality, but the verification of the condition on p, T and Γ is not easy in concrete cases. The approach by Carleman estimate ([1], [3], [5], [7]-[12]) can treat lower-order terms and moreover is essential for proving the uniqueness and the stability in an inverse problem of determining coefficients in (1.1) by Cauchy data on the boundary.

In the case where the principal parts are with variable coefficients, in order to estabslish a Carleman estimate, we have to assume extra conditions on the coefficients, and in many existing works (e.g. [3], [7], [8], [10], [12]), the following type of conditions (or similar) are assumed for p:

$$\text{there exists } y \in \mathbb{R}^n \text{ such that } \frac{(\nabla p(x,t)\cdot(x-y))}{2p(x,t)} < 1 \tag{1.4}$$

as long as $x \in \Omega$ is in a neighbourhood of Γ and $t \in (-t_0, t_0)$ with some $t_0 > 0$. Here and henceforth, $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$.

The physical meaning of such conditions as (1.4) in the existing works is not clear. The purpose of this paper is to establish a stability estimate for p satisfying a condition which is more general than (1.4) and can be interpreted physically.

Henceforth we set

$$B_\rho(x_0) = \{x \in \mathbb{R}^n;\ |x - x_0| < \rho\}$$

$$B_\rho(x_0, t_0) = \{(x,t) \in \mathbb{R}^{n+1};\ |x - x_0|^2 + |t - t_0|^2 < \rho^2\}$$

with $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $\rho > 0$. For $\beta > 0$, $r_1 > 0$ and $y \in \mathbb{R}^n$, we set

$$Q_\gamma(y) = Q_\gamma = \{(x,t) \in \Omega\times\mathbb{R};\ |x| < r_1,\ |x-y|^2 - \beta t^2 > \gamma\}.$$

Let $\nu = \nu(x)$ be the outward unit normal vector at x to $\partial\Omega$ and $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$.

Our main result is the stability which is local in (x,t).

Theorem (local stability). *Let $x_0 \in \partial\Omega$ be fixed and let us assume that there exists $\delta > 0$ such that $B_\delta(x_0) \cap \Omega$ is convex and*

$$\frac{\partial p}{\partial \nu}(x_0, 0) \le 0. \tag{1.5}$$

Let $T > 0$ be given arbitrarily. Then there exist $y \in \mathbb{R}^n$, $\beta > 0$, $r_1 > 0$, $\gamma > 0$, $C > 0$ and $\kappa \in (0, 1)$ such that

$$x_0 \in Q_\gamma(y) \tag{1.6}$$

and

$$\|u\|_{H^1(Q_\gamma(y))} \le C\mathcal{E}^\kappa(\mathcal{E}^{1-\kappa} + \|u\|^{1-\kappa}_{H^1(\Omega\times(-T,T))}) \tag{1.7}$$

where we set

$$\mathcal{E} = \|u\|_{H^{\frac{3}{2}}((\partial\Omega\cap B_\delta(x_0))\times(-T,T))} + \|u\|_{H^2(-T,T;L^2(\partial\Omega\cap B_\delta(x_0)))}$$
$$+ \left\|\frac{\partial u}{\partial \nu}\right\|_{H^2(-T,T;L^2(\partial\Omega\cap B_\delta(x_0)))} + \left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(-T,T;H^{\frac{1}{2}}(\partial\Omega\cap B_\delta(x_0)))}.$$

Corollary (unique continuation). *Let $x_0 \in \Gamma$ be fixed and let us assume that there exists $\delta > 0$ such that $B_\delta(x_0) \cap \Omega$ is convex and (1.5) holds. We assume that $u = |\nabla u| = 0$ on $\Gamma \times (-T, T)$. Then there exists a neighbourhood $\mathcal{U} \subset \mathbb{R}^n_x \times \mathbb{R}_t$ of $(x_0, 0)$ such that $u = 0$ in $\mathcal{U}$.*

Condition (1.5) means that the wave speed $\sqrt{p(x,t)}$ does not strictly decrease inward from $\partial\Omega$ at x_0, and (1.5) is interpreted as an acceptable sufficient condition for the unique continuation for example by tracing rays from x_0 in view of the classical law of refraction (Snell's law). In the case where $p(x, t)$ is t-independent or analytic in some component of (x, t), the uniqueness in the Cauchy problem is already proved for more general Γ (e.g., Hörmander [6], Theorem 3.4.1 (pp.59-60) in Isakov [7], Robbiano [19], Tataru [21]). However, for general $p = p(x,t)$ in the C^1-class, their results do not assert the unique continuation, and moreover the existing results by Carleman estimates, cannot guarantee the uniqueness without an extra condition such as (1.4).

When we assume (1.5) suitably for $x \in \Omega$ and $t \in (-T,T)$ with sufficiently large $T > 0$, we can establish the stability which is global in (x,t) and is what is called an observability inequality. However the proof requires extra continuation arguments, and in a succeeding paperm we will discuss the details.

Remark. For example, we consider $n = 2$ and $p(x) = (1 + x_1^2 + x_2^2)^2$. Then we have

$$\frac{\partial p}{\partial |x|} > 0, \qquad 0 < |x| < R,$$

and condition (1.5) does not hold for $x \neq 0$. Then, as is shown in Example 4.1 in Yao [22], there exists a closed geodesic in Ω by the Riemannian geometry by p. Our condition (1.5) is related with a condition which excludes closed geodesics. Moreover in the case where Γ is flat near x_0, condition (1.5) is a sufficient condition for the absence of boundary rays and waveguides (see pp.73–74 and (3.25) in Romanov [20]).

The proof of the theorem is done by choices of the point y in the Carleman weight function $|x-y|^2 - \beta t^2$.

§2. Key Carleman estimate.

Without loss of generality, we may set $x_0 = 0 \in \mathbb{R}^n$. We set $M_1 = \|p^{-\frac{1}{2}}\|_{C^1(\overline{\Omega}\times[-T,T])}$ and $M_2 = \|p^{\frac{1}{2}}\|_{C(\overline{\Omega}\times[-T,T])}$. We assume that there exist $r > 0$, $y \in \mathbb{R}^n \setminus \{0\}$ such that

$$\inf_{(x,t)\in B_r(0,0),x\in\Omega}\left\{1-\frac{(\nabla p(x,t)\cdot(x-y))}{2p(x,t)}\right\}\equiv \mu_0>0. \tag{2.1}$$

Let us choose $\beta>0$, $r_1\in(0,r)$ sufficiently small such that

$$\begin{cases}\beta(1+2rM_1M_2^2+rM_1M_2)<M_1^{-2}\mu_0,\\ \\ \beta<r^{-1}M_1^{-1}(|y|-r)\end{cases} \tag{2.2}$$

and

$$r<|y|,\quad r_1^2+2r_1|y|<\beta r^2. \tag{2.3}$$

Moreover we note that

$$|y|^2-(\beta r^2-r_1^2-2r_1|y|)<|y|^2 \tag{2.4}$$

because of (2.3). We set

$$\psi(x,t)=|x-y|^2-\beta t^2,$$

$$Q_\gamma=\{(x,t)\in\Omega\times\mathbb{R};\ |x|<r_1,\quad \psi(x,t)>\gamma\} \tag{2.5}$$

for

$$\gamma\in(|y|^2-(\beta r^2-r_1^2-2r_1|y|),\quad |y|^2).$$

Then, for

$$\gamma\in(|y|^2-(\beta r^2-r_1^2-2r_1|y|),\quad |y|^2),$$

we note that

$$(0,0)\in Q_\gamma \tag{2.6}$$

and that

$$(x,t)\in Q_\gamma \text{ implies } |t|<r. \tag{2.7}$$

In fact, (2.6) is straightforward from $|y|^2 > \gamma$. Next $(x,t) \in Q_\gamma$ implies that $|x| < r_1$, so that

$$|x-y|^2 \le |x|^2 + |y|^2 + 2|x||y| \le r_1^2 + |y|^2 + 2r_1|y|.$$

Therefore $|x-y|^2 - \gamma > \beta t^2$ yields that

$$t^2 < \frac{|x-y|^2-\gamma}{\beta} < \frac{r_1^2+|y|^2+2r_1|y|-\gamma}{\beta} < r^2$$

by $\gamma > |y|^2 - (\beta r^2 - r_1^2 - 2r_1|y|)$. Thus (2.6) and (2.7) follow.

Henceforth we set

$$\varphi(x,t) = e^{\sigma\psi(x,t)}$$

with parameter $\sigma > 0$. Then

Proposition 1 (Carleman estimate). *We assume (2.1). Let*

$$|y|^2 - (\beta r^2 - r_1^2 - 2r_1|y|) < \gamma < |y|^2.$$

Then there exists $\sigma_0 > 0$ such that for any $\sigma > \sigma_0$, we can take $s_0 = s_0(\sigma) > 0$ and $C = C(\sigma) > 0$ such that

$$\int_{Q_\gamma} (s^3|u|^2 + s|\partial_t u|^2 + s|\nabla u|^2)e^{2s\varphi}dxdt \le C\int_{Q_\gamma} |Pu|^2 e^{2s\varphi}dxdt \tag{2.8}$$

for $u \in H_0^1(Q_\gamma)$ with $Pu \in L^2(Q_\gamma)$ and all $s \ge s_0(\sigma)$.

Proof. We can prove Proposition 1 by means of Theorem 3.2.1 (p.49) in Isakov [7], but we will apply a simplified criterion (Theorem 2.1 in Isakov [8]). Let

$$x_{n+1} = t, \quad \xi' = (\xi_1, ..., \xi_n) \in \mathbb{R}^n,$$

$$\xi = (\xi_1, ..., \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1}, \quad \zeta = (\zeta_1, ..., \zeta_n, \zeta_{n+1}) \in \mathbb{C}^{n+1},$$

$$A(x,t,\zeta) = \zeta_{n+1}^2 - p(x,t)\sum_{j=1}^n \zeta_j^2,$$

$$\nabla = (\partial_1, ..., \partial_n), \quad \nabla_{x,t} = (\partial_1, ..., \partial_n, \partial_t).$$

First we will verify that

$$|\nabla_{x,t}\psi| > 0 \quad \text{on } \overline{Q_\gamma}. \tag{2.9}$$

In fact, we have

$$|\nabla\psi(x,t)| = 2|x-y| \ge 2(|y|-|x|) \ge 2(|y|-r_1) > 0$$

by (2.3). Second we will prove that

$$|A(x,t,\nabla_{x,t}\psi)| \ne 0, \quad (x,t) \in \overline{Q_\gamma}. \tag{2.10}$$

In fact, since $(x,t) \in \overline{Q_\gamma}$ implies that $|x| < r_1$ and $|t| < r$, we see that

$$\begin{aligned}&-\frac{1}{4}A(x,t,\nabla_{x,t}\psi) = p(x,t)|x-y|^2 - \beta^2 t^2\\ \ge &M_1^{-2}(|y|-r_1)^2 - \beta^2 r^2 > 0\end{aligned}$$

by means of (2.2).

Next we will prove the pseudoconvexity on $\overline{Q_\gamma}$: there exists a constant $C_0 > 0$ such that

$$\begin{aligned}J \equiv &\sum_{j,k=1}^{n+1}(\partial_j\partial_k\psi)\frac{\partial A}{\partial\xi_j}(x,t,\xi)\overline{\frac{\partial A}{\partial\xi_k}(x,t,\xi)}\\ +&\lim_{s\to 0}\frac{1}{s}\mathrm{Im}\sum_{k=1}^{n+1}(\partial_k A)(x,t,\xi+\sqrt{-1}s\nabla_{x,t}\psi)\overline{\frac{\partial A}{\partial\zeta_k}(x,t,\xi+\sqrt{-1}s\nabla_{x,t}\psi)} \ge C_0|\xi|^2\end{aligned} \tag{2.11}$$

for any $\xi \in \mathbb{R}^{n+1}$ and any $(x,t) \in \overline{Q_\gamma}$, provided that

$$A(x,t,\xi) = 0, \quad \sum_{j=1}^{n+1}\frac{\partial A}{\partial\xi_j}(x,t,\xi)\partial_j\psi(x,t) = 0. \tag{2.12}$$

Here and henceforth $\overline{c}$ denotes the complex conjugate of $c \in \mathbb{C}$.

Verification of (2.11). We see that (2.12) is equivalent to

$$\xi_{n+1}^2 = p|\xi'|^2, \quad p(\xi' \cdot (x-y)) + \beta t \xi_{n+1} = 0. \tag{2.13}$$

On the other hand,

$$\begin{aligned}&\sum_{j,k=1}^{n+1} (\partial_j \partial_k \psi) \frac{\partial A}{\partial \xi_j}(x,t,\xi)\overline{\frac{\partial A}{\partial \xi_k}(x,t,\xi)}\\ =&8(p^2(x,t)|\xi'|^2 - \beta\xi_{n+1}^2) = 8(p(x,t)-\beta)\xi_{n+1}^2\end{aligned}$$

by (2.13). Moreover

$$\begin{aligned}&\frac{1}{s}\mathrm{Im}\sum_{k=1}^{n+1}(\partial_k A)(x,t,\xi+\sqrt{-1}s\nabla_{x,t}\psi)\overline{\frac{\partial A}{\partial \zeta_k}(x,t,\xi+\sqrt{-1}s\nabla_{x,t}\psi)}\\ =&\frac{1}{s}\mathrm{Im}\left[\sum_{k=1}^{n}\{2(\partial_k p)p(\xi_k - \sqrt{-1}s\partial_k\psi)\} - 2(\partial_{n+1}p)(\xi_{n+1}-\sqrt{-1}s\partial_{n+1}\psi)\right]\sum_{j=1}^{n}(\xi_j+\sqrt{-1}s\partial_j\psi)^2\\ =&4(\xi'\cdot\nabla\psi)\{p(\nabla p\cdot\xi') - (\partial_{n+1}p)\xi_{n+1}\}\\ &+2\{-p(\nabla p\cdot\nabla\psi) + (\partial_{n+1}p)(\partial_{n+1}\psi)\}(|\xi'|^2 - s^2|\nabla\psi|^2).\end{aligned}$$

Therefore, by (2.13), we have

$$\begin{aligned}J =& 8(p(x,t)-\beta)\xi_{n+1}^2 + 4(\xi'\cdot\nabla\psi)\{p(\nabla p\cdot\xi') - (\partial_{n+1}p)\xi_{n+1}\}\\ &+2\{-p(\nabla p\cdot\nabla\psi) + (\partial_{n+1}p)(\partial_{n+1}\psi)\}|\xi'|^2\\ =&8(p(x,t)-\beta)\xi_{n+1}^2 + 8(\xi'\cdot(x-y))\{p(\nabla p\cdot\xi') - (\partial_{n+1}p)\xi_{n+1}\}\\ &-4p(\nabla p\cdot(x-y))|\xi'|^2 - 4(\partial_{n+1}p)\beta t|\xi'|^2\\ =&8p\left(1-\frac{(\nabla p\cdot(x-y))}{2p}\right)\xi_{n+1}^2 - 8\beta\xi_{n+1}^2\\ &-8\beta t(\nabla p\cdot\xi')\xi_{n+1} + \frac{4\beta t(\partial_t p)}{p}\xi_{n+1}^2.\end{aligned}$$

Here by (2.7), (2.13) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&|8\beta t(\nabla p\cdot\xi')\xi_{n+1}| = \left|8\beta t\frac{(\nabla p\cdot\xi')}{p\sqrt{p}}p\sqrt{p}\xi_{n+1}\right| \le 8\beta r\times 2|(\nabla(p^{-\frac{1}{2}})\cdot\xi')|p\sqrt{p}|\xi_{n+1}|\\ \le&8\beta rM_1(p|\xi'|^2+\xi_{n+1}^2)M_2^2 = 16\beta rM_1M_2^2\xi_{n+1}^2\end{aligned}$$

and

$$\left|\frac{4\beta t(\partial_t p)}{p}\xi_{n+1}^2\right| \le 4\beta r \times 2|\partial_t(p^{-\frac{1}{2}})|\sqrt{p}\xi_{n+1}^2 \le 8\beta r M_1 M_2 \xi_{n+1}^2.$$

Therefore

$$J \ge (8M_1^{-2}\mu_0 - 8\beta - 16\beta r M_1 M_2^2 - 8\beta r M_1 M_2)\xi_{n+1}^2.$$

By (2.13), we have $|\xi|^2 = \frac{p+1}{p}\xi_{n+1}^2$, and the first condition in (2.2) implies (2.8). Thus by Theorem 2.1 in Isakov [8], the proof of Proposition 1 is complete.

§3. Proof of Theorem 1.

If necessary, we rotate and translate Ω, so that without loss of generality, we can set $x_0 = 0$ and near x_0, the hypersurface $\partial\Omega$ can be represented by $x_1 = \gamma(x')$ with $x' = (x_2, ..., x_n)$ such that $\gamma(0, ..., 0) = 0$ and $\nabla\gamma(0, ..., 0) = 0$ and Ω is located at the same side of $x_1 < \gamma(x')$. By the convexity of Ω near 0, for small $\rho > 0$ we see that

$$\{x \in \Omega;\ |x| < \rho\} \subset \{(x_1, x') \in \mathbb{R}^n;\ x_1 < 0\}$$

and $\nu(0, ..., 0) = (1, 0, ..., 0)$. Then, by (1.5), we have

$$\partial_1 p(0, 0) \le 0. \tag{3.1}$$

We set

$$M = \max_{(x,t)\in\overline{\Omega}\times[-T,T]} |\nabla \log p(x,t)|, \qquad D_\rho = \{x \in \Omega;\ |x| < \rho\}.$$

For a small constant $\varepsilon > 0$, we choose $\delta(\varepsilon) > 0$ such that

$$\delta(\varepsilon) < \min\left\{\rho, T, \frac{1}{M}\right\} \tag{3.2}$$

$$|x|, |t| < \delta(\varepsilon) \text{ imply } \partial_1 \log p(x,t) \le \varepsilon. \tag{3.3}$$

We further choose large $R > 0$ and small $\delta_0 > 0$ such that

$$x \in \Omega \text{ and } |x - y| > R - \delta_0 \text{ imply } |x| < \rho \tag{3.4}$$

and

$$\begin{aligned}&\partial(D_\rho \cap \{x \in \mathbb{R}^n;\ |x - y| > R - \delta_0\})\\ =&(\partial\Omega \cap \overline{D_\rho}) \cup \{x \in \mathbb{R}^n;\ |x - y| = R - \delta_0\},\end{aligned} \tag{3.5}$$

where we set

$$y = (-R, 0, ..., 0).$$

In fact, (3.4) is possible because $\Omega \subset \{(x_1, x'); x_1 < 0\}$ and $\{x \in \mathbb{R}^n; |x - y| \le R\} \cap \overline{\Omega} = (0, 0)$, while (3.5) is satisfied for sufficiently large $R > 0$ and small $\rho > 0$, because Ω is convex near $\partial\Omega \cap \{x; |x| \le \rho\}$.

For this $R > 0$, we set $\varepsilon = \frac{1}{R}$. Then

$$\frac{1}{2} + \frac{1}{2} M\delta\left(\frac{1}{R}\right) < 1$$

by (3.2). Hence, recalling that $y = (-R, 0, ..., 0)$, by (3.3) we have

$$\begin{aligned}&\frac{(\nabla p(x,t) \cdot (x - y))}{2p(x,t)} = -\frac{1}{2}(\nabla \log p(x,t) \cdot y) + \frac{1}{2}(\nabla \log p(x,t) \cdot x)\\ =&\frac{1}{2}(\partial_1 \log p(x,t))R + \frac{1}{2}(\nabla \log p(x,t) \cdot x) \le \frac{1}{2} + \frac{1}{2}M|x|\\ \le&\frac{1}{2} + \frac{1}{2}M\delta\left(\frac{1}{R}\right) < 1, \qquad (x,t) \in B_{\delta(1/R)}(0,0), \quad x \in \Omega.\end{aligned}$$

Hence (2.1) holds true with $r = \delta \equiv \delta\left(\frac{1}{R}\right)$.

Next we choose $\beta > 0$ and $\delta_1 \in (0, \delta(1/R))$ by (2.2) and (2.3) where we set $y = (-R, 0, ..., 0)$ and $r = \delta\left(\frac{1}{R}\right)$. Furthermore for $\delta_0 > 0$, we can choose small $\delta, \delta_1 > 0$ again if necessary such that

$$\sqrt{R^2 - (\beta\delta^2 - \delta_1^2 - 2\delta_1 R)} > R - \delta_0.$$

Setting $r = \delta$ and $r_1 = \delta_1$, we define Q_γ by (2.5). Consequently we have Carleman estimate (2.8) for $R^2 - (\beta\delta^2 - \delta_1^2 - 2\delta_1 R) < \gamma < R^2$.

Now we will complete the proof of the theorem. We set $\Gamma = \partial\Omega \cap B_{\delta_1}(0)$. By the extension theorem, there exists $F \in H^2(D_{\delta_1} \times (-T, T))$ such that

$$\begin{cases} F = u, \quad \dfrac{\partial F}{\partial \nu} = \dfrac{\partial u}{\partial \nu} \quad \text{on } \Gamma \times (-T, T), \\ \\ \|F\|_{H^2(D_{\delta_1} \times (-T,T))} \le C\mathcal{E}. \end{cases} \tag{3.6}$$

Set $u - F = v$, and we have

$$\begin{cases} Pv = -PF \qquad \text{in } D_{\delta_1} \times (-T, T), \\ \\ v = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (-T, T). \end{cases} \tag{3.7}$$

Let us fix $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ such that

$$R^2 - (\beta\delta^2 - \delta_1^2 - 2\delta_1 R) < \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < R^2$$

and let us introduce a cut-off function $\chi = \chi(x, t) \in C_0^\infty(\mathbb{R}^{n+1})$ such that $0 \le \chi \le 1$ and

$$\chi(x, t) = \begin{cases} 1, & \psi(x, t) \ge \gamma_2, \\ 0, & \psi(x, t) \le \gamma_1. \end{cases} \tag{3.8}$$

We set

$$w = \chi v.$$

Then, noting (3.4) - (3.6), we see that

$$w \in H_0^2(Q_{\gamma_0}).$$

By (3.7), we have

$$\begin{aligned} &Pw = 2(\partial_t v)(\partial_t \chi) + v(\partial_t^2 \chi) \\ &-2p\nabla v \cdot \nabla \chi - pv\Delta\chi - \sum_{j=1}^{n+1} (q_j \partial_j \chi) v - \chi PF \quad \text{in } Q_{\gamma_0}. \end{aligned}$$

Henceforth $C > 0$ denotes generic constants which are independent of $s > 0$. Therefore, setting $r = \delta$ and $r_1 = \delta_1$, we can apply Proposition 1 to Pw, so that

$$\begin{aligned}
&\int_{Q_{\gamma_0}} (s^3|w|^2 + s|\nabla w|^2 + s|\partial_t w|^2)e^{2s\varphi} dxdt \\
\le& C\int_{Q_{\gamma_0}} \left|2(\partial_t v)(\partial_t \chi) + v(\partial_t^2 \chi) - 2p\nabla v \cdot \nabla \chi - pv\Delta\chi - \sum_{j=1}^{n+1}(q_j\partial_j\chi)v\right|^2 e^{2s\varphi} dxdt \\
&+C\int_{Q_{\gamma_0}} |PF|^2 e^{2s\varphi} dxdt.
\end{aligned}$$

By (3.8), the first integral at the right hand side is not zero only if $\gamma_1 \le \psi(x,t) \le \gamma_2$. Hence (3.6) yields

$$\begin{aligned}
&\int_{Q_{\gamma_0}} (s^3|w|^2 + s|\nabla w|^2 + s|\partial_t w|^2)e^{2s\varphi} dxdt \\
\le& C\|u\|^2_{H^1(Q_{\gamma_0})} \exp(2se^{\sigma\gamma_2}) + Ce^{2sC}\mathcal{E}^2
\end{aligned}$$

for all large $s > 0$. Since

$$\begin{aligned}
&\int_{Q_{\gamma_0}} (s^3|w|^2 + s|\nabla w|^2 + s|\partial_t w|^2)e^{2s\varphi} dxdt \\
\ge& \int_{Q_{\gamma_3}} (s^3|v|^2 + s|\nabla v|^2 + s|\partial_t v|^2)e^{2s\varphi} dxdt \\
\ge& \exp(2se^{\sigma\gamma_3}) \int_{Q_{\gamma_3}} (s^3|v|^2 + s|\nabla v|^2 + s|\partial_t v|^2) dxdt,
\end{aligned}$$

by means of (3.8), we obtain

$$\begin{aligned}
&\exp(2se^{\sigma\gamma_3}) \int_{Q_{\gamma_3}} (s^3|v|^2 + s|\nabla v|^2 + s|\partial_t v|^2) dxdt \\
\le& C\|u\|^2_{H^1(Q_{\gamma_0})} \exp(2se^{\sigma\gamma_2}) + Ce^{2sC}\mathcal{E}^2,
\end{aligned}$$

that is, there exists a constant $s_0 > 0$ such that

$$\|v\|^2_{H^1(Q_{\gamma_3})} \le C\|u\|^2_{H^1(Q_{\gamma_0})} e^{-sd} + Ce^{2sC}\mathcal{E}^2 \tag{3.9}$$

for all $s \ge s_0$. Here we set $d = 2(e^{\sigma\gamma_3} - e^{\sigma\gamma_2}) > 0$.

In (3.9), setting $s + s_0$ by s, we replace C by $C' = Ce^{2s_0C}$, so that we see that (3.9) holds for all $s \ge 0$. If $\mathcal{E} = 0$ in (3.9), then $u = v$ and

$$\|u\|^2_{H^1(Q_{\gamma_3})} \le C\|u\|^2_{H^1(Q_{\gamma_0})}e^{-sd}$$

for all $s > 0$, so that letting $s \longrightarrow \infty$, we have $u = 0$ in Q_{γ_3}. Therefore conclusion (1.7) holds. Next let $\mathcal{E} > 0$. If $\|u\|^2_{H^1(Q_{\gamma_0})} \le \mathcal{E}$, then conclusion (1.7) is obtained already.

If $\|u\|^2_{H^1(Q_{\gamma_0})} > \mathcal{E}$, then we can set

$$s = \frac{1}{2C+d}\log\frac{\|u\|^2_{H^1(Q_{\gamma_0})}}{\mathcal{E}} > 0.$$

Then (3.9) yields

$$\|v\|^2_{H^1(Q_{\gamma_3})} \le 2C\mathcal{E}^{\frac{d}{2C+d}}\|u\|^{\frac{4C}{2C+d}}_{H^1(Q_{\gamma_0})}.$$

Hence (1.7) follows. Thus the proof of the theorem is complete.

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