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THE GODBILLON-VEY CLASS OF TRANSVERSELY HOLOMORPHIC FOLIATIONS

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Abstract. Several examples of transversely holomorphic foliations with non-trivial Godbillon-Vey class are given. It is shown that if the complex codimension is odd, then there are at least two foliations which are distinct as real foliations. It is also shown that the Godbillon-Vey class is rigid under deformations in the category of transversely holomorphic foliations.

Introduction

The Godbillon-Vey class is the most important invariant in the theory of foliations and extensively studied. It is well-known that the Godbillon-Vey class admits continuous deformations, namely, there are families of foliations of which the Godbillon-Vey class varies continuously. However, when restricted to categories of foliations admitting certain transverse structures, the Godbillon-Vey class often become rigid or trivial [5],[35],[11],[7],[24],[30]. In this paper, we will study transversely holomorphic foliations and show the following non-triviality and the rigidity of the Godbillon-Vey class.

Theorem A.

1) *For each q, there are transversely holomorphic foliations of complex codimension q of which the Godbillon-Vey classes are non-trivial.*

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2) *If $q = 3$, then there are at least three transversely holomorphic foliations which are mutually distinct as real foliations. If q is odd and $q > 3$, then there are at least two transversely holomorphic foliations of complex codimension q which are distinct as real foliations of codimension $2q$.*

Moreover, these foliations can be realized as locally homogeneous foliations.

The Godbillon-Vey class of transversely holomorphic foliations seems to be firstly studied by Rasmussen [36], where some examples are given by using actions of complex Lie groups. Theorem A is shown by constructing examples obtained by clarifying and generalizing one of his examples.

Theorem B. *The Godbillon-Vey class is rigid under both actual and infinitesimal deformations in the category of transversely holomorphic foliations.*

Theorem B is in fact valid for classes which belong to the image of the natural map $H^*(\mathrm{WU}_{q+1}) \to H^*(\mathrm{WU}_q)$. By a deformation of transversely holomorphic foliation, we mean a smooth family of integrable distributions such that the resulting foliations are transversely holomorphic. On the other hand, the infinitesimal deformations will be introduced by following Heitsch [22], in which the rigidity under infinitesimal deformations has been shown for a certain type of cocycles. Theorem B is obtained as its generalization. See section 4 for more details.

This paper is organized as follows. First of all, basic notions and general constructions of secondary classes are recalled. In Section 2, Theorem A is shown in steps. Firstly, the theory of Kamber-Tondeur is recalled in the first part (§ 1). As a result, it will be shown that secondary classes of locally homogeneous foliations are realized in the Lie algebra cohomology. Some related known results in the real category are also recalled. Calculations of Lie algebra cohomology using the unitary trick will be explained in § 2. The construction of examples is carried out in § 3. They are constructed on the complex simple groups of type A_n, B_n, C_n and G_2. These examples will have some common properties and it will be shown that the groups of type D_n, E_n and F_4 cannot have foliations having the same properties.

In Section 3, relations with the residue of Heitsch [21],[23] are discussed. In Section 4, Theorem B is shown. The proof is separately given for smooth deformations and for infinitesimal deformations.

Section 5 is a review of Rasmussen's examples given in [36]. One of his results seemingly contradicts Theorem B. An explanation will be given.

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1. Definitions of Transversely holomorphic foliations and Complex secondary classes

Definition 1.1. A foliation $\mathcal{F}$ of real codimension $2q$ of a manifold M is said to be transversely holomorphic if there is a coordinate system $\{U_\alpha\}, \{\varphi_{\beta\alpha}\}$ of M satisfying the following conditions, namely,

1) $U_\alpha \cong V_\alpha \times D_\alpha$, where $V_\alpha \subset \boldsymbol{R}^{\dim M - 2q}$ and $D_\alpha \subset \boldsymbol{C}^q$ are open subsets, in a way such that if L is a leaf of $\mathcal{F}$, then the connected components of $L \cap U_\alpha$ are of the form $V_\alpha \times \{z\}$, $z \in D_\alpha$.
2) Under the above identification, $\varphi_{\beta\alpha}(x,z) = (\psi_{\beta\alpha}(x,z), \gamma_{\beta\alpha}(z))$, where each $\gamma_{\beta\alpha}$ is a biholomorphic local diffeomorphism.

The integer q is called the complex codimension $\mathcal{F}$ and denoted by $\operatorname{codim}_{\boldsymbol{C}}\mathcal{F}$.

There are some relevant complex vector bundles associated with transversely holomorphic foliations.

Definition 1.2. Let $T\mathcal{F}$ be the subbundle of TM spanned by the vectors tangent to the leaves of $\mathcal{F}$. Let $T_{\boldsymbol{C}}M = TM \otimes \boldsymbol{C}$ and let E be the subbundle of $T_{\boldsymbol{C}}M$ locally spanned over $\boldsymbol{C}$ by the vectors tangent to the leaves and transversely anti-holomorphic vectors $\dfrac{\partial}{\partial \bar{z}^1}, \dots, \dfrac{\partial}{\partial \bar{z}^q}$. The integrability condition for $\mathcal{F}$ implies that E is well-defined. Set $Q(\mathcal{F}) = T_{\boldsymbol{C}}M/E$ and call it the complex normal bundle.

Let $\boldsymbol{C}[v_1, \cdots, v_q]$ be the polynomial ring generated by $v_1, \dots, v_q$ with coefficients in $\boldsymbol{C}$. The degree of v_i is set to be $2i$. Let $\mathcal{I}_q$ be the ideal generated by the monomials of degree greater than $2q$, and set $\boldsymbol{C}_q[v_1, \cdots, v_q] = \boldsymbol{C}[v_1, \cdots, v_q]/\mathcal{I}_q$. Similarly, $\boldsymbol{C}_q[\bar{v}_1, \cdots, \bar{v}_q]$ is defined by replacing v_i with $\bar{v}_i$.

Definition 1.3. Let WU_q be the differential graded algebra defined by setting

$$\mathrm{WU}_q = \boldsymbol{C}_q[v_1, \cdots, v_q] \otimes \boldsymbol{C}_q[\bar{v}_1, \cdots, \bar{v}_q] \otimes \bigwedge[\widetilde{u}_1, \cdots, \widetilde{u}_q].$$

The exterior derivative on WU_q is defined by requiring $d\widetilde{u}_i = v_i - \bar{v}_i$ and $dv_i = d\bar{v}_i = 0$. The degree of $\widetilde{u}_i$ is set to be $2i-1$. The cohomology classes in $H^*(\mathrm{WU}_q)$ which involve $\widetilde{u}_i$'s are called complex secondary classes. Cochains in WU_q are denoted as follows. Let $I = \{i_1, i_2, \ldots, i_r\}$, where $i_1 < i_2 < \cdots < i_r$. Set then $\widetilde{u}_I = \widetilde{u}_{i_1}\widetilde{u}_{i_2}\cdots\widetilde{u}_{i_r}$. If I is empty, then set $\widetilde{u}_I = 1$. Let $J = (j_1, j_2, \ldots, j_q)$, where j_t's are non-negative integers. Set then $v_J = v_1^{j_1}v_2^{j_2}\cdots v_q^{j_q}$ and $|J| = j_1 + 2j_2 + \cdots + qj_q$. Similarly, $\bar{v}_J$ is defined for an index set J as above. Index sets for $\bar{v}_i$'s are usually denoted by K. Finally, the classes in $H^*(\mathrm{WU}_q)$ are usually denoted by their representatives by abuse of notation.

Real secondary classes are also considered by forgetting the transverse holomorphic structures. Set $\boldsymbol{R}_{2q}[c_1, \cdots, c_{2q}] = \boldsymbol{R}_{2q}[c_1, \cdots, c_{2q}]/\mathcal{I}'_{2q}$, where the degree of c_i is set to be $2i$ and $\mathcal{I}'_{2q}$ is the ideal generated by monomials of degree greater than $4q$.

Definition 1.3'. Set

$$\mathrm{WO}_{2q} = \boldsymbol{R}_{2q}[c_1, \cdots, c_{2q}] \otimes \bigwedge[h_1, h_3, \cdots, h_{2q-1}],$$

where the degree of h_i is $2i-1$ and the derivative is defined by requiring $dh_i = c_i$ and $dc_i = 0$. The cohomology of $H^*(\mathrm{WO}_{2q})$ which involve h_i's are called real secondary classes.

The following secondary classes are relevant [9],[10] (see also [34],[1]).

Definition 1.4.

1) The class $h_1c_1^{2q}$ in $H^{4q+1}(\mathrm{WO}_{2q})$ is called the Godbillon-Vey class and denoted by GV_{2q}.
2) The class $\sqrt{-1}\widetilde{u}_1(v_1^q + v_1^{q-1}\bar{v}_1 + \cdots + \bar{v}_1^q)$ in $H^{2q+1}(\mathrm{WU}_q)$ is called the imaginary part of the Bott class and denoted by ξ_q.

There is a natural map from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$ which corresponds to forgetting transverse complex structures. It is given as follows.

Theorem 1.5 [1]**.** *Let λ be the mapping from WO_{2q} to WU_q given by*

$$\lambda(c_k) = (\sqrt{-1})^k \sum_{j=0}^{k} (-1)^j v_{k-j}\bar{v}_j,$$

$$\lambda(h_{2k+1}) = \frac{(-1)^k}{2}\sqrt{-1}\sum_{j=0}^{2k+1}(-1)^j \widetilde{u}_{2k-j+1}(v_j + \bar{v}_j),$$

where v_0 and $\bar{v}_0$ are considered as 1. Then λ induces on the cohomology a mapping, denoted again by λ, which corresponds to forgetting transverse complex structures. In particular, The Godbillon-Vey class and the imaginary part of the Bott class are related by the formula

$$\lambda(\mathrm{GV}_{2q}) = \frac{(2q)!}{q!q!}\,\xi_q \cdot \mathrm{ch}_1^q,$$

where $\mathrm{ch}_1 = \dfrac{v_1 + \bar{v}_1}{2}$ corresponds the first Chern class of the complex normal bundle of the foliation. The image of GV_{2q} under λ is also called the Godbillon-Vey class.

Definition 1.6. A connection ∇ on $Q(\mathcal{F})$ is said to be a complex Bott connection if ∇ satisfies

$$\nabla_X Y = \mathcal{L}_X Y$$

for any sections X of E and Y of $Q(\mathcal{F})$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. It is equivalent to the condition $\nabla_X Y = \pi[X, \widetilde{Y}]$, where $\pi : T_{\boldsymbol{C}}M \to Q(\mathcal{F})$ is the natural projection and $\widetilde{Y}$ is any lift of Y to $T_{\boldsymbol{C}}M$.

Given a transversely holomorphic foliation $\mathcal{F}$ of M, the characteristic mapping $\chi_{\mathcal{F}} : H^*(\mathrm{WU}_q) \to H^*(M; \boldsymbol{C})$ is defined. First recall the definition of Chern-Simons forms [12].

Definition 1.7. Let ∇_0 and ∇_1 be connections on $Q(\mathcal{F})$ and let θ_0 and θ_1 are respective connection forms. Let f be an invariant polynomial on $\mathrm{GL}(q; \boldsymbol{C})$ of degree k. Set $\theta_t = (1-t)\theta_0 + t\theta_1$ and

$$\Delta_f(\theta_1, \theta_0) = \int_0^1 k f(\theta_1 - \theta_0, \Omega_t, \dots, \Omega_t)\,dt,$$

where $\Omega_t = d\theta_t + \theta_t \wedge \theta_t$ is the curvature form of θ_t.

It is well-known that $d\Delta_f(\theta_1, \theta_0) = f(\Omega_1) - f(\Omega_0)$ and $\Delta_f(\theta_0, \theta_1) = -\Delta_f(\theta_1, \theta_0)$. See Section 4 for more properties of $\Delta_f(\theta_1, \theta_0)$.

The characteristic mapping $\chi_{\mathcal{F}} : H^*(\mathrm{WU}_q) \to H^*(M; \boldsymbol{C})$ is defined on the form level as follows.

Definition 1.8. Let $\mathcal{F}$ be a transversely holomorphic foliation of complex codimension q of M. Let ∇ be a complex Bott connection on $Q(\mathcal{F})$ and let ∇^u be a unitary connection on $Q(\mathcal{F})$ with respect to some Hermitian metric on $Q(\mathcal{F})$. Denote by θ and θ^u the connection forms of ∇ and ∇^u, respectively. Let c_i be the Chern polynomial of degree i and set

$$v_i(\Omega) = c_i(\Omega),\ \bar{v}_i(\Omega) = \overline{c_i(\Omega)},$$
$$\widetilde{u}_i(\theta, \theta^u) = \Delta_{c_i}(\theta, \theta^u) - \overline{\Delta_{c_i}(\theta, \theta^u)},$$

where Ω is the curvature form of θ and $\overline{\omega}$ denotes the complex conjugate in value of ω for a differential form ω.

In what follows, Chern polynomials and Chern forms are denoted by v_i and $\bar{v}_i$ in order to avoid confusions with elements of WO_{2q}.

Theorem-Definition 1.9 (Bott [10])**.** *The correspondence which assigns v_i to $v_i(\Omega)$, $\bar{v}_i$ to $\bar{v}_i(\Omega)$ and $\widetilde{u}_i$ to $\widetilde{u}_i(\theta, \theta^u)$ induces a mapping from $H^*(\mathrm{WU}_q)$ to $H^*(M; \boldsymbol{C})$ independent of the choice of connections and metrics. This mapping is denoted by $\chi_{\mathcal{F}}$ and called the characteristic mapping. The image $\chi_{\mathcal{F}}(\omega)$ of $\omega \in H^*(\mathrm{WU}_q)$ is denoted also by $\omega(\mathcal{F})$.*

Remark 1.10. Let $B\Gamma_q^{\boldsymbol{C}}$ be the classifying space of transversely holomorphic foliations of complex codimension q, then $\chi_{\mathcal{F}}$ can be considered as a mapping from $H^*(\mathrm{WU}_q)$ to $H^*(B\Gamma_q^{\boldsymbol{C}}; \boldsymbol{C})$.

In what follows, the coefficients of cohomology groups are always chosen to be the complex numbers $\boldsymbol{C}$ unless otherwise stated.

2. Non-triviality of the Godbillon-Vey class

The aim of this section is to show the following

Theorem A.

1) *For each q, there are transversely holomorphic foliations of complex codimension q of which the Godbillon-Vey classes are non-trivial.*
2) *If $q = 3$, then there are at least three transversely holomorphic foliations which are mutually distinct as real foliations. If q is odd and $q > 3$, then there are at least two transversely holomorphic foliations of complex codimension q which are distinct as real foliations of codimension $2q$.*

Moreover, these foliations can be realized as locally homogeneous foliations.

For this purpose, we will first introduce locally homogeneous foliations and then explain how to compute their complex secondary classes. Theorem A is shown in § 3 by constructing examples.

§ 1. Locally homogeneous foliations and their complex secondary classes.

Notation 2.1.1. Given a Lie group, its Lie algebra is denoted by the corresponding german lower case letter, e.g., if G is a Lie group, then its Lie algebra is denoted by $\mathfrak{g}$.

Let G be a Lie group and let K be its connected closed Lie subgroup. Let H be a connected subgroup of G which contains K, and denote by $\widetilde{\mathcal{F}}$ the foliation of G whose leaves are $\{gH \mid g \in G\}$. This foliation induces a foliation $\widehat{\mathcal{F}}$ of G/K invariant under the left action of G. Assuming in addition that G/K admits a cocompact lattice $\varGamma$, a foliation $\mathcal{F}_\varGamma$ of $M = \varGamma\backslash G/K$ is induced.

Definition 2.1.2. The foliations of the form $\mathcal{F}_\varGamma$ for quadruplets $(G, H, K, \varGamma)$ as above are called locally homogenous foliations.

Definition 2.1.3. Assume that H is a closed Lie subgroup of G. A foliation $\mathcal{F}$ of M is said to be a (transversely) (G, H)-foliation if $\mathcal{F}$ admits a foliation coordinate system $\{V_\alpha \times D_\alpha\}$, $\{(\psi_{\beta\alpha}, \gamma_{\beta\alpha})\}$ as in Definition 1.1 such that D_α is an open subset of G/H and $\gamma_{\beta\alpha}$ is given by the natural left action of G on G/H.

Locally homogenous foliations are (G, H)-foliations if H is closed.

The following facts are already known for real secondary classes of transversely (G, H)-foliations.

Theorem 2.1.4 (Baker [5] for 1) and Pittie [35] for 2) and 3))**.** *In the category of (G, H)-foliations, we have the following.*

1) *If G is semisimple, then all real secondary classes are rigid.*
2) *If H is nilpotent, then all real secondary classes are trivial. If H is solvable, then only real secondary classes of the form $h_i c_J$ with $i + |J| = \operatorname{codim}_{\boldsymbol{R}} \mathcal{F} + 1$ can be non-trivial.*
3) *If (G, H) is a parabolic pair, namely, if G is semisimple and H is parabolic, then only real secondary classes of the form $h_I c_J$ with $i_1 + c_J = \operatorname{codim}_{\boldsymbol{R}} \mathcal{F} + 1$ can be non-trivial, where i_1 is the smallest entry of I. Moreover, such non-trivial classes are cohomologous to scalar multiples of $h_1 h_I c_1^q$.*

In the cases 2) and 3), there are non-trivial examples.

Assuming that $\mathfrak{g}/\mathfrak{h}$ admits complex structures, $\mathcal{F}_\varGamma$ is transversely holomorphic. It is in particular the case if G and H are complex Lie groups. In what follows, we pose the following

Assumption 2.1.5. Let G be a complex Lie group and let H be its closed connected complex subgroup. Assume that there is an Ad_K-invariant splitting $\sigma : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$, i.e., the image is invariant under the action of Ad_K. Assume also that there is an Ad_K-invariant Hermitian metric on $\mathfrak{g}/\mathfrak{h}$.

It is easy to verify that if σ is Ad_K-invariant, then $\mathrm{Ad}_k(\sigma(v)) = \sigma(\mathrm{Ad}_k(v))$ for

$v \in \mathfrak{g}/\mathfrak{h}$ and $k \in K$. Note that a splitting σ and a Hermitian metric as above can be always found if K is compact.

Let $\widehat{\mathcal{F}}$ be the foliation of G/K induced by the foliation $\widetilde{\mathcal{F}}$ of G as above, then the complex normal bundle $Q(\widehat{\mathcal{F}})$ of $\widehat{\mathcal{F}}$ is naturally isomorphic to $G \times_K (\mathfrak{g}/\mathfrak{h})$, where K acts on $G \times (\mathfrak{g}/\mathfrak{h})$ from the right by $(g, v) \cdot k = (gk, \mathrm{Ad}_{k^{-1}} v)$. Let $P(\widehat{\mathcal{F}})$ be the associated principal bundle of $Q(\widehat{\mathcal{F}})$, then $P \cong G \times_K \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$, where $(g, A) \cdot k = (gk, k^{-1}A)$ for $(g, A) \in G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$. Hence the normal bundle $Q(\mathcal{F}_\Gamma)$ is naturally isomorphic to $\Gamma \backslash G \underset{K}{\times} (\mathfrak{g}/\mathfrak{h})$. The following kind of connections are relevant.

Definition 2.1.6. A connection on $Q(\mathcal{F}_\Gamma)$ is said to be locally homogeneous if it is induced by a $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$-valued 1-form on the trivial bundle $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ which is invariant under the left G-action and the right K-action as above.

Under these assumptions, the following theorem is known.

Theorem 2.1.7 (Kamber-Tondeur [26], Baker [5], Pittie [35])**.** *Let (G, H, K, Γ) be as above and assume that there are an Ad_K-invariant splitting of $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ and an Ad_K-invariant Hermitian metric on $\mathfrak{g}/\mathfrak{h}$. Let $\mathfrak{g}_{\boldsymbol{R}}$ be the Lie algebra $\mathfrak{g}$ viewed as a real Lie algebra. By using locally homogenous connections, the characteristic mapping for $\mathcal{F}_\Gamma$ factors through $H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$. This mapping is independent of the choice of locally homogeneous connections.*

If $\mathfrak{g}'$ is a real Lie algebra and if $\mathfrak{k}'$ is a Lie subalgebra of $\mathfrak{g}'$, then the cohomology group $H^*(\mathfrak{g}', \mathfrak{k}')$ is by definition the cohomology of the complex

$$C^*(\mathfrak{g}', \mathfrak{k}') = \left\{\omega \in \textstyle\bigwedge^* \mathfrak{g}'^* \,\middle|\, i_K \omega = 0, i_K d\omega = 0 \text{ for all } K \in \mathfrak{k}'\right\},$$

where i_K denotes the interior product with K. We refer to [8] for more details.

Theorem 2.1.7 is particularly important by virtue of the following theorem by T. Kobayashi and K. Ono. We only need its quite reduced form, which is as follows;

Theorem 2.1.8 (Proposition 3.9 and Example 3.6 in [29])**.** *Let G' be a real connected semisimple Lie group and let K' be its compact subgroup. Let Γ' be a cocompact lattice of G'/K', then the natural mapping $H^*(\mathfrak{g}', \mathfrak{k}') \to H^*(\Gamma' \backslash G'/K')$ is injective.*

It follows that it suffices to study the characteristic classes in $H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$ rather than $H^*(\Gamma \backslash G/K)$ when examples as in § 3 are considered.

From now on, we will give a proof Theorem 2.1.7 in steps by following Baker [5]. We do not assume that G is semisimple nor K is compact until § 2.

Definition 2.1.9 (cf. Lemma 4.3 in [5])**.** Let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the projection and let σ be an Ad_K-invariant section to π. Set $\rho = \mathrm{id}_{\mathfrak{g}} - \sigma\pi$, and define a $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$-valued 1-form θ on $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ by setting

$$\theta_{(g,A)}(X,Y) = \mathrm{Ad}_{A^{-1}}(L^*_{g^{-1}}\rho^*\mathrm{ad})(X) + \tau_A(Y),$$

where $(X,Y) \in T_{(g,A)}(G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h}))$ and τ is the Maurer-Cartan form on $\mathrm{GL}(\mathfrak{g}/\mathfrak{h})$.

Note that ρ is also an Ad_K-invariant mapping from $\mathfrak{g}$ to $\mathfrak{h}$.

Lemma 2.1.10 [5]**.** *θ induces a connection on P invariant under the natural left action of G on P. Moreover, θ is associated with a Bott connection on the complex normal bundle $Q(\widehat{\mathcal{F}})$.*

Proof. **Claim 1.** θ projects down to P.
Let $k \in K$ and denote by R_k the right action on $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$, then

$$\begin{aligned}(R^*_k\theta)_{(g,A)} &= R^*_k\theta_{(gk,\mathrm{Ad}_{k^{-1}}A)} \\ &= R^*_k(\mathrm{Ad}_{A^{-1}}\mathrm{Ad}_{\mathrm{Ad}_k}(L^*_{g^{-1}}L^*_{k^{-1}}\rho^*\mathrm{ad}) + \tau_{k^{-1}A}) \\ &= \mathrm{Ad}_{A^{-1}}\mathrm{Ad}_{\mathrm{Ad}_k}(L^*_{g^{-1}}\mathrm{Ad}^*_{k^{-1}}\rho^*\mathrm{ad}) + \tau_A.\end{aligned}$$

Thus it suffices to show $\mathrm{Ad}^*_{k^{-1}}\rho^*\mathrm{ad} = \mathrm{Ad}_{\mathrm{Ad}_{k^{-1}}} \circ \rho^*\mathrm{ad}$. This follows from the following infinitesimal version.
Claim 2. $\mathrm{ad}^*_w\rho^*\mathrm{ad} = [\mathrm{ad}_w, \rho^*\mathrm{ad}]$ if $w \in \mathfrak{k}$, where the right is the Lie bracket of ad_w and $\rho^*\mathrm{ad}$ in $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$.
Indeed, for $X, Y \in \mathfrak{g}$, one has $(\mathrm{ad}^*_w\rho^*\mathrm{ad}(X))Y = \mathrm{ad}_{\rho[w,X]}Y$. Since $w \in \mathfrak{k}$ and ρ is Ad_K-invariant,

$$\begin{aligned}\mathrm{ad}_{\rho[w,X]}Y &= [[w,Y],\rho(X)] + [w,[\rho(X),Y]] \\ &= -\mathrm{ad}_{\rho(X)}(\mathrm{ad}_wY) + \mathrm{ad}_w(\mathrm{ad}_{\rho(X)}Y).\end{aligned}$$

Hence Claim 2 and Claim 1 are shown.

Let R_A denote the right action of $\mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ on P, and given a vector $v \in \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$, $\widetilde{v}$ denotes the vertical fundamental vector field induced by v.
Claim 3. $R^*_A\theta = \mathrm{Ad}_{A^{-1}}\theta$ and $\theta(\widetilde{v}) = v$.
Let $(X,Y) \in T_{(g,B)}(G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h}))$, then

$$\begin{aligned}(R^*_A\theta)_{(g,B)}(X,Y) &= \theta_{(g,BA)}(X, R_{A*}Y) \\ &= \mathrm{Ad}_{A^{-1}}\mathrm{Ad}_{B^{-1}}\mathrm{ad}_{\rho(L_{g^{-1}*}X)} + \tau_B(R_{A*}Y) \\ &= \mathrm{Ad}_{A^{-1}}(\theta_{(g,B)}(X,Y)).\end{aligned}$$

The second claim is clear.
Claim 4. θ is left invariant.
Let L_{g_1} denote the left action of g_1 on $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$, then

$$\begin{aligned}(L_{g_1}^*\theta)_{(g_2,A)}(X,Y) &= \mathrm{Ad}_{A^{-1}}(L_{g_1^{-1}}^* L_{g_2^{-1}}^* \rho^*\mathrm{ad})(L_{g_1 *}X) + \tau_A(Y) \\ &= \theta_{(g_2,A)}(X,Y).\end{aligned}$$

Claim 5. θ is a Bott connection.
Let $[g_0] \in G/K$ and choose a local decomposition $U_1 \times U_2$ of G around g_0, where U_1 and U_2 are open set such that $U_1 \subset K$ and U_2 is diffeomorphic to an open set of G/K containing $[g_0]$ (in terms of foliations, $U_1 \times U_2$ is a foliation chart for the foliation of G by cosets of K). Then define a local section of P around $[g_0]$ by setting $s([g]) = [g, \mathrm{id}_{\mathfrak{g}/\mathfrak{h}}]$, where $g \in U_2$. Let $X \in T_{[g_0]}(g_0H/K)$ and $Y \in Q(\widehat{\mathcal{F}})_{[g_0]}$, then one may assume that $L_{{g_0}^{-1}*}X \in \mathfrak{h}$ and $L_{{g_0}^{-1}*}Y \in \mathfrak{g}/\mathfrak{h}$. One has

$$(s^*\theta)_{[g_0]}(X)Y = \theta_{(g_0,\mathrm{id}_{\mathfrak{g}/\mathfrak{h}})}(s_*X)s_*Y = \mathrm{ad}_X Y.$$

This completes the proof. $\square$

Let $\{\omega^1, \ldots, \omega^q\}$ be a basis of $(\mathfrak{g}/\mathfrak{h})^*$ and consider ω^i's as elements of $\mathfrak{g}^*$ which vanish when restricted to $\mathfrak{h}$. Since H is a subgroup, there is a $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$-valued 1-form θ such that $d\omega = -\theta \wedge \omega$, where $\omega = {}^t(\omega^1, \ldots, \omega^q)$. Noticing that ω can be considered as an element of P, one has the following

Corollary 2.1.11. *Assume that $\theta = 0$ when restricted to the image of the Ad_K-invariant splitting σ as above, then θ can be regarded as a left invariant Bott connection on $Q(\widehat{\mathcal{F}})$.*

Fix now an Ad_K-invariant Hermitian metric on $\mathfrak{g}/\mathfrak{h}$ so that $\mathrm{Ad}_K \subset \mathrm{U}(\mathfrak{g}/\mathfrak{h})$. Let $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m}$ be an Ad_K-invariant splitting such that $\mathfrak{k} \oplus \mathfrak{n} = \mathfrak{k} + \ker \mathrm{ad}$ and $\mathrm{ad}_{\mathfrak{n}} = 0$, and denote by ρ' the projection from $\mathfrak{h}$ to $\mathfrak{k}$. Finally, choose an $\mathrm{Ad}_{\mathrm{Ad}_K}$-invariant splitting $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h}) = \mathrm{ad}_{\mathfrak{k}} \oplus \mathrm{ad}_{\mathfrak{m}} \oplus \mathfrak{l}$ and denote by p the projection to $\mathrm{ad}_{\mathfrak{k}}$.

Lemma 2.1.12 (cf. Lemma 4.4 in [5])**.** *Set $\rho_u = \rho'\rho : \mathfrak{g} \to \mathfrak{k}$, then we have the following properties:*

1) $p \circ \mathrm{ad}_{\rho(X)} = \mathrm{ad}_{\rho_u(X)}$ *for* $X \in \mathfrak{g}$.
2) *Set*

$$\theta^u_{(g,A)}(X,Y) = \mathrm{Ad}_{A^{-1}}(L_{g^{-1}}^*\rho_u^*\mathrm{ad})(X) + \tau_A(Y),$$

then θ^u is a unitary connection.

Proof. Let $X = X_1 + X_2 + X_3 \in \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m} = \mathfrak{h}$, then $\rho_u(X) = X_1$ and $\mathrm{ad}_X = \mathrm{ad}_{X_1} + \mathrm{ad}_{X_3}$. Thus $p \circ \mathrm{ad}_X = \mathrm{ad}_{X_1} = \mathrm{ad}_{\rho_u(X)}$. Since the mapping ρ_u is Ad_K-invariant, θ^u is shown to be a connection form on P as in Lemma 2.1.3. When restricted to $G \times_K \mathrm{U}(\mathfrak{g}/\mathfrak{h})$, θ^u is $\mathfrak{u}(\mathfrak{g}/\mathfrak{h})$-valued. Hence θ^u is unitary. $\square$

Proof of Theorem 2.1.7. Since the connections given by Lemmas 2.1.10 and 2.1.12 are left invariant, they induce connections on $Q(\mathcal{F})$. When calculated by these connections, the characteristic mapping factors through $H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$. The independence of the characteristic mapping from the choice of connections can be shown by standard arguments (cf. [10]). $\square$

Let $\mathcal{F}$ be a transversely holomorphic foliation of complex codimension q and suppose that $\bigwedge^q Q(\mathcal{F})$ is a trivial line bundle, then the Bott class is defined as follows [10]. Fix a trivialization s of $\bigwedge^q Q(\mathcal{F})$ and let ∇^s be the flat connection with respect to s and let θ^s be its connection form. Let ∇ be a Bott connection on $Q(\mathcal{F})$, then ∇ induces a connection on $\bigwedge^q Q(\mathcal{F})$, which we denote by ∇^b. Set now

$$u_1(\theta^b, \theta^s) = \Delta_{c_1}(\theta^b, \theta^s),$$

where θ^b is the connection form of ∇^b. Let Ω^b be the curvature form of θ^b, then $du_1(\theta^b, \theta^s) = v_1(\Omega^b)$ and thus $u_1(\theta^b, \theta^s)v_1(\Omega^b)^q$ is a closed form. It is known that the cohomology class represented by $u_1(\theta^b, \theta^s)v_1(\Omega^b)^q$ is independent of the choice of trivializations and connections [10].

Definition 2.1.13. The class represented by $u_1(\theta^b, \theta^s)v_1(\Omega^b)^q$ is called the Bott class and denoted by $\mathrm{Bott}_q(\mathcal{F})$.

Remark 2.1.14. Assuming furthermore that $Q(\mathcal{F})$ is trivial, differential forms u_i, $i \geq 2$, are well-defined and several characteristic classes are obtained. However, these classes depend on the choice of trivializations in general.

Let $\mathcal{F}_\Gamma$ be a locally homogeneous, transversely holomorphic foliation associated with (G, H, K, Γ). By repeating similar arguments as in the proof of Theorem 2.1.7, one has the following

Theorem 2.1.15. *Let $\mathcal{F}_\Gamma$ and (G, H, K, Γ) be as above. Assume that $\bigwedge^q Q(\mathcal{F}_\Gamma)$ is trivial as a homogeneous vector bundle. Then by choosing a left invariant trivialization and using a locally homogeneous Bott connection, the Bott class is realized as an element of $H^{2q+1}(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$. The Bott class is independent of the choice of invariant trivializations and locally homogeneous Bott connections.*

§ 2. Calculation of the Lie algebra cohomology.

In what follows, we assume that G is a complex semisimple Lie group and that K is a compact connected Lie subgroup of G. Hence there are always cocompact lattices of G/K and Theorem 2.1.8 is valid, and $\mathfrak{g}$ admits a compact real form $\mathfrak{g}_0$. We also assume that $\mathfrak{k} \subset \mathfrak{g}_0$. Under these assumptions, we will construct an isomorphism from $H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$ to $H^*(G_0 \times (G_0/K))$, where G_0 is a compact Lie group with Lie algebra $\mathfrak{g}_0$.

Notation 2.2.1. The complex Lie algebra $\mathfrak{g}$ considered as a real Lie algebra is denoted by $\mathfrak{g}_{\boldsymbol{R}}$. Let J be the complex structure of $\mathfrak{g}$ and let $\mathfrak{g}^-$ be the Lie algebra $\mathfrak{g}_{\boldsymbol{R}}$ equipped with the complex structure $-J$. The complex conjugate on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ is denoted by σ, namely, $\sigma(X + JY) = X - JY$ for $X, Y \in \mathfrak{g}_0$.

Definition 2.2.2. For $\omega \in \bigwedge^* \mathfrak{g}^*$, define $\overline{\omega} \in \bigwedge^* \mathfrak{g}^{-*}$ by taking complex conjugate in value. Their complexifications are denoted as follows;

$$\begin{aligned} \omega^{\boldsymbol{C}} &= \omega \otimes \boldsymbol{C} \in (\textstyle\bigwedge^* \mathfrak{g}_{\boldsymbol{R}}^*) \otimes \boldsymbol{C}, \\ \overline{\omega}^{\boldsymbol{C}} &= (\overline{\omega})^{\boldsymbol{C}}. \end{aligned}$$

Note that if ω restricted to $\mathfrak{g}_0$ is $\boldsymbol{R}$-valued (resp. $\sqrt{-1}\boldsymbol{R}$-valued), then $\overline{\omega} = \sigma^* \omega$ (resp. $\overline{\omega} = -\sigma^* \omega$).

Definition 2.2.3. Let $\kappa : \mathfrak{g}_0 \oplus \mathfrak{g}_0 \to \mathfrak{g}_0 \oplus \sqrt{-1} J \mathfrak{g}_0 \subset \mathfrak{g}_{\boldsymbol{R}} \otimes \boldsymbol{C}$ be the isomorphism of real Lie algebras defined by

$$\kappa(X_1, X_2) = \frac{1}{2}\left(X_1 - \sqrt{-1} J X_1\right) + \frac{1}{2}\left(X_2 + \sqrt{-1} J X_2\right).$$

Since $\mathfrak{g}_0 \oplus \sqrt{-1} J \mathfrak{g}_0$ is a real form of $\mathfrak{g}_{\boldsymbol{R}} \otimes \boldsymbol{C}$, θ induces an isomorphism from $\mathfrak{g} \oplus \mathfrak{g}$ to $\mathfrak{g}_{\boldsymbol{R}} \otimes \boldsymbol{C}$ by complexification. Denote the complexification again by κ, then

$$\kappa(X + JY, Z + JW) = \frac{1}{2}(X + JY + Z - JW) + \sqrt{-1}\,\frac{1}{2}(-JX + Y + JZ + W)$$

holds for $X, Y, Z, W \in \mathfrak{g}_0$.

Remark 2.2.4. For $X \in \mathfrak{g}_0$, one has

$$\begin{aligned} \kappa^{-1}(X) = (X, X) \quad & \kappa^{-1}(JX) = (JX, -JX) \\ \kappa^{-1}(\sqrt{-1}X) = (JX, JX) \quad & \kappa^{-1}(\sqrt{-1}JX) = (-X, X). \end{aligned}$$

These relations imply the following

Lemma 2.2.5. $\kappa^{-1}(\mathfrak{k}\otimes \boldsymbol{C}) = \{(k,k)\,|\,k\in\mathfrak{k}\otimes\boldsymbol{C}\}\subset\mathfrak{g}\oplus\mathfrak{g}$.

Let $\Delta\mathfrak{k}$ be the diagonal embedding of $\mathfrak{k}$ into $\mathfrak{g}\oplus\mathfrak{g}$, then $\Delta\mathfrak{k}=\kappa^{-1}(\mathfrak{k})$ and $\kappa^{-1}(\mathfrak{k}\otimes \boldsymbol{C})=\Delta\mathfrak{k}\otimes\boldsymbol{C}$.

As $\boldsymbol{C}$ is chosen as the coefficients, there is a natural isomorphism from $H^*(\mathfrak{g}_{\boldsymbol{R}},\mathfrak{k};\boldsymbol{C})$ to $H^*(\mathfrak{g}_{\boldsymbol{R}}\otimes\boldsymbol{C},\mathfrak{k}\otimes\boldsymbol{C};\boldsymbol{C})$. Hence κ induces an isomorphism

$$\kappa^*: H^*(\mathfrak{g}_{\boldsymbol{R}},\mathfrak{k})\to H^*(\mathfrak{g}\oplus\mathfrak{g},\Delta\mathfrak{k}\otimes\boldsymbol{C}).$$

Lemma 2.2.6. *Let $\omega\in\mathfrak{g}^*$ and set $\omega^1=(\omega,0)\in\mathfrak{g}^*\oplus\mathfrak{g}^*$ and $\omega^2=(0,\omega)\in\mathfrak{g}^*\oplus\mathfrak{g}^*$, then $\kappa^*(\omega^{\boldsymbol{C}})=\omega^1$. If $\omega|_{\mathfrak{g}_0}$ is $\boldsymbol{R}$-valued, then $\kappa^*(\overline{\omega}^{\boldsymbol{C}})=\omega^2$. If $\omega|_{\mathfrak{g}_0}$ is $\sqrt{-1}\boldsymbol{R}$-valued, then $\kappa^*(\overline{\omega}^{\boldsymbol{C}})=-\omega^2$.*

Proof. Let $X,Y,Z,W\in\mathfrak{g}_0$. Since $\omega\in\mathfrak{g}^*$, one has

$$\begin{aligned}
&\kappa^*(\omega^{\boldsymbol{C}})(X+JY,Z+JW)\\
=&\frac{1}{2}\left(\omega(X)+\omega(JY)+\omega(Z)-\omega(JW)\right)+\frac{1}{2}\left(\omega(X)+\omega(JY)-\omega(Z)+\omega(JW)\right)\\
=&\,\omega(X+JY).
\end{aligned}$$

Assume that $\omega|_{\mathfrak{g}_0}$ is $\boldsymbol{R}$-valued, then $\overline{\omega}=\sigma^*\omega$. Hence

$$\begin{aligned}
&\kappa^*(\overline{\omega}^{\boldsymbol{C}})(X+JY,Z+JW)\\
=&\frac{1}{2}\left(\overline{\omega}(X)+\overline{\omega}(JY)+\overline{\omega}(Z)-\overline{\omega}(JW)\right)+\frac{\sqrt{-1}}{2}\left(-\overline{\omega}(JX)+\overline{\omega}(Y)+\overline{\omega}(JZ)+\overline{\omega}(W)\right)\\
=&\frac{1}{2}\left(\omega(X)-\omega(JY)+\omega(Z)+\omega(JW)\right)+\frac{1}{2}\left(-\omega(X)+\omega(JY)+\omega(Z)+\omega(JW)\right)\\
=&\,\omega(Z+JW).
\end{aligned}$$

Similar calculations show that $\kappa^*(\overline{\omega}^{\boldsymbol{C}})=-\omega^2$ if $\omega|_{\mathfrak{g}_0}$ is $\sqrt{-1}\boldsymbol{R}$-valued. $\square$

Since $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, there are isomorphisms as follows:

$$\begin{aligned}
H^*(\mathfrak{g}\oplus\mathfrak{g},\Delta(\mathfrak{k}\otimes\boldsymbol{C}))\cong&H^*((\mathfrak{g}_0\otimes\boldsymbol{C})\oplus(\mathfrak{g}_0\otimes\boldsymbol{C}),\Delta\mathfrak{k}\otimes\boldsymbol{C})\\
\cong&H^*((\mathfrak{g}_0\oplus\mathfrak{g}_0)\otimes\boldsymbol{C},\Delta\mathfrak{k}\otimes\boldsymbol{C})\\
\cong&H^*((G_0\times G_0)/K),
\end{aligned}$$

where K acts on $G_0\times G_0$ diagonally from the right. The diffeomorphism $\tau: G_0\times(G_0/K)\to(G_0\times G_0)/K$ given by $\tau(g_1,[g_2])=[g_1g_2,g_2]$ induces an isomorphism

$$\tau^*: H^*((\mathfrak{g}_0\oplus\mathfrak{g}_0)\otimes\boldsymbol{C},\Delta\mathfrak{k}\otimes\boldsymbol{C})\to H^*(\mathfrak{g}_0\otimes\boldsymbol{C})\otimes H^*(\mathfrak{g}_0\otimes\boldsymbol{C},\mathfrak{k}\otimes\boldsymbol{C})$$

given by $\tau^*([\alpha,\beta])=([\alpha],[\alpha+\beta])$. Note that $H^*(\mathfrak{g}_0\otimes\boldsymbol{C})\otimes H^*(\mathfrak{g}_0\otimes\boldsymbol{C},\mathfrak{k}\otimes\boldsymbol{C})\cong H^*(G_0\times(G_0/K))$. Summing up, we obtained the following

Proposition 2.2.7. *Let κ and τ as above, then*

$$\tau^*\kappa^* : H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k}) \to H^*(\mathfrak{g}_0) \otimes H^*(\mathfrak{g}_0, \mathfrak{k}) \cong H^*(G_0 \times (G_0/K)).$$

is induced by the correspondence $\omega \mapsto (\omega, \omega)$, $\overline{\omega} \mapsto (0, \omega)$ (resp. $\overline{\omega} \mapsto (0, -\omega)$) for $\omega \in \mathfrak{g}^*$ such that $\omega|_{\mathfrak{g}_0}$ is $\boldsymbol{R}$-valued (resp. $\sqrt{-1}\boldsymbol{R}$-valued).

§ 3. Examples.

First we make some remarks.

Notation 2.3.1. Cochains in WO_{2q} are regarded as cochains in WU_q via the mapping λ in Theorem 1.5. If $\alpha \in H^*(\mathrm{WU}_q)$, then the image of α under $\chi_{\mathcal{F}_\Gamma}$ as an element of $H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$ is denoted by $\alpha(K)$.

Lemma 2.3.2. *Suppose that $\alpha(K)$ is non-trivial in $H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$. If K' is a closed subgroup such that $K \subset K' \subset H$, then $\alpha(K')$ is non-trivial in $H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k}')$.*

Proof. We have a natural mapping $r : H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k}') \to H^*(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$. By the functoriality of the characteristic mapping, $r(\alpha(K')) = \alpha(K)$. □

Thus it is preferable to show the non-triviality of $\mathrm{GV}_{2q}(K)$ for small K. But there is the following

Proposition 2.3.3. *One has $v_i(\{e\}) = \bar{v}_i(\{e\}) = 0$ for all i. In particular, $\mathrm{GV}_{2q}(\{e\}) = 0$.*

Proof. The bundle $Q(\widehat{\mathcal{F}})$ admits a G-invariant trivialization because it is isomorphic to $G \times (\mathfrak{g}/\mathfrak{h})$. Hence $v_i(\{e\}) = \bar{v}_i(\{e\}) = 0$. The triviality of the Godbillon-Vey class follows from Theorem 1.5. □

We recall the definition of several Lie algebras to fix notations. We denote by I_q the identity matrix of $M(q; \boldsymbol{C})$, and set $J_q = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix} \in M(2q; \boldsymbol{C})$.

Definition 2.3.4.

1) $\mathfrak{sl}(q+1; \boldsymbol{C}) = \{X \in M(q+1; \boldsymbol{C}) \ ; \ \operatorname{tr} X = 0\}$
2) $\mathfrak{su}(q+1) = \left\{X \in \mathfrak{sl}(q+1; \boldsymbol{C}) \ ; \ X + {}^t\overline{X} = 0\right\}$
3) $\mathfrak{so}(q; F) = \{X \in M(q; F) \ ; \ X + {}^tX = 0\}$, where $F = \boldsymbol{R}$ or $F = \boldsymbol{C}$,
4) $\mathfrak{sp}(q; \boldsymbol{C}) = \{X \in M(2q; \boldsymbol{C}) \ ; \ {}^tXJ_q + J_qX = 0\}$,
5) $\mathfrak{sp}(q) = \mathfrak{sp}(q; \boldsymbol{C}) \cap \mathfrak{su}(2q)$

$\mathfrak{sp}(q)$ is also denoted by $\mathfrak{sp}(q; \boldsymbol{R})$.

For more details including the topology of homogeneous spaces, we refer to [32].

Notation 2.3.5. We denote by E_{ij} $(0 \leq i, j \leq q)$ the standard basis of the Lie algebra $\mathfrak{gl}(q+1; \boldsymbol{C})$. Rows and columns of matrices are always counted from zero.

Example 2.3.6. Let $\mathfrak{g} = \mathfrak{sl}(q+1; \boldsymbol{C})$ and $\mathfrak{g}_0 = \mathfrak{su}(q+1)$, and construct an $(\mathrm{SL}(q+1; \boldsymbol{C}), \boldsymbol{C}P^q)$-foliation. Let T^q be the maximal torus of G standardly realized as a subset of diagonal matrices, and let U_q, SU_q and H be the subgroups of G defined by

$$\begin{aligned} U_q &= \left\{ \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} \middle| B \in \mathrm{U}(q), a = (\det B)^{-1} \right\}, \\ SU_q &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \middle| B \in \mathrm{SU}(q) \right\}, \\ H &= \left\{ \begin{pmatrix} a & * \\ 0 & B \end{pmatrix} \middle| B \in \mathrm{GL}(q; \boldsymbol{C}), a = (\det B)^{-1} \right\}. \end{aligned}$$

We denote U_q and SU_q again by $\mathrm{U}(q)$ and $\mathrm{SU}(q)$, respectively. The subgroup $\mathrm{U}(q)$ is also denoted by $T^1 \times \mathrm{SU}(q)$. Let K be a compact connected subgroup of G contained in $\mathrm{U}(q)$ and containing T^q, hence $G \supset H \supset \mathrm{U}(q) \supset K \supset T^q$.

Let ω_{ij} be the dual of E_{ij} restricted to $\mathfrak{g}$, then $\sum\limits_{i=0}^{q} \omega_{ii} = 0$ and $d\omega_{ij} = -\sum\limits_{k=0}^{q} \omega_{ik} \wedge \omega_{kj}$. Set $\omega = {}^t(\omega_{10}, \omega_{20}, \cdots, \omega_{q0})$, then $\mathfrak{h} = \ker \omega$ and $d\omega = -\theta \wedge \omega$, where

$$\theta = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1q} \\ \vdots & \ddots & \vdots \\ \omega_{q1} & \cdots & \omega_{qq} \end{pmatrix} - \omega_{00} I_q.$$

Here I_q denotes the identity matrix of dimension q. Since θ restricted to $\sigma(\mathfrak{g}/\mathfrak{h})$ is 0, Corollary 2.1.11 implies that θ can be seen as a Bott connection with respect to the basis $\{[E_{i0}]\}_i$ of $\mathfrak{g}/\mathfrak{h}$. On the other hand, define a splitting $\sigma : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ by the formula $\sigma([E_{i0}]) = E_{i0}$, where $i > 0$, then σ is $\mathrm{Ad}_{\mathrm{U}(q)}$-invariant. Let g be the Hermitian metric on $\mathfrak{g}/\mathfrak{h}$ given by $g([X], [Y]) = \mathrm{tr}\, {}^t\sigma([X])\overline{\sigma([Y])}$ for $[X], [Y] \in \mathfrak{g}/\mathfrak{h}$, then g is $\mathrm{Ad}_{\mathrm{U}(q)}$-invariant and $\{[E_{i0}]\}_i$ is an orthonormal basis. Hence the connection form of the unitary connection θ^u given by Lemma 2.1.12 with respect to $\{[E_{i0}]\}_i$ is skew-Hermitian.

Denote cochains in WO_{2q} and WU_q evaluated by the Bott connection θ and the unitary connection θ^u again by their own letters, then

$$\begin{aligned} h_1 &= \sqrt{-1}\widetilde{u}_1 = \frac{q+1}{2\pi}(\omega_{00} + \overline{\omega_{00}}), \\ c_1 &= dh_1 = \sqrt{-1}(v_1 - \bar{v}_1) \\ &= -\frac{q+1}{2\pi} \sum_{i=0}^{q} (\omega_{0i} \wedge \omega_{i0} + \overline{\omega_{0i}} \wedge \overline{\omega_{i0}}). \end{aligned}$$

It follows from Theorem 2.1.7 that

$$\mathrm{GV}_{2q}(K) = h_1 c_1^{2q} = \epsilon\,(\omega_{00} + \overline{\omega_{00}}) \wedge \left(\bigwedge_{i=1}^{q} \omega_{0i} \wedge \omega_{i0} \wedge \overline{\omega_{0i}} \wedge \overline{\omega_{i0}} \right)$$

as an element of $H^{2q+1}(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$, where $\epsilon = (2q)! \left(\dfrac{q+1}{2\pi} \right)^{2q+1}$.

Set now $X_{ij} = E_{ij} - E_{ji}$, $Y_{ij} = \sqrt{-1}(E_{ij} + E_{ji})$ and $K_k = \sqrt{-1}(E_{00} - E_{kk})$, where $0 \le i < j \le q$ and $1 \le k \le q$. These vectors form a basis of $\mathfrak{g}_0 = \mathfrak{su}(q+1)$. Let α_k, β_{ij}, γ_{ij} be the dual of K_k, X_{ij}, Y_{ij}, respectively. Set $X_{ji} = -X_{ij}$ and $Y_{ji} = Y_{ij}$ if $i > j$ and denote their dual forms by β_{ji} and γ_{ji}, then $-\beta_{ji} = \beta_{ij}$ and $\gamma_{ji} = \gamma_{ij}$. Denote their extension to $\mathfrak{g}$ by complexification again by the same letters, then

$$\begin{aligned}
\omega_{00} &= \sqrt{-1}(\alpha_1 + \cdots + \alpha_q), \\
\omega_{ij} &= \beta_{ij} + \sqrt{-1}\gamma_{ij}, \ \text{where } i \ne j.
\end{aligned}$$

By Lemma 2.2.6,

$$\begin{aligned}
&\kappa^* \left(\bigwedge_{i=1}^{q} \omega_{0i} \wedge \omega_{i0} \wedge \overline{\omega_{0i}} \wedge \overline{\omega_{i0}} \right) \\
&= \bigwedge_{i=1}^{q} (\beta_{0i}^1 + \sqrt{-1}\gamma_{0i}^1) \wedge (\beta_{i0}^1 + \sqrt{-1}\gamma_{i0}^1) \wedge (\beta_{0i}^2 - \sqrt{-1}\gamma_{0i}^2) \wedge (\beta_{i0}^2 - \sqrt{-1}\gamma_{i0}^2) \\
&= \bigwedge_{i=1}^{q} (\beta_{0i}^1 + \sqrt{-1}\gamma_{0i}^1) \wedge (\beta_{0i}^1 - \sqrt{-1}\gamma_{0i}^1) \wedge (\beta_{0i}^2 - \sqrt{-1}\gamma_{0i}^2) \wedge (\beta_{0i}^2 + \sqrt{-1}\gamma_{0i}^2) \\
&= \bigwedge_{i=1}^{q} (4\beta_{0i}^1 \wedge \gamma_{0i}^1 \wedge \beta_{0i}^2 \wedge \gamma_{0i}^2).
\end{aligned}$$

Here the superscripts are as in Lemma 2.2.6. Hence the equation

$$\begin{aligned}
\kappa^*(\mathrm{GV}_{2q}(K)) &= \epsilon\,\sqrt{-1}(\alpha_0^1 - \alpha_0^2) \wedge \bigwedge_{i=1}^{q} (4\beta_{0i}^1 \wedge \gamma_{0i}^1 \wedge \beta_{0i}^2 \wedge \gamma_{0i}^2) \\
&= (2q)! \left(\frac{q+1}{\pi} \right)^{2q+1} \frac{\sqrt{-1}}{2} (\alpha_0^1 - \alpha_0^2) \wedge \bigwedge_{i=1}^{q} (\beta_{0i}^1 \wedge \gamma_{0i}^1 \wedge \beta_{0i}^2 \wedge \gamma_{0i}^2)
\end{aligned}$$

holds in $H^*(\mathfrak{g} \oplus \mathfrak{g}, \Delta\mathfrak{k} \otimes \boldsymbol{C}) \cong H^*(\mathfrak{g}_0 \oplus \mathfrak{g}_0, \Delta\mathfrak{k})$, where $\alpha_0 = \alpha_1 + \cdots + \alpha_q$.

Finally, we have by Proposition 2.2.7 the following equation;

$$\begin{aligned}
&\tau^*\kappa^*(\mathrm{GV}_{2q}(K)) \\
=&(2q)! \left(\frac{q+1}{\pi} \right)^{2q+1} \frac{\sqrt{-1}}{2} \alpha_0^1 \wedge \left(\bigwedge_{i=1}^{q} (\beta_{0i}^1 + \beta_{0i}^2) \wedge (\gamma_{0i}^1 + \gamma_{0i}^2) \right) \wedge \left(\bigwedge_{i=1}^{q} \beta_{0i}^2 \wedge \gamma_{0i}^2 \right) \\
=&(2q)! \left(\frac{q+1}{\pi} \right)^{2q+1} \frac{\sqrt{-1}}{2} \alpha_0^1 \wedge \left(\bigwedge_{i=1}^{q} \beta_{0i}^1 \wedge \gamma_{0i}^1 \right) \wedge \left(\bigwedge_{i=1}^{q} \beta_{0i}^2 \wedge \gamma_{0i}^2 \right).
\end{aligned}$$

The non-triviality of $\mathrm{GV}_{2q}(K)$ is shown as follows. It suffices to show by Lemma 2.3.2 the non-triviality of $\mathrm{GV}_{2q}(T^q)$ in $H^*(\mathrm{SU}(q+1)) \otimes H^*(\mathrm{SU}(q+1)/T^q)$. It is clear that $\alpha_0^1 \wedge \left(\bigwedge_{j=1}^q \beta_{0j}^1 \wedge \gamma_{0j}^1\right)$ and $\bigwedge_{j=1}^q \beta_{0j}^2 \wedge \gamma_{0j}^2$ are non-zero multiple of the volume forms of $S^{2q+1} = \mathrm{SU}(q+1)/\mathrm{SU}(q)$ and $\boldsymbol{CP}^q = \mathrm{SU}(q+1)/(T^1 \times \mathrm{SU}(q))$, respectively. Since the natural mappings $\pi_1 : \mathrm{SU}(q+1) \to S^{2q+1} = \mathrm{SU}(q+1)/\mathrm{SU}(q)$ and $\pi_2 : \mathrm{SU}(q+1)/T^q \to \boldsymbol{CP}^q = \mathrm{SU}(q+1)/(T^1 \times \mathrm{SU}(q))$ induce injective mappings on the cohomology, $\mathrm{GV}_{2q}(T^q)$ is non-trivial in the cohomology.

On the other hand, $\mathrm{GV}_{2q}(K)$ is trivial if K is contained in $\mathrm{SU}(q)$. By Lemma 2.3.2, it suffices to show the claim for $K = \mathrm{SU}(q)$. Then, the characteristic mapping factors through $H^*(\mathrm{SU}(q+1)) \otimes H^*(\mathrm{SU}(q+1)/\mathrm{SU}(q))$, which is trivial in degree 2. Therefore $\mathrm{GV}_{2q}(K)$ is trivial by Theorem 1.5 because $\mathrm{ch}_1(K)$ is trivial.

Let us verify now the relation between ξ_q and GV_{2q} given in Theorem 1.5. Under the notations as above, one has

$$v_1 = -\frac{q+1}{2\pi\sqrt{-1}} \sum_{i=1}^q \omega_{0i} \wedge \omega_{i0} \text{ and } \bar{v}_1 = \frac{q+1}{2\pi\sqrt{-1}} \sum_{i=1}^q \overline{\omega_{0i}} \wedge \overline{\omega_{i0}}.$$

Hence

$$\begin{aligned}
\kappa^* v_1 &= -\frac{q+1}{2\pi\sqrt{-1}} \sum_{i=1}^q (\beta_{0i}^1 + \sqrt{-1}\gamma_{0i}^1) \wedge (-\beta_{0i}^1 + \sqrt{-1}\gamma_{0i}^1) \\
&= -\frac{q+1}{\pi} \sum_{i=1}^q \beta_{0i}^1 \wedge \gamma_{0i}^1, \\
\kappa^* \bar{v}_1 &= \frac{q+1}{2\pi\sqrt{-1}} \sum_{i=1}^q (\beta_{0i}^2 - \sqrt{-1}\gamma_{0i}^2) \wedge (-\beta_{0i}^2 - \sqrt{-1}\gamma_{0i}^2) \\
&= -\frac{q+1}{\pi} \sum_{i=1}^q \beta_{0i}^2 \wedge \gamma_{0i}^2.
\end{aligned}$$

It follows that

$$\tau^* \kappa^* \mathrm{ch}_1(K)^q = q! \left(-\frac{q+1}{\pi}\right)^q \bigwedge_{i=1}^q (\beta_{0i}^2 \wedge \gamma_{0i}^2)$$

as an element of $H^{2q}(\mathrm{SU}(q+1) \times (\mathrm{SU}(q+1)/K))$. On the other hand,

$$\begin{aligned}
\xi_q(K) &= \sqrt{-1}\widetilde{u}_1(v_1^q + v_1^{q-1}\bar{v}_1 + \cdots + \bar{v}_1^q) \\
&= \sqrt{-1}\left(\frac{q+1}{2\pi}\right) q! \left(-\frac{q+1}{\pi}\right)^q \alpha_0^1 \wedge \bigwedge_{i=1}^q (\beta_{0i}^1 \wedge \gamma_{0i}^1) + \sum_{i=1}^q \omega_i \wedge \beta_{0i}^2 \wedge \gamma_{0i}^2
\end{aligned}$$

for some ω_i, $i = 1, \ldots, q$. Hence the equation $\mathrm{GV}_{2q}(K) = \frac{(2q)!}{q!q!}\xi_q(K)\mathrm{ch}_1(K)^q$ certainly holds. Remark that $\xi_q(K)$ is non-trivial even if $K = \{e\}$.

Remark 2.3.7.

1) As explained, the non-triviality of $\mathrm{GV}_{2q}(T^q)$ follows from the non-triviality of $\mathrm{GV}_{2q}(T^1 \times \mathrm{SU}(q))$. On the other hand, $\mathrm{GV}_{2q}(\mathrm{SU}(q))$ is trivial. In other words, $\mathrm{GV}_{2q}(T^1 \times \mathrm{SU}(q))$ become trivial when pulled-back by the S^1-bundle $p : \Gamma\backslash\mathrm{SL}(q+1;\boldsymbol{C})/\mathrm{SU}(q) \to \Gamma\backslash\mathrm{SL}(q+1;\boldsymbol{C})/(T^1 \times \mathrm{SU}(q))$, where Γ is a cocompact lattice of $\mathrm{SL}(q+1;\boldsymbol{C})/(T^1 \times \mathrm{SU}(q))$. This is related to the Hopf fibration as follows. Recall that $\mathrm{GV}_{2q}(K)$ is decomposed into the product of $\xi_q(K)$ and $\mathrm{ch}_1(K)^q$. By the last part of Example 2.3.6, the both $\xi_q(\mathrm{SU}(q))$ and $\xi_q(T^1 \times \mathrm{SU}(q))$ are non-trivial. On the other hand, $\mathrm{ch}_1(\mathrm{SU}(q))^q$ is trivial while $\mathrm{ch}_1(T^1 \times \mathrm{SU}(q))^q$ is non-trivial. Now consider the following diagram;
$$\begin{tikzcd}
S^{2q+1} \arrow[r] \arrow[d] & \Gamma\backslash\mathrm{SL}(q+1;\boldsymbol{C})/\mathrm{SU}(q) \arrow[r] \arrow[d, "p"'] & \Gamma\backslash\mathrm{SL}(q+1;\boldsymbol{C})/\mathrm{SU}(q+1) \arrow[d, equal] \\
\boldsymbol{C}P^q \arrow[r] & \Gamma\backslash\mathrm{SL}(q+1;\boldsymbol{C})/(T^1 \times \mathrm{SU}(q)) \arrow[r] & \Gamma\backslash\mathrm{SL}(q+1;\boldsymbol{C})/\mathrm{SU}(q+1),
\end{tikzcd}$$
where the first column is the Hopf fibration. Hence one can consider the non-triviality of $\mathrm{ch}_1(T^1 \times \mathrm{SU}(q))^q$ is derived from the Hopf fibration.

2) By Proposition 2.3.3, $\mathrm{GV}_{2q}(\{e\}) = 0$ in $H^*(\mathfrak{sl}(q+1;\boldsymbol{C})_{\boldsymbol{R}})$ because $v_1(\{e\}) = 0$. Since the complex normal bundle is trivial, the Bott class $u_1v_1^q(\{e\})$ is well-defined (Definition 2.1.13). The product of the Bott class and its complex conjugate is $u_1\bar{u}_1v_1^q\bar{v}_1^q(\{e\})$. By a similar calculation as in Example 2.3.6, one has
$$u_1\bar{u}_1v_1^q\bar{v}_1^q(\{e\}) = \left(\frac{q+1}{2\pi}\right)^{2q+2} \omega_{00} \wedge \overline{\omega_{00}} \wedge (d\omega_{00})^q \wedge (d\overline{\omega_{00}})^q.$$
The mapping $\tau^*\kappa^*$ is now an isomorphism from $H^*(\mathfrak{sl}(q+1;\boldsymbol{C})_{\boldsymbol{R}})$ to $H^*(\mathrm{SU}(q+1)) \otimes H^*(\mathrm{SU}(q+1))$. The image of $u_1\bar{u}_1v_1^q\bar{v}_1^q(\{e\})$ under $\tau^*\kappa^*$ is equal to
$$\frac{q!q!}{4}\left(\frac{q+1}{\pi}\right)^{2q+2} \alpha_0^1 \wedge \alpha_0^2 \wedge \left(\bigwedge_{j=1}^q \beta_{0j}^1 \wedge \gamma_{0j}^1\right) \wedge \left(\bigwedge_{j=1}^q \beta_{0j}^2 \wedge \gamma_{0j}^2\right).$$
By repeating the argument as in Example 2.3.6, this class is seen to be non-trivial. It is easy to show that the Bott class is well-defined and non-trivial if K is contained in $\mathrm{SU}(q)$. See Section 3 for related constructions.

3) Another complex secondary class can be computed as follows. By Lemma 2.2.11, the matrix valued 1-form $\frac{\theta - {}^t\overline{\theta}}{2}$ induces a unitary connection. Set $\widehat{\omega_{ij}} = \omega_{ij} + \overline{\omega_{ji}}$, then one has

$$\begin{aligned}\widetilde{u}_2 =& \frac{1}{8\pi^2}\left((5\widehat{\omega_{00}} + \widehat{\omega_{11}}) \wedge (\omega_{01} \wedge \omega_{10} - \overline{\omega_{01}} \wedge \overline{\omega_{10}})\right. \\ &+ (5\widehat{\omega_{00}} + \widehat{\omega_{22}}) \wedge (\omega_{02} \wedge \omega_{20} - \overline{\omega_{02}} \wedge \overline{\omega_{20}}) \\ &+ (\widehat{\omega_{11}} - \widehat{\omega_{22}}) \wedge \widehat{\omega_{21}} \wedge \widehat{\omega_{12}} \\ &\left.-\widehat{\omega_{21}} \wedge (\omega_{10} \wedge \omega_{02} - \overline{\omega_{20}} \wedge \overline{\omega_{01}}) - \widehat{\omega_{12}} \wedge (\omega_{20} \wedge \omega_{01} - \overline{\omega_{10}} \wedge \overline{\omega_{02}})\right).\end{aligned}$$

Hence

$$\widetilde{u}_1\widetilde{u}_2 v_1^q \bar{v}_1^q(K) = \epsilon\, \widehat{\omega_{11}} \wedge \widehat{\omega_{22}} \wedge \omega_{01} \wedge \omega_{10} \wedge \omega_{02} \wedge \omega_{20} \wedge \widehat{\omega_{21}} \wedge \widehat{\omega_{12}},$$

where ϵ is a non-zero constant. As the above differential form is a non-zero multiple of the volume form of $\mathrm{SU}(3)/(T^1 \times \mathrm{SU}(2))$, it is non-trivial.

Example 2.3.6, continued. Other real secondary classes also can be computed. As an example, consider the case where $q = 2$. Noticing that these classes can be realized as classes in $H^*(\mathrm{SU}(3)) \otimes H^*(\mathrm{SU}(3)/(T^1 \times \mathrm{SU}(2)))$, it suffices to compute the classes of degree $4q + 1 = 9$ by a theorem of Pittie [35] referred in § 1 as 3) of Theorem 2.1.4. Indeed, if $h_I c_J(T^1 \times \mathrm{SU}(2))$ is non-trivial, then $i_1 + |J| = 2q+1 = 5$. Thus the degree of $h_I c_J(T^1 \times \mathrm{SU}(2))$ is $9 + (2i_2 - 1) + \cdots + (2i_r - 1)$, where $I = \{i_1, \cdots, i_r\}$. Noticing that $i_2 \geq 3$, the only possibility is $I = \{i_1\}$.

The classes of degree 9 are $h_1c_1^4$, $h_1c_1^2c_2$, $h_1c_1c_3$, h_1c_4, $h_1c_2^2$ and h_3c_2. By Theorem 1.5, the following formulae hold for c_2, c_3, c_4 and h_3;

$$c_2 = -(v_2 - v_1\bar{v}_1 + \bar{v}_2), \quad c_3 = -\sqrt{-1}(-v_2\bar{v}_1 + v_1\bar{v}_2), \quad c_4 = v_2\bar{v}_2,$$
$$h_3 = -\frac{\sqrt{-1}}{2}(-\widetilde{u}_2(v_1 + \bar{v}_1) + \widetilde{u}_1(v_2 + \bar{v}_2)).$$

Hence

$$h_1c_1^4 = 6\sqrt{-1}\widetilde{u}_1 v_1^2\bar{v}_1^2, \quad h_1c_1^2c_2 = \sqrt{-1}\widetilde{u}_1(v_1^2\bar{v}_2 + 2v_1^2\bar{v}_1^2 + v_2\bar{v}_1^2),$$
$$h_1c_1c_3 = \sqrt{-1}\widetilde{u}_1(v_1^2\bar{v}_2 + v_2\bar{v}_1^2), \quad h_1c_4 = \sqrt{-1}\widetilde{u}_1 v_2\bar{v}_2,$$
$$h_1c_2^2 = \sqrt{-1}\widetilde{u}_1(2v_2\bar{v}_2 + v_1^2\bar{v}_1^2),$$

and

$$\begin{aligned}
h_3c_2 &= \frac{\sqrt{-1}}{2}(-\widetilde{u}_2(v_1+\bar{v}_1)+\widetilde{u}_1(v_2+\bar{v}_2))(v_2-v_1\bar{v}_1+v_2)\\
&= \frac{\sqrt{-1}}{2}(-\widetilde{u}_2(-v_1^2\bar{v}_1+v_1\bar{v}_2+v_2\bar{v}_1-v_1\bar{v}_1^2)+2\widetilde{u}_1v_2\bar{v}_2)\\
&= \frac{\sqrt{-1}}{2}(-\widetilde{u}_2(v_1-\bar{v}_1)(v_1^2+\bar{v}_2-v_2-\bar{v}_1^2)+2\widetilde{u}_1v_2\bar{v}_2)\\
&\equiv \frac{\sqrt{-1}}{2}(-\widetilde{u}_1(v_2-\bar{v}_2)(v_1^2+\bar{v}_2-v_2-\bar{v}_1^2)+2\widetilde{u}_1v_2\bar{v}_2)\\
&= \frac{\sqrt{-1}}{2}\widetilde{u}_1(v_2\bar{v}_1^2+v_1^2\bar{v}_2),
\end{aligned}$$

where '$\equiv$' means that the equality holds in $H^*(\mathrm{WU}_2)$.

On the other hand, the curvature form of the Bott connection θ is given by

$$\begin{aligned}
&d\theta+\theta\wedge\theta\\
&=\begin{pmatrix} d\omega_{11}-d\omega_{00}+\omega_{12}\wedge\omega_{21} & d\omega_{12}+\omega_{11}\wedge\omega_{12}+\omega_{12}\wedge\omega_{22}\\ d\omega_{21}+\omega_{21}\wedge\omega_{11}+\omega_{22}\wedge\omega_{21} & d\omega_{22}-d\omega_{00}+\omega_{21}\wedge\omega_{12}\end{pmatrix}\\
&=\begin{pmatrix} 2\omega_{01}\wedge\omega_{10}+\omega_{02}\wedge\omega_{20} & -\omega_{10}\wedge\omega_{02}\\ -\omega_{20}\wedge\omega_{01} & \omega_{01}\wedge\omega_{10}+2\omega_{02}\wedge\omega_{20}\end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
v_1 &= -\frac{3}{2\pi\sqrt{-1}}(\omega_{01}\wedge\omega_{10}+\omega_{02}\wedge\omega_{20}),\\
v_2 &= \left(\frac{-1}{2\pi\sqrt{-1}}\right)^2 6\omega_{01}\wedge\omega_{10}\wedge\omega_{02}\wedge\omega_{20}\\
\widetilde{u}_1 &= \frac{3}{2\pi\sqrt{-1}}(\omega_{00}+\overline{\omega_{00}}).
\end{aligned}$$

Define a differential form (gv) by setting

$$(\mathrm{gv})=\frac{3}{(2\pi)^5}(\omega_{00}+\overline{\omega_{00}})\wedge\omega_{01}\wedge\omega_{10}\wedge\omega_{02}\wedge\omega_{20}\wedge\overline{\omega_{01}}\wedge\overline{\omega_{10}}\wedge\overline{\omega_{02}}\wedge\overline{\omega_{20}},$$

then

$$\begin{aligned}
\mathrm{GV}_4 = h_1c_1^4 &= 6\cdot(2\cdot 3^2)^2(\mathrm{gv}) = 2^3\cdot 3^5\,(\mathrm{gv}),\\
h_1c_1^2c_2 &= \big((2\cdot 3^2)\cdot 6+2(2\cdot 3^2)^2+6\cdot(2\cdot 3^2)\big)\,(\mathrm{gv}) = 2^5\cdot 3^3\,(\mathrm{gv}),\\
h_1c_1c_3 &= \big((2\cdot 3^2)\cdot 6+6\cdot(2\cdot 3^2)\big)\,(\mathrm{gv}) = 2^3\cdot 3^3\,(\mathrm{gv}),\\
h_1c_4 &= 6^2\,(\mathrm{gv}) = 2^2\cdot 3^2\,(\mathrm{gv}),\\
h_1c_2^2 &= \big(2\cdot 6^2+(2\cdot 3^2)^2\big)\,(\mathrm{gv}) = 2^2\cdot 3^2\cdot 11\,(\mathrm{gv}),\\
h_3c_2 &= \frac{1}{2}\big(6\cdot(2\cdot 3^2)+(2\cdot 3^2)\cdot 6\big)\,(\mathrm{gv}) = 2^2\cdot 3^3(\mathrm{gv}).
\end{aligned}$$

Hence

$$
\begin{gathered}
h_1c_1^2c_2(K) = \frac{4}{9}\mathrm{GV}_4(K), \quad h_1c_1c_3(K) = \frac{1}{9}\mathrm{GV}_4(K), \\
h_1c_4(K) = \frac{1}{54}\mathrm{GV}_4(K), \quad h_1c_2^2(K) = \frac{11}{54}\mathrm{GV}_4(K), \\
h_3c_2(K) = \frac{1}{18}\mathrm{GV}_4(K)
\end{gathered}
$$

in $H^9(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k}; \boldsymbol{C})$ if K is as in Example 2.3.6. These classes satisfy the following relations;

$$
\begin{aligned}
h_3c_2 &= \frac{1}{2}h_1c_1c_3, \\
h_1c_4 &= \frac{1}{2}h_1c_2^2 - \frac{1}{12}h_1c_1^4, \\
h_1c_1c_3 &= h_1c_1^2c_2 - \frac{1}{3}h_1c_1^4.
\end{aligned}
$$

These equations hold in fact for any transversely holomorphic foliations [2].

Example 2.3.8. Let $G = \mathrm{SO}(q+2;\boldsymbol{C})$, $\mathfrak{g} = \mathfrak{so}(q+2;\boldsymbol{C})$ and $\mathfrak{g}_0 = \mathfrak{so}(q+2;\boldsymbol{R})$. Denote by $T^{\left[\frac{q+2}{2}\right]}$ the maximal torus realized as $\mathrm{SO}(2;\boldsymbol{R})\oplus\cdots\oplus\mathrm{SO}(2;\boldsymbol{R})$ ($(q+2)/2$-times) if q is even, and $\mathrm{SO}(2;\boldsymbol{R}) \oplus \cdots \oplus \mathrm{SO}(2;\boldsymbol{R}) \oplus \{1\}$ ($(q+1)/2$-times) if q is odd. Set $X_{ij} = E_{ij} - E_{ji}$, where E_{ij} is defined as in Notation 2.3.5, then $\{X_{ij} \ ; \ 0 \le i < j \le q+1\}$ is a basis for $\mathfrak{g}$. Note that this is also a basis for $\mathfrak{g}_0 = \mathfrak{so}(q+2;\boldsymbol{R})$ over $\boldsymbol{R}$.

Let $\mathfrak{h}^\pm$ be the Lie subalgebras of $\mathfrak{g}$ defined by

$$
\mathfrak{h}^\pm = \left\langle X_{01}, X_{0k} \pm \sqrt{-1}X_{1k}, X_{ij} \ ; \ 2 \le k \le q+1, \, 2 \le i < j \le q+1 \right\rangle_{\boldsymbol{C}},
$$

and let $H^\pm$ be the corresponding Lie subgroups. Let K be a connected compact Lie subgroup of G such that $T^{\left[\frac{q+2}{2}\right]} \subset K \subset T^1 \times \mathrm{SO}(q;\boldsymbol{R}) = \mathrm{SO}(2;\boldsymbol{R}) \oplus \mathrm{SO}(q;\boldsymbol{R})$. We will show that $\mathrm{GV}_{2q}(K)$ is non-trivial if and only if q is odd. In what follows, the quadruplet (G, H^+, K, Γ) is considered and $\mathfrak{h}^+$ and H^+ are simply denoted by $\mathfrak{h}$ and H, respectively, because the argument for (G, H^-, K, Γ) is completely parallel.

Let ω_{ij} be the dual of X_{ij} $(i \ne j)$, then $d\omega_{ij} = -\sum\limits_{0 \le k \le q+1} \omega_{ik} \wedge \omega_{kj}$, where $\omega_{ij} = -\omega_{ji}$ and $\omega_{ii} = 0$. It is easy to see that $\mathfrak{h} = \ker\left\langle \omega_{0i} + \sqrt{-1}\omega_{1i} \ ; \ 2 \le i \le q+1 \right\rangle$, and one has

$$
d(\omega_{0i} + \sqrt{-1}\omega_{1i}) = \sqrt{-1}\omega_{01} \wedge (\omega_{0i} + \sqrt{-1}\omega_{1i}) + \sum_{l=2}^{q+1} \omega_{li} \wedge (\omega_{0l} + \sqrt{-1}\omega_{1l}).
$$

By setting $\omega = {}^t(\omega_{02} + \sqrt{-1}\omega_{12}, \ldots, \omega_{0,q+1} + \sqrt{-1}\omega_{1,q+1})$, the above equation implies that $d\omega = -\theta \wedge \omega$, where

$$\theta = -\begin{pmatrix} \sqrt{-1}\omega_{01} & -\omega_{23} & -\omega_{24} & \cdots & -\omega_{2,q+1} \\ \omega_{23} & \sqrt{-1}\omega_{01} & -\omega_{34} & \cdots & -\omega_{3,q+1} \\ \vdots & & \ddots & & \vdots \\ \omega_{2,q+1} & \omega_{3,q+1} & \cdots & \cdots & \sqrt{-1}\omega_{01} \end{pmatrix}.$$

On the other hand, let $\sigma : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ be the splitting defined by $\sigma([X_{0j} - \sqrt{-1}X_{1j}]) = X_{0j} - \sqrt{-1}X_{1j}$, where $j = 2, \ldots, q+1$, then σ is $\mathrm{Ad}_{T^1 \times \mathrm{SO}(q+2;\boldsymbol{R})}$-invariant. To see this, notice first that

$$[X_{ij}, X_{kl}] = \delta_{jk}X_{il} + \delta_{il}X_{jk} - \delta_{ik}X_{jl} - \delta_{jl}X_{ik},$$

where δ_{ij} is the Kronecker delta. Since the Lie algebra of $T^1 \times \mathrm{SO}(q;\boldsymbol{R})$ is generated by X_{01} and X_{ij}, $2 \leq i < j \leq (q+1)$, over $\boldsymbol{R}$, it suffices to verify that $[X_{01}, X_{0l} - \sqrt{-1}X_{1l}]$ and $[X_{ij}, X_{0l} - \sqrt{-1}X_{1l}]$ belong to the image of σ, where $l = 2, \ldots, q+1$. If $l \geq 2$, then

$$[X_{01}, X_{0l} - \sqrt{-1}X_{1l}] = -X_{1l} - \sqrt{-1}X_{0l} = -\sqrt{-1}(X_{0l} - \sqrt{-1}X_{1l}).$$

On the other hand,

$$\begin{aligned}[X_{ij}, X_{0l} - \sqrt{-1}X_{1l}] &= (\delta_{il}X_{j0} - \delta_{jl}X_{i0}) - \sqrt{-1}(\delta_{il}X_{j1} - \delta_{jl}X_{i1}) \\ &= -\delta_{il}(X_{0j} - \sqrt{-1}X_{1j}) + \delta_{jl}(X_{0i} - \sqrt{-1}X_{1i}).\end{aligned}$$

Thus σ is $\mathrm{Ad}_{T^1 \times \mathrm{SO}(q+2;\boldsymbol{R})}$-invariant.

As θ is equal to 0 when restricted to the image of σ, the above θ can be used as a Bott connection by Corollary 2.1.11. Moreover, let g be the Hermitian metric on $\mathfrak{g}/\mathfrak{h}$ defined by $g([X],[Y]) = \frac{1}{4}\mathrm{tr}\,{}^t\sigma([X])\overline{\sigma([Y])}$ for $[X],[Y] \in \mathfrak{g}/\mathfrak{h}$, then g is an $\mathrm{Ad}_{T^1 \times \mathrm{SO}(q+2;\boldsymbol{R})}$-invariant metric with respect to which $\{[X_{0j} - \sqrt{-1}X_{1j}]\}$ is an orthonormal basis. Hence we may use a unitary connection represented by a skew-Hermitian matrix.

Then by Theorem 2.1.7, one has the following equalities;

$$\begin{aligned} h_1 &= \frac{q\sqrt{-1}}{2\pi}(\omega_{01} - \overline{\omega_{01}}), \\ c_1 &= \frac{q\sqrt{-1}}{2\pi}\sum_{k=2}^{q+1}(\omega_{0k} \wedge \omega_{1k} - \overline{\omega_{0k}} \wedge \overline{\omega_{1k}}). \end{aligned}$$

It follows that

$$\mathrm{GV}_{2q}(K) = \sqrt{-1}\left(\frac{q}{2\pi}\right)^{2q+1}(\omega_{01} - \overline{\omega_{01}}) \wedge \bigwedge_{l=2}^{q+1}(\omega_{0l} \wedge \omega_{1l} \wedge \overline{\omega_{0l}} \wedge \overline{\omega_{1l}})$$

as an element of $H^{4q+1}(\mathfrak{g}_{\boldsymbol{R}}, \mathfrak{k})$. As $\{X_{ij}\}$ is also a basis of $\mathfrak{g}_0$ over $\boldsymbol{R}$, it follows from Lemma 2.2.6 that

$$\kappa^*\mathrm{GV}_{2q}(K) = \sqrt{-1}\left(\frac{q}{2\pi}\right)^{2q+1}(\omega^1_{01} - \omega^2_{01}) \wedge \bigwedge_{l=2}^{q+1}(\omega^1_{0l} \wedge \omega^1_{1l} \wedge \omega^2_{0l} \wedge \omega^2_{1l}),$$

where $\{\omega_{ij}\}$ is considered as the dual basis of $\mathfrak{g}_0^*$.

Finally by Proposition 2.2.7,

$$\tau^*\kappa^*\mathrm{GV}_{2q}(K) = \sqrt{-1}\left(\frac{q}{2\pi}\right)^{2q+1}\omega^1_{01} \wedge \bigwedge_{l=2}^{q+1}(\omega^1_{0l} \wedge \omega^1_{1l} \wedge \omega^2_{0l} \wedge \omega^2_{1l}).$$

In what follows, denote $\mathrm{SO}(m;\boldsymbol{R})$ simply by $\mathrm{SO}(m)$ as usual. Denote by $\mathrm{SO}(q)$ the subgroup $\{1\}\oplus\{1\}\oplus\mathrm{SO}(q)$ of $\mathrm{SO}(q+2)$. Recall that $T^1\times\mathrm{SO}(q) = \mathrm{SO}(2)\oplus\mathrm{SO}(q)$. Let $\pi_1 : \mathrm{SO}(q+2) \to \mathrm{SO}(q+2)/\mathrm{SO}(q)$ and $\pi_2 : \mathrm{SO}(q+2) \to \mathrm{SO}(q+2)/(T^1 \times \mathrm{SO}(q))$ be the natural projections. Then $\tau^*\varphi^*\mathrm{GV}_{2q}(K)$ is a non-zero multiple of the pull-back of the volume form of $(\mathrm{SO}(q+2)/\mathrm{SO}(q)) \times \bigl(\mathrm{SO}(q+2)/(T^1 \times \mathrm{SO}(q))\bigr)$ by (π_1, π_2) to $H^*(\mathrm{SO}(q+2)) \otimes H^*(\mathrm{SO}(q+2)/(T^1 \times \mathrm{SO}(q)))$. It is classically known that $\pi_1^*(\mathrm{vol}_{\mathrm{SO}(q+2)/\mathrm{SO}(q)})$ is non-trivial if and only if q is odd. Assume that q is odd and write $q = 2m-1$, then $\pi_2^*(\mathrm{vol}_{\mathrm{SO}(2m+1)/(T^1\times\mathrm{SO}(2m-1))})$ is non-trivial in $H^*(\mathrm{SO}(2m+1)/T^m)$. Therefore, $\tau^*\kappa^*(\mathrm{GV}_{2q}(K))$ is non-trivial if q is odd and $T^{\left[\frac{q+2}{2}\right]} \subset K \subset T^1 \times \mathrm{SO}(q)$, and $\tau^*\kappa^*(\mathrm{GV}_{2q}(K'))$ is trivial for any closed subgroup K' of $T^1 \times \mathrm{SO}(q)$ if q is even. $\square$

It is well-known that $\mathrm{SL}(2;\boldsymbol{C})$ is a double (the universal) covering of $\mathrm{SO}(3;\boldsymbol{C})$. This is still true as foliated spaces, namely, we have the following

Proposition 2.3.9. *There is a covering map* $\mathrm{SL}(2;\boldsymbol{C})$ *to* $\mathrm{SO}(3;\boldsymbol{C})$ *which preserves the foliations defined in Examples 2.3.6 and 2.3.8.*

Proof. First recall a description of a covering map by following [16] and [33]. Let $\{X_0, X_1, X_2\}$, where

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

be a basis of $\mathfrak{sl}(2;\boldsymbol{C})$, and denote by $\mathcal{F}^+$ the foliation of $\mathrm{SL}(2;\boldsymbol{C})$ induced by X_1 and denote by $\mathcal{F}^-$ the foliation induced by X_2. Let $\{X_{ij} = E_{ij} - E_{ji} \,|\, 0 \le i < j \le 2\}$

be a basis of $\mathfrak{so}(3;\boldsymbol{C})$ and denote by $\mathcal{G}^{\pm}$ the foliation of $\mathrm{SO}(3;\boldsymbol{C})$ induced from $\mathfrak{h}^{\pm}$ given in Example 2.3.8. Let φ be the linear isomorphism from $\mathfrak{sl}(2;\boldsymbol{C})$ to $\boldsymbol{C}^3$ given by $\varphi(aX_0+bX_1+cX_2) = {}^t\left(\frac{1}{2\sqrt{-1}}(b-c), \frac{1}{2}(b+c), -a\right)$. For $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, define $\iota(g) \in \mathrm{GL}(3;\boldsymbol{C})$ by $\iota(g)^t(z_1,z_2,z_3) = \varphi \circ \mathrm{Ad}_g \circ \varphi^{-1}({}^t(z_1,z_2,z_3))$, then

$$\iota(g) = \begin{pmatrix} \frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{2} & \frac{\alpha^2-\beta^2+\gamma^2-\delta^2}{2\sqrt{-1}} & -\sqrt{-1}(\alpha\beta+\gamma\delta) \\ \frac{\alpha^2+\beta^2-\gamma^2-\delta^2}{-2\sqrt{-1}} & \frac{\alpha^2-\beta^2-\gamma^2+\delta^2}{2} & \alpha\beta-\gamma\delta \\ \sqrt{-1}(\alpha\gamma+\beta\delta) & \alpha\gamma-\beta\delta & \alpha\delta+\beta\gamma \end{pmatrix}.$$

It follows that ι is a homomorphism from $\mathrm{SL}(2;\boldsymbol{C})$ to $\mathrm{SO}(3;\boldsymbol{C})$. The differential $\iota_* : \mathfrak{sl}(2;\boldsymbol{C}) \to \mathfrak{so}(3;\boldsymbol{C})$ is given by $\iota_*(X_0) = -2\sqrt{-1}X_{01}$, $\iota_*(X_1) = -\sqrt{-1}X_{02}+X_{12}$ and $\iota_*(X_2) = -\sqrt{-1}X_{02} - X_{12}$. Thus ι is a local isomorphism which maps $\mathcal{F}^{\pm}$ to $\mathcal{G}^{\pm}$, respectively. Since $\ker\iota = \{\pm I_2\}$, each leaf of $\mathcal{G}^{\pm}$ is doubly covered by a leaf of $\mathcal{F}^{\pm}$. Thus ι is certainly a required covering map. $\square$

Example 2.3.10. Let $\mathfrak{g} = \mathfrak{sp}(n+1;\boldsymbol{C})$, $G = \mathrm{Sp}(n+1;\boldsymbol{C})$, and $\mathfrak{g}_0 = \mathfrak{sp}(n+1;\boldsymbol{R}) = \mathfrak{sp}(n+1) \cap \mathfrak{su}(2n+2)$. Note that

$$\mathfrak{sp}(n+1;\boldsymbol{C}) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \,\middle|\, A,B,C \in M(n+1;\boldsymbol{C}), B = {}^tB \text{ and } C = {}^tC \right\},$$

$$\begin{aligned}\mathfrak{sp}(n+1;\boldsymbol{R}) &= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \,\middle|\, {}^t\overline{A} + A = 0, B = {}^tB, C = {}^tC \text{ and } B + {}^t\overline{C} = 0 \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \,\middle|\, A = -{}^t\overline{A}, B = {}^tB \right\},\end{aligned}$$

Let $X_{ij} = E_{ij} - E_{j+n,i+n}$, $Y_{kk} = E_{k,k+n}$, $Y_{kl} = E_{k,l+n} + E_{l,k+n}$, $Z_{k'k'} = E_{k'+n,k'}$ and $Z_{k'l'} = E_{k'+n,l'} + E_{l'+n,k'}$, where $0 \le i,j \le n$, $0 \le k < l \le n$ and $0 \le k' < l' \le n$, then $\{X_{ij}, Y_{kl}, Z_{k'l'}\}_{0\le i,j\le n, 0\le k\le l\le n, 0\le k'\le l'\le n}$ is a basis of $\mathfrak{g}$ over $\boldsymbol{C}$. Consider $\mathfrak{sp}(n;\boldsymbol{C})$ as a Lie subalgebra of $\mathfrak{sp}(n+1;\boldsymbol{C})$ by setting

$$\mathfrak{sp}(n;\boldsymbol{C}) = \left\langle X_{ij}, Y_{kl}, Z_{k'l'} \,\middle|\, 1 \le i,j \le n,\ 1 \le k \le l \le n,\ 1 \le k' \le l' \le n \right\rangle_{\boldsymbol{C}},$$

then $\mathfrak{sp}(n;\boldsymbol{R})$ is also realized as a real Lie subalgebra of $\mathfrak{sp}(n+1;\boldsymbol{C})$ via inclusion to $\mathfrak{sp}(n;\boldsymbol{C})$. Let T^{n+1} be the maximal torus generated by $\sqrt{-1}X_{ii}$, $0 \le i \le n$, over $\boldsymbol{R}$, and let $T^1 \times \mathrm{Sp}(n;\boldsymbol{R})$ be the real subgroup of $\mathrm{Sp}(n+1;\boldsymbol{C})$ whose Lie algebra is generated over $\boldsymbol{R}$ by $\sqrt{-1}X_{00}$ and $\mathfrak{sp}(n;\boldsymbol{R})$. Note that $T^{n+1} \subset T^1 \times \mathrm{Sp}(n;\boldsymbol{R}) \subset \mathrm{Sp}(n+1;\boldsymbol{C})$.

In what follows, K is assumed to be a compact connected real Lie subgroup such that $T^1 \times \mathrm{Sp}(n;\boldsymbol{R}) \supset K \supset T^{n+1}$. Let ω_{ij}, η_{kl} and ζ_{kl} be the dual of X_{ij}, Y_{kl} and

Z_{kl}, respectively, where $\eta_{lk} = \eta_{kl}$ and $\zeta_{lk} = \zeta_{kl}$, then

$$\begin{aligned} d\omega_{ij} &= -\sum_{s=0}^{n} \omega_{is} \wedge \omega_{sj} - \sum_{t=0}^{n} \eta_{it} \wedge \zeta_{tj}, \\ d\eta_{kl} &= -\sum_{s=0}^{n} \omega_{ks} \wedge \eta_{sl} + \sum_{t=0}^{n} \eta_{kt} \wedge \omega_{lt}, \\ d\zeta_{k'l'} &= -\sum_{s=0}^{n} \zeta_{k's} \wedge \omega_{sl'} + \sum_{t=0}^{n} \omega_{tk'} \wedge \zeta_{tl'}. \end{aligned}$$

Let $\mathfrak{h} = \ker \langle \omega_{i0}, \zeta_{0j} \rangle_{1 \le i \le n,\ 0 \le j \le n}$, then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and

$$\mathfrak{h} = \left\langle X_{00}, X_{ij}, Y_{kl}, Z_{k'l'} \;\middle|\; 1 \le i \le n,\, 0 \le j \le n,\, 0 \le k \le l \le n,\, 1 \le k' \le l' \le n \right\rangle.$$

The foliation induced from $\mathfrak{h}$ is of complex codimension $q = 2n+1$.

Let $\sigma : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ be the splitting defined by

$$\sigma([X_{i0}]) = X_{i0},\ \sigma([Z_{0j}]) = Z_{0j},$$

then σ is $\mathrm{Ad}_{T^1 \times \mathrm{Sp}(n;\boldsymbol{R})}$-invariant. An $\mathrm{Ad}_{T^1 \times \mathrm{Sp}(n;\boldsymbol{R})}$-invariant Hermitian metric g on $\mathfrak{g}/\mathfrak{h}$ is defined by setting $g([X],[Y]) = \mathrm{tr}\, {}^t\sigma([X])\overline{\sigma([Y])}$, and an orthonormal basis for g is $\{\frac{1}{\sqrt{2}}[X_{i0}], [Z_{0j}]\}$.

Let $\omega = {}^t(\sqrt{2}\omega_{10}, \sqrt{2}\omega_{20}, \ldots, \sqrt{2}\omega_{n0}, \zeta_{00}, \zeta_{01}, \ldots, \zeta_{0n})$ and set

$$\widetilde{\theta} = \begin{pmatrix} \omega_{11}-\omega_{00} & \omega_{12} & \cdots & \omega_{1n} & \sqrt{2}\eta_{10} & \sqrt{2}\eta_{11} & \cdots & \sqrt{2}\eta_{1n} \\ \omega_{21} & \omega_{22}-\omega_{00} & \cdots & \omega_{2n} & \sqrt{2}\eta_{20} & \sqrt{2}\eta_{21} & \cdots & \sqrt{2}\eta_{2n} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \omega_{n1} & \cdots & \omega_{n,n-1} & \omega_{nn}-\omega_{00} & \sqrt{2}\eta_{n0} & \sqrt{2}\eta_{n1} & \cdots & \sqrt{2}\eta_{nn} \\ \frac{1}{\sqrt{2}}\zeta_{01} & \frac{1}{\sqrt{2}}\zeta_{02} & \cdots & \frac{1}{\sqrt{2}}\zeta_{0n} & -2\omega_{00} & -\omega_{10} & \cdots & -\omega_{n0} \\ \frac{1}{\sqrt{2}}\zeta_{11} & \frac{1}{\sqrt{2}}\zeta_{12} & \cdots & \frac{1}{\sqrt{2}}\zeta_{1n} & -\omega_{01} & -\omega_{11}-\omega_{00} & \cdots & -\omega_{n1} \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\ \frac{1}{\sqrt{2}}\zeta_{n1} & \frac{1}{\sqrt{2}}\zeta_{n2} & \cdots & \frac{1}{\sqrt{2}}\zeta_{nn} & -\omega_{0n} & -\omega_{1n} & \cdots & -\omega_{nn}-\omega_{00} \end{pmatrix},$$

then $d\omega = -\widetilde{\theta} \wedge \omega$. By Definition 2.1.9 and Corollary 2.1.11, a Bott connection is given by

$$\theta = \begin{pmatrix} \omega_{11}-\omega_{00} & \omega_{12} & \cdots & \omega_{1n} & \sqrt{2}\eta_{10} & \sqrt{2}\eta_{11} & \cdots & \sqrt{2}\eta_{1n} \\ \omega_{21} & \omega_{22}-\omega_{00} & \cdots & \omega_{2n} & \sqrt{2}\eta_{20} & \sqrt{2}\eta_{21} & \cdots & \sqrt{2}\eta_{2n} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \omega_{n1} & \cdots & \omega_{n,n-1} & \omega_{nn}-\omega_{00} & \sqrt{2}\eta_{n0} & \sqrt{2}\eta_{n1} & \cdots & \sqrt{2}\eta_{nn} \\ 0 & 0 & \cdots & 0 & -2\omega_{00} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2}}\zeta_{11} & \frac{1}{\sqrt{2}}\zeta_{12} & \cdots & \frac{1}{\sqrt{2}}\zeta_{1n} & -\omega_{01} & -\omega_{11}-\omega_{00} & \cdots & -\omega_{n1} \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\ \frac{1}{\sqrt{2}}\zeta_{n1} & \frac{1}{\sqrt{2}}\zeta_{n2} & \cdots & \frac{1}{\sqrt{2}}\zeta_{nn} & -\omega_{0n} & -\omega_{1n} & \cdots & -\omega_{nn}-\omega_{00} \end{pmatrix}.$$

Hence

$$\begin{aligned} h_1 &= \frac{2n+2}{2\pi}(\omega_{00}+\overline{\omega_{00}}) = \frac{q+1}{2\pi}(\omega_{00}+\overline{\omega_{00}}), \\ v_1 &= \frac{q+1}{2\pi\sqrt{-1}} d\omega_{00}. \end{aligned}$$

It follows that

$$\begin{aligned} &\mathrm{GV}_{2q}(K) \\ =&\frac{(2q)!}{q!q!}\left(\frac{q+1}{2\pi}\right)^{2q+1}(\omega_{00}+\overline{\omega_{00}})\wedge(d\omega_{00})^q\wedge(d\overline{\omega_{00}})^q \\ =&(2q)!\left(\frac{q+1}{2\pi}\right)^{2q+1}(\omega_{00}+\overline{\omega}_{00})\wedge(\bigwedge_{j=1}^n \omega_{0j}\wedge\omega_{j0})\wedge(\bigwedge_{j=0}^n \eta_{0j}\wedge\zeta_{0j}) \\ &\qquad\qquad\wedge(\bigwedge_{j=1}^n \overline{\omega}_{0j}\wedge\overline{\omega}_{j0})\wedge(\bigwedge_{j=0}^n \overline{\eta}_{0j}\wedge\overline{\zeta}_{0j}). \end{aligned}$$

Choose $\{\sqrt{-1}X_{ii}, X_{jk}-X_{kj}, \sqrt{-1}(X_{jk}+X_{kj}), Y_{ij}-Z_{ij}, \sqrt{-1}(Y_{ij}+Z_{ij})\}$ as a basis of $\mathfrak{sp}(n+1;\boldsymbol{R})$ and let α_{ii}, β_{jk}, γ_{jk}, μ_{jk}, ν_{jk} $(0\le i\le n,\ 0\le j<k\le n)$ be their respective dual forms. Their extension to $\mathfrak{g}$ by complexification satisfy the following equations;

$$\begin{aligned} \omega_{ii} = \sqrt{-1}\alpha_{ii},\quad \omega_{jk} = \beta_{jk}+\sqrt{-1}\gamma_{jk},\quad \omega_{kj} = -\beta_{jk}+\sqrt{-1}\gamma_{jk}, \\ \eta_{ij} = \mu_{ij}+\sqrt{-1}\nu_{ij},\quad \zeta_{ij} = -\mu_{ij}+\sqrt{-1}\nu_{ij}. \end{aligned}$$

Hence

$$\begin{aligned} \tau^*\kappa^*\mathrm{GV}_{2q}(K) = \left(\frac{q+1}{2\pi}\right)^{2q+1} 2^{2q-2}(2q)!\,\alpha^1_{00}\wedge(\bigwedge_{j=1}^n \beta^1_{0j}\wedge\gamma^1_{0j})\wedge(\bigwedge_{j=0}^n \mu^1_{0j}\wedge\nu^1_{0j}) \\ \wedge(\bigwedge_{j=1}^n \beta^2_{0j}\wedge\gamma^2_{0j})\wedge(\bigwedge_{j=0}^n \mu^2_{0j}\wedge\nu^2_{0j}). \end{aligned}$$

Finally, as in the previous examples, the mappings $\pi_1 : \mathrm{Sp}(n+1)\to \mathrm{Sp}(n+1)/\mathrm{Sp}(n) = S^{2q+1}$ and $\pi_2 : \mathrm{Sp}(n+1)/T^{n+1}\to \mathrm{Sp}(n+1)/(T^1\times\mathrm{Sp}(n)) = \boldsymbol{C}P^q$ (note that $q=2n+1$), induce injective maps on the cohomology, where $\mathrm{Sp}(n;\boldsymbol{R})$ is simply denoted by $\mathrm{Sp}(n)$. Hence $\mathrm{GV}_{2q}(K)$ is non-trivial. □

The following proposition is obvious from the construction.

Proposition 2.3.11. *The foliation of* $\mathrm{Sp}(n+1;\boldsymbol{C})$ *given by Example 2.3.10 is the pull-back of the foliation of* $\mathrm{SL}(2n+2;\boldsymbol{C})$ *given by Example 2.3.6 by the natural inclusion.*

Hence the foliations of $\mathrm{Sp}(n+1; \boldsymbol{C})$ and $\mathrm{SL}(2n+2; \boldsymbol{C})$ are derived from the same complex Γ-structure. In particular, the foliations we constructed on $\mathrm{Sp}(1; \boldsymbol{C})$ and on $\mathrm{SL}(2; \boldsymbol{C})$ are isomorphic as foliated spaces. Consequently, there is a double covering map from $\mathrm{Sp}(1; \boldsymbol{C})$ to $\mathrm{SO}(3; \boldsymbol{C})$ as foliated spaces.

On the other hand, the foliations obtained by using $\mathrm{SL}(q+1; \boldsymbol{C})$ and $\mathrm{SO}(q+2; \boldsymbol{C})$ are distinct if q is an odd integer greater than 1. Denote by V_2 the second Chern character of the complex normal bundle, then $V_2 = v_1^2 - 2v_2$ and we have the following

Proposition 2.3.12. *If $q > 1$, then V_2 and v_1^2 are related as follows;*

$V_2 = \dfrac{1}{q+1} v_1^2$ *for the foliations constructed using* $\mathrm{SL}(q+1; \boldsymbol{C})$ *in Example 2.3.6,*

$V_2 = \dfrac{q-2}{q^2} v_1^2$ *for the foliations constructed using* $\mathrm{SO}(q+2; \boldsymbol{C})$ *in Example 2.3.8,*

when evaluated by the Bott connections as in Examples 2.3.6 and 2.3.8.

Proof. Denote by θ_1 the Bott connection in Example 2.3.6 for $\mathrm{SL}(q+1; \boldsymbol{C})$. Recall that

$$\theta_1 = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1q} \\ \vdots & \ddots & \vdots \\ \omega_{q1} & \cdots & \omega_{qq} \end{pmatrix} - \omega_{00} I_q.$$

Denote by $R_1 = d\theta_1 + \theta_1 \wedge \theta_1$ the curvature form of θ_1, then

$$R_1 = \begin{pmatrix} -d\omega_{00} - \omega_{10} \wedge \omega_{01} & -\omega_{10} \wedge \omega_{02} & \cdots & -\omega_{10} \wedge \omega_{0q} \\ -\omega_{20} \wedge \omega_{01} & -d\omega_{00} - \omega_{20} \wedge \omega_{02} & \cdots & -\omega_{20} \wedge \omega_{0q} \\ \vdots & & \ddots & \vdots \\ -\omega_{q0} \wedge \omega_{01} & -\omega_{q0} \wedge \omega_{02} & \cdots & -d\omega_{00} - \omega_{q0} \wedge \omega_{0q} \end{pmatrix}.$$

Hence

$$\begin{aligned} v_1 &= \frac{-1}{2\pi\sqrt{-1}} \operatorname{tr} R_1 = \frac{(q+1)}{2\pi\sqrt{-1}} d\omega_{00}, \\ V_2 &= \frac{-1}{4\pi^2} \operatorname{tr} R_1^2 = \frac{-1}{4\pi^2} (q+1)(d\omega_{00})^2. \end{aligned}$$

Thus $V_2 = \dfrac{1}{q+1} v_1^2$.

On the other hand, let θ_2 be the Bott connection in Example 2.3.8 for $\mathrm{SO}(q+2; \boldsymbol{C})$, then

$$\theta_2 = -\begin{pmatrix} \sqrt{-1}\omega_{01} & -\omega_{23} & -\omega_{24} & \cdots & -\omega_{2,q+1} \\ \omega_{23} & \sqrt{-1}\omega_{01} & -\omega_{34} & \cdots & -\omega_{3,q+1} \\ \vdots & & \ddots & & \vdots \\ \omega_{2,q+1} & \omega_{3,q+1} & \cdots & \cdots & \sqrt{-1}\omega_{01} \end{pmatrix}.$$

The curvature matrix R_2 of θ_2 is given by

$$R_2 = \begin{pmatrix} -\sqrt{-1}d\omega_{01} & -\omega_{20}\wedge\omega_{03}-\omega_{21}\wedge\omega_{13} & \cdots & -\omega_{20}\wedge\omega_{0(q+1)}-\omega_{21}\wedge\omega_{1(q+1)} \\ \omega_{20}\wedge\omega_{03}+\omega_{21}\wedge\omega_{13} & -\sqrt{-1}d\omega_{01} & \cdots & -\omega_{30}\wedge\omega_{0(q+1)}-\omega_{31}\wedge\omega_{1(q+1)} \\ \vdots & & \ddots & \vdots \\ \omega_{20}\wedge\omega_{0(q+1)}+\omega_{21}\wedge\omega_{1(q+1)} & \omega_{30}\wedge\omega_{0(q+1)}+\omega_{31}\wedge\omega_{1(q+1)} & \cdots & -\sqrt{-1}d\omega_{01}. \end{pmatrix}$$

Hence

$$\begin{aligned} v_1 &= \frac{-1}{2\pi\sqrt{-1}}\operatorname{tr} R_2 = \frac{q}{2\pi}d\omega_{01}, \\ V_2 &= \frac{-1}{4\pi^2}\operatorname{tr} R_2^2 = \frac{q-2}{4\pi^2}(d\omega_{01})^2. \end{aligned}$$

Thus $V_2 = \dfrac{q-2}{q^2}v_1^2$. $\square$

Corollary 2.3.13. *The foliations obtained by using* $\mathrm{SL}(q+1;\boldsymbol{C})$ *and* $\mathrm{SO}(q+2;\boldsymbol{C})$ *determine distinct real* Γ*-structures if* q *is an odd integer greater than* 1.

Proof. By Theorem 1.5, $c_2 = -v_2 + v_1\bar{v}_1 - \bar{v}_2$ holds in WU_q. Assume that $v_1^{q-2}V_2 = kv_1^q$ when evaluated by a Bott connection, then

$$\begin{aligned} \mathrm{GV}_{2q} - 2h_1c_1^{2q-2}c_2 &= (\sqrt{-1})^{2q-1}\tilde{u}_1(v_1-\bar{v}_1)^{2q-2}(v_1^2 - 2v_2 + \bar{v}_1^2 - 2\bar{v}_2) \\ &= \sqrt{-1}\frac{(2q-2)!}{q!(q-2)!}\tilde{u}_1(v_1^q\bar{v}_1^{q-2}(\bar{v}_1^2-2\bar{v}_2) + v_1^{q-2}(v_1^2-2v_2)\bar{v}_1^q) \\ &= \sqrt{-1}\frac{(2q-2)!}{q!(q-2)!}(2k)\tilde{u}_1v_1^q\bar{v}_1^q \\ &= \frac{k(q-1)}{2q-1}\mathrm{GV}_{2q}. \quad \square \end{aligned}$$

Corollary 2.3.13 implies that even though there is a double covering $\mathrm{Sp}(2;\boldsymbol{C}) \to \mathrm{SO}(5;\boldsymbol{C})$, the foliation obtained by using $\mathrm{Sp}(2;\boldsymbol{C})$ as in Example 2.3.10 is not isomorphic to the pull-back of the foliation obtained by using $\mathrm{SO}(5;\boldsymbol{C})$ as in Example 2.3.8.

Other classes are also compared as follows when $q=3$.

Example 2.3.14. We compare the previous examples constructed by using $\mathrm{SL}(4;\boldsymbol{C})$, $\mathrm{SO}(5;\boldsymbol{C})$ and $\mathrm{Sp}(2;\boldsymbol{C})$ by examining the secondary classes of degree 13. It is known that the following classes in $H^{13}(\mathrm{WO}_6)$ form the so-called Vey basis [18]; $h_1c_1^6$, $h_1c_1^4c_2$, $h_1c_1^3c_3$, $h_1c_1^2c_4$, $h_1c_1^2c_2^2$, $h_1c_1c_5$, $h_1c_1c_2c_3$, $h_1c_2c_4$, $h_1c_2^3$, $h_1c_3^2$, h_1c_6, h_3c_4,

$h_3c_2^2$, where $h_1c_1^6 = \mathrm{GV}_6$. Among them, the following classes form a basis of its image in $H^{13}(\mathrm{WU}_3)$ [2]: $h_1c_3^2, h_1c_1c_2c_3, h_1c_1^3c_3, h_1c_1^4c_2, h_1c_1^2c_2^2, h_1c_1^6, h_3c_2^2$. On the other hand, by Theorem 1.5, the ratio of these classes to the Godbillon-Vey class can be calculated as in Corollary 2.3.13 if there are relations of the form $v_1v_2 = \alpha v_1^3$, $v_3 = \beta v_1^3$ as differential forms. These values (α, β) are respectively $(2^{-3}\cdot 3, 2^{-4})$, $(2^2\cdot 3^{-2}, 2\cdot 3^{-3})$, $(2^{-3}\cdot 3, 2^{-4})$ for $\mathrm{SL}(4;\boldsymbol{C})$, $\mathrm{SO}(5;\boldsymbol{C})$ and $\mathrm{Sp}(2;\boldsymbol{C})$. Thus we have the following table;

	$\mathrm{SL}(4;\boldsymbol{C})$	$\mathrm{SO}(5;\boldsymbol{C})$	$\mathrm{Sp}(2;\boldsymbol{C})$
$h_1c_1^6$	1	1	1
$h_1c_1^4c_2$	$2^{-2}\cdot 3^2\cdot 5^{-1}$	$2^{-1}\cdot 3^{-2}\cdot 5^{-1}\cdot 43$	$2^{-2}\cdot 3^2\cdot 5^{-1}$
$h_1c_1^3c_3$	$2^{-5}\cdot 5^{-1}\cdot 19$	$3^{-3}\cdot 5^{-1}\cdot 19$	$2^{-5}\cdot 5^{-1}\cdot 19$
$h_1c_1^2c_2^2$	$2^{-6}\cdot 13$	$2^{-1}\cdot 3^{-4}\cdot 37$	$2^{-6}\cdot 13$
$h_1c_1c_2c_3$	$2^{-8}\cdot 3\cdot 5^{-1}\cdot 23$	$2\cdot 3^{-5}\cdot 5^{-1}\cdot 41$	$2^{-8}\cdot 3\cdot 5^{-1}\cdot 23$
$h_1c_3^2$	$2^{-9}\cdot 5^{-1}\cdot 37$	$2\cdot 3^{-6}\cdot 5^{-1}\cdot 37$	$2^{-9}\cdot 5^{-1}\cdot 37$
$h_3c_2^2$	$2^{-11}\cdot 5^{-1}\cdot 11\cdot 23$	$2^{-1}\cdot 3^{-7}\cdot 5^{-1}\cdot 709$	$2^{-11}\cdot 5^{-1}\cdot 11\cdot 23$

Here the values in the table are the ratio to GV_6, for example, $h_1c_1^4c_2 = 2^{-2}\cdot 3^2\cdot 5^{-1}h_1c_1^6$ for $\mathrm{SL}(4;\boldsymbol{C})$. From these tables and formulae in [2], one can see that if ω is a member of the Vey basis of $H^{13}(\mathrm{WO}_6)$ as above, the ratio of ω to $\mathrm{GV}_6 = h_1c_1^6$ is always less than 1 (except the ratio to GV_6 itself), for which we have no explanation.

By Proposition 2.3.11, the foliations of $\mathrm{SL}(4;\boldsymbol{C})$ and that of $\mathrm{Sp}(2;\boldsymbol{C})$ are essentially the same at least on the Lie algebra level. On the other hand, it is clear that the ratios of the classes to GV_6 are already determined on the Lie algebra level. Therefore the ratios in the table are identical.

Foliations with non-trivial Godbillon-Vey class can be also constructed by using an exceptional Lie group.

Example 2.3.15. Let G be the exceptional complex simple Lie group G_2. Let $\mathfrak{g}_2$ be the Lie algebra of G_2, then as found in [16],

$$\mathfrak{g}_2 = \left\langle Z_i, X_i, Y_i \ ;\ 1 \leq i \leq 6 \ \middle|\ \begin{array}{l} Z_3 = Z_1 + 3Z_2,\ Z_4 = 2Z_2 + 3Z_2, \\ Z_5 = Z_1 + Z_2,\ Z_6 = Z_1 + 2Z_2, \\ [X_i, Y_i] = Z_i,\ [Z_i, X_i] = 2X_i,\ [Z_i, Y_i] = -2Y_i \end{array} \right\rangle_{\boldsymbol{C}}.$$

Let γ_i, α_i, β_i be the dual of Z_i, X_i, Y_i, respectively, then they satisfy the following relations, namely,

$$\begin{aligned}
d\gamma_1 &= -\alpha_1\wedge\beta_1 - \alpha_3\wedge\beta_3 - 2\alpha_4\wedge\beta_4 - \alpha_5\wedge\beta_5 - \alpha_6\wedge\beta_6,\\
d\gamma_2 &= -\alpha_2\wedge\beta_2 - 3\alpha_3\wedge\beta_3 - 3\alpha_4\wedge\beta_4 - \alpha_5\wedge\beta_5 - 2\alpha_6\wedge\beta_6,\\
d\alpha_1 &= -2\gamma_1\wedge\alpha_1 + \gamma_2\wedge\alpha_1 + \beta_2\wedge\alpha_3 + 2\beta_3\wedge\alpha_4 - \beta_4\wedge\alpha_5,\\
d\alpha_2 &= 3\gamma_1\wedge\alpha_2 - 2\gamma_2\wedge\alpha_2 - 3\beta_1\wedge\alpha_3 - \beta_5\wedge\alpha_6,\\
d\alpha_3 &= -\alpha_1\wedge\alpha_2 + \gamma_1\wedge\alpha_3 - \gamma_2\wedge\alpha_3 - 2\beta_1\wedge\alpha_4 - \beta_4\wedge\alpha_6,\\
d\alpha_4 &= -2\alpha_1\wedge\alpha_3 - \gamma_1\wedge\alpha_4 + \beta_1\wedge\alpha_5 + \beta_3\wedge\alpha_6,\\
d\alpha_5 &= 3\alpha_1\wedge\alpha_4 - 3\gamma_1\wedge\alpha_5 + \gamma_2\wedge\alpha_5 + \beta_2\wedge\alpha_6,\\
d\alpha_6 &= 3\alpha_3\wedge\alpha_4 + \alpha_2\wedge\alpha_5 - \gamma_2\wedge\alpha_6,\\
d\beta_1 &= 2\gamma_1\wedge\beta_1 - \gamma_2\wedge\beta_1 - \alpha_2\wedge\beta_3 - 2\alpha_3\wedge\beta_4 + \alpha_4\wedge\beta_5,\\
d\beta_2 &= -3\gamma_1\wedge\beta_2 + 2\gamma_2\wedge\beta_2 + 3\alpha_1\wedge\beta_3 + \alpha_5\wedge\beta_6,\\
d\beta_3 &= \beta_1\wedge\beta_2 - \gamma_1\wedge\beta_3 + \gamma_2\wedge\beta_3 + 2\alpha_1\wedge\beta_4 + \alpha_4\wedge\beta_6,\\
d\beta_4 &= 2\beta_1\wedge\beta_3 + \gamma_1\wedge\beta_4 - \alpha_1\wedge\beta_5 - \alpha_3\wedge\beta_6,\\
d\beta_5 &= -3\beta_1\wedge\beta_4 + 3\gamma_1\wedge\beta_5 - \gamma_2\wedge\beta_5 - \alpha_2\wedge\beta_6,\\
d\beta_6 &= -3\beta_3\wedge\beta_4 - \beta_2\wedge\beta_5 + \gamma_2\wedge\beta_6.
\end{aligned}$$

It is known that the following real Lie subalgebra $\mathfrak{g}_0$ is a compact real form of $\mathfrak{g}_2$, namely,

$$\mathfrak{g}_0 = \left\langle \sqrt{-1}Z_i, X_i - Y_i, \sqrt{-1}(X_i + Y_i) \right\rangle_{\boldsymbol{R}}.$$

The compactness can be shown by verifying that the Killing form restricted on $\mathfrak{g}_0$ is negative definite.

Let ζ_i, λ_i and μ_i be the dual of $\sqrt{-1}Z_i$, $(X_i - Y_i)$, $\sqrt{-1}(X_i + Y_i)$, respectively, and denote again by the same symbol their extension to $\mathfrak{g}_2$ by complexification, then $\gamma_i = \sqrt{-1}\zeta_i$, $\alpha_i = \lambda_i + \sqrt{-1}\mu_i$, and $\beta_i = -\lambda_i + \sqrt{-1}\mu_i$.

Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be complex Lie subalgebras of $\mathfrak{g}_2$ defined respectively as follows;

$$\mathfrak{h}_1 = \ker\langle\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\rangle, \ \mathfrak{h}_2 = \ker\langle\beta_1,\beta_3,\beta_4,\beta_5,\beta_6\rangle.$$

They are easily seen to be Lie subalgebras. Let i be either 1 or 2, and let H_i be the Lie subgroup whose Lie algebra is $\mathfrak{h}_i$, then H_i contains the maximal torus T^2 generated by Z_1 and Z_2. Set $\mathfrak{su}(2)_i = \left\langle \sqrt{-1}Z_i, (X_i - Y_i), \sqrt{-1}(X_i + Y_i) \right\rangle_{\boldsymbol{R}}$ and let dw_i be the inclusion of $\mathfrak{su}(2)_i$ into $\mathfrak{g}_2$, then dw_i induces an embedding of SU(2) into G_2, which is denoted by w_i. Denote the image of w_i by $\mathrm{SU}(2)_i$. Similarly, let $\mathfrak{u}(2)_i = \left\langle \sqrt{-1}Z_1, \sqrt{-1}Z_2, (X_i - Y_i), \sqrt{-1}(X_i + Y_i) \right\rangle_{\boldsymbol{R}}$ and repeat the same construction. Denote the image by $\mathrm{U}(2)_i$, then $\mathrm{U}(2)_i$ is isomorphic to $\mathrm{U}(2)$. Note that $\mathrm{SU}(2)_i \subset \mathrm{U}(2)_i \subset G_2^{\boldsymbol{R}}$, where $G_2^{\boldsymbol{R}}$ is the compact real form of G_2 whose Lie algebra is $\mathfrak{g}_0$. In what follows, K_i is assumed to be a compact connected Lie subgroup such that $T^2 \subset K_i \subset \mathrm{U}(2)_i$ when the foliation induced by $\mathfrak{h}_i$ is considered.

First we study the foliation induced by $\mathfrak{h}_1$. In order to apply Theorem 2.1.7, let $\sigma_1 : \mathfrak{g}_2/\mathfrak{h}_1 \to \mathfrak{g}_2$ be the section defined by

$$\sigma_1([Y_i]) = Y_i,\ \ i = 2,3,4,5,6,$$

then σ_1 is $\mathrm{Ad}_{\mathrm{U}(2)_1}$-invariant. The Hermitian metric g_1 with respect to which $\{\sqrt{3}[Y_2], [Y_3], [Y_4], \sqrt{3}[Y_5], [Y_6]\}$ is an orthonormal basis is $\mathrm{Ad}_{\mathrm{U}(2)_1}$-invariant. This is shown by direct calculations, for example,

$$g_1([X_1 - Y_1, Y_2], Y_3) + g_1(Y_2, [X_1 - Y_1, Y_3]) = g_1(Y_3, Y_3) + g_1(Y_2, -3Y_2) = 0.$$

Set $\omega_1 = {}^t(\frac{1}{\sqrt{3}}\beta_2, \beta_3, \beta_4, \frac{1}{\sqrt{3}}\beta_5, \beta_6)$, then $d\omega_1 = -\widetilde{\theta}_1 \wedge \omega_1$, where

$$\widetilde{\theta}_1 = \begin{pmatrix} 3\gamma_1 - 2\gamma_2 & -\sqrt{3}\alpha_1 & 0 & 0 & -\frac{1}{\sqrt{3}}\alpha_5 \\ -\sqrt{3}\beta_1 & \gamma_1 - \gamma_2 & -2\alpha_1 & 0 & -\alpha_4 \\ 0 & -2\beta_1 & -\gamma_1 & \sqrt{3}\alpha_1 & \alpha_3 \\ 0 & 0 & \sqrt{3}\beta_1 & -3\gamma_1 + \gamma_2 & \frac{1}{\sqrt{3}}\alpha_2 \\ 0 & 0 & 3\beta_3 & \sqrt{3}\beta_2 & -\gamma_2 \end{pmatrix}.$$

By Definition 2.1.9 and Lemma 2.1.10, the $\mathfrak{gl}(5;\boldsymbol{C})$-valued 1-form

$$\theta_1 = \begin{pmatrix} 3\gamma_1 - 2\gamma_2 & -\sqrt{3}\alpha_1 & 0 & 0 & -\frac{1}{\sqrt{3}}\alpha_5 \\ -\sqrt{3}\beta_1 & \gamma_1 - \gamma_2 & -2\alpha_1 & 0 & -\alpha_4 \\ 0 & -2\beta_1 & -\gamma_1 & \sqrt{3}\alpha_1 & \alpha_3 \\ 0 & 0 & \sqrt{3}\beta_1 & -3\gamma_1 + \gamma_2 & \frac{1}{\sqrt{3}}\alpha_2 \\ 0 & 0 & 0 & 0 & -\gamma_2 \end{pmatrix}$$

is a Bott connection. Hence $h_1 = \dfrac{3}{2\pi}(\gamma_2 + \overline{\gamma_2})$ and $v_1 = \dfrac{3}{2\pi\sqrt{-1}} d\gamma_2$. Since

$$d\gamma_2 = -\alpha_2 \wedge \beta_2 - 3\alpha_3 \wedge \beta_3 - 3\alpha_4 \wedge \beta_4 - \alpha_5 \wedge \beta_5 - 2\alpha_6 \wedge \beta_6$$

and since $\mathrm{GV}_{10} = \frac{10!}{5!5!} h_1 v_1^5 \bar{v}_1^5$,

$$\begin{aligned} \mathrm{GV}_{10}(\mathfrak{h}_1, K_1) &= \left(\frac{3}{2\pi}\right)^{11} (2 \cdot 3^2 \cdot (5!))^2 (\gamma_2 + \overline{\gamma_2}) \wedge \bigwedge_{i=2}^{6} (\alpha_i \wedge \beta_i) \wedge \bigwedge_{i=2}^{6} (\overline{\alpha_i} \wedge \overline{\beta_i}) \\ &= \frac{2^8 \cdot 3^{17} \cdot 5^2}{(2\pi)^{11}} (\gamma_2 + \overline{\gamma_2}) \wedge \bigwedge_{i=2}^{6} (\alpha_i \wedge \beta_i) \wedge \bigwedge_{i=2}^{6} (\overline{\alpha_i} \wedge \overline{\beta_i}), \end{aligned}$$

where $\mathrm{GV}_{10}(\mathfrak{h}_1, K_1)$ denotes the Godbillon-Vey class of the foliation given by the quadruplet (G_2, H_1, K_1, Γ), where Γ is any cocompact lattice of G_2/K_1. By Proposition 2.2.7,

$$\tau^*\kappa^*\mathrm{GV}_{10}(\mathfrak{h}_1, K_1) = \frac{2^{18}\cdot 3^{17}\cdot 5^2}{(2\pi)^{11}}\sqrt{-1}\zeta_2^1 \wedge \bigwedge_{i=2}^{6}(\lambda_i^1 \wedge \mu_i^1) \wedge \bigwedge_{i=2}^{6}(\lambda_i^2 \wedge \mu_i^2).$$

It is clear that $\zeta_2^1 \wedge \bigwedge_{i=2}^{6}(\lambda_i^1 \wedge \mu_i^1)$ and $\bigwedge_{i=2}^{6}(\lambda_i^1 \wedge \mu_i^1)$ are the volume forms of $G_2^{\boldsymbol{R}}/\mathrm{SU}(2)_1$ and $G_2^{\boldsymbol{R}}/\mathrm{U}(2)_1$, respectively, where $G_2^{\boldsymbol{R}}$ is the compact Lie group with Lie algebra $\mathfrak{g}_0$. Then by Lemma 2.3.16 below, $\tau^*\kappa^*\mathrm{GV}_{10}(\mathfrak{h}_1, K_1)$ is non-trivial.

The foliation induced by $\mathfrak{h}_2$ can be studied in a similar way. Define $\sigma_2 : \mathfrak{g}_2/\mathfrak{h}_2 \to \mathfrak{g}_2$ by setting $\sigma_2([Y_j]) = Y_j$, $j = 1, 3, 4, 5, 6$, then σ_2 is $\mathrm{Ad}_{\mathrm{U}(2)_2}$-invariant. Let g_2 be the Hermitian metric on $\mathfrak{g}_2/\mathfrak{h}_2$ with respect to which $\{[Y_1], [Y_3], [Y_4], [Y_5], [Y_6]\}$ is an orthonormal basis, then g_2 is $\mathrm{Ad}_{\mathrm{U}(2)_2}$-invariant.

Set $\omega_2 = {}^t(\beta_1, \beta_3, \beta_4, \beta_5, \beta_6)$ and

$$\widetilde{\theta}_2 = \begin{pmatrix} -2\gamma_1+\gamma_2 & \alpha_2 & 2\alpha_3 & -\alpha_4 & 0 \\ \beta_2 & \gamma_1-\gamma_2 & -2\alpha_1 & 0 & -\alpha_4 \\ 0 & -2\beta_1 & -\gamma_1 & \alpha_1 & \alpha_3 \\ 0 & 0 & 3\beta_1 & -3\gamma_1+\gamma_2 & \alpha_2 \\ 0 & 0 & 3\beta_3 & \beta_2 & -\gamma_2 \end{pmatrix},$$

then $d\omega_2 = -\widetilde{\theta}_2 \wedge \omega_2$. Hence

$$\theta_2 = \begin{pmatrix} -2\gamma_1+\gamma_2 & \alpha_2 & 2\alpha_3 & -\alpha_4 & 0 \\ \beta_2 & \gamma_1-\gamma_2 & -2\alpha_1 & 0 & -\alpha_4 \\ 0 & 0 & -\gamma_1 & \alpha_1 & \alpha_3 \\ 0 & 0 & 0 & -3\gamma_1+\gamma_2 & \alpha_2 \\ 0 & 0 & 0 & \beta_2 & -\gamma_2 \end{pmatrix}$$

induces a Bott connection. The characteristic homomorphism is calculated as follows. Firstly, one has

$$\begin{aligned} h_1 &= \frac{5}{2\pi}(\gamma_1 + \overline{\gamma_1}), \\ v_1 &= \frac{5}{2\pi\sqrt{-1}}d\gamma_1 \\ &= \frac{5}{2\pi\sqrt{-1}}(-\alpha_1\wedge\beta_1 - \alpha_3\wedge\beta_3 - 2\alpha_4\wedge\beta_4 - \alpha_5\wedge\beta_5 - \alpha_6\wedge\beta_6). \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{GV}_{10}(\mathfrak{h}_2, K_2) &= \left(\frac{5}{2\pi}\right)^{11}(2\cdot 5!)^2(\gamma_1+\overline{\gamma_1}) \wedge \bigwedge_{i\neq 2}(\alpha_i\wedge\beta_i) \wedge \bigwedge_{i\neq 2}(\overline{\alpha_i}\wedge\overline{\beta_i}) \\ &= \frac{2^6\cdot 3^2\cdot 5^{13}}{(2\pi)^{11}}(\gamma_1+\overline{\gamma_1}) \wedge \bigwedge_{i\neq 2}(\alpha_i\wedge\beta_i) \wedge \bigwedge_{i\neq 2}(\overline{\alpha_i}\wedge\overline{\beta_i}). \end{aligned}$$

By Proposition 2.2.7,

$$\tau^*\kappa^*\mathrm{GV}_{10}(\mathfrak{h}_2, K_2) = \sqrt{-1}\,\frac{2^{16}\cdot 3^2\cdot 5^{13}}{(2\pi)^{11}}\gamma_1^1 \wedge \bigwedge_{i\neq 2}(\lambda_i^1\wedge\mu_i^1)\wedge\bigwedge_{i\neq 2}(\lambda_i^2\wedge\mu_i^2).$$

As in the previous case, this is the product of the volume forms of $G_2^{\boldsymbol{R}}/\mathrm{SU}(2)_2$ and $G_2^{\boldsymbol{R}}/\mathrm{U}(2)_2$. Hence $\tau^*\kappa^*\mathrm{GV}_{10}(\mathfrak{h}_2, K_2)$ is non-trivial by Lemma 2.3.16.

The foliations defined by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are derived from distinct real Γ-structures. Let $R(\theta)$ be the curvature form of θ, then $v_1^2 - 2v_2 = \left(\frac{-1}{2\pi\sqrt{-1}}\right)^2 \operatorname{tr} R(\theta)^2$. Hence

$$\begin{aligned} v_1^3(v_1^2 - 2v_2)(\mathfrak{h}_1, K_1) &= \frac{1}{27}v_1^5(\mathfrak{h}_1, K_1),\\ v_1^3(v_1^2 - 2v_2)(\mathfrak{h}_2, K_2) &= \frac{3}{25}v_1^5(\mathfrak{h}_2, K_2). \end{aligned}$$

These relations are shown as follows by using the curvature matrices $R(\theta_1)$ and $R(\theta_2)$ presented after Lemma 2.3.16. Set $[i,j,k] = \alpha_i\wedge\beta_i\wedge\alpha_j\wedge\beta_j\wedge\alpha_k\wedge\beta_k$, and define the symbols $[i,j]$ and $[i,j,k,l,m]$ in the same way. When $\theta = \theta_1$, one has

$$\begin{aligned} \operatorname{tr} R(\theta_1) &= 3\alpha_2\wedge\beta_2 + 9\alpha_3\wedge\beta_3 + 9\alpha_4\wedge\beta_4 + 3\alpha_5\wedge\beta_5 + 6\alpha_6\wedge\beta_6,\\ (\operatorname{tr} R(\theta_1))^3 &= 3^3\cdot 6\,(9[2,3,4] + 3[2,3,5] + 6[2,3,6] + 3[2,4,5] + 6[2,4,6]\\ &\qquad + 2[2,5,6] + 9[3,4,5] + 18[3,4,6] + 6[3,5,6] + 6[4,5,6]),\\ (\operatorname{tr} R(\theta_1))^5 &= 3^5\cdot 5!\cdot 2\cdot 3^2[2,3,4,5,6] = 2^4\cdot 3^8\cdot 5[2,3,4,5,6]. \end{aligned}$$

Let $\operatorname{tr}' R(\theta_1)^2$ be the terms of $\operatorname{tr} R(\theta_1)^2$ which contain $[l,m]$, then it is clear that $(\operatorname{tr} R(\theta_1))^3 \operatorname{tr} R(\theta_1)^2 = (\operatorname{tr} R(\theta_1))^3 \operatorname{tr}' R(\theta_1)^2$. One has

$$\begin{aligned} \operatorname{tr}' R(\theta_1)^2 =& 2[2,3] + 2[2,4] - 6[2,5] + 8[2,6] - 54[3,4]\\ &+ 2[3,5] + 24[3,6] + 2[4,5] + 24[4,6] + 8[5,6]. \end{aligned}$$

Hence

$$(\mathrm{tr}R(\theta_1))^3\mathrm{tr}R(\theta_1)^2 = 2^4\cdot 3^5\cdot 5[2,3,4,5,6] = 3^{-3}(\mathrm{tr}R(\theta_1))^5.$$

When $\theta = \theta_2$, calculations of the same kind show that

$$(\mathrm{tr}R(\theta_2))^3\mathrm{tr}R(\theta_2)^2 = 3\cdot 5^{-2}(\mathrm{tr}R(\theta_2))^5.$$

Hence the normal bundles associated with $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are not isomorphic as complex vector bundles. Moreover, they determine distinct Γ-structures as real foliations. Indeed, by repeating the proof of Corollary 2.3.13, one has

$$\begin{aligned} h_1c_1^{10}(\mathfrak{h}_1, K_1) - 2h_1c_1^8c_2(\mathfrak{h}_1, K_1) &= 2^2\cdot 3^{-5}h_1c_1^{10}(\mathfrak{h}_1, K_1),\\ h_1c_1^{10}(\mathfrak{h}_2, K_2) - 2h_1c_1^8c_2(\mathfrak{h}_2, K_2) &= 2^2\cdot 3^{-1}\cdot 5^{-2}h_1c_1^{10}(\mathfrak{h}_2, K_2). \end{aligned}$$

Rewriting these relations, one obtains

$$
\begin{aligned}
h_1c_1^8c_2(\mathfrak{h}_1, K_1) &= 2^{-1} \cdot 3^{-5} \cdot 239\,\mathrm{GV}_{10}(\mathfrak{h}_1, K_1), \\
h_1c_1^8c_2(\mathfrak{h}_2, K_2) &= 2^{1} \cdot 3^{-1} \cdot 5^{-2} \cdot 71\,\mathrm{GV}_{10}(\mathfrak{h}_2, K_2).
\end{aligned}
$$

Hence these foliations are distinct as real foliations. Note that the foliation induced by $\mathfrak{h}_1$ is distinct also from the foliations of $\mathrm{SL}(6;\boldsymbol{C})$ and of $\mathrm{SO}(7;\boldsymbol{C})$ by Proposition 2.3.12 and Corollary 2.3.13. Indeed,

$$
\begin{aligned}
h_1c_1^8c_2 &= 2^{-1} \cdot 3^{-3} \cdot 5\,\mathrm{GV}_{10} \quad \text{for } \mathrm{SL}(6;\boldsymbol{C}), \\
h_1c_1^8c_2 &= 2^{1} \cdot 3^{-1} \cdot 5^{-2} \cdot 71\,\mathrm{GV}_{10} \quad \text{for } \mathrm{SO}(7;\boldsymbol{C}).
\end{aligned}
$$

On the other hand, the foliation induced by $\mathfrak{h}_2$ is obtained from the foliation of $\mathrm{SO}(7;\boldsymbol{C})$ at least on the Lie algebra level. This is shown as follows. Let $i : \mathfrak{g}_2 \hookrightarrow \mathfrak{so}(7;\boldsymbol{C})$ be the inclusion of Lie algebra determined by requiring

$$
\begin{aligned}
i(Z_1) &= -\sqrt{-1}(X_{01} - 2X_{23} + X_{45}), \\
i(Z_2) &= -\sqrt{-1}(X_{23} - X_{45}), \\
i(X_1) &= \frac{1}{2}\left((X_{05} + X_{14} - 2X_{36}) - \sqrt{-1}(X_{04} - X_{15} + 2X_{26})\right), \\
i(Y_1) &= \frac{1}{2}\left(-(X_{05} + X_{14} - 2X_{36}) - \sqrt{-1}(X_{04} - X_{15} + 2X_{26})\right), \\
i(X_2) &= \frac{1}{2}\left(-(X_{25} - X_{34}) - \sqrt{-1}(X_{24} + X_{35})\right), \\
i(Y_2) &= \frac{1}{2}\left((X_{25} - X_{34}) - \sqrt{-1}(X_{24} + X_{35})\right),
\end{aligned}
$$

then $i^*(\mathfrak{h}^+) = \mathfrak{h}_2$, where $\mathfrak{h}^+$ and X_{ij} are as in Example 2.3.8.

The proof of non-triviality of the Godbillon-Vey class is completed by the following lemma.

Lemma 2.3.16. *We retain the notations in Example 2.3.15.*

1) *The pull-back of the volume forms of $G_2^{\boldsymbol{R}}/\mathrm{SU}(2)_i$, $i = 1, 2$, are non-trivial in $H^*(G_2^{\boldsymbol{R}})$.*
2) *The classes represented by $\bigwedge_{i=2}^{6}(\lambda_i \wedge \mu_i)$ and $\bigwedge_{i\neq 2}(\lambda_i \wedge \mu_i)$ are non-trivial in $H^*(G_2^{\boldsymbol{R}}/T^2)$.*

Proof. First we show 2). The equation

$$
d\zeta_1 = -2\lambda_1 \wedge \mu_1 - 2\lambda_3 \wedge \mu_3 - 4\lambda_4 \wedge \mu_4 - 2\lambda_5 \wedge \mu_5 - 2\lambda_6 \wedge \mu_6,
$$

implies that $d\zeta_1$ determines a class in $H^2(G_2^{\boldsymbol{R}}/T^2)$. The product $(d\zeta_1)\wedge\bigwedge_{i=2}^{6}(\lambda_i\wedge\mu_i)$ is easily seen to be a non-zero multiple of the volume form of $G_2^{\boldsymbol{R}}/T^2$. Therefore $\bigwedge_{i=2}^{6}(\lambda_i\wedge\mu_i)$ is non-trivial in $H^*(G_2^{\boldsymbol{R}}/T^2)$. The non-triviality of $\bigwedge_{i\neq 2}(\lambda_i\wedge\mu_i)$ is shown by considering the product with the class represented by $d\zeta_2$.

In order to show 1), we define ω_1 and ω_2 by setting

$$\omega_i = \zeta_{3-i}\wedge\bigwedge_{j\neq i}(\lambda_j\wedge\mu_j),$$

where $i=1,2$, and show that $[\sigma]\cup[\omega_i]\neq 0$ for some $[\sigma]\in H^3(\mathfrak{g}_0;\boldsymbol{R})$. First note that we may work on $\mathfrak{g}$ because $H^3(\mathfrak{g}_0;\boldsymbol{C})\cong H^3(\mathfrak{g}_0;\boldsymbol{R})\otimes\boldsymbol{C}\cong H^3(\mathfrak{g}_2;\boldsymbol{C})$. Define $\sigma'\in(\mathfrak{g}_2^3)^*$ by setting $\sigma(X,Y,Z)=\mathrm{tr}(\mathrm{ad}_{[X,Y]}\mathrm{ad}_Z)$, then by the proof of Theorem 21.1 in [13], σ' is a cocycle representing a non-trivial class in $H^3(\mathfrak{g}_2;\boldsymbol{C})$. Up to multiplication of a non-zero constant, σ' is of the form

$$\begin{aligned}\sigma' = &-9(2\gamma_1-\gamma_2)\wedge\alpha_1\wedge\beta_1+3(3\gamma_1-2\gamma_2)\wedge\alpha_2\wedge\beta_2+9(\gamma_1-\gamma_2)\wedge\alpha_3\wedge\beta_3\\ &-9\gamma_1\wedge\alpha_4\wedge\beta_4-3(3\gamma_1-\gamma_2)\wedge\alpha_5\wedge\beta_5-3\gamma_2\wedge\alpha_6\wedge\beta_6\\ &+(\text{terms not involving }\gamma_i).\end{aligned}$$

On the other hand, ω_i, when complexified, is a non-zero multiple of

$$\gamma_{3-i}\wedge\bigwedge_{j\neq i}(\alpha_j\wedge\beta_j).$$

Hence $[\sigma']\cup[\omega_i]$ is represented by a non-zero multiple of

$$\gamma_1\wedge\gamma_2\wedge\bigwedge_{j=1}^{6}(\alpha_j\wedge\beta_j).\quad\square$$

Remark 2.3.17. By following [14], one can show that the above σ' is in fact as follows;

$$\begin{aligned}\sigma' = {}&6\gamma_1\wedge d\gamma_1-3\gamma_1\wedge d\gamma_2-3\gamma_2\wedge d\gamma_1+2\gamma_2\wedge d\gamma_2\\ &+3\alpha_1\wedge d\beta_1+3\beta_1\wedge d\alpha_1+\alpha_2\wedge d\beta_2+\beta_2\wedge d\alpha_2+3\alpha_3\wedge d\beta_3+3\beta_3\wedge d\alpha_3\\ &+3\alpha_4\wedge d\beta_4+3\beta_4\wedge d\alpha_4+\alpha_5\wedge d\beta_5+\beta_5\wedge d\alpha_5+\alpha_6\wedge d\beta_6+\beta_6\wedge d\alpha_6.\end{aligned}$$

This follows from the fact that

$$3\gamma_1^2-3\gamma_1\gamma_2+\gamma_2^2+3\alpha_1\beta_1+\alpha_2\beta_2+3\alpha_3\beta_3+3\alpha_4\beta_4+\alpha_5\beta_5+\alpha_6\beta_6$$

is a primitive element of $I(\mathfrak{g})$, where $I(\mathfrak{g})$ is the set of left invariant symmetric polynomials invariant also under the adjoint action. Note also that $H^3(\mathfrak{g}_2;\boldsymbol{C})$ is in fact isomorphic to $\boldsymbol{C}$.

The curvature matrices $R(\theta_1)$ and $R(\theta_2)$ are presented in the next page.

The matrices $R(\theta_1)$ and $R(\theta_2)$ are as follows;

$R(\theta_1) = d\theta_1 + \theta_1 \wedge \theta_1$

$$
= \begin{pmatrix}
\begin{array}{l} 2\alpha_2 \wedge \beta_2 + 3\alpha_3 \wedge \beta_3 \\ -\alpha_5 \wedge \beta_5 + \alpha_6 \wedge \beta_6 \end{array} & \begin{array}{l} -\sqrt{3}(\beta_2 \wedge \alpha_3 \\ \quad +2\beta_3 \wedge \alpha_4 \\ \quad\quad -\beta_4 \wedge \alpha_5 \end{array} & 0 & 0 & -\frac{1}{\sqrt{3}}\beta_2 \wedge \alpha_6 \\
\begin{array}{l} \sqrt{3}(\alpha_2 \wedge \beta_3 \\ \quad +2\alpha_3 \wedge \beta_4 \\ \quad\quad -\alpha_4 \wedge \beta_5) \end{array} & \begin{array}{l} \alpha_2 \wedge \beta_2 + 2\alpha_3 \wedge \beta_3 \\ +\alpha_4 \wedge \beta_4 + \alpha_6 \wedge \beta_6 \end{array} & \begin{array}{l} -2(\beta_2 \wedge \alpha_3 \\ \quad +2\beta_3 \wedge \alpha_4 \\ \quad\quad -\beta_4 \wedge \alpha_5) \end{array} & 0 & -\beta_3 \wedge \alpha_6 \\
0 & \begin{array}{l} 2(\alpha_2 \wedge \beta_3 \\ \quad +2\alpha_3 \wedge \beta_4 \\ \quad\quad -\alpha_4 \wedge \beta_5) \end{array} & \begin{array}{l} \alpha_3 \wedge \beta_3 + 2\alpha_4 \wedge \beta_4 \\ +\alpha_5 \wedge \beta_5 + \alpha_6 \wedge \beta_6 \end{array} & \begin{array}{l} \sqrt{3}(\beta_2 \wedge \alpha_3 \\ \quad +2\beta_3 \wedge \alpha_4 \\ \quad\quad -\beta_4 \wedge \alpha_5) \end{array} & -\beta_4 \wedge \alpha_6 \\
0 & 0 & \begin{array}{l} -\sqrt{3}(\alpha_2 \wedge \beta_3 \\ \quad +2\alpha_3 \wedge \beta_4 \\ \quad\quad -\alpha_4 \wedge \beta_5) \end{array} & \begin{array}{l} -\alpha_2 \wedge \beta_2 + 3\alpha_4 \wedge \beta_4 \\ +2\alpha_5 \wedge \beta_5 + \alpha_6 \wedge \beta_6 \end{array} & -\frac{1}{\sqrt{3}}\beta_5 \wedge \alpha_6 \\
0 & 0 & 0 & 0 & \begin{array}{l} \alpha_2 \wedge \beta_2 + 3\alpha_3 \wedge \beta_3 \\ \quad +3\alpha_4 \wedge \beta_4 + \alpha_5 \wedge \beta_5 \\ \quad\quad +2\alpha_6 \wedge \beta_6 \end{array}
\end{pmatrix}
$$

$R(\theta_2) = d\theta_2 + \theta_2 \wedge \theta_2$

$$
= \begin{pmatrix}
\begin{array}{l} 2\alpha_1 \wedge \beta_1 - \alpha_3 \wedge \beta_3 \\ +\alpha_4 \wedge \beta_4 + \alpha_5 \wedge \beta_5 \end{array} & -3\beta_1 \wedge \alpha_3 - \beta_5 \wedge \alpha_6 & -4\beta_1 \wedge \alpha_4 - 2\beta_4 \wedge \alpha_6 & -\beta_1 \wedge \alpha_5 - \beta_3 \wedge \alpha_6 & 0 \\
3\alpha_1 \wedge \beta_3 + \alpha_5 \wedge \beta_6 & \begin{array}{l} -\alpha_1 \wedge \beta_1 + 2\alpha_3 \wedge \beta_3 \\ +\alpha_4 \wedge \beta_4 + \alpha_6 \wedge \beta_6 \end{array} & -4\beta_3 \wedge \alpha_4 + 2\beta_4 \wedge \alpha_5 & 0 & -\beta_1 \wedge \alpha_5 - \beta_3 \wedge \alpha_6 \\
0 & 0 & \begin{array}{l} \alpha_1 \wedge \beta_1 + \alpha_3 \wedge \beta_3 \\ \quad +2\alpha_4 \wedge \beta_4 + \alpha_5 \wedge \beta_5 \\ \quad\quad +\alpha_6 \wedge \beta_6 \end{array} & 2\beta_3 \wedge \alpha_4 - \beta_4 \wedge \alpha_5 & -2\beta_1 \wedge \alpha_4 - \beta_4 \wedge \alpha_6 \\
0 & 0 & 0 & \begin{array}{l} 3\alpha_1 \wedge \beta_1 + 3\alpha_4 \wedge \beta_4 \\ +2\alpha_5 \wedge \beta_5 + \alpha_6 \wedge \beta_6 \end{array} & -3\beta_1 \wedge \alpha_3 - \beta_5 \wedge \alpha_6 \\
0 & 0 & 0 & 3\alpha_1 \wedge \beta_3 + \alpha_5 \wedge \beta_6 & \begin{array}{l} 3\alpha_3 \wedge \beta_3 + 3\alpha_4 \wedge \beta_4 \\ +\alpha_5 \wedge \beta_5 + 2\alpha_6 \wedge \beta_6 \end{array}
\end{pmatrix}
$$

Remark 2.3.18. Some other foliations G_2 with non-trivial Godbillon-Vey class can be obtained by considering the action of the Weyl group. Let σ_1 be the automorphism of G_2 which maps (Z_1, Z_2) to $(2Z_1 + 3Z_2, -Z_1 - Z_2)$, and let σ_2 be the automorphism which maps (Z_1, Z_2) to $(Z_1, -Z_1 - Z_2)$, then they generate the Weyl group. On the other hand, set

$$\begin{aligned}
\omega_1 &= {}^t(\beta_2, \beta_3, \beta_4, \beta_5, \beta_6), & \omega'_1 &= {}^t(\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6),\\
\omega_2 &= {}^t(\beta_1, \beta_3, \beta_4, \beta_5, \beta_6), & \omega'_2 &= {}^t(\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6),\\
\omega_3 &= {}^t(\beta_1, \alpha_2, \beta_4, \beta_5, \beta_6), & \omega'_3 &= {}^t(\alpha_1, \beta_2, \alpha_4, \alpha_5, \alpha_6),\\
\omega_4 &= {}^t(\beta_1, \alpha_2, \alpha_3, \beta_5, \alpha_6), & \omega'_4 &= {}^t(\alpha_1, \beta_2, \beta_3, \alpha_5, \beta_6),\\
\omega_5 &= {}^t(\beta_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6), & \omega'_5 &= {}^t(\alpha_1, \beta_2, \beta_3, \beta_4, \beta_6),\\
\omega_6 &= {}^t(\beta_1, \alpha_2, \alpha_3, \beta_4, \beta_5), & \omega'_6 &= {}^t(\alpha_1, \beta_2, \beta_3, \alpha_4, \alpha_5),
\end{aligned}$$

and set $\mathfrak{h}_i = \ker \omega_i$ and $\mathfrak{h}'_i = \ker \omega'_i$, then they are Lie subalgebras of $\mathfrak{g}_2$. First consider the action of σ_1. From $\mathfrak{h}_1$, one obtains $\mathfrak{h}'_4$, $\mathfrak{h}'_3$, $\mathfrak{h}_1$, $\mathfrak{h}_4$, $\mathfrak{h}_3$ and then $\mathfrak{h}_1$ again. From $\mathfrak{h}_2$, one obtains $\mathfrak{h}'_5$, $\mathfrak{h}'_6$, $\mathfrak{h}'_2$, $\mathfrak{h}_5$, $\mathfrak{h}_6$ and then $\mathfrak{h}_2$ again. On the other hand, under the action of σ_2, one obtains $\mathfrak{h}'_1$ from $\mathfrak{h}_1$ and $\mathfrak{h}_5$ from $\mathfrak{h}_2$, respectively.

The examples constructed using $A_q = \mathrm{SL}(q+1; \boldsymbol{C})$, $B_m = \mathrm{SO}(2m+1; \boldsymbol{C})$ $(q = 2m-1)$, $C_{n+1} = \mathrm{Sp}(n+1; \boldsymbol{C})$ $(q = 2n+1)$ and G_2 $(q = 5)$ have certain common properties. Denote by X_n one of these groups, and let X_n^{crf} be the compact real form of X_n as in the above examples, then

$$T \subset K \subset T^1 \times X_{n-1}^{\mathrm{crf}} \subset T^1 \times X_{n-1} \subset H \subset X_n,$$

where T is the maximal torus realized as above. The inclusion of X_{n-1} into X_n is realized by considering the inclusion of corresponding Dynkin diagrams. Hence we regard $G_1 = \mathrm{SL}(2; \boldsymbol{C})$.

The Lie algebra $\mathfrak{h}$ defining the leaves can be described as follows. Let $\mathfrak{x}_n$ be the Lie algebra of X_n and set $\widetilde{\mathfrak{x}}_{n-1} = \mathfrak{t}^1 \oplus \mathfrak{x}_{n-1}$, then there is a splitting $\mathfrak{x}_n = \widetilde{\mathfrak{x}}_{n-1} \oplus \mathfrak{a}$ as complex vector spaces so that one can find a decomposition $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ such that the both $\widetilde{\mathfrak{x}}_{n-1} \oplus \mathfrak{a}^\pm$ are complex Lie subalgebras. These subalgebras are $\mathfrak{h}$ as above. Finally, the Godbillon-Vey class is realized as the pull-back of the product of the volume forms of $X_n^{\mathrm{crf}}/X_{n-1}^{\mathrm{crf}}$ and $X_n^{\mathrm{crf}}/(T^1 \times X_{n-1}^{\mathrm{crf}})$.

The construction using $\mathrm{SO}(q+2; \boldsymbol{C})$ in Example 2.3.8 was not successful if q is even. In fact, we have the following

Proposition 2.3.19. *Assume that $T^1 \times \mathrm{SO}(2n-2;\boldsymbol{C})$ and the maximal torus T^n are realized as in Example 2.3.8. If $n > 2$, then there is no Lie subalgebra $\mathfrak{h}$ of $\mathfrak{so}(2n;\boldsymbol{C})$ with the following properties:*

1) *$\mathfrak{h}$ contains $\mathfrak{t}^1 \oplus \mathfrak{so}(2n-2;\boldsymbol{C})$.*
2) *The Godbillon-Vey class of the foliation of $\Gamma\backslash\mathrm{SO}(2n;\boldsymbol{C})/T^n$ defined by $\mathfrak{h}$ is non-trivial as an element of $H^{4q+1}(\mathfrak{so}(2n;\boldsymbol{C}),\mathfrak{t}^n;\boldsymbol{C})$.*

Proof. We retain the notations in Example 2.3.8. Set $Y_{0i} = X_{0i} + \sqrt{-1}X_{1i}$ and $Z_{0i} = X_{0i} - \sqrt{-1}X_{1i}$ for $i \geq 2$. Denote by $\mathfrak{k}$ the Lie subalgebra $\mathfrak{t}^1 \oplus \mathfrak{so}(2n-2;\boldsymbol{C})$. Let $\mathfrak{h}$ be a Lie subalgebra having the properties 1) and 2), then $\mathfrak{h}/\mathfrak{k}$ is invariant under the action of $\mathrm{ad}_{\mathfrak{k}}$. Define linear subspaces $\mathfrak{a}^\pm$ of $\mathfrak{h}/\mathfrak{k}$ by setting

$$\begin{aligned}\mathfrak{a}^+ &= (\langle Y_{02}, Y_{03}, \ldots, Y_{0,2n-1}\rangle + \mathfrak{k})/\mathfrak{k},\\ \mathfrak{a}^- &= (\langle Z_{02}, Z_{03}, \ldots, Z_{0,2n-1}\rangle + \mathfrak{k})/\mathfrak{k},\end{aligned}$$

then $\mathfrak{h}/\mathfrak{k} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$. Denote by $i^\pm$ the inclusions of $\mathfrak{a}^\pm$ to $\mathfrak{h}/\mathfrak{k}$, and denote by $p^\pm$ the projections from $\mathfrak{h}/\mathfrak{k}$ to $\mathfrak{a}^\pm$ corresponding to the above direct sum. Since $\mathrm{ad}_{X_{01}}Y_{0i} = \sqrt{-1}Y_{0i}$ and $\mathrm{ad}_{X_{01}}Z_{0i} = -\sqrt{-1}Z_{0i}$, $\mathfrak{h}/\mathfrak{k} = i^+p^+(\mathfrak{h}/\mathfrak{k}) \oplus i^-p^-(\mathfrak{h}/\mathfrak{k})$. Thus it suffices to study invariant subspaces of $\mathfrak{a}^\pm$.

Assume that $\mathfrak{a}'$ is an invariant subspace of $\mathfrak{a}^+$. Fix integers i, j such that $2 \leq i < j < 2n$, and set $V_{ij}^\pm = Y_{0i} \pm \sqrt{-1}Y_{0j}$. Set $\mathfrak{b}^\pm = \boldsymbol{C}V_{ij}^\pm$ and $\mathfrak{z} = \langle Y_{0k} \,|\, k \neq i, j\rangle$, then $\mathfrak{a}^+ = \mathfrak{b}^+ \oplus \mathfrak{b}^- \oplus \mathfrak{z}$. Let $\iota^\pm$ and $\iota_{\mathfrak{z}}$ be the inclusions to $\mathfrak{a}^+$, and let $\pi^\pm$ and $\pi_{\mathfrak{z}}$ be the corresponding projections from $\mathfrak{a}^+$ to $\mathfrak{b}^\pm$ and $\mathfrak{z}$, then $\mathfrak{a}' = \iota_+\pi_+(\mathfrak{a}') \oplus \iota_-\pi_-(\mathfrak{a}') \oplus \iota_{\mathfrak{z}}\pi_{\mathfrak{z}}(\mathfrak{a}')$. If $\iota_\pm\pi_\pm(\mathfrak{a}') = \{0\}$ for any pair (i,j), then $\mathfrak{a}' = \{0\}$. Assume that $\iota_+\pi_+(\mathfrak{a}') \neq \{0\}$ for a pair (i,j), then $\iota_+\pi_+(\mathfrak{a}') = \boldsymbol{C}V_{ij}^+$. In particular $Y_{0i} + \sqrt{-1}Y_{0j} \in \mathfrak{a}'$. Noticing that $n > 2$, choose an integer k distinct from i, j and such that $2 \leq k \leq 2n-1$. For such a k, $\mathrm{ad}_{X_{ik}}(\mathrm{ad}_{X_{ik}}Y_{0i} + \sqrt{-1}Y_{0j}) = -Y_{0i}$ and thus $Y_{0i} \in \mathfrak{a}'$. Since $\mathrm{ad}_{X_{ik}}Y_{0i} = -Y_{0k}$ for $k \geq 2$, $k \neq i$, and $\mathrm{ad}_{X_{01}}Y_{0i} = -Y_{1i}$, this implies that $\mathfrak{a}' = \mathfrak{a}^+$.

By the same argument, $(\mathfrak{h}/\mathfrak{k}) \cap \mathfrak{a}^-$ is either $\{0\}$ or $\mathfrak{a}^-$. Hence $\mathfrak{h}$ is either $\mathfrak{t}^1 \times \mathfrak{so}(2n-2;\boldsymbol{C})$, $\mathfrak{so}(2n;\boldsymbol{C})$ or the Lie algebras $\mathfrak{h}^\pm$ defined in Example 2.3.8. It is easy to show that the Godbillon-Vey class of the foliation induced by $\mathfrak{t}^1 \times \mathfrak{so}(2n-2;\boldsymbol{C})$ is trivial. We have already shown in Example 2.3.8 that the Godbillon-Vey classes of the foliations induced respectively by $\mathfrak{h}^\pm$ are trivial. Thus the proposition is proved. $\square$

Remark 2.3.20. It is well-known that $\mathfrak{so}(4;\boldsymbol{C}) \cong \mathfrak{sl}(2;\boldsymbol{C}) \oplus \mathfrak{sl}(2;\boldsymbol{C})$ and $\mathfrak{so}(6;\boldsymbol{C}) \cong \mathfrak{sl}(4;\boldsymbol{C})$. Hence it is possible despite Proposition 2.3.19 to construct foliations with non-trivial Godbillon-Vey classes at least on the Lie algebra level.

As already remarked, the Godbillon-Vey class is realized as the product of volume forms of $X_n^{\mathrm{crf}}/(T^1 \times X_{n-1}^{\mathrm{crf}})$ and $X_n^{\mathrm{crf}}/X_{n-1}^{\mathrm{crf}}$. Hence if the Godbillon-Vey class is non-trivial, then the image of the volume form of $X_n^{\mathrm{crf}}/X_{n-1}^{\mathrm{crf}}$ remains non-trivial when pulled back to X_n^{crf}. On this line, we have the following. Recall that the inclusion of X_{n-1} into X_n is realized via the inclusion of corresponding Dynkin diagrams.

Proposition 2.3.21. *The mapping $\pi^* H^*(X_n^{\mathrm{crf}}/X_{n-1}^{\mathrm{crf}}) \to H^*(X_n^{\mathrm{crf}})$ annihilates the volume if the pair (X_n, X_{n-1}) is one of $(F_4, \mathrm{Sp}(3;\boldsymbol{C}))$, $(F_4, \mathrm{SO}(7;\boldsymbol{C}))$, $(E_6, \mathrm{SL}(6;\boldsymbol{C}))$, $(E_6, \mathrm{SO}(10;\boldsymbol{C}))$, (E_7, E_6) or (E_8, E_7). Hence examples of the same kind as in this article cannot be constructed for these pairs.*

Proof. It is known the cohomology of these groups are as follows [14]:

$$
\begin{aligned}
H^*(\mathfrak{f}_4) &\cong \textstyle\bigwedge[e_3, e_{11}, e_{15}, e_{23}],\\
H^*(\mathfrak{e}_6) &\cong \textstyle\bigwedge[e_3, e_9, e_{11}, e_{15}, e_{17}, e_{23}],\\
H^*(\mathfrak{e}_7) &\cong \textstyle\bigwedge[e_3, e_{11}, e_{15}, e_{19}, e_{23}, e_{27}, e_{35}],\\
H^*(\mathfrak{e}_8) &\cong \textstyle\bigwedge[e_3, e_{15}, e_{23}, e_{27}, e_{35}, e_{39}, e_{47}, e_{59}],
\end{aligned}
$$

where e_i denotes the generators of degree i. The dimensions of $\mathrm{Sp}(3;\boldsymbol{C})$ (or $\mathrm{SO}(7;\boldsymbol{C})$) and F_4 are 21 and 52, respectively. However, $H^{31}(\mathfrak{f}_4) = \{0\}$. In order to prove the claim for E_6, first consider $E_5 = \mathrm{SL}(6;\boldsymbol{C})$. Then $H^*(\mathfrak{sl}(6;\boldsymbol{C})) \cong \bigwedge[e_3, e_5, e_7, e_9, e_{11}]$. Since the embedding is induced from the inclusion of corresponding Dynkin diagrams, we may assume that the image of e_i under π_* is again e_i if and only if e_i is non-trivial in the image. If π^* does not annihilate the volume form, there is a non-trivial class in $H^{43}(\mathfrak{e}_6)$ written in terms of e_{15} and e_{23}. It is clearly impossible. If E_5 is considered as $\mathrm{SO}(10;\boldsymbol{C})$, the proof is done simply by counting dimension as in the case of F_4. The claim for other groups are also shown in this way. $\square$

More systematic treatment seems appropriate for examining all possible pairs (X_n, X_{n-1}). We will not pursue it here.

3. Relation with the Residue theory

The examples considered in the previous section are related to Heitsch's residue [21] as follows.

Let $\widetilde{\mathcal{F}}$ be the foliation of $\widetilde{M} = \mathrm{SL}(2;\boldsymbol{C}) \times \boldsymbol{C}^2$ whose leaves are the orbits of the right action of $\mathrm{SL}(2;\boldsymbol{C})$ on $\widetilde{M}$ given by $(g_0, v_0) \cdot g = (g_0 g, g^{-1} v_0)$. Let T be the holomorphic vector field on $\boldsymbol{C}^2$ defined by $T = z\dfrac{\partial}{\partial z} + w\dfrac{\partial}{\partial w}$, where (z, w) is the natural coordinate of $\boldsymbol{C}^2$. Since the natural left action of $\mathrm{SL}(2;\boldsymbol{C})$ on $\boldsymbol{C}^2$ preserves T, $\widetilde{\mathcal{F}}$ and T induce a foliation $\widetilde{\mathcal{G}}$ of $\widetilde{M}^* = \mathrm{SL}(2;\boldsymbol{C}) \times (\boldsymbol{C}^2 \setminus \{0\})$. Moreover, as these foliations are invariant under the natural left action of $\mathrm{SL}(2;\boldsymbol{C})$ on $\widetilde{M}$ and $\widetilde{M}^*$, we have foliations $\widetilde{\mathcal{F}}_\Gamma$ of $\Gamma\backslash\widetilde{M}$ and $\widetilde{\mathcal{G}}_\Gamma$ of $\Gamma\backslash\widetilde{M}^*$ for any cocompact lattice Γ of $\mathrm{SL}(2;\boldsymbol{C})$. The original construction of the residue in [21],[23] applied to these foliations is as follows. Let $\iota : \Gamma\backslash\mathrm{SL}(2;\boldsymbol{C}) \times S^3 \to \Gamma\backslash\widetilde{M}^*$ be the inclusion, then ι^* is an isomorphism. Moreover, since ι is transversal to $\widetilde{\mathcal{G}}_\Gamma$, there is a natural foliation $\iota^*\widetilde{\mathcal{G}}_\Gamma$ of $\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C}) \times S^3$. Since the normal bundles of $\widetilde{\mathcal{G}}_\Gamma$ and $\iota^*\widetilde{\mathcal{G}}_\Gamma$ are trivial, the Bott classes $u_1v_1(\widetilde{\mathcal{G}}_\Gamma)$ and $u_1v_1(\iota^*\widetilde{\mathcal{G}}_\Gamma)$ are defined. Let $H_c^*(\Gamma\backslash\widetilde{M})$ be the cohomology of $\Gamma\backslash\widetilde{M}$ compactly supported in the direction of $\boldsymbol{C}^2$, then we have the following sequence, namely,

$$H^*(\Gamma\backslash\widetilde{M}^*) \xrightarrow{\partial} H^{*+1}(\Gamma\backslash\widetilde{M}, \Gamma\backslash\widetilde{M}^*) \xrightarrow{i} H_c^{*+1}(\Gamma\backslash\widetilde{M}) \xrightarrow{\int_{\boldsymbol{C}^2}} H^{*-3}(\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})),$$

where ∂ is the boundary homomorphism, i is the natural mapping and $\int_{\boldsymbol{C}^2}$ is the integration of compactly supported 4-forms on $\boldsymbol{C}^2$. It is shown by Heitsch [21],[23] that there is a natural choice of a Bott connection for $\widetilde{\mathcal{F}}_\Gamma$ adapted to T and that if we denote by $v_1^2(\widetilde{\mathcal{F}}_\Gamma, T)$ the differential form v_1^2 calculated by this Bott connection, then $v_1^2(\widetilde{\mathcal{F}}_\Gamma, T)$ is of compact support and it represents $i(\partial(u_1v_1(\widetilde{\mathcal{G}}_\Gamma)))$. The image of $v_1^2(\widetilde{\mathcal{F}}_\Gamma, T)$ under $\int_{\boldsymbol{C}^2}$ is by definition the residue of $v_1^2(\widetilde{\mathcal{F}}_\Gamma)$ with respect to T. Since $u_1v_1(\iota^*\widetilde{\mathcal{G}}_\Gamma) = \iota^*u_1v_1(\widetilde{\mathcal{G}}_\Gamma)$, this class is seen to be non-trivial if the residue of v_1^2 is non-trivial (which is indeed the case).

On the other hand, there is a following relation between the Bott class and the Godbillon-Vey class (Theorem 2.3 in [4]). Let $\int_{S^1} : H^*(\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C}) \times S^3) \to H^{*-1}(\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C}) \times \boldsymbol{C}P^1)$ be the integration along the fiber of the Hopf fibration, then $u_1v_1(\iota^*\widetilde{\mathcal{G}}_\Gamma)\bar{u}_1\bar{v}_1(\iota^*\widetilde{\mathcal{G}}_\Gamma)$ is mapped to a non-zero multiple of $\mathrm{GV}_1(\widetilde{\mathcal{H}}_\Gamma)$, where $\widetilde{\mathcal{H}}_\Gamma$ is the foliation of $\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C}) \times \boldsymbol{C}P^1$ obtained by taking the quotient by the natural S^1 action along the fibers. Composing with the integration on the fibers of $\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C}) \times \boldsymbol{C}P^1 \to \Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})$, one obtains the following commutative diagram;

$$\begin{array}{ccccc}
H^*(\Gamma\backslash\widetilde{M}^*) & \xrightarrow{\partial} & H^{*+1}(\Gamma\backslash\widetilde{M}, \Gamma\backslash\widetilde{M}^*) & \xrightarrow{i} & H_c^{*+1}(\Gamma\backslash\widetilde{M}) \\
\iota^* \big\downarrow \wr & & & & \int_{\boldsymbol{C}^2} \big\downarrow \\
H^*(\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C}) \times S^3) & \xrightarrow{\int_{S^1}} & H^{*-1}(\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C}) \times \boldsymbol{C}P^1) & \longrightarrow & H^{*-3}(\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})).
\end{array}$$

However, the image of $u_1v_1(\widetilde{\mathcal{G}}_\Gamma)\bar{u}_1\bar{v}_1(\widetilde{\mathcal{G}}_\Gamma)$ under $i \circ \partial$ is trivial because

$$\begin{aligned}
i \circ \partial(u_1v_1(\widetilde{\mathcal{G}}_\Gamma)\bar{u}_1\bar{v}_1(\widetilde{\mathcal{G}}_\Gamma)) =& - u_1v_1\bar{v}_1^2(\widetilde{\mathcal{F}}_\Gamma) + \bar{u}_1\bar{v}_1v_1^2(\widetilde{\mathcal{F}}_\Gamma) \\
=& d(u_1\bar{u}_1v_1\bar{v}_1(\widetilde{\mathcal{F}}_\Gamma, T)),
\end{aligned}$$

where $u_1\bar{u}_1v_1\bar{v}_1(\widetilde{\mathcal{F}}_\Gamma,T)$ is the compactly supported differential form calculated in the same way as calculating $v_1^2(\widetilde{\mathcal{F}}_\Gamma,T)$. It is relevant here that u_1 and $\bar{u}_1$ are well-defined for $\widetilde{\mathcal{F}}_\Gamma$.

Take now the quotient of the above foliations by SU(2) acting on $\widetilde{M}$ and $\widetilde{M}^*$ from the right via the inclusion to $\mathrm{SL}(2;\boldsymbol{C})$. Set $M=\mathrm{SL}(2;\boldsymbol{C})\underset{\mathrm{SU}(2)}{\times}\boldsymbol{C}^2$ and $M^*=\mathrm{SL}(2;\boldsymbol{C})\underset{\mathrm{SU}(2)}{\times}(\boldsymbol{C}^2\setminus\{0\})$, then the foliations as above induce foliations $\mathcal{F}_\Gamma$ of $\Gamma\backslash M$, $\mathcal{G}_\Gamma$ of $\Gamma\backslash M^*$, $\iota^*\mathcal{G}_\Gamma$ of $\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})\underset{\mathrm{SU}(2)}{\times}S^3$ and $\mathcal{H}_\Gamma$ of $\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})\underset{\mathrm{SU}(2)}{\times}\boldsymbol{C}P^1$, where Γ is now a cocompact lattice of $\mathrm{SL}(2;\boldsymbol{C})/\mathrm{SU}(2)$. The diagram is as follows, namely,

$$\begin{array}{ccccc}
H^*(\Gamma\backslash M^*) & \xrightarrow{\partial} & H^{*+1}(\Gamma\backslash M,\Gamma\backslash M^*) & \xrightarrow{i} & H_c^{*+1}(\Gamma\backslash M)\\
{\scriptstyle\iota^*}\downarrow\wr & & & & {\scriptstyle\int_{\boldsymbol{C}^2}}\downarrow\\
H^*(\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})\underset{\mathrm{SU}(2)}{\times}S^3) & \xrightarrow{\int_{S^1}} & H^{*-1}(\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})\underset{\mathrm{SU}(2)}{\times}\boldsymbol{C}P^1) & \longrightarrow & H^{*-3}(\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})/\mathrm{SU}(2)),
\end{array}$$

where $\int_{\boldsymbol{C}^2}$ is the integration along the fiber. The image of $u_1v_1(\widetilde{\mathcal{G}}_\Gamma)\bar{u}_1\bar{v}_1(\widetilde{\mathcal{G}}_\Gamma)$ under $\int_{\boldsymbol{C}^2}\circ i\circ\partial$ is a non-zero multiple of the volume form of $\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})/\mathrm{SU}(2)$ and non-trivial. It is well-known that the foliation $\mathcal{H}_\Gamma$ is isomorphic to the foliation given in Example 2.3.6. The fact that $\mathrm{GV}_2(\mathcal{H}_\Gamma)$ become trivial when pulled back to $\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})\underset{\mathrm{SU}(2)}{\times}S^3$ corresponds to 1) of Remark 2.3.7. We remark that if $\pi:\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})\underset{\mathrm{SU}(2)}{\times}\boldsymbol{C}P^1\to\Gamma\backslash\mathrm{SL}(2;\boldsymbol{C})/\mathrm{SU}(2)$ is the projection, then $\pi^*\circ\int_{\boldsymbol{C}^2}\circ i\circ\partial(u_1v_1(\widetilde{\mathcal{G}}_\Gamma)\bar{u}_1\bar{v}_1(\widetilde{\mathcal{G}}_\Gamma))$ is a non-zero multiple of the imaginary part of the Bott class $\xi_1(\mathcal{H}_\Gamma)$.

The same kind of construction can be done in higher codimensional cases. It is also possible to apply this construction to other examples involving $\mathrm{SO}(2n+1;\boldsymbol{C})$, $\mathrm{Sp}(n;\boldsymbol{C})$ and G_2 by using the Iwasawa decomposition and naturally associated S^1-bundles. Thus we can still say that all the known examples of transversely holomorphic foliations with non-vanishing secondary classes can be obtained through the residue theory as pointed out in [25] by Hurder.

4. The rigidity theorem and Infinitesimal derivatives

In this section, infinitesimal derivatives of secondary classes are introduced by following Heitsch [22]. It will be shown that complex secondary classes determined by the image of $H^*(\mathrm{WU}_{q+1})$ under the natural mapping to $H^*(\mathrm{WU}_q)$ are rigid under actual and infinitesimal deformations. In particular, the Godbillon-Vey class is shown to be rigid in the category of transversely holomorphic foliations. On the other hand, there are classes in $H^*(\mathrm{WU}_q)$ such as the imaginary part of the Bott class which admit continuous deformations. These classes are called variable classes. Heitsch introduced in [22] the infinitesimal derivatives for *cocycles* in WU_q which represent variable classes of lowest degree. In the same paper, the infinitesimal derivatives for any *classes* in $H^*(\mathrm{WO}_q)$ were also introduced. The most of this section will be devoted to completing Heitsch's construction by defining the infinitesimal derivatives for any *classes* in $H^*(\mathrm{WU}_q)$. The construction seems known for specialists, indeed, the most of the definitions and the proofs are only small modifications of Heitsch's in [22] using notions in [15]. However, we give the details for completeness and for their importance.

We follow Heitsch's line of arguments for easy comparison. When definitions and theorems are given, the number of corresponding statements in [22] will be also given so far as possible. Finally, we remark that infinitesimal deformations are also discussed by Girbau, Haefliger and Sundararaman [17].

§ 1. Definitions and Statement of results.

The following mapping from $H^*(\mathrm{WU}_{q+1})$ to $H^*(\mathrm{WU}_q)$ is relevant. It is an analog to the real case [19].

Definition 4.1.1. Let ρ be the DGA-homomorphism from WU_{q+1} to WU_q defined by the following formulae:

$$\rho(\widetilde{u}_i) = \begin{cases} \widetilde{u}_i & \text{if } i \neq q+1 \\ 0 & \text{if } i = q+1 \end{cases}$$

$$\rho(v_i) = \begin{cases} v_i & \text{if } i \neq q+1 \\ 0 & \text{if } i = q+1 \end{cases}, \quad \rho(\bar{v}_i) = \begin{cases} \bar{v}_i & \text{if } i \neq q+1 \\ 0 & \text{if } i = q+1 \end{cases}$$

We denote by ρ_* the induced homomorphism from $H^*(\mathrm{WU}_{q+1})$ to $H^*(\mathrm{WU}_q)$.

The mapping ρ_* is induced by the standard inclusion of $\boldsymbol{C}^q$ into $\boldsymbol{C}^{q+1}$.

Definition 4.1.2. Let $\{\mathcal{F}_t\}$ be a family of transversely holomorphic foliations of fixed complex codimension, of a fixed manifold M. Then $\{\mathcal{F}_t\}$ is said to be a continuous deformation of $\mathcal{F}_0$ if $\{\mathcal{F}_t\}$ is a continuous family as integrable distributions. If the family is in fact smooth, it is said to be smooth.

1) If there exists a smooth family of diffeomorphisms which conjugate each $\mathcal{F}_t$ to $\mathcal{F}_0$, then we call such a $\{\mathcal{F}_t\}$ as deformations preserving the diffeomorphism type.
2) Particularly if $\mathcal{F}_t$ is identical to $\mathcal{F}_0$ when the transverse holomorphic structures are forgotten, the family $\{\mathcal{F}_t\}$ is called as a deformation of transverse holomorphic structures.

The following theorems are the main results in this section.

Theorem B1. *The secondary classes defined by $H^*(\mathrm{WU}_q)$ is rigid under smooth deformations if they belong to the image of ρ_*. More precisely, let $\{\mathcal{F}_s\}$ be a smooth family of transversely holomorphic foliations of complex codimension q and let ω be an element of $\rho_*(H^*(\mathrm{WU}_{q+1}))$, then $\omega(\mathcal{F}_s) = \omega(\mathcal{F}_t)$ as elements of $H^*(M)$ for any s and t.*

Given a transversely holomorphic foliation $\mathcal{F}$ of M, infinitesimal deformations are elements of $H^1(M; \Theta_{\mathcal{F}})$ (see Definition 4.3.3 for details). The infinitesimal derivatives of elements of $H^*(\mathrm{WU}_q)$ are given by the mapping

$$D_{\cdot}(\,\cdot\,) : H^1(M; \Theta_{\mathcal{F}}) \times H^*(\mathrm{WU}_q) \to H^*(M; \boldsymbol{C})$$

in Definition 4.3.11. It will be shown that a smooth family $\{\mathcal{F}_s\}$ as above determines a natural infinitesimal derivative $\beta \in H^1(M; \Theta_{\mathcal{F}})$ such that $D_\beta(\omega) = \left.\frac{\partial}{\partial s}\omega(\mathcal{F}_s)\right|_{s=0}$ for $\omega \in H^*(\mathrm{WU}_q)$ (Theorem 4.3.25). The infinitesimal version of Theorem B1 is as follows.

Theorem B2. *The image of $H^1(M; \Theta_{\mathcal{F}}) \times (\rho_* H^*(\mathrm{WU}_{q+1}))$ under the above mapping $D_{\cdot}(\,\cdot\,)$ is $\{0\}$.*

Theorems B1 and B2 will be proved in steps. Before beginning the proof, we present an important

Corollary 4.1.3. *The Godbillon-Vey class is rigid under both smooth and infinitesimal deformations in the category of transversely holomorphic foliations.*

Proof. Let q be the codimension of the foliations, then the equation

$$\mathrm{GV}_{2q} = \rho_* \left(\frac{(2q)!}{q!q!} \xi_{q+1} \cdot \mathrm{ch}_1^{q-1} \right) = \frac{(2q)!}{q!q!} \sqrt{-1} \widetilde{u}_1 v_1^q \bar{v}_1^q$$

holds in $H^{4q+1}(\mathrm{WU}_{q+1})$, where ξ_{q+1} is defined in Definition 1.4. $\square$

The following corollary follows from Corollary 4.1.3 and Theorem 1.5.

Corollary 4.1.4. *Let $\mathcal{F}_s$ be a smooth family of transversely holomorphic foliations of codimension q, then the product of $\mathrm{ch}_1(\mathcal{F}_0)^q$ and $\frac{d}{ds}\xi(\mathcal{F}_s)$ is identically equal to zero. (Note that $\mathrm{ch}_1(\mathcal{F}_s)^q$ is independent of s.) Similarly, for any infinitesimal deformation β of $\mathcal{F}$, $D_\beta(\xi_q)(\mathcal{F})\mathrm{ch}_1(\mathcal{F})^q = 0$ holds for the infinitesimal derivative $D_\beta(\xi_q)(\mathcal{F})$ of ξ_q with respect to β.*

Let $\mathcal{F}_s$ be a smooth family of transversely holomorphic foliations of codimension q and assume that $\mathrm{GV}_{2q}(\mathcal{F}_s)$ is non-trivial, then $\mathrm{ch}_1(\mathcal{F}_0)^q$ is non-trivial by Theorem 1.5. Assume that the mapping $\cup\,\mathrm{ch}_1(\mathcal{F}_0)^q : H^*(M;\boldsymbol{C}) \to H^{*+2q}(M;\boldsymbol{C})$ is injective, then $\dfrac{d}{ds}\xi_q(\mathcal{F}_s)$ is trivial because $\dfrac{d}{ds}\mathrm{GV}_{2q}(\mathcal{F}_s) = \dfrac{d}{ds}\xi_q(\mathcal{F}_s)\,\mathrm{ch}_1(\mathcal{F}_0)^q = 0$ by Theorem B1. This implies that the class ξ_q is in fact rigid in such a case. So far as we know, any smooth family $\{\mathcal{F}_s\}$ such that $\xi_q(\mathcal{F}_s)$ varies continuously has trivial first Chern class. In this line, we have the following

Question 4.1.5. *Is there a smooth family of transversely holomorphic foliations for which the imaginary part of the Bott class varies continuously and the first Chern class of the complex normal bundle is non-trivial?*

The infinitesimal version of this question can be also asked.

§ **2. Rigidity under smooth deformations.**

The aim of this section is to prove Theorem B1. We begin with some definitions.

Definition 4.2.1. Let $\{\mathcal{F}_s\}$ be a smooth deformation of transversely holomorphic foliations. Noticing that the complex normal bundles of the foliations remain isomorphic, denote them by Q and consider the same unitary connection θ_0 for some Hermitian metric on Q. Let θ_1^s be a smooth family of complex Bott connections of $\mathcal{F}_s$ and denote by ψ_s its differential with respect to s, namely, $\psi_s = \dfrac{\partial}{\partial s}\theta_1^s$. Let f be a homogeneous polynomial of degree $2k$ in v_i and $\bar{v}_j$, then define differential forms Δ_f and V as follows. First set $\theta_t^s = t\theta_1^s + (1-t)\theta_0$ and denote by Ω_t^s its curvature, then set

$$\begin{aligned}
\Delta_f(\theta_1^s,\theta_0) &= k\int_0^1 f(\theta_1^s-\theta_0,\Omega_t^s,\dots,\Omega_t^s)dt\\
V_f(\theta_1^s,\theta_0) &= \int_0^1 tf(\psi_s,\theta_1^s-\theta_0,\Omega_t^s,\dots,\Omega_t^s)dt.
\end{aligned}$$

The following formulae are shown in [19];

(4.2.2a) $\quad \dfrac{\partial}{\partial s}(\Delta_f(\theta_1^s,\theta_0)) = k(k-1)dV_f(\theta_1^s,\theta_0) + kf(\psi_s,\Omega_1^s,\dots,\Omega_1^s)$, and

(4.2.2b) $\quad \dfrac{\partial}{\partial s}d(\Delta_f(\theta_1^s,\theta_0)) = \dfrac{\partial}{\partial s}f(\Omega_1^s,\dots,\Omega_1^s) = kdf(\psi_s,\Omega_1^s,\dots,\Omega_1^s)$,

where Ω_1^s denotes the curvature form of the connection θ_1^s and the the exterior derivative is considered only on M, namely, along the fibers of $M\times\boldsymbol{R}\to\boldsymbol{R}$.

The following auxiliary definition will be convenient.

Definition 4.2.3. Set $\widetilde{\mathrm{WU}}_q = \bigwedge[\widetilde{u}_1, \ldots, \widetilde{u}_q] \otimes \boldsymbol{C}[v_1, \ldots, v_q] \otimes \boldsymbol{C}[\bar{v}_1, \ldots, \bar{v}_q]$ and equip $\widetilde{\mathrm{WU}}_q$ with the differential $\widetilde{d}$ by requiring $\widetilde{d}\widetilde{u}_i = v_i - \bar{v}_i$ and $\widetilde{d}v_i = \widetilde{d}\bar{v}_i = 0$. Let $\widetilde{\mathcal{I}}_q$ be the ideal of $\widetilde{\mathrm{WU}}_q$ generated by cochains of the form $\widetilde{u}_I v_J \bar{v}_K$ with $|J| > q$ or $|K| > q$. Note then that $\mathrm{WU}_q = \widetilde{\mathrm{WU}}_q/\widetilde{\mathcal{I}}_q$. If φ is a cochain in WU_q, then the natural lift of φ is the cochain in $\widetilde{\mathrm{WU}}_q$ obtained by representing φ as the linear combination of cochains of the form $\widetilde{u}_I v_J \bar{v}_K$ with $|J| \leq q$ and $|K| \leq q$.

These $\widetilde{\mathrm{WU}}_q$ and $\widetilde{d}$ correspond to WU_q but the Bott vanishing is ignored. It is easy to verify the equation $\widetilde{d} \circ \widetilde{d} = 0$. In other words, $\widetilde{d}\widetilde{d}\widetilde{\varphi}$ is exactly equal to 0 for any $\widetilde{\varphi} \in \widetilde{\mathrm{WU}}_q$. This simple property is frequently used in what follows.

The following differential form is significant.

Definition 4.2.4. Let θ^u and θ be a unitary connection and a Bott connection on $Q(\mathcal{F})$, respectively. Let θ' be a derivative of a family of Bott connections or an infinitesimal derivative of a Bott connection which will be introduced in Definition 4.3.7, or a certain matrix valued function which will appear in proving Theorem 4.3.17. For $\widetilde{\varphi} \in \widetilde{\mathrm{WU}}_q$, define a differential form $\Delta\widetilde{\varphi}(\theta^u, \theta, \theta')$ as follows. Firstly, when $\widetilde{\varphi} = \widetilde{u}_I v_J \bar{v}_K$, set

$$\delta(\widetilde{\varphi})(\theta^u, \theta, \theta') = (|J| + |K|) v_J \bar{v}_K(\theta', \Omega)\widetilde{u}_I(\theta, \theta^u),$$

where Ω is the curvature form of θ. Set now

$$\Delta\widetilde{\varphi}(\theta^u, \theta, \theta') = \delta(\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \theta').$$

We extend δ and Δ to the whole $\widetilde{\mathrm{WU}}_q$ by linearity.

If $\widetilde{\varphi} = \widetilde{u}_I v_J \bar{v}_K \in \widetilde{\mathrm{WU}}_q$ and $I = \{i_1, \ldots, i_t\}$ with $i_1 < i_2 < \cdots < i_t$, then

$$\begin{aligned}&\Delta\widetilde{\varphi}(\theta^u, \theta, \theta') \\ &= \sum_l (-1)^{l-1} (|J| + |K| + i_l) \left(v_J \bar{v}_K (v_{i_l} - \bar{v}_{i_l})\right) (\theta', \Omega) \widetilde{u}_{I(l)}(\theta, \theta^u),\end{aligned}$$

where $I(l) = I \setminus \{i_l\}$.

Remark 4.2.5. We have the following formulae;

$$(|J| + |K|)(v_J v_K)(\theta', \Omega) = |J|\, v_J(\theta', \Omega) v_K(\Omega) + |K|\, v_J(\Omega) v_K(\theta', \Omega), \tag{4.2.5a}$$

$$\begin{cases} v_J(\theta', \Omega) = 0 & \text{as differential forms if } |J| > q + 1, \\ \bar{v}_K(\theta', \Omega) = 0 & \text{as differential forms if } |K| > q + 1. \end{cases} \tag{4.2.5b}$$

Theorem B1 will follow from the following

Proposition 4.2.6. *Let $\varphi \in \mathrm{WU}_q$ be a cocycle, then $\dfrac{\partial}{\partial s}\chi_s(\varphi)$ is represented by $\Delta\widetilde{\varphi}(\theta_0, \theta_1^s, \psi_s)$, where $\widetilde{\varphi}$ is any lift of φ to $\widetilde{\mathrm{WU}}_q$.*

Proof. Let φ be a cocycle in WU_q and compute $\dfrac{\partial}{\partial s}\chi_s(\varphi)$. For each i $(1 \le i \le q)$, there are elements α_i and β_i of WU_q which do not involve $\widetilde{u}_i$ and such that $\varphi = \widetilde{u}_i\alpha_i + \beta_i$. Note that α_i is closed because φ is closed.

In the rest of the proof, we adopt the following notations, namely, $\widetilde{u}_i(\theta_1^s, \theta_0)$, $v_j(\Omega_1^s)$ and $\bar{v}_k(\Omega_1^s)$ are simply denoted by $\widetilde{u}_i(s)$, $v_j(s)$ and $\bar{v}_k(s)$, respectively. The differential form $v_j(\psi_s, \Omega_1^s)$ is denoted by $w_j(s)$, and $\bar{v}_k(\psi_s, \Omega_1^s)$ is denoted by $\bar{w}_k(s)$. Denote $V_{v_i}(\theta_1^s, \theta_0)$ and $\overline{V}_{v_i}(\theta_1^s, \theta_0)$ simply by V_i and $\overline{V}_i$, respectively. Finally, we set $\widetilde{V}_i = V_i - \overline{V}_i$ and $\widetilde{w}_i(s) = w_i(s) - \bar{w}_i(s)$. Thus $\dfrac{\partial}{\partial s}\widetilde{u}_i(s) = i(i-1)d\widetilde{V}_i + i(w_i(s) - \bar{w}_i(s)) = i(i-1)d\widetilde{V}_i + i\widetilde{w}_i$.

Let $\dfrac{\partial}{\partial s_i}$ be the differential operator obtained by applying $\dfrac{\partial}{\partial s}$ only to $\widetilde{u}_i(\theta_1^s, \theta_0)$, $v_i(\theta_1^s)$ and $\bar{v}_i(\theta_1^s)$, then $\dfrac{\partial}{\partial s}$ is decomposed as $\dfrac{\partial}{\partial s} = \dfrac{\partial}{\partial s_1} + \cdots + \dfrac{\partial}{\partial s_q}$. In order to compute $\dfrac{\partial}{\partial s_i}\chi_s(\varphi)$, write $\alpha_i = \sum_{j,k} v_i^j \bar{v}_i^k a_{j,k}^i$ and $\beta_i = \sum_{j,k} v_i^j \bar{v}_i^k b_{j,k}^i$ so that neither $a_{j,k}^i$ nor $b_{j,k}^i$ involves v_i and $\bar{v}_i$. Then we have the following equation, namely,

$$\begin{aligned}
\frac{\partial}{\partial s_i}\chi_s(\varphi) =& \frac{\partial}{\partial s_i}\chi_s(\widetilde{u}_i\alpha_i + \beta_i) \\
=& \sum_{j,k} \Big(i(i-1)d\widetilde{V}_i + i\widetilde{w}_i(s)\Big)\, v_i^j(s)\bar{v}_i^k(s)a_{j,k}^i(s) \\
&+ \sum_{j,k} ij\widetilde{u}_i(s)v_i^{j-1}(s)dw_i(s)\bar{v}_i^k(s)a_{j,k}^i(s) \\
&+ \sum_{j,k} ik\widetilde{u}_i(s)v_i^j(s)\bar{v}_i^{k-1}(s)d\bar{w}_i(s)a_{j,k}^i(s) \\
&+ \sum_{j,k} ijv_i^{j-1}(s)dw_i(s)\bar{v}_i^k(s)b_{j,k}^i(s) \\
&+ \sum_{j,k} ikv_i^j(s)\bar{v}_i^{k-1}(s)d\bar{w}_i(s)b_{j,k}^i(s).
\end{aligned}$$

The first term is equal to

$$i(i-1)d\widetilde{V}_i\alpha_i(s) + \sum_{j,k} i\widetilde{w}_i(s)v_i^j(s)\bar{v}_i^k(s)a_{j,k}^i(s).$$

Note that $d\widetilde{V}_i\alpha_i(s) = d(\widetilde{V}_i\alpha_i(s))$ because α_i is closed. The second term is cohomologous to

$$\sum_{j,k} ij(v_i(s)-\bar{v}_i(s))v_i^{j-1}(s)w_i(s)\bar{v}_i^k(s)a^i_{j,k}(s)$$
$$+\sum_{j,k} ij\widetilde{u}_i(s)v_i^{j-1}(s)w_i(s)\bar{v}_i^k(s)da^i_{j,k}(s),$$

which is equal to

$$\sum_{j,k} ijv_i^j(s)\bar{v}_i^k(s)w_i(s)a^i_{j,k}(s)-\sum_{j,k} ijv_i^{j-1}(s)\bar{v}_i^k w_i(s)a^i_{j,k-1}(s)$$
$$-\sum_{j,k} ijv_i^{j-1}(s)w_i(s)\bar{v}_i^k(s)\widetilde{u}_i(s)da^i_{j,k}(s),$$

where $a^i_{j,-1}$ is understood to be zero. Similarly, the third term is cohomologous to

$$-\sum_{j,k} ikv_i^j(s)\bar{v}_i^k(s)\bar{w}_i(s)a^i_{j,k}(s)+\sum_{j,k} ikv_i^j(s)\bar{v}_i^{k-1}\bar{w}_i(s)a^i_{j-1,k}(s)$$
$$-\sum_{j,k} ikv_i^j(s)\bar{v}_i^{k-1}(s)w_i(s)\widetilde{u}_i(s)da^i_{j,k}(s),$$

where $a^i_{-1,k}=0$. The fourth and fifth terms are respectively cohomologous to

$$\sum_{j,k} ijv_i^{j-1}(s)\bar{v}_i^k(s)w_i(s)db^i_{j,k}(s), \text{ and}$$
$$\sum_{j,k} ikv_i^j(s)\bar{v}_i^{k-1}(s)\bar{w}_i(s)db^i_{j,k}(s).$$

Hence we have the following equalities modulo exact terms, namely,

$$\begin{aligned}
&\frac{\partial}{\partial s_i}\chi_s(\varphi)\\
=&\sum_{j,k} i(j+1)v_i^j(s)\bar{v}_i^k(s)w_i(s)a^i_{j,k}(s)-\sum_{j,k} i(k+1)v_i^j(s)\bar{v}_i^k(s)\bar{w}_i(s)a^i_{j,k}(s)\\
&-\sum_{j,k} ijv_i^{j-1}(s)\bar{v}_i^k w_i(s)a^i_{j,k-1}(s)+\sum_{j,k} ikv_i^j(s)\bar{v}_i^{k-1}\bar{w}_i(s)a^i_{j-1,k}(s)\\
&-\sum_{j,k} ijv_i^{j-1}(s)w_i(s)\bar{v}_i^k(s)\widetilde{u}_i(s)da^i_{j,k}(s)-\sum_{j,k} ikv_i^j(s)\bar{v}_i^{k-1}(s)w_i(s)\widetilde{u}_i(s)da^i_{j,k}(s)\\
&+\sum_{j,k} ijv_i^{j-1}(s)\bar{v}_i^k(s)w_i(s)db^i_{j,k}(s)+\sum_{j,k} ikv_i^j(s)\bar{v}_i^{k-1}(s)\bar{w}_i(s)db^i_{j,k}(s)\\
=&\sum_{j,k} ijv_i^{j-1}(s)\bar{v}_i^k w_i(s)\left(a^i_{j-1,k}(s)-a^i_{j,k-1}(s)+db^i_{j,k}(s)-\widetilde{u}_i(s)da^i_{j,k}(s)\right)\\
&+\sum_{j,k} ikv_i^j(s)\bar{v}_i^{k-1}\bar{w}_i(s)\left(-a^i_{j,k-1}(s)+a^i_{j-1,k}(s)+db^i_{j,k}(s)-\widetilde{u}_i(s)da^i_{j,k}(s)\right).
\end{aligned}$$

On the other hand, if $\widetilde{\varphi}$ is the natural lift of φ, then one has the following equation for each i, namely,

$$\begin{aligned}
\widetilde{d}\widetilde{\varphi} &= ((v_i - \bar{v}_i)\alpha_i - \widetilde{u}_i d\alpha_i + d\beta_i) \\
&= \sum_{j,k}(v_i - \bar{v}_i)v_i^j \bar{v}_i^k a_{j,k}^i + \sum_{j,k} v_i^j \bar{v}_i^k db_{j,k}^i - \sum_{j,k} \widetilde{u}_i v_i^j \bar{v}_i^k da_{j,k}^i \\
&= \sum_{j,k} v_i^j \bar{v}_i^k (a_{j-1,k}^i - a_{j,k-1}^i + db_{j,k}^i - \widetilde{u}_i da_{j,k}^i).
\end{aligned}$$

Proposition 4.2.6 for this choice of $\widetilde{\varphi}$ now follows from (4.2.5a). In order to show the proposition for other choices, it suffices to show that $\Delta(\widetilde{d}\widetilde{\alpha} + \widetilde{\beta})(\theta_0, \theta_1^s, \psi_s)$ is exact for $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{\mathrm{WU}}_q$, where $\widetilde{\beta} \in \widetilde{\mathcal{I}}_q$. Firstly, one has $\Delta(\widetilde{d}\widetilde{\alpha})(\theta_0, \theta_1^s, \psi_s) = \delta(\widetilde{d}(\widetilde{d}\widetilde{\alpha}))(\theta_0, \theta_1^s, \psi_s) = 0$. On the other hand, let $\widetilde{\beta} = \widetilde{u}_I v_J \bar{v}_K$ with $|J| > q$. If $I = \phi$, then $\Delta(v_J \bar{v}_K)(\theta_0, \theta_1^s, \psi_s) = 0$ because $\widetilde{d}(v_J \bar{v}_K) = 0$. If $I \neq \phi$, then the following equation holds, namely,

$$\begin{aligned}
&\Delta(\widetilde{u}_I v_J \bar{v}_K)(\theta_0, \theta_1^s, \psi_s) \\
=&\sum_l (-1)^{l-1}(|J| + |K| + i_l) v_J \bar{v}_K (v_{i_l} - \bar{v}_{i_l})(\psi_s, \Omega^s)\widetilde{u}_{I(l)}(\theta_1^s, \theta_0) \\
=& -\sum_l (-1)^{l-1} |J|\, v_J(\psi_s, \Omega^s)\bar{v}_K \bar{v}_{i_l}(\Omega^s)\widetilde{u}_{I(l)}(\theta_1^s, \theta_0) \\
=& d\left(|J|\, v_J(\psi_s, \Omega^s)\bar{v}_K(\Omega^s)\widetilde{u}_I(\theta_1^s, \theta_0)\right).
\end{aligned}$$

Here the second equality holds for $v_J(\Omega^s) = 0$ and $v_J(\psi_s, \Omega^s)v_{i_l}(\Omega^s) = 0$ by the Bott vanishing. The last equality follows from (4.2.2b) and $dv_J(\psi_s, \Omega^s) = 0$.

Finally, $\dfrac{\partial}{\partial s}\chi_s(\varphi)$ is closed as $\chi_s(\varphi)$ is closed independent of s. □

Proof of Theorem B1. Let φ be a cocycle in WU_{q+1} and let $\widetilde{\varphi}$ be any lift of φ to $\widetilde{\mathrm{WU}}_{q+1}$, then $\widetilde{d}\widetilde{\varphi}$ is the linear combination of the monomials of the form $\widetilde{u}_I v_J \bar{v}_K$ with $|J| > q+1$ or $|K| > q+1$. Hence $\Delta(\rho\widetilde{\varphi})(\theta_0, \theta_1^s, \varphi_s)$ identically vanishes by (4.2.5b). □

Compared with the real case, the space $H^*(\mathrm{WU}_q)$ and the cokernel of ρ_* are rather complicated. For example, we have the following.

Proposition 4.2.7 (cf. Theorem 1.8 in [2])**.** *In the lower codimensional cases, the cokernel of ρ_* is described as follows:*

$q = 1$: $\operatorname{coker} \rho_*$ *is generated by* $\widetilde{u}_1(v_1 + \bar{v}_1)$.

$q=2$: $\operatorname{coker}\rho_*$ *is generated by* $v_1+\bar{v}_1$, $v_1^2+v_2+2v_1\bar{v}_1+\bar{v}_1^2+\bar{v}_2$ *and the classes in* $H^*(\mathrm{WU}_2)$ *of degree* 5, 10 *or* 12, *namely, the following classes:*

5	$\widetilde{u}_1(v_1^2+v_1\bar{v}_1+\bar{v}_1^2), \widetilde{u}_1(v_2+\bar{v}_2)+\widetilde{u}_2(v_1+\bar{v}_1)$
10	$\widetilde{u}_1\widetilde{u}_2v_1\bar{v}_1(v_1+\bar{v}_1)$
12	$\widetilde{u}_1\widetilde{u}_2v_1^2\bar{v}_1^2, \widetilde{u}_1\widetilde{u}_2v_1^2\bar{v}_2, \widetilde{u}_1\widetilde{u}_2v_2\bar{v}_1^2, \widetilde{u}_1\widetilde{u}_2v_2\bar{v}_2$

Here the number in the left column stands for the degree of the classes in the same row.

The examples of Baum and Bott in [6],[10] show that the classes of the lowest degree can vary. We do not know if the classes of higher degree can vary.

§ 3. Infinitesimal deformations, infinitesimal derivatives and rigidity under infinitesimal deformations.

Recall that $T_{\boldsymbol{C}}M$ denotes the complexified tangent bundle $TM\otimes\boldsymbol{C}$ of M and that E is the complex vector bundle locally spanned by $T\mathcal{F}$ and the transverse antiholomorphic vectors $\dfrac{\partial}{\partial\bar{z}_i}$. The complex normal bundle is then defined by setting $Q(\mathcal{F})=T_{\boldsymbol{C}}M/E$. Let ∇ be a Bott connection on $Q(\mathcal{F})$.

Definition 4.3.1 (1.4)**.** Define a derivation d_∇ defined on $\Gamma^\infty(\bigwedge^pE^*\otimes Q(\mathcal{F}))$ with values in $\Gamma^\infty(\bigwedge^{p+1}E^*\otimes Q(\mathcal{F}))$ by setting

$$\begin{aligned}
&d_\nabla\sigma(X_0,\dots,X_p)\\
&=\sum_{0\le i\le p}(-1)^i\nabla_{X_i}\sigma(X_0,\dots,\widehat{X_i},\dots,X_p)\\
&\quad+\sum_{0\le i<j\le p}(-1)^{i+j}\sigma([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_p),
\end{aligned}$$

where $\sigma\in\Gamma^\infty(\bigwedge^pE^*\otimes Q(\mathcal{F}))$, $X_i\in\Gamma^\infty(E)$ and the symbol '$\widehat{\ }$' means omission.

A local description of d_∇ is given in [15], where d_∇ is denoted by d_Q. A section σ of $Q(\mathcal{F})$ is said to be foliated and transversely holomorphic if $\mathcal{L}_X\sigma=0$ for $X\in E$. In other words, σ is foliated and transversely holomorphic if σ is locally constant along the leaves and transversely holomorphic. The following fact can be found in the proof of Theorem 1.27 of [15].

Lemma 4.3.2. *Let* $\Theta_{\mathcal{F}}$ *be the sheaf of germs of foliated transversely holomorphic vector fields, then* $d_\nabla\circ d_\nabla=0$ *and*

$$0\longrightarrow\Theta_{\mathcal{F}}\longrightarrow\Gamma^\infty(\textstyle\bigwedge^0E^*\otimes Q(\mathcal{F}))\xrightarrow{d_\nabla}\Gamma^\infty(\textstyle\bigwedge^1E^*\otimes Q(\mathcal{F}))\xrightarrow{d_\nabla}\cdots$$

is a resolution of $\Theta_{\mathcal{F}}$.

We do not give a proof here but simply remark that the equation $d_\nabla \circ d_\nabla = 0$ follows from the fact that $\nabla_{X_i}\nabla_{X_j} - \nabla_{X_j}\nabla_{X_i} - \nabla_{[X_i,X_j]} = 0$ for sections X_i, X_j of E because ∇ is a Bott connection.

Definition 4.3.3. Let $\Theta_{\mathcal{F}}$ be the sheaf of germs of foliated transversely holomorphic vector fields. We denote by $H^*(M;\Theta_{\mathcal{F}})$ the cohomology of $(\varGamma^\infty(\bigwedge^* E^* \otimes Q(\mathcal{F})), d_\nabla)$.

Remark 4.3.4. It is shown in [15] (Theorem 1.27) that $H^*(M;\Theta_{\mathcal{F}})$ is of finite dimension.

The infinitesimal derivative of secondary classes will be given as a mapping from $H^1(M;\Theta_{\mathcal{F}}) \times H^*(\mathrm{WU}_q)$ to $H^*(M;\boldsymbol{C})$. In what follows, we follow the conventions as in [22] but Bott connections on $Q(\mathcal{F})$ instead of $Q(\mathcal{F})^*$ are used.

Let P be the principal bundle associated to $Q(\mathcal{F})^*$ with projection π. Let $\widehat{\mathcal{F}}$ be the lift of $\mathcal{F}$ to P. If ω is an element of P, then ω is a q-tuple of linearly independent elements of $Q(\mathcal{F})^*$ at $\pi(\omega)$. For a vector X in $T_\omega P$, set $\omega(X) = {}^t(\omega^1(\pi_* X), \dots, \omega^q(\pi_* X))$ and call this ω as the canonical form. The differential forms ω^i are considered as components of ω and we denote $\omega = {}^t(\omega^1, \dots, \omega^q)$. Note that the connection form θ of any Bott connection ∇ satisfies $d\omega = -\theta \wedge \omega$ (Here the sign is opposite due to convention when compared with [22]). Let $\Omega = d\theta + \theta \wedge \theta$ be the curvature form of ∇, then $\Omega \wedge \omega = 0$.

Now let $\beta \in H^1(M;\Theta_{\mathcal{F}})$ and let σ' be a representative of β. Such a σ' is a section of $E^* \otimes Q(\mathcal{F})$ with $d_\nabla \sigma' = 0$. By arbitrary extending, σ' can be regarded as a section of $T^*_{\boldsymbol{C}} M \otimes Q(\mathcal{F})$. On the other hand, by pulling back to P and considering the horizontal lifts in value, a section of $\bigwedge^k T^*_{\boldsymbol{C}} M \otimes Q(\mathcal{F})$ is considered as a section of $\bigwedge^k P^* \otimes Q(\widehat{\mathcal{F}})$. Furthermore, a section of $\bigwedge^k P^* \otimes Q(\widehat{\mathcal{F}})$ can be considered as a $\boldsymbol{C}^q$-valued k-form on P by composing with the canonical form ω. A section of $\bigwedge^k P^* \otimes Q(\widehat{\mathcal{F}})$ is *always* considered as a $\boldsymbol{C}^q$-valued k-form in this way and represented in columns. On the contrary, a section σ projects down to a section σ'' of $\bigwedge^k T^*_{\boldsymbol{C}} M \otimes Q(\mathcal{F})$ if and only if

1) σ is horizontal, that is, $\sigma(X_1, \dots, X_k) = 0$ if $\pi_*(X_i) = 0$ for some i,
2) $L_a^* \sigma = a\sigma$ for $a \in \mathrm{GL}(q;\boldsymbol{C})$, where L_a is the left action of $\mathrm{GL}(q;\boldsymbol{C})$ on P.

Thus the above section σ' can be viewed as a section of $P^* \otimes Q(\widehat{\mathcal{F}})$ satisfying the conditions 1), 2) and

3) $d\sigma + \theta \wedge \sigma = 0$ when restricted to $\pi^* E$.

Let $\mathcal{I}(\omega)$ be the ideal generated by $\omega^1, \ldots, \omega^q$ in the space of differential forms on P, then 3) is equivalent to $d\sigma + \theta \wedge \sigma \in \mathcal{I}(\omega)$.

Definition 4.3.5 (Definition 3.8)**.** Let β be an element of $H^1(M; \Theta_{\mathcal{F}})$ and let σ be a representative of β as a $\boldsymbol{C}^q$-valued 1-form on P. The derivative ω' of the canonical form ω with respect to σ is given by

$$\omega' = -\sigma.$$

It follows from the condition 3) above that $d\omega' + \theta \wedge \omega' \in \mathcal{I}(\omega)$. Let θ' be a $\mathfrak{gl}_q\boldsymbol{C}$-valued 1-form on P such that

$$d\omega' + \theta \wedge \omega' = -\theta' \wedge \omega. \tag{4.3.6}$$

Definition 4.3.7 (Definition 3.10)**.** Any $\mathfrak{gl}_q\boldsymbol{C}$-valued 1-form θ' on P satisfying (4.3.6) is called an infinitesimal derivative of θ with respect to σ.

If θ'_0 and θ'_1 are two infinitesimal derivatives of θ with respect to σ, then $(\theta'_1 - \theta'_0) \wedge \omega = 0$. Hence

$$(\theta'_1 - \theta'_0)^i_j = \sum_k \lambda^i_{j,k}\omega^k \tag{4.3.8}$$

for some $\boldsymbol{C}$-valued functions $\lambda^i_{j,k}$ on P satisfying $\lambda^i_{j,k} = \lambda^i_{k,j}$.

Lemma 4.3.9 (Lemma 2.12)**.** *If θ' is an infinitesimal derivative of θ, then*

i) *θ' is horizontal,*
ii) *θ' is tensorial of type* ad *modulo ω, namely, $L_a^*\theta' - a\theta' a^{-1} \in \mathcal{I}(\omega)$.*

Proof. i) Let $X \in T_\omega P$ such that $\pi_* X = 0$, then $\omega(X) = 0$. As ω' is horizontal, one has also $\omega'(X) = 0$. Let $\widetilde{X}$ be a vector field such that $\widetilde{X}_\omega = X$. Now choose vector fields Y_j, $j = 1, \ldots, q$ which are equivariant under the left action and such that $\omega^k((Y_k)_\omega) = 1$ and $\omega^k((Y_j)_\omega) = 0$ if $j \neq k$. Set $\alpha = \omega' - A\omega$, where A is a matrix valued function defined by settling $A = (\omega'(Y_1) \ \ldots \ \omega'(Y_q))$, then α is horizontal. Note that $XA = 0$ because the both Y_j and ω' are equivariant and $\pi_* X = 0$. Then $\alpha(Y) = 0$ and $\omega(Y)$ is the identity matrix. One has by (4.3.6)

$$\begin{aligned} d\alpha &= -\theta \wedge \omega' - \theta' \wedge \omega - dA \wedge \omega + A\theta \wedge \omega \\ &= -\theta \wedge \alpha - \theta' \wedge \omega - dA \wedge \omega + A\theta \wedge \omega - \theta \wedge A\omega. \end{aligned}$$

As $d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) = 0$ because $\pi_*[X,Y] = 0$, the following equation holds, namely,

$$\begin{aligned}
\theta'(X) &= \theta'(X)\omega(Y) \\
&= (\theta' \wedge \omega)(X,Y) \\
&= (-\theta \wedge \alpha - dA \wedge \omega + A\theta \wedge \omega - \theta \wedge A\omega)(X,Y) \\
&= -dA(X) + A\theta(X) - \theta(X)A.
\end{aligned}$$

Since $dA(X) = X(A) = 0$, it suffices to show that $\theta(X) = 0$. This follows from the equations $\theta(X) = \theta(X)\omega(Y) = (\theta \wedge \omega)(X,Y) = -d\omega(X,Y)$ and $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = 0$.
ii) One has $L_a^*\theta = a\theta a^{-1}$, $L_a^*\omega = a\omega$ and $L_a^*\omega' = \omega'$. Applying L_a^* to (4.3.6) one has $-L_a^*\theta' \wedge a\omega = ad\omega' + a\theta a^{-1} \wedge a\omega'$. The right hand side is equal to $-a\theta' a^{-1} \wedge a\omega$ again by (4.3.6). $\square$

Definition 4.3.10. Let $\widetilde{\varphi} \in \widetilde{\mathrm{WU}}_q$ be a lift of a cocycle φ in WU_q and let $\beta \in H^1(M;\Theta_{\mathcal{F}})$ be represented by σ. Let θ^u be a unitary connection for some Hermitian metric on $Q(\mathcal{F})$ and let θ be a Bott connection. Let Ω be the curvature form of θ, and let θ' be an infinitesimal derivative of θ with respect to σ. Define a differential form on P then by setting

$$D_\sigma(\widetilde{\varphi}) = \Delta\widetilde{\varphi}(\theta^u, \theta, \theta'),$$

where the right hand side is defined in Definition 4.2.4.

We will show that $D_\sigma(\widetilde{\varphi})$ projects down to a closed form on M, and that its cohomology class depends only on $[\varphi] \in H^*(\mathrm{WU}_q)$ and $\beta \in H^1(M;\Theta_{\mathcal{F}})$. Then the following definition is justified.

Definition 4.3.11. For $f \in H^*(\mathrm{WU}_q)$ and $\beta \in H^1(M;\Theta_{\mathcal{F}})$, choose representatives φ of f and σ of β. Set $D_\beta(f) = [D_\sigma(\widetilde{\varphi})]$, where $\widetilde{\varphi}$ is any lift of φ to $\widetilde{\mathrm{WU}}_q$, and call it the infinitesimal derivative of f with respect to β.

Remark 4.3.12. If $\varphi = \widetilde{u}_{i_1} v_{i_2} \dots v_{i_k} + \bar{v}_{i_1} \widetilde{u}_{i_2} v_{i_3} \dots v_{i_k} + \dots + \bar{v}_{i_1} \dots \bar{v}_{i_{k-1}} \widetilde{u}_{i_k}$, then $D_\beta(\widetilde{\varphi})$ coincides with the one given by Definition 3.14 of [22], where this kind of φ is denoted by hc_I.

Remark 4.3.13. Taking (4.2.5b) into account, Definition 4.3.11 can be seen as a complex version of (2.15) in [22]. Indeed, if one begins with the cocycles of the form $h_I c_J \in \mathrm{WO}_q$, the same differential forms are obtained by following the construction in this paper.

Proof of Theorem B2. Once well-definedness is established, the theorem follows from Definition 4.3.10 by using (4.2.5b). $\square$

We come back to show the well-definedness of infinitesimal derivatives.

Lemma 4.3.14 (Theorem 3.17)**.** *The differential form $D_\sigma(\widetilde{\varphi})$ in Definition 4.3.10 projects down to a well-defined closed form on M which depends on σ, θ, θ^u and the choice of the lift $\widetilde{\varphi}$.*

Proof. (a) $D_\sigma(\widetilde{\varphi})$ is independent of the choice of θ'.

Let θ'_0 and θ'_1 be infinitesimal derivatives of θ with respect to σ and let g be a monomial in $v_1, \ldots, v_q$ and $\bar{v}_1, \ldots, \bar{v}_q$. Since $\theta'_1 - \theta'_0 = \lambda\omega$ by (4.3.8), $g(\theta'_1, \Omega) - g(\theta'_0, \Omega) = g(\lambda\omega, \Omega)$. As $\widetilde{\varphi}$ is a lift of a cocycle, $\widetilde{d}\widetilde{\varphi}$ is the linear combination of cochains in $\widetilde{\mathcal{I}}_q$. It follows that $\Delta\widetilde{\varphi}(\theta^u, \theta, \theta'_1) - \Delta\widetilde{\varphi}(\theta^u, \theta, \theta'_0) \in \mathcal{I}(\omega)^{q+1} \cup \overline{\mathcal{I}(\omega)}^{q+1} = \{0\}$.

(b) $D_\sigma(\widetilde{\varphi})$ projects down to M.

It suffices to show $v_J\bar{v}_K(v_{i_l} - \bar{v}_{i_l})(\theta', \Omega)$ projects down to M. We have

$$\begin{aligned} &L_a^*\left(v_J\bar{v}_K(v_{i_l} - \bar{v}_{i_l})(\theta', \Omega)\right) - v_J\bar{v}_K(v_{i_l} - \bar{v}_{i_l})(\theta', \Omega) \\ &= v_J\bar{v}_K(v_{i_l} - \bar{v}_{i_l})(L_a^*\theta', a\Omega a^{-1}) - v_J\bar{v}_K(v_{i_l} - \bar{v}_{i_l})(a\theta' a^{-1}, a\Omega a^{-1}) \\ &= v_J\bar{v}_K(v_{i_l} - \bar{v}_{i_l})(L_a^*\theta' - a\theta' a^{-1}, a\Omega a^{-1}). \end{aligned}$$

It follows that $L_a^*\Delta\widetilde{\varphi}(\theta^u, \theta, \theta') = \Delta\widetilde{\varphi}(\theta^u, \theta, \theta')$ from Lemma 4.3.9 ii) and an argument as in the proof of (a).

(c) $D_\sigma(\widetilde{\varphi})$ is closed.

Recall that $\widetilde{d}\widetilde{\varphi}$ is the linear combination of cochains of the form $\widetilde{u}_I v_J \bar{v}_K$ with $|J| > q$ or $|K| > q$. Since $D_\sigma(\widetilde{\varphi}) = \Delta\widetilde{\varphi}(\theta^u, \theta, \theta') = \delta(\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \theta')$ and since $\widetilde{d}(\widetilde{d}\widetilde{\varphi}) = 0$, $D_\sigma(\widetilde{\varphi})$ is closed by the following Lemma 4.3.16. This completes the proof of Lemma 4.3.14. $\square$

Before completing the proof of Lemma 4.3.14 by giving Lemma 4.3.16, we introduce the following differential form.

Definition 4.3.15. Let θ_0^u and θ_1^u be unitary connections (not necessarily with respect to the same Hermitian metric), and let $\widetilde{u}_I v_J \bar{v}_K \in \widetilde{\mathrm{WU}}_q$. Decompose $I = I_1 \cup I_2$ so that I_1 consists only of indices less than or equal to i, and I_2 consists only of indices greater than i, then set $\widetilde{u}_I^{(i)}(\theta, \theta_0^u, \theta_1^u) = \widetilde{u}_{I_1}(\theta, \theta_0^u)\widetilde{u}_{I_2}(\theta, \theta_1^u)$. Finally, set

$$\delta_i(\widetilde{u}_I v_J \bar{v}_K)(\theta_0^u, \theta_1^u, \theta, \theta') = (|J| + |K|) v_J \bar{v}_K(\theta', \Omega)\widetilde{u}_I^{(i)}(\theta, \theta_0^u, \theta_1^u).$$

We extend δ_i to the whole $\widetilde{\mathrm{WU}}_q$ by linearity.

Lemma 4.3.16. *Let $\widetilde{\varphi} \in \widetilde{\mathrm{WU}}_q$ such that $\widetilde{d}\widetilde{\varphi} = 0$. Assume that $\widetilde{\varphi} \in \mathcal{I}_q \cup \overline{\mathcal{I}_q}$, then $\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')$ is closed. In particular, $v_J\bar{v}_K(\theta', \Omega)$ is closed if $|J| > q$ or $|K| > q$.*

Proof. Assume first that $|J| > q$ and show that $v_J(\theta', \Omega)$ is closed. Write $\Omega_j^i = \sum\limits_k \Gamma_{j,k}^i \wedge \omega^k$ and set $\Omega_j'^i = \sum\limits_k \Gamma_{j,k}^i \wedge \omega'^k$, then $\Omega \wedge \omega' = -\Omega' \wedge \omega$. On the other hand, one obtains from the exterior derivative of (4.3.6), the equations $\Omega = d\theta + \theta \wedge \theta$, $d\omega = -\theta \wedge \omega$ and (4.3.6) itself the equation

$$-\Omega \wedge \omega' = (d\theta' + [\theta, \theta']) \wedge \omega,$$

where $[\theta, \theta'] = \theta \wedge \theta' + \theta' \wedge \theta$. Hence $v_J(d\theta' + [\theta, \theta'], \Omega) = v_J(\Omega', \Omega)$.

Let $\mathcal{I}_s(\omega)$ be the ideal of differential forms on P generated by $\omega^1 + s\omega'^1, \ldots, \omega^q + s\omega'^q$, then $\mathcal{I}_s(\omega)^{q+1} = \{0\}$ independent of s. Set $\Omega(s) = \Omega + s\Omega'$, then $\Omega(s) \in \mathcal{I}_s(\omega)$ because $(\Omega(s))_j^i = \sum\limits_k \Gamma_{j,k}^i \wedge (\omega^k + s\omega'^k)$. One has now the following equation;

$$\begin{aligned}
d\left(v_J(\theta', \Omega)\right) &= v_J(d\theta', \Omega) - (|J| - 1)v_J(\theta', d\Omega, \Omega) \\
&= v_J(d\theta', \Omega) + (|J| - 1)v_J(\theta', [\theta, \Omega], \Omega) \\
&= v_J(d\theta' + [\theta, \theta'], \Omega) \\
&= v_J(\Omega', \Omega) \\
&= \frac{1}{|J|} \frac{\partial}{\partial s}\bigg|_{s=0} v_J(\Omega(s)) \\
&= 0.
\end{aligned}$$

The last equality holds because $v_J(\Omega(s))$ is identically zero. On the other hand, by (4.2.5b),

$$(|J| + |K|)v_J\bar{v}_K(\theta', \Omega) = |J|\, v_J(\theta', \Omega)\bar{v}_K(\Omega) + |K|\, v_J(\Omega)\bar{v}_K(\theta', \Omega)$$

and $v_J(\Omega) = 0$ because $|J| > q$. Hence $v_J\bar{v}_K(\theta', \Omega)$ is closed. Similarly, $v_J\bar{v}_K(\theta', \Omega)$ is also closed if $|K| > q$.

Assume now that $\widetilde{\varphi} = \sum\limits_t x_t \widetilde{u}_{I_t} v_{J_t} \bar{v}_{K_t}$, where $x_t \in \boldsymbol{C}$. We may assume that the number of elements of I_t are constant, which is denoted by $\#I$. If $\#I = 0$, then $\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')$ is already shown to be closed. If $\#I > 0$, then one has

$$\begin{aligned}
&d\left(\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')\right) \\
&= \sum_{t,l} (-1)^l x_t(|J_t| + |K_t|) v_{J_t}\bar{v}_{K_t}(\theta', \Omega)(v_{i_l} - \bar{v}_{i_l})(\Omega)\widetilde{u}_{I_t(l)}^{(i)}(\theta, \theta_0^u, \theta_1^u).
\end{aligned}$$

Since $|J_t| > q$ or $|K_t| > q$, $v_{J_t}(\theta', \Omega) v_{i_l}(\Omega) \bar{v}_{K_t}(\Omega) = v_{J_t}(\Omega) \bar{v}_{K_t}(\theta', \Omega) \bar{v}_{i_l}(\Omega) = 0$. Hence

$$\begin{aligned} d\left(\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')\right) = & \sum_{t,l} -(-1)^l x_t \, |J_t| \, v_{J_t}(\theta', \Omega) \bar{v}_{K_t}(\Omega) \bar{v}_{i_l}(\Omega) \widetilde{u}^{(i)}_{I_t(l)}(\theta^u, \theta) \\ & + \sum_{t,l} (-1)^l x_t \, |K_t| \, v_{J_t}(\Omega) v_{i_l}(\Omega) \bar{v}_{K_t}(\theta', \Omega) \widetilde{u}^{(i)}_{I_t(l)}(\theta^u, \theta). \end{aligned}$$

Now by (4.2.5a) and (4.2.5b),

$$\begin{aligned} &(|J_t| + |K_t| + i_l) v_{J_t} \bar{v}_{K_t} \bar{v}_{i_l}(\theta', \Omega) = |J_t| \, v_{J_t}(\theta', \Omega) \bar{v}_{K_t} \bar{v}_{i_l}(\Omega), \text{ and} \\ &(|J_t| + |K_t| + i_l) v_{J_t} v_{i_l} \bar{v}_{K_t}(\theta', \Omega) = |K_t| \, v_{J_t} v_{i_l}(\Omega) \bar{v}_{K_t}(\theta', \Omega). \end{aligned}$$

Thus

$$\begin{aligned} d\left(\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')\right) = & \sum_{t,l} -(-1)^l x_t (|J_t| + |K_t| + i_l) v_{J_t} \bar{v}_{K_t} \bar{v}_{i_l}(\theta', \Omega) \widetilde{u}^{(i)}_{I_t(l)}(\theta^u, \theta) \\ & + \sum_{t,l} (-1)^l x_t (|J_t| + |K_t| + i_l) v_{J_t} v_{i_l} \bar{v}_{K_t}(\theta', \Omega) \widetilde{u}^{(i)}_{I_t(l)}(\theta^u, \theta) \\ = & \, \delta_i(\widetilde{d\widetilde{\varphi}})(\theta_0^u, \theta_1^u, \theta, \theta') \\ = & \, 0 \end{aligned}$$

because $\widetilde{d\widetilde{\varphi}} = 0$. This completes the proof of Lemma 4.3.16. $\square$

Theorem 4.3.17 (cf. Theorem 3.17)**.** *For $f \in H^*(\mathrm{WU}_q)$ and $\beta \in H^1(M; \Theta_{\mathcal{F}})$, choose representatives φ of f and σ of β. Let $\widetilde{\varphi}$ be any lift of φ to $\widetilde{\mathrm{WU}}_q$. Then the cohomology class $[D_\sigma(\widetilde{\varphi})]$ is independent of the choice of representatives and lifts.*

Proof. Let θ^u, θ, θ', Ω be as in Definition 4.3.10.
(a) $[D_\sigma(\widetilde{\varphi})]$ is independent of the choice of the Bott connection θ.
Let θ_0 and θ_1 be Bott connections and choose their infinitesimal derivatives θ_0' and θ_1' with respect to σ. Note that $D_\sigma(\widetilde{\varphi})$ is independent of the choice of infinitesimal derivatives by Lemma 4.3.14. Set $\theta_t = \theta_0 + t(\theta_1 - \theta_0)$, then θ_t is also a Bott connection and one of its infinitesimal derivatives is given by $\theta_t' = \theta_0' + t(\theta_1' - \theta_0')$. Let Ω_t be the connection form of θ_t, and we will show that $\dfrac{\partial}{\partial t} \Delta\widetilde{\varphi}(\theta^u, \theta_t, \theta_t')$ is exact. Recalling that $\Delta\widetilde{\varphi}(\theta^u, \theta_t, \theta_t')$ is calculated by evaluating $\widetilde{d\widetilde{\varphi}} \in \widetilde{\mathcal{I}}_q$, first we show the claim when $\widetilde{d\widetilde{\varphi}}$ does not involve any $\widetilde{u}_i$. One has

$$\begin{aligned} d\left(v_J \bar{v}_K(\theta_t', \theta_1 - \theta_0, \Omega_t)\right) = & \, v_J \bar{v}_K(d\theta_t', \theta_1 - \theta_0, \Omega_t) - v_J \bar{v}_K(\theta_t', d\theta_1 - d\theta_0, \Omega_t) \\ & - (|J| + |K| - 2) v_J \bar{v}_K(\theta_t', \theta_1 - \theta_0, [\theta_t, \Omega_t], \Omega_t) \\ = & \, v_J \bar{v}_K(d\theta_t' + [\theta_t, \theta_t'], \theta_1 - \theta_0, \Omega_t) \\ & - v_J \bar{v}_K(\theta_t', d\theta_1 - d\theta_0 + [\theta_t, \theta_1 - \theta_0], \Omega_t). \end{aligned}$$

Note that each of the differential forms in the above equation projects down to M. On the other hand,

$$\begin{aligned}&\frac{\partial}{\partial t} v_J \bar{v}_K(\theta'_t, \Omega_t)\\ &= v_J \bar{v}_K(\theta'_1 - \theta'_0, \Omega_t) + (|J| + |K| - 1) v_J \bar{v}_K(\theta'_t, d(\theta_1 - \theta_0) + [\theta_t, \theta_1 - \theta_0], \Omega_t).\end{aligned}$$

Hence

$$\begin{aligned}&\frac{\partial}{\partial t} v_J \bar{v}_K(\theta'_t, \Omega_t) + (|J| + |K| - 1) d\left(v_J \bar{v}_K(\theta'_t, \theta_1 - \theta_0, \Omega_t)\right)\\ &= v_J \bar{v}_K(\theta'_1 - \theta'_0, \Omega_t) + (|J| + |K| - 1) v_J \bar{v}_K(d\theta'_t + [\theta_t, \theta'_t], \theta_1 - \theta_0, \Omega_t).\end{aligned}$$

As in the proof of Lemma 4.3.16, write $(\Omega_t)^i_j = \sum_k \Gamma^i_{j,k} \wedge \omega^k$ and set $(\Omega'_t)^i_j = \sum_k \Gamma^i_{j,k} \wedge \omega'^k$. Then $\Omega'_t \wedge \omega = (d\theta'_t + [\theta_t, \theta'_t]) \wedge \omega$. Since $\theta_0 \wedge \omega = \theta_1 \wedge \omega = -d\omega$, $(\theta_1 - \theta_0) \wedge \omega = 0$. Hence $(\theta_1 - \theta_0)^i_j = \sum_k \lambda^i_{j,k} \omega^k$ for some $\lambda^i_{j,k}$. Now by (4.3.6) one has $(\theta'_1 - \theta'_0) \wedge \omega = -(\lambda\omega) \wedge \omega' = (\lambda\omega') \wedge \omega$. Set $\Omega(s,t) = \Omega_t + s\Omega'_t$, $\theta(s) = (\theta_1 - \theta_0) + s\lambda\omega'$ and $\mathcal{I}_s(\omega) = \mathcal{I}(\omega^1 + s\omega'^1, \ldots, \omega^q + s\omega'^q)$, then $\Omega(s,t), \theta(s) \in \mathcal{I}_s(\omega)$. Thus $v_J \bar{v}_K(\theta(s), \Omega(s,t)) = 0$ if $|J| > q$ or $|K| > q$. Differentiating with respect to s and setting $s = 0$, one obtains

$$v_J \bar{v}_K(\lambda\omega', \Omega_t) + (|J| + |K| - 1) v_J \bar{v}_K(\theta_1 - \theta_0, \Omega'_t, \Omega_t) = 0.$$

As the left hand side is equal to $v_J \bar{v}_K((\theta'_1 - \theta'_0, \Omega_t) + (|J| + |K| - 1) v_J \bar{v}_K(\theta_1 - \theta_0, d\theta'_t + [\theta_t, \theta'_t], \Omega_t)$,

$$\frac{\partial}{\partial t} v_J \bar{v}_K(\theta'_t, \Omega_t) = -(|J| + |K| - 1) d\left(v_J \bar{v}_K(\theta'_t, \theta_1 - \theta_0, \Omega_t)\right)$$

if $|J| > q$ or $|K| > q$.

If $\widetilde{d}\widetilde{\varphi}$ involves some of $\widetilde{u}_i$'s, write $\widetilde{d}\widetilde{\varphi} = \sum_i x_i v_{J_i} \bar{v}_{K_i} \widetilde{u}_{I_i}$, where $|J_i| > q$ or $|K_i| > q$, and $x_i \in \boldsymbol{C}$. By definition,

$$\Delta\widetilde{\varphi}(\theta^u, \theta_t, \theta'_t) = \sum_i x_i (|J_i| + |K_i|) v_{J_i} \bar{v}_{K_i}(\theta'_t, \Omega_t) \widetilde{u}_{I_i}(\theta_t, \theta^u).$$

Hence

$$\begin{aligned}&\frac{\partial}{\partial t} \Delta\widetilde{\varphi}(\theta^u, \theta_t, \theta'_t)\\ &= -\sum_i x_i (|J_i| + |K_i|)(|J_i| + |K_i| - 1) d\left(v_{J_i} \bar{v}_{K_i}(\theta'_t, \theta_1 - \theta_0, \Omega_t)\right) \widetilde{u}_{I_i}(\theta_t, \theta^u)\\ &\quad + \sum_{i,l} x_i (|J_i| + |K_i|) v_{J_i} \bar{v}_{K_i}(\theta'_t, \Omega_t) (-1)^{l-1} i_l (i_l - 1) d\widetilde{V}_{i_l}(\theta_t, \theta^u) \widetilde{u}_{I_i(l)}(\theta_t, \theta^u)\\ &\quad + \sum_{i,l} x_i (|J_i| + |K_i|) v_{J_i} \bar{v}_{K_i}(\theta'_t, \Omega_t) (-1)^{l-1} i_l \widetilde{v}_{i_l}(\theta_1 - \theta_0, \Omega_t) \widetilde{u}_{I_i(l)}(\theta_t, \theta^u),\end{aligned}$$

where $I_i(l) = I_i \setminus \{i_l\}$. Fix now an integer k and rewrite $\widetilde{d}\widetilde{\varphi}$ as $\widetilde{d}\widetilde{\varphi} = \widetilde{u}_k\alpha_k + \beta_k$ so that α_k and β_k do not involve $\widetilde{u}_k$, then $\widetilde{d}(\widetilde{d}\widetilde{\varphi}) = 0$ implies that $\widetilde{d}\alpha_k = 0$. Hence

$$\begin{aligned}
&\sum_{\substack{i,l\\ i_l=k}} x_i(|J_i| + |K_i|)v_{J_i}\bar{v}_{K_i}(\theta'_t, \Omega_t)(-1)^{l-1}i_l(i_l - 1)d\widetilde{V}_{i_l}(\theta_t, \theta^u)\widetilde{u}_{I_i(l)}(\theta_t, \theta^u)\\
=&\, k(k-1)d(\widetilde{V}_k(\theta_t, \theta^u)\delta(\alpha_k)(\theta^u, \theta_t, \theta'_t))
\end{aligned}$$

because $\delta(\alpha_k)(\theta^u, \theta_t, \theta'_t)$ is closed by Lemma 4.3.16. Thus $\dfrac{\partial}{\partial t}\Delta\widetilde{\varphi}(\theta^u, \theta_t, \theta'_t)$ is cohomologous to R, where

$$\begin{aligned}
R = &-\sum_i x_i(|J_i| + |K_i|)(|J_i| + |K_i| - 1)d\left((v_{J_i}\bar{v}_{K_i}(\theta'_t, \theta_1 - \theta_0, \Omega_t))\right)\widetilde{u}_{I_i}(\theta_t, \theta^u)\\
&+\sum_{i,l} x_i(|J_i| + |K_i|)v_{J_i}\bar{v}_{K_i}(\theta'_t, \Omega_t)(-1)^{l-1}i_l\widetilde{v}_{i_l}(\theta_1 - \theta_0, \Omega_t)\widetilde{u}_{I_i(l)}(\theta_t, \theta^u).
\end{aligned}$$

It suffices to show that R is exact. This is indeed done as follows, namely, by (A.6b) one has the following equation;

$$\begin{aligned}
&-(|J_i| + |K_i|)(|J_i| + |K_i| - 1)d\left(v_{J_i}\bar{v}_{K_i}(\theta'_t, \theta_1 - \theta_0, \Omega_t)\right)\widetilde{u}_{I_i}(\theta_t, \theta^u)\\
\equiv&(|J_i| + |K_i|)(|J_i| + |K_i| - 1)v_{J_i}\bar{v}_{K_i}(\theta'_t, \theta_1 - \theta_0, \Omega_t)d\widetilde{u}_{I_i}(\theta_t, \theta^u)\\
=&\sum_l(|J_i| + |K_i|)(|J_i| + |K_i| - 1)v_{J_i}\bar{v}_{K_i}(\theta'_t, \theta_1 - \theta_0, \Omega_t)(-1)^{l-1}(v_{i_l} - \bar{v}_{i_l})(\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t, \theta^u)\\
=&-\sum_l |J_i|\left(|J_i| - 1\right)v_{J_i}(\theta'_t, \theta_1 - \theta_0, \Omega_t)\bar{v}_{K_i}(\Omega_t)(-1)^{l-1}\bar{v}_{i_l}(\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t, \theta^u)\\
&-\sum_l |J_i|\,|K_i|\,v_{J_i}(\theta'_t, \Omega_t)\bar{v}_{K_i}(\theta_1 - \theta_0, \Omega_t)(-1)^{l-1}\bar{v}_{i_l}(\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t, \theta^u)\\
&-\sum_l |J_i|\,|K_i|\,v_{J_i}(\theta_1 - \theta_0, \Omega_t)v_{i_l}(\Omega_t)\bar{v}_{K_i}(\theta'_t, \Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t, \theta^u)\\
&+\sum_l |K_i|\left(|K_i| - 1\right)v_{J_i}(\Omega_t)v_{i_l}(\Omega_t)\bar{v}_{K_i}(\theta'_t, \theta_1 - \theta_0, \Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t, \theta^u),
\end{aligned}$$

where the symbol '$\equiv$' means that the equality holds modulo exact forms. On the other hand,

$$\begin{aligned}
&(|J_i| + |K_i|)v_{J_i}\bar{v}_{K_i}(\theta'_t, \Omega_t)(-1)^{l-1}i_l\widetilde{v}_{i_l}(\theta_1 - \theta_0, \Omega_t)\widetilde{u}_{I_i(l)}(\theta_t, \theta^u)\\
=&-|J_i|\,v_{J_i}(\theta'_t, \Omega_t)\bar{v}_{K_i}(\Omega_t)(-1)^{l-1}i_l\bar{v}_{i_l}(\theta_1 - \theta_0, \Omega_t)\widetilde{u}_{I_i(l)}(\theta_t, \theta^u)\\
&-|K_i|\,v_{J_i}(\Omega_t)v_{i_l}(\theta_1 - \theta_0, \Omega_t)\bar{v}_{K_i}(\theta'_t, \Omega_t)(-1)^{l-1}i_l\widetilde{u}_{I_i(l)}(\theta_t, \theta^u).
\end{aligned}$$

Therefore, one has

$$
\begin{aligned}
R \equiv & -\sum_{i,l} x_i \left|J_i\right| (\left|J_i\right|-1) v_{J_i}(\theta_t', \theta_1-\theta_0, \Omega_t) \bar{v}_{K_i}(\Omega_t) \bar{v}_{i_l}(\Omega_t)(-1)^{l-1} \widetilde{u}_{I_i(l)}(\theta_t, \theta^u) \\
& -\sum_{i,l} x_i \left|J_i\right| (\left|K_i\right|+i_l) v_{J_i}(\theta_t', \Omega_t) \bar{v}_{K_i} \bar{v}_{i_l}(\theta_1-\theta_0, \Omega_t)(-1)^{l-1} \widetilde{u}_{I_i(l)}(\theta_t, \theta^u) \\
& -\sum_{i,l} x_i (\left|J_i\right|+i_l) \left|K_i\right| v_{J_i} v_{i_l}(\theta_1-\theta_0, \Omega_t) \bar{v}_{K_i}(\theta_t', \Omega_t)(-1)^{l-1} \widetilde{u}_{I_i(l)}(\theta_t, \theta^u) \\
& +\sum_{i,l} x_i \left|K_i\right| (\left|K_i\right|-1) v_{J_i}(\Omega_t) v_{i_l}(\Omega_t) \bar{v}_{K_i}(\theta_t', \theta_1-\theta_0, \Omega_t)(-1)^{l-1} \widetilde{u}_{I_i(l)}(\theta_t, \theta^u).
\end{aligned}
$$

Denote by R' the right hand side of the above equation. Now by (A.6b),

$$
\begin{aligned}
& (\left|J_i\right|+\left|K_i\right|+i_l) v_{J_i} v_{i_l} \bar{v}_{K_i}(\theta_t', \theta_1-\theta_0, \Omega_t) \\
= & -(\left|J_i\right|+i_l) \left|K_i\right| v_{J_i} v_{i_l}(\theta_1-\theta_0, \Omega_t) \bar{v}_{K_i}(\theta_t', \Omega_t) \\
& +\left|K_i\right| (\left|K_i\right|-1) v_{J_i} v_{i_l}(\Omega_t) \bar{v}_{K_i}(\theta_t', \theta_1-\theta_0, \Omega_t),
\end{aligned}
$$

and

$$
\begin{aligned}
& (\left|J_i\right|+\left|K_i\right|+i_l) v_{J_i} \bar{v}_{K_i} \bar{v}_{i_l}(\theta_t', \theta_1-\theta_0, \Omega_t) \\
= & \left|J_i\right| (\left|J_i\right|-1) v_{J_i}(\theta_t', \theta_1-\theta_0, \Omega_t) \bar{v}_{K_i} \bar{v}_{i_l}(\Omega_t) \\
& +\left|J_i\right| (\left|K_i\right|+i_l) v_{J_i}(\theta_t', \Omega_t) \bar{v}_{K_i} \bar{v}_{i_l}(\theta_1-\theta_0, \Omega_t).
\end{aligned}
$$

Thus

$$
\begin{aligned}
R' = & \sum_{i,l} x_i (\left|J_i\right|+\left|K_i\right|+i_l) v_{J_i} v_{i_l} \bar{v}_{K_i}(\theta_t', \theta_1-\theta_0, \Omega_t)(-1)^{l-1} \widetilde{u}_{I_i(l)}(\theta_t, \theta^u) \\
& -\sum_{i,l} x_i (\left|J_i\right|+\left|K_i\right|+i_l) v_{J_i} \bar{v}_{K_i} \bar{v}_{i_l}(\theta_t', \theta_1-\theta_0, \Omega_t)(-1)^{l-1} \widetilde{u}_{I_i(l)}(\theta_t, \theta^u) \\
= & \, \delta(\widetilde{d}\widetilde{d}\widetilde{\varphi})(\theta^u, \theta_t, \theta_t') \\
= & \, 0.
\end{aligned}
$$

This completes the proof of (a).
(b) $[D_\sigma(\widetilde{\varphi})]$ is independent of the choice of the unitary connection θ^u.

We first admit the fact that $\widetilde{u}_i(\theta, \theta_1^u) - \widetilde{u}_i(\theta, \theta_0^u) = d\widetilde{V}_i'$ for some differential form $\widetilde{V}_i'$ for two unitary connections θ_0^u and θ_1^u. Let $\widetilde{\varphi} \in \widetilde{\mathrm{WU}}_q$ be the natural lift of φ and define α_i and β_i by requiring that $\widetilde{d\varphi} = \widetilde{u}_i\alpha_i + \beta_i$ and that α_i and β_i do not involve $\widetilde{u}_i$. Then $\Delta\widetilde{\varphi}(\theta_1^u, \theta, \theta') = \delta_0(\widetilde{d\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')$ and $\Delta\widetilde{\varphi}(\theta_1^u, \theta, \theta') =$

$\delta_q(\widetilde{d\widetilde{\varphi}})(\theta_0^u, \theta_1^u, \theta, \theta')$. Thus it suffices to show that $\delta_{k-1}(\widetilde{d\widetilde{\varphi}})(\theta_0^u, \theta_1^u, \theta, \theta')$ is cohomologous to $\delta_k(\widetilde{d\widetilde{\varphi}})(\theta_0^u, \theta_1^u, \theta, \theta')$ for each k. Since β_k does not involve $\widetilde{u}_k$,

$$
\begin{aligned}
&\delta_{k-1}(\widetilde{d\widetilde{\varphi}})(\theta_0^u, \theta_1^u, \theta, \theta') \\
=& -\widetilde{u}_k(\theta, \theta_1^u)\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') + \delta_{k-1}(\beta_k)(\theta_0^u, \theta_1^u, \theta, \theta') \\
=& -\widetilde{u}_k(\theta, \theta_1^u)\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') + \delta_k(\beta_k)(\theta_0^u, \theta_1^u, \theta, \theta').
\end{aligned}
$$

On the other hand, $\widetilde{d}\widetilde{\alpha}_i = 0$ because $\widetilde{d}\widetilde{d\widetilde{\varphi}} = 0$. It follows that $d\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') = 0$ by Lemma 4.3.16. Hence

$$
\begin{aligned}
&\delta_{k-1}(\widetilde{d\widetilde{\varphi}})(\theta_0^u, \theta_1^u, \theta, \theta') + d\left(\widetilde{V}_k'\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta')\right) \\
=& -\widetilde{u}_k(\theta, \theta_0^u)\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') + \delta_k(\beta_k)(\theta_0^u, \theta_1^u, \theta, \theta') \\
=& -\widetilde{u}_k(\theta, \theta_0^u)\delta_k(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') + \delta_k(\beta_k)(\theta_0^u, \theta_1^u, \theta, \theta') \\
=& \,\delta_k(\widetilde{d\widetilde{\varphi}})(\theta_0^u, \theta_1^u, \theta, \theta')
\end{aligned}
$$

because α_k does not involve $\widetilde{u}_k$.

Thus it suffices to find a differential form $\widetilde{V}_i'$ such that $\widetilde{u}_i(\theta, \theta_1^u) - \widetilde{u}_i(\theta, \theta_0^u) = d\widetilde{V}_i'$ for each i. First fix a Hermitian metric on $Q(\mathcal{F})$ and let θ_0^u and θ_1^u be unitary connections. Apply (4.2.2a) after setting $f = \widetilde{v}_i = v_i - \bar{v}_i$, $\theta_1^s = \theta + s(\theta_1^u - \theta)$ and $\theta_0 = \theta_0^u$, then integrating it with respect to s, one obtains the equation

$$\Delta_{\widetilde{v}_i}(\theta_1^u, \theta_0^u) - \Delta_{\widetilde{v}_i}(\theta, \theta_0^u) = k(k-1)dW_{\widetilde{v}_i}(\theta_1^u, \theta_0^u) + \Delta_{\widetilde{v}_i}(\theta_1^u, \theta),$$

where $W_{\widetilde{v}_i}(\theta_1^u, \theta_0^u) = \displaystyle\int_0^1 V_{\widetilde{v}_i}(\theta_1^s, \theta_0^u)ds$. Hence

$$\Delta_{\widetilde{v}_i}(\theta, \theta_1^u) - \Delta_{\widetilde{v}_i}(\theta, \theta_0^u) + \Delta_{\widetilde{v}_i}(\theta_1^u, \theta_0^u) = k(k-1)dW_{\widetilde{v}_i}(\theta_1^u, \theta_0^u). \tag{4.3.18}$$

Set $\theta_t^u = \theta_0^u + t(\theta_1^u - \theta_0^u)$, then by (4.2.2a),

$$\frac{\partial}{\partial t}\Delta_{\widetilde{v}_i}(\theta_t^u, \theta_0^u) = k(k-1)dV_{\widetilde{v}_i}(\theta_t^u, \theta_0) + k\widetilde{v}_i(\theta_1^u - \theta_0^u, \Omega_t^u, \ldots, \Omega_t^u).$$

Since θ_0^u and θ_1^u are unitary and since $V_{\widetilde{v}_i} = \widetilde{V}_i$,

$$\frac{\partial}{\partial t}\Delta_{\widetilde{v}_i}(\theta_t^u, \theta_0^u) = k(k-1)d\widetilde{V}_i(\theta_t^u, \theta_0).$$

Hence $\widetilde{u}_i(\theta, \theta_1^u) - \widetilde{u}_i(\theta, \theta_0^u)$ is exact if θ_0^u and θ_1^u are unitary connections for a fixed Hermitian metric.

Let now h_0 and h_1 be Hermitian metrics on $Q(\mathcal{F})$ and let θ_0^u and θ_1^u be unitary connections for h and h', respectively. The equation (4.3.18) is also valid so that it suffices to show that $\Delta_f(\theta_1^u,\theta_0^u)$ is exact for $f=\widetilde{v}_i=v_i-\bar{v}_i$. Denote by ι_t the natural isomorphism from M to $M\times\{t\}$ and by π the projection from $M\times\boldsymbol{R}$ to $\boldsymbol{R}$. Consider then the foliation $\mathcal{F}\times\boldsymbol{R}$ of $M\times\boldsymbol{R}$ whose leaves are given by $L\times\boldsymbol{R}$, where L is a leaf of $\mathcal{F}$. Let $\widetilde{\theta}^u$ be a unitary connection on $Q(\mathcal{F}\times\boldsymbol{R})$ for some Hermitian metric such that $\theta_t^u=\theta_0^u$ for $t\le 0$ and $\theta_t^u=\theta_1^u$ for $t\ge 1$, where $\theta_t^u=\iota_t^*\widetilde{\theta}^u$. Now write $\Delta_{\widetilde{v}_i}(\widetilde{\theta}^u,\pi^*\theta_0^u)=\lambda+\mu\wedge dt$, where λ, μ do not involve dt, and define a differential form $\widetilde{V}_i'(\theta_1^u,\theta_0^u)$ on M by setting

$$\widetilde{V}_i'(\theta_1^u,\theta_0^u)=-\int_0^1\mu dt,$$

then $d\widetilde{V}_i'(\theta_1^u,\theta_0^u)=\Delta_{\widetilde{v}_i}(\theta_1^u,\theta_0^u)$. This is shown as follows. First,

$$d_{M\times\boldsymbol{R}}\Delta_{\widetilde{v}_i}(\widetilde{\theta}^u,\pi^*\theta_0)=\Big(v_i(\widetilde{\theta}^u)-\pi^*v_i(\theta_0^u)\Big)-\Big(\bar{v}_i(\widetilde{\theta}^u)-\pi^*\bar{v}_i(\theta_0^u)\Big)=0.$$

Hence $\dfrac{\partial\lambda}{\partial t}+d_M\mu=0$, where d_M denotes the exterior derivative along the fiber of $\pi:M\times\boldsymbol{R}\to\boldsymbol{R}$. On the other hand, one has

$$d\widetilde{V}_i'(\theta_1^u,\theta_0^u)=-\int_0^1 d_M\mu dt=\int_0^1\frac{\partial\lambda}{\partial t}dt=\lambda(1)-\lambda(0)$$

and $\lambda(t)=\iota_t^*\lambda=\iota_t^*\Delta_{\widetilde{v}_i}(\widetilde{\theta}^u,\pi^*\theta_0)=\Delta_{\widetilde{v}_i}(\theta_t^u,\theta_0)$. Finally, $\lambda(1)=\Delta_{\widetilde{v}_i}(\theta_1^u,\theta_0^u)$ and $\lambda(0)=\Delta_{\widetilde{v}_i}(\theta_0^u,\theta_0^u)=0$.

(c) $[D_\sigma(\widetilde{\varphi})]$ is independent of the choice of representative of β.

Recall that representatives of β are by definition sections of $E^*\otimes Q(\mathcal{F})$. They are considered as $\boldsymbol{C}^q$-valued 1-forms on P by extending them arbitrary to sections of $T^*_{\boldsymbol{C}}M\otimes Q(\mathcal{F})$ and then lifted to P.

We first show that $[D_\sigma(\widetilde{\varphi})]$ is independent of extensions as above. Suppose that σ_0 and σ_1 are representatives of β and assume that $\sigma_0=\sigma_1$ when restricted to π^*E, where $\pi:P\to M$ is the projection, then $\sigma_1-\sigma_0=\mu\omega$ for some matrix valued function μ. Let θ_0' and θ_1' be corresponding derivatives of θ, then by (4.3.6),

$$(\theta_1'-\theta_0')\wedge\omega=d(\sigma_1-\sigma_0)+\theta\wedge(\sigma_1-\sigma_0)=(d\mu+[\theta,\mu])\wedge\omega.$$

Hence

$$\begin{aligned}v_J\bar{v}_K(\theta_1',\Omega)-v_J\bar{v}_K(\theta_0',\Omega)&=v_J\bar{v}_K(d\mu+[\theta,\mu],\Omega)\\&=v_J\bar{v}_K(d\mu,\Omega)+(|J|+|K|-1)v_J\bar{v}_K(\mu,-[\theta,\Omega])\\&=v_J\bar{v}_K(d\mu,\Omega)+(|J|+|K|-1)v_J\bar{v}_K(\mu,d\Omega)\\&=d\left(v_J\bar{v}_K(\mu,\Omega)\right).\end{aligned}$$

Let $\widetilde{u}_I v_J \bar{v}_K$ be an element of $\widetilde{\widetilde{\mathrm{WU}}}_q$ such that $|J| > q$, then

$$\begin{aligned}
&(|J| + |K|) v_J \bar{v}_K(\theta'_1, \Omega) \widetilde{u}_I(\theta, \theta^u) - v_J \bar{v}_K(\theta'_0, \Omega) \widetilde{u}_I(\theta, \theta^u) \\
=& \, d\left(((|J| + |K|) v_J \bar{v}_K(\mu, \Omega)\right) \widetilde{u}_I(\theta, \theta^u) \\
=& \, d\left(|J| \, v_J(\mu, \Omega) \bar{v}_K(\Omega)\right) \widetilde{u}_I(\theta, \theta^u) \\
\equiv& - |J| \, v_J(\mu, \Omega) \bar{v}_K(\Omega) d\widetilde{u}_I(\theta, \theta^u) \\
=& - |J| \sum_t (-1)^{t-1} v_J(\mu, \Omega) \bar{v}_K(\Omega)(v_{i_t} - \bar{v}_{i_t})(\Omega) \widetilde{u}_{I(t)}(\theta, \theta^u) \\
=& \, |J| \sum_t (-1)^{t-1} v_J(\mu, \Omega) \bar{v}_K(\Omega) \bar{v}_{i_t}(\Omega) \widetilde{u}_{I(t)}(\theta, \theta^u) \\
=& (|J| + |K| + i_t) \sum_t (-1)^{t-1} v_J \bar{v}_K \bar{v}_{i_t}(\mu, \Omega) \widetilde{u}_{I(t)}(\theta, \theta^u) \\
=& \, \delta(\widetilde{d}(\widetilde{u}_I v_J \bar{v}_K))(\theta^u, \theta, \mu).
\end{aligned}$$

Similarly,

$$(|J| + |K|) v_J \bar{v}_K(\theta'_1, \Omega) \widetilde{u}_I(\theta, \theta^u) - v_J \bar{v}_K(\theta'_0, \Omega) \widetilde{u}_I(\theta, \theta^u) \equiv \delta(\widetilde{d}(\widetilde{u}_I v_J \bar{v}_K))(\theta^u, \theta, \mu)$$

if $|K| > q$. Hence

$$\delta(\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \theta'_1) - \delta(\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \theta'_0) \equiv \delta(\widetilde{d}\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \mu) = 0.$$

In order to complete the proof of (c), it suffices to show that $D_\sigma(\widetilde{\varphi})$ is exact for sections σ corresponding to $d_\nabla \gamma$, $\gamma \in \Gamma^\infty Q(\mathcal{F})$, because $D_{\sigma_0 + \sigma_1}(\widetilde{\varphi}) = D_{\sigma_0}(\widetilde{\varphi}) + D_{\sigma_1}(\widetilde{\varphi})$, where d_∇ is as in Definition 4.3.1. Recall briefly how such a σ is obtained. Choose a lift Y of γ to $T_{\boldsymbol{C}}M$ and let $\widehat{Y}$ be its horizontal lift. Define a function g on P by setting $g(\omega) = \omega(\widehat{Y})$. Then $dg + \theta g$ can be chosen as σ. By definition $\omega' = -dg - \theta g$. An infinitesimal derivative θ' with respect to σ is by definition a $\mathfrak{gl}(q; \boldsymbol{C})$-valued 1-form satisfying $\theta' \wedge \omega = -d\omega' - \theta \wedge \omega'$. The right hand side is now equal to $d\theta g - \theta \wedge dg + \theta \wedge dg + \theta \wedge \theta g = \Omega g$. Let $\{\Gamma_k\}$ be a family of matrix valued 1-forms such that $\Omega = \sum\limits_k \Gamma_k \wedge \omega^k$, then $\Omega g = \sum\limits_k \Gamma_k g \wedge \omega^k$. Note that writing $\Gamma_k = (\Gamma^i_{j,k})$, one has $\Gamma^i_{j,k} = \Gamma^i_{k,j}$ and hence $(\sum\limits_k \Gamma_k \omega^k(\widehat{Y})) \wedge \omega = (\sum\limits_k \Gamma_k g^k) \wedge \omega = \sum\limits_j \Gamma_j g \wedge \omega^j = \Omega g$ and $(\sum\limits_k \Gamma_k(\widehat{Y})\omega^k) \wedge \omega = 0$. Thus by setting $\theta' = -i_{\widehat{Y}}\Omega$,

$$\theta' \wedge \omega = \sum_k (\Gamma_k(\widehat{Y})\omega)\omega^k + (\sum_k \Gamma_k \omega^k(\widehat{Y})) \wedge \omega = \Omega g.$$

Therefore, for this choice of θ',

$$v_J \bar{v}_K(\theta', \Omega) = -\frac{1}{|J| + |K|} i_{\widehat{Y}} v_J \bar{v}_K(\Omega) = 0$$

if $|J|>q$ or $|K|>q$. Hence $\delta(\widetilde{d}\widetilde{\varphi})(\theta^u,\theta,\theta')=0$ if φ is closed in WU_q.

(d) $[D_\sigma(\widetilde{\varphi})]$ is independent of the choice of φ and its lift $\widetilde{\varphi}$.

It suffices to show that $D_\sigma(\widetilde{d}\widetilde{\varphi}+\alpha)$ is exact, where $\widetilde{\varphi}\in\widetilde{\mathrm{WU}}_q$ and $\alpha\in\widetilde{\mathcal{I}}_q$. First, $D_\sigma(\widetilde{d}\widetilde{\varphi})=0$ because $\widetilde{d}(\widetilde{d}\widetilde{\varphi})=0$. In order to show that $D_\sigma(\alpha)$ is exact for $\alpha\in\widetilde{\mathcal{I}}_q$, first show the claim for $\alpha=\widetilde{u}_I v_J \bar{v}_K$ with $|J|>q$. If I is empty, then $\widetilde{d}\alpha=0$ so that $D_\sigma(\alpha)=\Delta\alpha(\theta^u,\theta,\theta')=0$. Suppose that I is nonempty, then by using the equations $v_J(\Omega)=0$ and $v_J(\theta',\Omega)v_{i_l}(\Omega)=0$ one obtains the following equalities;

$$\begin{aligned}
D_\sigma(\alpha) &= \Delta\alpha(\theta^u,\theta,\theta') \\
&= \sum_l (-1)^{l-1}(|J|+|K|+i_l)\left(v_J\bar{v}_K(v_{i_l}-\bar{v}_{i_l})\right)(\theta',\Omega)\widetilde{u}_{I(l)}(\theta^u,\theta) \\
&= \sum_l (-1)^l\,|J|\,v_J(\theta',\Omega)\bar{v}_K(\Omega)\bar{v}_{i_l}(\Omega)\widetilde{u}_{I(l)}(\theta^u,\theta) \\
&= d\left(|J|\,v_J(\theta',\Omega)\bar{v}_K(\Omega)\widetilde{u}_I(\theta^u,\theta)\right).
\end{aligned}$$

The last equality holds because $v_J(\theta',\Omega)$ is closed by Lemma 4.3.16. Similarly, $D_\sigma(\alpha)$ is exact if $|K|>q$. This completes the proof. □

Finally we show that the infinitesimal derivative of secondary classes coincide with the actual derivative when there is an actual deformation realizing the infinitesimal derivative.

Definition 4.3.19 (Definition 2.7 in [20]). Let $\{\mathcal{F}_s\}$ be a smooth deformation of transversely holomorphic foliations of M. Denote by π_s the projection from $T_{\boldsymbol{C}}M$ to $Q(\mathcal{F}_s)$. Fix a Riemannian metric on $T_{\boldsymbol{C}}M$ which is transversely Hermitian. Assuming that s is small if necessary, one can find by using the metric as above a smooth family of splittings $T_{\boldsymbol{C}}M=E_s\oplus\nu_s$, where $\nu_s\cong Q(\mathcal{F}_s)$. Denote by π'_s the projection from $T_{\boldsymbol{C}}M$ to ν_s. The infinitesimal deformation σ associated to $\mathcal{F}_s$ is the smooth section σ of $E_0^*\otimes Q(\mathcal{F}_0)$ defined by

$$\sigma(X)=-\pi_0\left(\left.\frac{\partial}{\partial s}\pi'_s(X)\right|_{s=0}\right).$$

Lemma 4.3.20 (Lemma 2.8 in [20]). *σ does not depend on the choice of the splitting.*

Proof. It suffices to work in a foliation chart. Let $\{e_1,\ldots,e_q\}$ be a local frame of $Q(\mathcal{F}_0)$, Fix a splitting as above and let $\{\widetilde{e}_1,\ldots,\widetilde{e}_q\}$ be the lift of $\{e_1,\ldots,e_q\}$ to $T_{\boldsymbol{C}}M$. We may assume that there is a smooth family of frames $\{\widetilde{e}_1(s),\ldots,\widetilde{e}_q(s)\}$ of ν_s such that $\widetilde{e}_i(0)=\widetilde{e}_i$, $i=1,\ldots,q$. Given $X\in E_0$, write $\pi'_s(X)=\sum\limits_{i=1}^q f_i(X,s)\widetilde{e}_i(s)$, where f_i's depend on X and s but are independent of the splitting in the following sense; choose another splitting $T_{\mathbf{C}}M = E_s \oplus \nu'_s$ and let $\{\widetilde{e}'_1(s), \ldots, \widetilde{e}'_q(s)\}$ be the family of frames of ν'_s such that $\pi_s \widetilde{e}'_i(s) = \pi_s \widetilde{e}_i(s) \in Q(\mathcal{F}_s)$. If one writes the projection of X to ν'_s as $\sum_{i=1}^{q} g_i(X,s)\widetilde{e}'_i(s)$, then $g_i = f_i$.

On the other hand, $\left.\frac{\partial}{\partial s}\widetilde{e}_i(s)\right|_{s=0} \in E_0$. Indeed, $\pi'_s \circ \pi'_s = \pi'_s$ implies that

$$\left(\frac{\partial}{\partial s}\pi'_s\right)\pi'_s + \pi'_s\left(\frac{\partial}{\partial s}\pi'_s\right) = \frac{\partial}{\partial s}\pi'_s. \tag{4.3.21}$$

Thus

$$\frac{\partial}{\partial s}\pi'_s\widetilde{e}_i(s) + \pi'_s\left(\frac{\partial}{\partial s}\pi'_s\widetilde{e}_i(s)\right) = \frac{\partial}{\partial s}\pi'_s\widetilde{e}_i(s).$$

Hence $\pi'_0\left(\left.\frac{\partial}{\partial s}\pi'_s\widetilde{e}_i(s)\right|_{s=0}\right) = 0$. It follows that $\left.\frac{\partial}{\partial s}\pi'_s(X)\right|_{s=0} = \sum_{i=1}^{q}\frac{\partial f_i}{\partial s}(X,0)\widetilde{e}_i(0)$. Therefore, $\pi_0\left(\left.\frac{\partial}{\partial s}\pi'_s(X)\right|_{s=0}\right) = \sum_{i=1}^{q}\frac{\partial f_i}{\partial s}(X,0)e_i$. Since f_i's are independent of splittings, we are done. $\square$

Lemma 4.3.22 (Corollary 2.11 in [20])**.** *$d_\nabla \sigma = 0$.*

Proof. Let $X, Y \in E_0$, then $\nabla_X Z = \pi_0[X, \widetilde{Z}]$ for $Z \in Q(\mathcal{F})$, where $\widetilde{Z}$ is any lift of Z to $T_{\mathbf{C}}M$. Hence

$$\begin{aligned}
d_\nabla\sigma(X,Y) &= \nabla_X\sigma(Y) - \nabla_Y\sigma(X) - \sigma([X,Y]) \\
&= \pi_0([X, \widetilde{\sigma(Y)}]) - \pi_0([Y, \widetilde{\sigma(X)}]) - \sigma([X,Y]), \\
&= \pi_0\left(\left[X, -\pi'_0\left.\frac{\partial}{\partial s}\pi'_s(Y)\right|_{s=0}\right]\right) - \pi_0\left(\left[Y, -\pi'_0\left.\frac{\partial}{\partial s}\pi'_s(X)\right|_{s=0}\right]\right) \\
&\quad + \pi_0\left(\left.\frac{\partial}{\partial s}\pi'_s[X,Y]\right|_{s=0}\right).
\end{aligned}$$

If $v \in E_s$, then $\pi'_s\frac{\partial}{\partial s}\pi'_s(v) = \frac{\partial}{\partial s}\pi'_s(v)$ by (4.3.21). Thus $\frac{\partial}{\partial s}\pi'_s(v) \in \nu_s$. Hence

$$\pi'_0\left(\left[X, -\pi'_0\left.\frac{\partial}{\partial s}\pi'_s(Y)\right|_{s=0}\right]\right) = -\pi'_0\left(\left[X, \left.\frac{\partial}{\partial s}\pi'_s(Y)\right|_{s=0}\right]\right).$$

Similarly,

$$\pi'_0\left(\left[Y, -\pi'_0\left.\frac{\partial}{\partial s}\pi'_s(X)\right|_{s=0}\right]\right) = -\pi'_0\left(\left[Y, \left.\frac{\partial}{\partial s}\pi'_s(X)\right|_{s=0}\right]\right).$$

On the other hand,

$$\begin{aligned}
&\frac{\partial}{\partial s}\pi'_s\left[X-\pi'_s(X),Y-\pi'_s(Y)\right]\Big|_{s=0}\\
&=\frac{\partial}{\partial s}\pi'_s\Big|_{s=0}\left[X-\pi'_0(X),Y-\pi'_0(Y)\right]\\
&\quad+\pi'_0\left[-\frac{\partial}{\partial s}\pi'_s(X)\Big|_{s=0},Y-\pi'_0(Y)\right]+\pi'_0\left[X-\pi'_0(X),-\frac{\partial}{\partial s}\pi'_s(Y)\Big|_{s=0}\right]\\
&=\frac{\partial}{\partial s}\pi'_s\Big|_{s=0}[X,Y]-\pi'_0\left[\frac{\partial}{\partial s}\pi'_s(X)\Big|_{s=0},Y\right]-\pi'_0\left[X,\frac{\partial}{\partial s}\pi'_s(Y)\Big|_{s=0}\right]
\end{aligned}$$

because $X,Y\in E$. Thus $d_\nabla\sigma(X,Y)=\frac{\partial}{\partial s}\pi'_s\left[X-\pi'_s(X),Y-\pi'_s(Y)\right]\Big|_{s=0}$. Noticing that $X-\pi'_s(X),Y-\pi'_s(Y)\in E_s$ and E_s is integrable, $d_\nabla\sigma(X,Y)=0$. $\square$

Remark 4.3.23. For deformations $\{E_s\}$ of E_0 not necessarily integrable, $d_\nabla\sigma$ is called the integrability tensor in [20].

Definition 4.3.24. Let $\{\mathcal{F}_s\}$ be a smooth family of transversely holomorphic foliations of M and let σ be as above. The element $[\sigma]$ in $H^1(M;\Theta_{\mathcal{F}})$ is also called the infinitesimal deformation associated to $\{\mathcal{F}_s\}$.

Theorem 4.3.25 (Theorem 3.23). *Let $\{\mathcal{F}_s\}_{s\in\boldsymbol{R}}$ be a differential family of transversely holomorphic foliations of M, of complex codimension q. Let $\beta\in H^1(M;\Theta_{\mathcal{F}})$ be the infinitesimal deformation of $\mathcal{F}_0$ determined by $\{\mathcal{F}_s\}$, then*

$$D_\beta(f)=\frac{\partial}{\partial s}f(\mathcal{F}_s)\Big|_{s=0}$$

for $f\in H^*(\mathrm{WU}_q)$.

Proof. Let P_s be the principal bundle associated with $Q(\mathcal{F}_s)^*$. We may assume that s is small so that P_s is canonically isomorphic to P_0. Hence there are families of canonical forms ω_s and complex Bott connections θ_s on $Q(\mathcal{F}_s)$ such that $d\omega_s=-\theta_s\wedge\omega_s$. Setting $\dot\omega_s=\frac{\partial}{\partial s}\omega_s\Big|_{s=0}$ and $\dot\theta_s=\frac{\partial}{\partial s}\theta_s\Big|_{s=0}$, one has

$$d\dot\omega_s=-\dot\theta_s\wedge\omega_s-\theta_s\wedge\dot\omega_s.$$

On the other hand, if σ is the infinitesimal deformation associated to $\{\mathcal{F}_s\}$, then a 1-form $\widehat\sigma$ on P representing σ is given as follows. Let P_s be the principal bundle associated to $Q(\mathcal{F})^*$ and let $Q(\widehat{\mathcal{F}_s})$ be the pull-back of $Q(\mathcal{F}_s)$ by the projection to M. Let $\omega_s={}^t(\omega_s^1,\dots,\omega_s^q)$ be the canonical form on $Q(\widehat{\mathcal{F}_s})$, then

$$\widehat\sigma(\widehat X)=-\pi_0\left(\frac{\partial}{\partial s}(\omega_s^1(X)\widetilde e_1(s)+\cdots+\omega_s^q(X)\widetilde e_q(s))\Big|_{s=0}\right),$$

where $\widetilde{e}_i(s)$ denotes the horizontal lift of $\widetilde{e}_i(s)$ as in the proof of Lemma 4.3.20. Since $\left.\dfrac{\partial}{\partial s}\widetilde{e}_i(s)\right|_{s=0}$ belongs to the kernel of π_0, one has

$$\begin{aligned}
\widehat{\sigma}(\widehat{X}) &= -\pi_0\left(\left.\frac{\partial}{\partial s}\omega_s^1(X)\right|_{s=0}\widetilde{e}_1(0)+\cdots+\left.\frac{\partial}{\partial s}\omega_s^q(X)\right|_{s=0}\widetilde{e}_q(0)\right)\\
&= -\left.\frac{\partial}{\partial s}\omega_s^1(X)\right|_{s=0}e_1-\cdots-\left.\frac{\partial}{\partial s}\omega_s^q(X)\right|_{s=0}e_q\\
&= -\dot{\omega}(\widehat{X}).
\end{aligned}$$

It follows that $\dot{\theta}_s$ can be chosen as an infinitesimal derivative of θ_0 along $[\sigma]$. Thus Theorem 4.3.25 follows from Proposition 4.2.6. $\square$

5. A review of Rasmussen's examples

The following statements with several examples are presented in [36] by Rasmussen;

1) There are transversely holomorphic foliations of which the Godbillon-Vey class and some other classes are non-trivial.
2) There are smooth families of transversely holomorphic foliations which realizes a continuous variation of the Godbillon-Vey class.

Our Theorem B contradicts 2). We will study the reason by reviewing examples in [36]. The part 2) is shown in § 4 of [36]. The examples given there involve an action of $\boldsymbol{C}^k$ on $\boldsymbol{C}P^n$ defined by $(t_1,\dots,t_k)\cdot[z_0:\cdots:z_n]=e^{\lambda_1t_1+\cdots+\lambda_kt_k}[z_0:\cdots:z_n]$, where $(\lambda_1,\dots,\lambda_k)\neq(0,\dots,0)$ parametrizes the variation. This action is always trivial so that the foliations are independent of these parameters. In addition, those foliations are constructed on a fiber bundle over $M\times T^{2k}$ with fiber $\boldsymbol{C}P^n\times\boldsymbol{C}P^m$ for some k, n and m, where $T^{2k}=(\boldsymbol{C}^k\setminus\{0\})/(t_1,\dots,t_k)\sim(\lambda t_1,\dots,\lambda t_k)$ for some $\lambda\in\boldsymbol{C}$ such that $|\lambda|\neq 0,1$. The variation of the Godbillon-Vey class is claimed to be realized by the variation of $|\lambda|$. Indeed, the Godbillon-Vey class is the multiple of a volume form on T^{2k} by $|\lambda|^{2k}$. However, this does not imply the variation of the Godbillon-Vey class because the term $|\lambda|^{2k}$ vanishes after normalizing the volume of T^{2k}. Hence thus constructed foliation and its Godbillon-Vey class remain the same.

On the other hand, the examples showing non-triviality are essentially the same as ours. In fact, our construction is motivated by these examples as mentioned in Introduction, in particular, the examples in the first part of § 3 of [36] coincides with Example 2.3.6 after taking the quotient by cocompact lattices. However, at the last part of § 3 [36] (page 163), locally homogeneous spaces such as $\Gamma\backslash\mathrm{SL}(n;\boldsymbol{C})/\mathrm{SL}(n-1;\boldsymbol{C})$ are considered. Cocompact lattices for this kind of homogeneous spaces do not exist in general (e.g. Example 5.21 in [28] and references there).

Appendix

Some common materials used in this article are presented in this Appendix for completeness. Most of these can be found in Kobayashi-Nomizu [27] but they are modified by following the convention in Matsushima [31]. Differences appear in coefficients, for example, $\omega \wedge \eta = \dfrac{1}{p!q!}\mathrm{Alt}(\omega \otimes \eta)$ for a p-form ω and a q-form η, where Alt stands for the alternizer. Another example is the formula $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$.

1. Invariant Polynomials.

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Denote by $I^k(G)$ be the set of invariant polynomials of degree k.

Definition A.1. Let $f \in I^k(G)$ and let $\varphi_1, \dots, \varphi_k$ be $\mathfrak{g}$-valued differential forms of degree $q_1, \dots, q_k$, respectively. Define a $(q_1 + \cdots + q_k)$-form $f(\varphi_1, \dots, \varphi_k)$ as follows. First choose a basis $\{E_1, \dots, E_r\}$ for $\mathfrak{g}$ and write $\varphi_i = \sum_{j=1}^{r} E_j \varphi_i^j$. Set then

$$f(\varphi_1, \dots, \varphi_k) = \sum_{j_1, \dots, j_k = 1}^{r} f(E_{j_1}, \dots, E_{j_k}) \varphi_1^{j_1} \wedge \cdots \wedge \varphi_k^{j_k}.$$

Notation A.2 (Chern convention)**.** Let $f \in I^k(G)$ and let $\varphi_1, \dots, \varphi_l$ be $\mathfrak{g}$-valued differential forms as above. If $l < k$, then set

$$f(\varphi_1, \dots, \varphi_l) = f(\varphi_1, \dots, \underbrace{\varphi_l, \dots, \varphi_l}_{k-l+1 \text{ times}}).$$

Definition A.3. Let $f : \mathfrak{gl}(n;C) \to \boldsymbol{C}$ be a multilinear mapping which is invariant under the adjoint action. The polarization of f is the unique element $\widehat{f}$ of $I^k(\mathrm{GL}(n;\boldsymbol{C}))$ such that

$$\widehat{f}(X, X, \dots, X) = f(X)$$

for any $X \in \mathfrak{gl}(n;\boldsymbol{C})$, where k is the degree of f as a polynomial. By abuse of notation, $\widehat{f}$ is denoted again by f.

Remark A.4. The polarization is compatible with the Chern convention, namely, one has

$$\widehat{f}(\Omega, \dots, \Omega) = f(\Omega)$$

for any even form Ω and any multilinear mapping f.

Definition A.5. Let $f \in I^k(G)$, $g \in I^l(G)$, define $fg \in I^{k+l}(G)$ by setting

$$\begin{aligned} &fg(X_1, X_2, \ldots, X_{k+l}) \\ =&\frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} f(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) g(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}). \end{aligned}$$

Lemma A.6. *Let $f \in I^k(G)$ and $g \in I^l(G)$. If θ, η are of odd degree and if Ω is of even degree, then*

$$(k+l) fg(\theta, \Omega) = k f(\theta, \Omega) g(\Omega) + l f(\Omega) g(\theta, \Omega) \tag{A.6a}$$

$$\begin{aligned} &(k+l)(k+l-1) fg(\theta, \eta, \Omega) \\ =&k(k-1) f(\theta, \eta, \Omega) \wedge g(\Omega) + kl f(\theta, \Omega) \wedge g(\eta_2, \Omega) \\ &- kl f(\eta, \Omega) \wedge g(\theta, \Omega) + l(l-1) f(\Omega) \wedge g(\theta, \eta_2, \Omega). \end{aligned} \tag{A.6b}$$

Proof. The formula (A.6a) is easy. Let $E_1, \ldots, E_r$ be a basis for $\mathfrak{g}$ and write $\theta = \sum E_j \theta^j$, $\eta = \sum E_j \eta^j$ and $\Omega = \sum E_j \Omega^j$, then

$$fg(\theta, \eta, \Omega) = \sum_{j_1, j_2, \ldots, j_{k+l}} fg(E_{j_1}, \ldots, E_{j_{k+l}}) \theta^{j_1} \wedge \eta^{j_2} \wedge \Omega^{j_3} \wedge \cdots \wedge \Omega^{j_{k+l}},$$

where

$$fg(E_{j_1}, \ldots, E_{j_{k+l}}) = \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} f(E_{j_{\sigma(1)}}, \ldots, E_{j_{\sigma(k)}}) g(E_{j_{\sigma(k+1)}}, \ldots, E_{j_{\sigma(k+l)}}).$$

Set $\omega(j_1, j_2, \ldots, j_{k+l}) = \theta^{j_1} \wedge \eta^{j_2} \wedge \Omega^{j_3} \wedge \cdots \wedge \Omega^{j_{k+l}}$, then

$$\begin{aligned} &(k+l)!\, fg(\theta, \eta, \Omega) \\ =&(k+l)! \sum_{j_1, j_2, \ldots, j_{k+l}} fg(E_{j_1}, \ldots, E_{j_{k+l}}) \omega(j_1, j_2, \ldots, j_{k+l}) \\ =& \sum_{j_1, j_2, \ldots, j_{k+l}} \sum_{\sigma \in \mathfrak{S}_{k+l}} f(E_{j_{\sigma(1)}}, \ldots, E_{j_{\sigma(k)}}) g(E_{j_{\sigma(k+1)}}, \ldots, E_{j_{\sigma(k+l)}}) \omega(j_1, j_2, \ldots, j_{k+l}) \end{aligned}$$

Elements of $\mathfrak{S}_{k+l}$ are divided into four types, namely,

1) $\sigma(1), \sigma(2) \leq k$,
2) $\sigma(1) \leq k < \sigma(2)$,
3) $\sigma(2) \leq k < \sigma(1)$,
4) $k < \sigma(1), \sigma(2)$.

The number of such elements are $(k+l-2)!k(k-1)$, $(k+l-2)!kl$, $(k+l-2)!kl$, $(k+l-2)!l(l-1)$, respectively. The formula (A.6b) follows from this. $\square$

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