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by

Yoshihiro SAWANO and Hitoshi TANAKA

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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Yoshihiro Sawano and Hitoshi Tanaka

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Abstract

We give a natural definition of the Morrey spaces for Radon measures which may be non-doubling but satisfy the growth condition. In these spaces we investigate the behavior of the maximal operator, the fractional integral operator, the singular integral operator and their vector-valued extensions.

1 Introduction

For $1 \le q \le p < \infty$ the (classical) Morrey spaces are defined as

$$\mathcal{M}^p_q(\mathbf{R}^d) := \left\{ f \in L^q_{loc}(\mathbf{R}^d) : \|f \,|\, \mathcal{M}^p_q(\mathbf{R}^d)\| < \infty \right\},$$

where the norm $\|f \,|\, \mathcal{M}^p_q(\mathbf{R}^d)\|$ is given by

$$\|f \,|\, \mathcal{M}^p_q(\mathbf{R}^d)\| := \sup_{x \in \mathbf{R}^d,\, l>0} |B(x,l)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,l)} |f|^q \, dy \right)^{\frac{1}{q}}.$$

Here, $B(x,l)$ is a closed ball with its center x and radius l as usual and $|B(x,l)|$ denotes its volume. The Morrey spaces describe local regularity more precisely than the Lebesgue spaces $L^p(\mathbf{R}^d)$ (c.f. [6]). A Radon measure μ on $\mathbf{R}^d$ is said to be doubling if there exists some constant C such that $\mu(B(x,2l)) \le C\,\mu(B(x,l))$ for all $x \in \mathrm{supp}(\mu)$ and $l > 0$. In most results of classical Carderón-Zygmund theory the doubling condition on μ seems to be

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an essential assumption (c.f. [3], [9]). However recently it has been shown that many results in this theory also hold without the doubling assumption, as is found in [7], [10] and many other literatures. In this paper we shall define the Morrey spaces with non-doubling Radon measures and investigate the properties of them. Throughout this paper μ will be a (positive) Radon measure on $\mathbf{R}^d$ satisfying the growth condition

$$\mu(B(x,l)) \le c_0\, l^n \text{ for all } x \in \operatorname{supp}(\mu) \text{ and } l > 0, \tag{1}$$

where c_0 and n, $0 < n \le d$, are some fixed numbers.

Firstly, let us give notations and definitions. By "cube" we mean a closed cube whose edges are parallel to the coordinate axes. Its side length will be denoted by $\ell(Q)$ and its center by $z(Q)$. The set of all cubes $Q \subset \mathbf{R}^d$ satisfying $\mu(Q) > 0$ will be denoted by $\mathcal{Q}(\mu)$. For $c > 0$, $c\,Q$ will denote a cube concentric to Q with its sidelength $c\,\ell(Q)$.

Let $k > 1$ and $1 \le q \le p < \infty$. We define a Morrey space $\mathcal{M}^p_q(k,\mu)$ as

$$\mathcal{M}^p_q(k,\mu) := \left\{ f \in L^q_{loc}(\mu) : \| f \,|\, \mathcal{M}^p_q(k,\mu) \| < \infty \right\},$$

where the norm $\| f \,|\, \mathcal{M}^p_q(k,\mu) \|$ is given by

$$\| f \,|\, \mathcal{M}^p_q(k,\mu) \| := \sup_{Q \in \mathcal{Q}(\mu)} \mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q \, d\mu \right)^{\frac{1}{q}}. \tag{2}$$

Clearly, we see that $\mathcal{M}^p_q(k,\mu) \supset L^p(\mu)$ and by using Hölder's inequality to (2) we have that $\| f \,|\, \mathcal{M}^p_{q_1}(k,\mu) \| \ge \| f \,|\, \mathcal{M}^p_{q_2}(k,\mu) \|$ for all $p \ge q_1 \ge q_2 \ge 1$. This tells us that the following inclusion holds:

$$L^p(\mu) = \mathcal{M}^p_p(k,\mu) \subset \mathcal{M}^p_{q_1}(k,\mu) \subset \mathcal{M}^p_{q_2}(k,\mu).$$

As is easily seen, the space $\mathcal{M}^p_q(k,\mu)$ is a Banach space with its norm.

The parameter $k > 1$ appearing in the definition does not affect the definition of the space. More precisely, we have the following proposition, which will be a key to our arguments throughout this paper.

Proposition 1.1. *Let $k_1, k_2 > 1$. Then we have $\mathcal{M}^p_q(k_1,\mu) \approx \mathcal{M}^p_q(k_2,\mu)$ in the sense of the equivalent norms.*

Proof. Let $k_1 \le k_2$. Then the inclusion $\mathcal{M}^p_q(k_1,\mu) \subset \mathcal{M}^p_q(k_2,\mu)$ is trivial by the definition of the norms. Let us show the reverse inclusion. Let $f \in \mathcal{M}^p_q(k_2,\mu)$ and $Q \in \mathcal{Q}(\mu)$. Then we have to estimate

$$\mu(k_1 Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q \, d\mu \right)^{\frac{1}{q}}.$$

Simple geometric observation shows that there exist some cubes Q_1, Q_2, ..., Q_N with the same sidelength such that

$$Q \subset \bigcup_{i=1}^{N} Q_i, \quad k_2 Q_i \subset k_1 Q \, (i = 1, 2, \ldots, N) \text{ and } N \leq C \left(\frac{k_2 - 1}{k_1 - 1} \right)^d .$$

Using this covering, we easily obtain

$$\begin{aligned} & \mu(k_1 Q)^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q |f|^q \, d\mu \right)^{\frac{1}{q}} \\ \leq & \sum_{i=1}^{N} \mu(k_1 Q)^{\frac{1}{p} - \frac{1}{q}} \left(\int_{Q_i} |f|^q \, d\mu \right)^{\frac{1}{q}} \\ \leq & \sum_{Q_i \in \mathcal{Q}(\mu)} \mu(k_2 Q_i)^{\frac{1}{p} - \frac{1}{q}} \left(\int_{Q_i} |f|^q \, d\mu \right)^{\frac{1}{q}} \\ \leq & N \| f \, | \, \mathcal{M}_q^p(k_2, \mu) \| . \quad \blacksquare \end{aligned}$$

In view of this proposition we sometimes omit parameter k in $\mathcal{M}_q^p(k, \mu)$.

The boundedness of fractional integral operators on the (classical) Morrey spaces $\mathcal{M}_q^p(\mathbf{R}^d)$ was studied by Adams ([1]), Chiarenza and Frasca ([2]) etc. Chiarenza and Frasca showed that the Hardy-Littlewood maximal operator is bounded on the Morrey spaces ([2, Theorem 1]). By establishing a pointwise estimate of fractional integrals involved with the Hardy-Littlewood maximal function, they showed the boundedness of fractional integral operators on the Morrey spaces ([2, Theorem 2]). Boundedness of the singular integral is also proved there ([2, Theorem 3]). In this paper we shall recover these results in the setting of non-doubling measures. Moreover, we shall prove the vector-valued maximal inequality.

Main theorems are stated in each section. Section 2 is devoted to the study of the maximal operators, including a Fefferman-Stein type inequality, where we will see our definition of the space goes well. Section 3 and 4 contain fractional integral operators. Finally in Section 5 we investigate the boundedness of the singular integral.

In what follows the letter C will be used for constants that may change from one occurrence to another. $A \sim B$ is used to indicate that $C^{-1} A \leq B \leq C \, A$ for some $C > 0$ independent on (for example) the functions f.

2 Maximal inequalities

In this section we shall investigate some maximal inequalities. In proving the maximal inequalities we do not have to pose the growth condition on μ. For $\kappa > 1$ and $f \in L^1_{loc}(\mu)$ we use the following modified maximal operator:

$$M_\kappa f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu(\kappa Q)} \int_Q |f| \, d\mu.$$

We use the next results of this operator in our theory.

Proposition 2.1 ([8],[10]). *If $\kappa > 1$ and $1 < p \le \infty$, then we have*

$$\|M_\kappa f \mid L^p(\mu)\| \le C_{d,p,\kappa} \|f \mid L^p(\mu)\|.$$

We also have the inequality of Fefferman-Stein type.

Proposition 2.2 ([8]). *If $\kappa > 1$, $1 < p < \infty$ and $1 < q \le \infty$, then we have the vector-valued maximal inequality* :

$$\left\| \left(\sum_{j \in \mathbf{N}} (M_\kappa f_j)^q \right)^{1/q} \mid L^p(\mu) \right\| \le C_{d,p,q,\kappa} \left\| \left(\sum_{j \in \mathbf{N}} |f_j|^q \right)^{1/q} \mid L^p(\mu) \right\|.$$

In this section we shall extend these results to the Morrey spaces $\mathcal{M}^p_q(\mu)$.

Theorem 2.1. *If $k, \kappa > 1$ and $1 < q \le p < \infty$, then we have*

$$\|M_\kappa f \mid \mathcal{M}^p_q(k,\mu)\| \le C_{d,p,q,\kappa,k} \|f \mid \mathcal{M}^p_q(k,\mu)\|.$$

Proof. Fix $Q_0 \in \mathcal{Q}(\mu)$ and put $L := \ell(Q_0)/2$. Let $f_1 := \chi_{\frac{\kappa+7}{\kappa-1} Q_0} f$ and $f_2 := f - f_1$. Then for all $y \in Q_0$ we have

$$M_\kappa f(y) \le M_\kappa f_1(y) + M_\kappa f_2(y). \tag{3}$$

It follows from the definition of M_κ that

$$M_\kappa f_2(y) \le \sup_{y \in Q \in \mathcal{Q}(\mu),\, \ell(Q) \ge 8L/(\kappa-1)} \frac{1}{\mu(\kappa Q)} \int_Q |f| \, d\mu.$$

Suppose that $y \in Q_0$, $y \in Q \in \mathcal{Q}(\mu)$ and $\ell(Q) \ge 8L/(\kappa - 1)$. Then simple calculus yields $Q_0 \subset \dfrac{1+\kappa}{2} Q$. Thus, we obtain

$$M_\kappa f_2(y) \le \sup_{Q_0 \subset Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{2\kappa}{\kappa+1} Q\right)} \int_Q |f| \, d\mu. \tag{4}$$

Proposition 2.1, (3), (4) and Hölder's inequality yield

$$\begin{aligned}
&\mu\left(\frac{2\kappa(\kappa+7)}{\kappa^2-1}\,Q_0\right)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{Q_0}(M_\kappa f)^q\,d\mu\right)^{\frac{1}{q}}\\
&\quad\le \mu\left(\frac{2\kappa(\kappa+7)}{\kappa^2-1}\,Q_0\right)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\mathbf{R}^d}(M_\kappa f_1)^q\,d\mu\right)^{\frac{1}{q}}\\
&\qquad+\mu(Q_0)^{\frac{1}{p}-\frac{1}{q}}\cdot\left(\int_{Q_0}(M_\kappa f_2)^q\,d\mu\right)^{\frac{1}{q}}\\
&\quad\le \mu\left(\frac{2\kappa(\kappa+7)}{\kappa^2-1}\,Q_0\right)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\mathbf{R}^d}(M_\kappa f_1)^q\,d\mu\right)^{\frac{1}{q}}\\
&\qquad+\mu(Q_0)^{\frac{1}{p}}\cdot\sup_{Q_0\subset Q\in\mathcal{Q}(\mu)}\frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}\,Q\right)}\int_Q|f|\,d\mu\\
&\quad\le C\,\mu\left(\frac{2\kappa(\kappa+7)}{\kappa^2-1}\,Q_0\right)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\frac{\kappa+7}{\kappa-1}\,Q_0}|f|^q\,d\mu\right)^{\frac{1}{q}}\\
&\qquad+C'\sup_{Q_0\subset Q\in\mathcal{Q}(\mu)}\mu\left(\frac{2\kappa}{\kappa+1}\,Q\right)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q|f|^q\,d\mu\right)^{\frac{1}{q}}\\
&\quad\le C\|f\,|\,\mathcal{M}^p_q(2\kappa/(\kappa+1),\mu)\|.
\end{aligned}$$

Hence we have

$$\|M_\kappa f\,|\,\mathcal{M}^p_q(2\kappa(\kappa+7)/(\kappa^2-1),\mu)\|\le C\|f\,|\,\mathcal{M}^p_q(2\kappa/(\kappa+1),\mu)\|.$$

Using Proposition 1.1, we obtain the theorem. ∎

Furthermore we have the following vector-valued version.

Theorem 2.2. *If $k,\kappa>1$, $1<q\le p<\infty$ and $1<r\le\infty$, then we have*

$$\left\|\left(\sum_{j\in\mathbf{N}}(M_\kappa f_j)^r\right)^{1/r}\,\Big|\,\mathcal{M}^p_q(k,\mu)\right\|\le C_{d,p,q,r,\kappa,k}\left\|\left(\sum_{j\in\mathbf{N}}|f_j|^r\right)^{1/r}\,\Big|\,\mathcal{M}^p_q(k,\mu)\right\|.$$

To prove this theorem we need a covering lemma.

Lemma 2.1. *For all $\rho>1$ there exists an integer ν, depending only on ρ and d, which satisfies the following condition*:

Let $\{Q_j\}_{j\in J}$ be a finite family of cubes in $\mathbf{R}^d$. Suppose that all cubes Q_j contain a fixed point x. Then we can select a set $J'\subset J$, $\#J'\le$

ν, such that any cube Q_j can be covered by some ρQ_k, $k \in J'$. Here, we use $\#A$ to denote the cardinality of a set A. (We will see that $\nu \sim \max\left\{1,\, 16^d\, |\log(\rho-1)|\, (\rho-1)^{-d}\right\}$.)

Proof. Let $2L := \max_j \ell(Q_j)$. We shall choose a cube inductively. First, choose a cube Q_{j_1} so that $\ell(Q_{j_1}) = 2L$. Suppose that $Q_{j_1}, \ldots, Q_{j_{k-1}}$ are selected. Consider the set

$$J_k := \{j \in J : \text{none of } \rho Q_{j_m},\, m = 1, \ldots, k-1, \text{ contains } Q_j\}.$$

If $J_k = \emptyset$, then we do not select cubes any more. If $J_k \neq \emptyset$, we choose a cube Q_{j_k}, $j_k \in J_k$, so that Q_{j_k} maximizes $\ell(Q_j)$ with $j \in J_k$. We now proceed this step and obtain a set $J' := \{j_1, j_2, \ldots\} \subset J$.

Simple geometric observation shows that if $k, k' \in J'$, then we have

$$|z(Q_k) - z(Q_{k'})| \geq \frac{\rho-1}{2} \max(\ell(Q_k), \ell(Q_{k'})).$$

Recall that all Q_j's contain x. This shows that the number of Q_k, $k \in J'$, such that $2^{m-1}L < \ell(Q_k) \leq 2^m L$ is less than or equal to $\max(1, 16^d(\rho-1)^{-d})$ for all $m = -1, -2, \ldots$. Noticing that $\ell(Q_k) > (\rho-1)\, L$ for all $k \in J'$, we obtain the lemma. ∎

Proof of Theorem 2.2. Fix $Q_0 \in \mathcal{Q}(\mu)$ and put $L := \ell(Q_0)/2$. Let $f_{j,1} := \chi_{\frac{\kappa+7}{\kappa-1} Q_0} f_j$ and $f_{j,2} := f_j - f_{j,1}$. Then, in the same way as in that of the proof of Theorem 2.1, for $y \in Q_0$ we have that

$$M_\kappa f_j(y) \leq M_\kappa f_{j,1}(y) + M_\kappa f_{j,2}(y)$$

and that

$$M_\kappa f_{j,2}(y) \leq \sup_{Q_0 \subset Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}\, Q\right)} \int_Q |f_j|\, d\mu.$$

Using Proposition 2.2 and the above estimates, we see that

$$\begin{aligned}
&\mu\left(\frac{4\kappa}{3\kappa+1} \cdot \frac{\kappa+7}{\kappa-1} Q_0\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_0} \|M_\kappa f_j \mid l^r\|^q\, d\mu\right)^{\frac{1}{q}} \\
&\quad \leq \; C\,\mu\left(\frac{4\kappa}{3\kappa+1} \cdot \frac{\kappa+7}{\kappa-1} Q_0\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbf{R}^d} \|M_\kappa f_{j,1} \mid l^r\|^q\, d\mu\right)^{\frac{1}{q}} \\
&\qquad + C'\,\mu(Q_0)^{\frac{1}{p}} \cdot \left\| \sup_{Q_0 \subset Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}\, Q\right)} \int_Q |f_j|\, d\mu \;\Big|\; l^r \right\|.
\end{aligned}$$

By Proposition 2.2, the first term of the right-hand side of the above relation can be bounded by $\big\| \|f_j \,|\, l^r\| \,|\, \mathcal{M}^p_q(4\kappa/(3\kappa+1),\mu)\big\|$. So we shall concentrate ourselves on estimating the second term.

Let $\{Q_j\}_{j\in\mathbf{N}}$ be a family of cubes satisfying $Q_j \supset Q_0$. Then by a simple limitting argument it suffices to verify that for any $N \in \mathbf{N}$

$$\begin{aligned} &\mu(Q_0)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{N} \left(\frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}\, Q_j\right)} \int_{Q_j} |f_j|\, d\mu \right)^r \right)^{\frac{1}{r}} \\ \le\;\; & C\, \big\| \|f_j \,|\, l^r\| \,|\, \mathcal{M}^p_q(4\kappa/(3\kappa+1),\mu)\big\|, \end{aligned} \tag{5}$$

where C is a constant independent on N. By duality argument (5) is reduced to prove the following inequality:

$$\begin{aligned} &\mu(Q_0)^{\frac{1}{p}} \cdot \sum_{j=1}^{N} a_j \left(\frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}\, Q_j\right)} \int_{Q_j} |f_j|\, d\mu \right) \\ \le\;\; & C\, \big\| \|f_j \,|\, l^r\| \,|\, \mathcal{M}^p_q(4\kappa/(3\kappa+1),\mu)\big\| \end{aligned} \tag{6}$$

for a non-negative sequence $\{a_j\} \in l^{r'}$ with $\|a_j \,|\, l^{r'}\| = 1$. (r' is a conjugate exponent of r.)

To prove (6), we put for $i = 1, 2, \dots$

$$J_i := \left\{ j \in \mathbf{N} \cap [1, N] \,:\, 2^{i-1}\mu(Q_0) \le \mu\left(\frac{2\kappa}{\kappa+1}\, Q_j\right) < 2^i \mu(Q_0) \right\}.$$

Then we have

$$\begin{aligned} &\mu(Q_0)^{\frac{1}{p}} \cdot \sum_{j=1}^{N} a_j \left(\frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}\, Q_j\right)} \int_{Q_j} |f_j|\, d\mu \right) \\ &\quad = \mu(Q_0)^{\frac{1}{p}} \cdot \sum_{i} \sum_{j\in J_i} a_j \left(\frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}\, Q_j\right)} \int_{Q_j} |f_j|\, d\mu \right). \end{aligned} \tag{7}$$

We now use Lemma 2.1

for the family of the cubes $\{Q_j\}_{j\in J_i}$ with $\rho = \dfrac{3\kappa+1}{2(\kappa+1)}$, to obtain an integer ν and a set $J_i' \subset J_i$, $\# J_i' \le \nu$, satisfying that all cubes Q_j, $j \in J_i$, can be covered by some $\rho\, Q_k$, $k \in J_i'$. Using this observation, we can proceed further

$$
\begin{aligned}
&\mu(Q_0)^{\frac{1}{p}} \cdot \sum_{j \in J_i} a_j \left(\frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}\, Q_j\right)} \int_{Q_j} |f_j|\, d\mu \right) \\
&\leq \mu(Q_0)^{\frac{1}{p}} \cdot \frac{1}{2^{i-1}\mu(Q_0)} \cdot \sum_{j \in J_i} a_j \int_{Q_j} |f_j|\, d\mu \\
&= 2\,\mu(Q_0)^{\frac{1}{p}} \cdot \frac{1}{2^{i}\mu(Q_0)} \cdot \sum_{k \in J'_i} \sum_{j \in J_i \colon Q_j \subset \rho\, Q_k} a_j \int_{Q_j} |f_j|\, d\mu \\
&\leq 2\,\mu(Q_0)^{\frac{1}{p}} \cdot \sum_{k \in J'_i} \frac{1}{\mu\left(\frac{2\kappa}{\kappa+1}\, Q_k\right)} \int_{\rho\, Q_k} \left(\sum_{j \in J_i \colon Q_j \subset \rho\, Q_k} a_j |f_j| \right) d\mu \\
&\leq 2^{-(i-1)/p+1} \nu \cdot \left(2^{i-1}\mu(Q_0)\right)^{1/p} \\
&\qquad \cdot \left(\frac{1}{\mu(\frac{2\kappa}{\kappa+1}\rho^{-1}\, \rho\, Q_k)} \right)^{\frac{1}{q}} \left(\int_{\rho\, Q_k} \|f_j \mid l^r\|^q\, d\mu \right)^{\frac{1}{q}} \\
&\leq 2^{-(i-1)/p+1} \nu \left\| \|f_j \mid l^r\| \mid \mathcal{M}^p_q(4\kappa/(3\kappa+1), \mu) \right\|. \qquad (8)
\end{aligned}
$$

From (7), (8) we arrived at the desired inequality (6) and obtain the theorem. ∎

3 Boundedness of the fractional integral operator

In this section we investigate the fractional integral operator I_α defined by García Cuerva and Eduardo Gatto.

Definition ([4]). For α with $0 < \alpha < n$, we define a fractional integral operator as

$$I_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\, d\mu(y),$$

where n is a constant in the growth condition of μ.

The following result is known due to Garcia and Eduardo [4].

Proposition 3.1 ([4]). *Let $1 < p < n/\alpha$ and $1/s = 1/p - \alpha/n$. Then I_α is bounded from $L^p(\mu)$ to $L^s(\mu)$.*

In this section we shall extend this result to the Morrey spaces $\mathcal{M}^p_q(\mu)$. As is the case with the classical one ([2, Theorem 2]), we have the following theorem.

Theorem 3.1. *Suppose that the parameters satisfy*

$$1 < q \le p < \infty,\ 1 < t \le s < \infty,\ t/s = q/p,\ 1/s = 1/p - \alpha/n.$$

Then we have I_α is bounded from $\mathcal{M}^p_q(\mu)$ to $\mathcal{M}^t_s(\mu)$:

$$\|I_\alpha f \,|\, \mathcal{M}^s_t(k,\mu)\| \le C_{p,q,s,t,\alpha,k} \|f \,|\, \mathcal{M}^p_q(k,\mu)\|, \quad k > 1.$$

The proof of this theorem follows the argument in [2], except for certain technical modifications. We first prove a pointwise estimate using the maximal operator, which immediately leads us to vector-valued improvement.

Lemma 3.1. *If $1 < q \le p < \infty$, $1 < p < n/\alpha$ and $1/s = 1/p - \alpha/n$, then we have a pointwise estimate*

$$|I_\alpha f(x)| \le C_{p,q,\alpha,s} \|f \,|\, \mathcal{M}^p_q(2,\mu)\|^{1-p/s} \cdot (M_2 f(x))^{p/s}.$$

Proof. We may assume that f is positive. Fix $x \in \mathbf{R}^d$. We put for $l > 0$

$$f_l(x) := \frac{1}{l^n} \int_{B(x,l)} f \, d\mu. \tag{9}$$

For all $y \in \mathbf{R}^d$, $y \ne x$, we have an identity

$$\int_0^\infty \frac{\chi_{B(x,l)}(y)}{l^n} l^{\alpha-1} \, dl = \int_{|x-y|}^\infty l^{\alpha-n-1} \, dl = \frac{C}{|x-y|^{n-\alpha}}.$$

This identity and Fubini's theorem yield

$$\begin{aligned} I_\alpha f(x) &= C \int_{\mathbf{R}^d} \left(\int_0^\infty \frac{\chi_{B(x,l)}(y)}{l^n} l^{\alpha-1} \, dl \right) f(y) \, d\mu(y) \\ &= C \int_0^\infty f_l(x) l^{\alpha-1} \, dl. \end{aligned} \tag{10}$$

Take $\epsilon > 0$ which will be determined later on. We separate the above integral into $\mathrm{I} := \int_0^\epsilon f_l(x) l^{\alpha-1} \, dl$ and $\mathrm{II} := \int_\epsilon^\infty f_l(x) l^{\alpha-1} \, dl$. By the growth condition (1) noticing that $f_l(x) \le C\, M_2 f(x)$, we have

$$\mathrm{I} \le C \int_0^\epsilon M_2 f(x) l^{\alpha-1} \, dl = C\, M_2 f(x)\, \epsilon^\alpha.$$

Let $Q(x,l)$ be a cube whose center is x and sidelength is $2l$. Then taking into account of the growth condition, we see that

$$\left(c_0 \sqrt{d} \cdot 2l \right)^{\frac{n}{p} - \frac{n}{q}} \le \mu(Q(x,2l))^{\frac{1}{p} - \frac{1}{q}}.$$

Using this and Hölder's inequality and the growth condition once more, we have the following formula, which will be also used later,

$$\begin{aligned}
f_l(x) &= \frac{1}{l^n}\int_{B(x,l)} f\ d\mu \\
&\le \frac{\mu(B(x,l))^{1-1/q}}{l^n}\left(\int_{B(x,l)} f^q\,d\mu\right)^{1/q} \\
&\le C\,l^{-\frac{n}{p}}\,l^{\frac{n}{p}-\frac{n}{q}}\left(\int_{B(x,l)} f^q\,d\mu\right)^{\frac{1}{q}} \\
&\le C\,l^{-\frac{n}{p}}\,\mu(Q(x,2l))^{\frac{1}{p}-\frac{1}{q}}\left(\int_{Q(x,l)} f^q\,d\mu\right)^{\frac{1}{q}} \qquad (11)\\
&\le C\,l^{-\frac{n}{p}}\,\|f\,|\,\mathcal{M}^p_q(2,\mu)\|. \qquad (12)
\end{aligned}$$

Inserting this, II can be estimated by $C\,\|f\,|\,\mathcal{M}^p_q(2,\mu)q\|\,\epsilon^{-n/s}$.

Thus, we obtain

$$I_\alpha f(x) \le C\,\left(M_2 f(x)\epsilon^\alpha + \|f\,|\,\mathcal{M}^p_q(2,\mu)\|\epsilon^{-n/s}\right).$$

Putting $\epsilon = \left(\frac{\|f\,|\,\mathcal{M}^p_q(2,\mu)\|}{M_2 f(x)}\right)^{p/n}$, we obtain the desired estimate. ∎

Using this lemma and Theorem 2.1, we can easily prove the theorem.

Corollary 3.1. *If we assume further that $1 < r \le \infty$, then we have*

$$\|\|I_\alpha f_j\,|\,l^r\|\,|\,\mathcal{M}^s_t(k,\mu)\| \le C_{p,q,r,s,t,k}\|\|f_j\,|\,l^r\|\,|\,\mathcal{M}^p_q(k,\mu)\|.$$

4 Regularity of the fractional integral operator

In this section we investigate another type of the fractional integral operator K_α also defined by García-Cuerva and Eduardo Gatto.

Definition ([4]). Let n be a constant appearing in the growth condition.

(1) Let $0 < \alpha < n$ and $0 < \epsilon \le 1$. A function $k_\alpha : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{C}$ is said to be a fractional kernel of order α, if it satisfies that

$$|k_\alpha(x,y)| \le \frac{C}{|x-y|^{n-\alpha}} \text{ for all } x \ne y$$

and that

$$|k_\alpha(x,y) - k_\alpha(x',y)| \le C\,\frac{|x-x'|^\epsilon}{|x-y|^{n-\alpha+\epsilon}}, \text{ if } |x-y| \ge 2|x'-x|.$$

(2) We define an operator K_α for the kernel in (1):

$$K_\alpha f(x) := \int_{\mathbf{R}^d} k_\alpha(x,y) f(y)\, d\mu(y).$$

(3) A function space $Lip(\alpha)$ is always considered as a space modulo constant. Its norm is given by

$$\|f \mid Lip(\alpha)\| := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

As is listed in [4], the typical example of the kernel k_α with $\epsilon = 1$ is $k_\alpha(x,y) = \dfrac{1}{|x-y|^{n-\alpha}}$.

As for this fractional integral operator K_α, the following result is known.

Proposition 4.1 ([4]). *Let k_α be a fractional kernel with regularity ϵ. Suppose that $0 < \alpha - n/p < \epsilon$. Then, K_α is a bounded operator from $L^p(\mu)$ to $Lip(\alpha - n/p)$.*

In this section we shall extend this result to the Morrey spaces $\mathcal{M}^p_q(\mu)$.

Theorem 4.1. *Let k_α be a fractional kernel with regularity ϵ. Suppose that $1 \le q \le p < \infty$ and that $0 < \alpha - n/p < \epsilon$. Then, K_α is a bounded operator from $\mathcal{M}^p_q(k,\mu)$ to $Lip(\alpha - n/p)$.*

Proof. Let $x \neq y$ and $r = |x-y|$. Then we have by the definition

$$\begin{aligned}
&|K_\alpha f(x) - K_\alpha f(y)| \\
&\le \int_{\mathbf{R}^d} |k_\alpha(x,z) - k_\alpha(y,z)|\,|f(z)|\, d\mu(z) \\
&\le \int_{B(x,2r)} |k_\alpha(x,z)|\,|f(z)|\, d\mu(z) + \int_{B(x,2r)} |k_\alpha(y,z)|\,|f(z)|\, d\mu(z) \\
&\quad + \int_{B(x,2r)^c} |k_\alpha(x,z) - k_\alpha(y,z)|\,|f(z)|\, d\mu(z) \\
&\le C \int_{B(x,2r)} \frac{|f(z)|}{|z-x|^{n-\alpha}}\, d\mu(z) + C \int_{B(y,3r)} \frac{|f(z)|}{|z-y|^{n-\alpha}}\, d\mu(z) \\
&\quad + C'\,|x-y|^\epsilon \int_{B(x,2r)^c} \frac{|f(z)|}{|z-x|^{n-\alpha+\epsilon}}\, d\mu(z) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad (13)
\end{aligned}$$

It is the same as (9) that we put $f_l(x) = \dfrac{1}{l^n} \displaystyle\int_{B(x,l)} |f| \, d\mu$. Firstly, in the same way as (10), we have that

$$\int_{B(x,2r)} \frac{|f(z)|}{|z-x|^{n-\alpha}} \, d\mu(z) = C \int_0^{2r} f_l(x) l^{\alpha-1} \, dl$$

and, using the formula (11), that

$$\mathrm{I} \le C \, \| f \, | \, \mathcal{M}^p_q(2,\mu) \| \int_0^{2r} l^{\alpha-n/p-1} \, dl = C \, \| f \, | \, \mathcal{M}^p_q(2,\mu) \| \, |x-y|^{\alpha-n/p}. \quad (14)$$

Similarly, noting $r = |x-y|$, we see that

$$\mathrm{II} \le C \, \| f \, | \, \mathcal{M}^p_q(2,\mu) \| \, |x-y|^{\alpha-n/p}. \quad (15)$$

Lastly, it follows that

$$\int_{B(x,2r)^c} \frac{|f(z)|}{|z-x|^{n-\alpha+\epsilon}} \, d\mu(z) = C \int_{2r}^{\infty} f_l(x) \, l^{\alpha-\epsilon-1} \, dl$$

and that

$$\begin{aligned} \mathrm{III} \;&\le\; C \, \| f \, | \, \mathcal{M}^p_q(2,\mu) \| \, |x-y|^{\epsilon} \int_{2r}^{\infty} l^{\alpha-n/p-\epsilon-1} \, dl \\ &=\; C \, \| f \, | \, \mathcal{M}^p_q(2,\mu) \| \, |x-y|^{\alpha-n/p}. \end{aligned} \quad (16)$$

From (13) and (14)–(16) we obtain the theorem. ∎

5 Boundedness of the singular integral operator

Finally, we investigate the boundedness of the singular integral operator whose definition is listed in [7].

Definition. Let μ and n be as above. The singular integral operator T is a bounded linear operator from $L^2(\mu)$ to $L^2(\mu)$ that satisfies the following:

There exists a function K that satisfies three properties listed below.

(1) There exists $C > 0$ such that $|K(x,y)| \le \dfrac{C}{|x-y|^n}$.

(2) There exist $\epsilon > 0$ and $C > 0$ such that

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le C\frac{|x-z|^\epsilon}{|x-y|^{n+\epsilon}},$$

if $|x-y| \ge 2|x-z|$.

(3) If f is a bounded μ-measurable function with a bounded support, then we have

$$Tf(x) = \int_{\mathbf{R}^d} K(x,y) f(y)\, d\mu(y) \text{ for all } x \notin \operatorname{supp}(f).$$

As for this singular integral operator T, the following result is known due to Nazarov, Treil and Volberg.

Proposition 5.1 ([7]). *T is a bounded operator from $L^p(\mu)$ to itself, if $1 < p < \infty$.*

In this section we shall extend this result to the Morrey spaces $\mathcal{M}_q^p(\mu)$.

Theorem 5.1. *For $k > 1$, T is a bounded operator from $\mathcal{M}_q^p(k,\mu) \cap L^2(\mu)$ to itself, if $1 < q \le p < \infty$.*

Proof. Fix $B := B(x,r) \subset \mathbf{R}^d$, $r > 0$, and take $f \in \mathcal{M}_q^p(k,\mu) \cap L^2(\mu)$. Decompose f according to $2B$, that is, $f = f_1 + f_2$ where $f_1 = \chi_{2B} f$.

For $y \in B$ we see by the definition that

$$|Tf_2(y)| \le \int_{(2B)^c} |K(y,z)|\,|f(z)|\, d\mu(z) \le C \int_{(2B)^c} \frac{|f(z)|}{|z-x|^n}\, d\mu(z).$$

Recall again that $f_l(x) = \dfrac{1}{l^n} \displaystyle\int_{B(x,l)} |f|\, d\mu$. Then we have that

$$\int_{(2B)^c} \frac{|f(z)|}{|z-x|^n}\, d\mu(z) = C \int_{2r}^\infty f_l(x) l^{-1}\, dl$$

and, using formula (11), that

$$|Tf_2(y)| \le C\, \|f \mid \mathcal{M}_q^p(2,\mu)\| \int_{2r}^\infty l^{-n/p-1}\, dl = C\, \|f \mid \mathcal{M}_q^p(2,\mu)\|\, r^{-n/p}. \quad (17)$$

Using Proposition 5.1 and (17), we obtain

$$\begin{aligned}
&\mu(Q(x,3r))^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |Tf(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\le \mu(Q(x,3r))^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbf{R}^d} |Tf_1(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\quad + C\,\mu(Q(x,3r))^{\frac{1}{p}-\frac{1}{q}} \mu(Q(x,r))^{\frac{1}{q}} r^{-n/p} \|f \mid \mathcal{M}^p_q(2,\mu)\| \\
&\le C\,\mu(Q(x,3r))^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q(x,2r)} |f|^q \, d\mu \right)^{\frac{1}{q}} \\
&\quad + C'\,\mu(Q(x,r))^{\frac{1}{p}} r^{-n/p} \cdot \|f \mid \mathcal{M}^p_q(2,\mu)\| \\
&\le C\,\|f \mid \mathcal{M}^p_q(k,\mu)\|.
\end{aligned}$$

(In the last relation we use the growth condition.) Hence, we obtain the theorem. ∎

This theorem can be easily extended to vector-valued one.

Corollary 5.1. *Suppose that $k > 1$, $1 < q \le p < \infty$ and $1 < r < \infty$. Then we have*

$$\left\| \|Tf_j \mid l^r\| \mid \mathcal{M}^p_q(k,\mu) \right\| \le C_{d,p,q,r,k,n} \left\| \|f_j \mid l^r\| \mid \mathcal{M}^p_q(k,\mu) \right\|.$$

Proof. To prove this, we proceed as in the last theorem. We indicate the necessary change.

For all j we decompose $f_j = f_{j,1} + f_{j,2}$ where $f_{j,1} = \chi_{2B} f_j$. The estimate for $f_{j,1}$ is quite the same, where we use the vector-valued version of Proposition 5.1 proved in [5]. For the estimate of $f_{j,2}$, we proceed in the same way as (17) and have

$$|T(f_{j,2})(y)| \le C \int_{(2B)^c} \frac{|f_j(z)|}{|z-x|^n} \, d\mu(z).$$

It follows from Minkowski's inequality that

$$\left(\sum_{j \in \mathbf{N}} |T(f_{j,2})(y)|^r \right)^{\frac{1}{r}} \le C \int_{(2B)^c} \frac{\left(\sum_{j \in \mathbf{N}} |f_j(z)|^r \right)^{\frac{1}{r}}}{|z-x|^n} \, d\mu(z).$$

The rest being the same, we omit the detail. ∎

Remark. Once these type of the estimates are obtained, those of the maximal operator of the truncated singular integral are easily obtained by Cotlar's inequality [7].

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Yoshihiro Sawano
Department of Graduate School of Mathematical Sciences,
University of Tokyo,
3-8-1 Komaba,
Tokyo 153-8914
JAPAN
E-email: yosihiro@ms.u-tokyo.ac.jp

Hitoshi Tanaka
Department of Graduate School of Mathematical Sciences,
University of Tokyo,
3-8-1 Komaba,
Tokyo 153-8914
JAPAN
E-email: htanaka@ms.u-tokyo.ac.jp

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012