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Abstract. In this paper we show that, for each Ekedahl-Oort stratum contained in the supersingular locus, the number of its irreducible components is equal to a class number of a quaternion unitary group.

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1 Introduction

Let $\mathcal{A}_g$ be the coarse moduli space $\mathcal{A}_{g,1,1} \otimes \mathbb{F}_p$ of principally polarized abelian varieties over a field of characteristic p.

F. Oort and T. Ekedahl defined a stratification on $\mathcal{A}_g$ called the Ekedahl-Oort stratification. Namely, $\mathcal{A}_g$ is divided into locally closed subschemes S_φ determined by elementary series φ of length g. An elementary series φ of length g is a map

$$\varphi : \{1, 2, \cdots, g\} \to \{0, 1, 2, \cdots, g\}$$

satisfying $\varphi(i-1) \leq \varphi(i) \leq \varphi(i-1)+1$ $(i = 1, 2, \cdots, g)$ with $\varphi(0) = 0$. Two principally polarized abelian varieties X and Y are in the same stratum if and only if there exists an isomorphism between their p-kernels $X[p]$ and $Y[p]$. (See [17] for the definition and fundamental theorems.)

A conjecture of F. Oort says that S_φ is irreducible unless S_φ is contained in the supersingular locus W_σ. And this became a theorem by combining a recent result of G. van der Geer and T. Ekedahl with a criterion of F. Oort: $\varphi([(g+1)/2]) = 0$ if and only if $S_\varphi \subset W_\sigma$. See [18, 7.5] for an exposition on this development.

On the other hand, F. Oort also expected in loc. cit. that the strata S_φ contained in W_σ are reducible for large p's. In this paper, we confirm this in an explicit way.

To describe our main theorem, we need some notations. We choose a supersingular elliptic curve E defined over $\mathbb{F}_p$ ([3] and also see [12, 1.2]). For each integer c with $0 \leq c \leq [g/2]$, we denote by Λ_c the set of the equivalence classes of polarizations μ on E^g such that $\ker \mu \simeq \alpha_p^{\oplus 2c}$. Here α_p is the kernel of the Frobenius map $F : \mathbb{G}_a \to \mathbb{G}_a$. For an element $\mu \in \Lambda_c$, let $\mathcal{T}_\mu$ be the fine moduli scheme of isogenies

$$\rho : (E^g, \mu) \to (Y, \lambda)$$

of polarized supersingular abelian varieties such that λ is a principal polarization. The moduli space $\mathcal{T}_\mu$ turns out to be non-singular irreducible of dimension $c(c+1)/2$ (Corollary 4.10).

Here are the main results proved in this paper (Theorems 5.6 and 6.19):

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1. For any integer c with $0 \leq c \leq [g/2]$, there exists a canonical quasi-finite surjective morphism
$$\Psi_c : \coprod_{\mu \in \Lambda_c} \mathcal{T}_\mu \longrightarrow \coprod_{\varphi(g-c)=0} S_\varphi.$$
2. For any elementary series φ with $\varphi(g-c) = 0$ and $\varphi(g-c+1) \neq 0$, the number of irreducible components of S_φ is equal to $\sharp\Lambda_c$. Moreover $\sharp\Lambda_c$ equals a class number $H_{g,c}$ of a quaternion unitary group (see Definition 5.8).

Since $\lim_{p\to\infty} \sharp\Lambda_c = \infty$ (Lemma 5.10), it follows that S_φ with $\varphi([(g+1)/2]) = 0$ is reducible for large p's.

The outline of this paper is as follows. In Section 2, we prepare some notations of Dieudonné modules and the Ekedahl-Oort stratification. After introducing good symplectic bases of supersingular Dieudonné modules, we show in Section 3 that the first jumping number of elementary series is given by an invariant $c(N)$ (Definition 3.7) of $N = M/pM$ in a certain class of Dieudonné modules M.

In Section 4, we investigate the moduli space $\mathcal{T}_\mu$ mentioned above. We shall construct finite étale morphisms from the affine open subschemes of $\mathcal{T}_\mu$ to the spaces $N_{g,c}$ of matrices.

After these preparations, by proving the equality of the invariant $c(N)$ and the height of minimal isogenies, we obtain the above morphism Ψ_c (Section 5).

In the last section, we investigate more closely the structures of the spaces $N_{g,c}$ of matrices. In particular, we can compute the codimension of the a-number locus $\overline{S}_\varphi(a)$ in the Zariski closure $\overline{S}_\varphi$ of S_φ. The main theorem follows from these calculations and some technical lemmas.

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2 Preliminaries

2.1 Dieudonné modules over a perfect field

We fix once and for all a rational prime p. Let K be a perfect field of characteristic p. We define a non-commutative ring A by the p-adic completion of
$$W(K)[F,V]/(FV-p, VF-p, Fa-a^\sigma F, Va-a^{\sigma^{-1}}V, \forall a \in W(K)).$$
Here σ is the Frobenius map on K.

Definition 2.1. *A Dieudonné module* is a left A-module M which is finitely generated as $W(K)$-module. If M is free as $W(K)$-module, we call M free. Two free Dieudonné module M and N are said to be isogenous if there is an A-homomorphism from M to N with torsion cokernel. We define a-number of M as
$$a(M) = \dim_K M/(F,V)M.$$
A free Dieudonné module M is called *supersingular* (resp. *superspecial*) if M is isogenous (resp. isomorphic) to $A_{1,1}^{\oplus g}$ for some g. Here $A_{1,1} := A/(F-V)$ and g is called the genus of M.

Definition 2.2. (1) Assume $g \geq 2$. *A superspecial abelian variety over K* is an abelian variety Y over K such that there is an isomorphism between Y and E^g over algebraically closed field $\overline{K}$ with supersingular elliptic curve E. This definition does not depend on choices of E as follows from results by Deligne, Ogus and Shioda ([15, Theorem 6.2] and [20, Theorem 3.5]). Also see [12, 1.6].

(2) An abelian variety X over K is said to be *supersingular* if and only if there exists an isogeny from E^g to X over algebraically closed field $\overline{K}$.

For an abelian variety over K, we have a free Dieudonné module $M := \mathbb{D}(X)$ of genus g by the covariant Dieudonné functor $\mathbb{D}$. Then the a-number of X:

$$a(X) := \dim_K \operatorname{Hom}(\alpha_p, X)$$

is equal to $a(M)$ (see [12, 5.2]).

A. Ogus proved the following important theorem, which he called supersingular Torelli's theorem ([15, Theorem 6.2]).

Theorem 2.3. *Assume that K is algebraically closed. Let $\mathcal{S}_g(K)$ be the category of supersingular abelian varieties over K. Assume $g \geq 2$. The functor $(\mathbb{D}, \mathrm{tr})$ gives a bijection between the set of isomorphism classes of $\mathcal{S}_g(K)$ and the set of supersingular Dieudonné modules M of genus g with trace map $\mathrm{tr} : \overset{2g}{\wedge} M \overset{\simeq}{\longrightarrow} W(K)$. Besides, for two objects X, Y of $\mathcal{S}_g(K)$, we have an isomorphism*

$$\operatorname{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \operatorname{Hom}_A(\mathbb{D}(X), \mathbb{D}(Y)).$$

The next lemma will be frequently used.

Lemma 2.4 (Lemma 3.1 in [11]). *For a supersingular Dieudonné module M, there exists a smallest superspecial Dieudonné module $S^0(M)$ in $M \otimes \operatorname{frac} W(K)$ containing M, and dually there is a biggest superspecial Dieudonné module $S_0(M)$ contained in M. Here $\operatorname{frac} W(K)$ stands for the field of fractions of $W(K)$.*

Corresponding to $S_0(M)$ of this lemma, for a supersingular abelian variety X over K, there exists a superspecial abelian variety Y over K and a K-isogeny $\rho : Y \to X$ such that for any superspecial abelian variety Y' over K and any K-isogeny $\rho' : Y' \to X$, there is a unique K-isogeny $\phi : Y' \to Y$ such that $\rho' = \rho \circ \phi$. We denote by $S^0(X)$ the pair (Y, ρ). The isogeny ρ is called minimal isogeny. Dually $S_0(X)$ are also defined. See [12, 1.8].

If X has a polarization $\lambda : X \to X^t$, we get the non-degenerate $W(K)$-bilinear alternating form

$$\langle\ ,\ \rangle : M \otimes_{W(K)} M \to W(K),$$

which satisfies $\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma$ by [14, p.101]. We call such an alternating form a quasi-polarization of M. If λ is principal, then $\langle\ ,\ \rangle$ is a perfect pairing.

2.2 Dieudonné modules over a general base

Let R be an $\mathbb{F}_p$-algebra and $S = \operatorname{Spec}(R)$.

Definition 2.5. *A Dieudonné module over S* is a locally free $W(R)$-module M with $W(R)$-linear homomorphism

$$F : M^{(p)} \to M, \qquad V : M \to M^{(p)}.$$

Here $M^{(p)}$ stands for the base change of M by the p-th power homomorphism $R \to R$. *A quasi-polarization on M* is a $W(R)$-bilinear alternating form

$$\langle\ ,\ \rangle : M \otimes_{W(R)} M \to W(R)$$

satisfying

$$\langle Fx, y\rangle = \langle x, Vy\rangle^{\sigma}$$

for $x \in M^{(p)}$ and $y \in M$.

2.3 Ekedahl-Oort stratification

In this subsection we give definitions and theorems related to the Ekedahl-Oort stratification used later on. For the details, see [17]. For brevity, we restrict ourselves to the case of principally quasi-polarized Dieudonné modules $(M, \langle\ ,\ \rangle)$ over perfect fields K.

Put $N = M/pM$, which is equipped with the $K[F, V]$-module structure. For a submodule S of N, we denote by $V^{-1}S$ the submodule $V^{-1}(S \cap VN)$ of N. We denote by $\mathcal{W}$ be the set of finite words of F and V^{-1}. It follows that the set

$$\{\varpi N \mid \varpi \in \mathcal{W}\}$$

consists of finite elements N_i $(i = 0, 1, \cdots, 2r)$ satisfying

$$0 = N_0 \subset \cdots \subset N_r \subset \cdots \subset N_{2r} = N \tag{1}$$

with $N_r = FN$. The filtration (1) is called *a canonical filtration of N* ([17, §2.2]).

Let us set $\rho(i) := \dim_k N_i$, $FN_i = N_{v(i)}$ and $V^{-1}N_i = N_{f(i)}$. Then we have $v(i)+f(i) = r+i$.

Definition 2.6. *The elementary series φ of M*, noted by $ES(M)$, is the map

$$\varphi : \{1, \cdots, g\} \to \{0, 1, \cdots, g\}$$

defined inductively as follows: For each $i = 1, \cdots, r$ and for all $\rho(i-1) < j \leq \rho(i)$,

$$\varphi(j) = \begin{cases} \varphi(j-1) + 1 & \text{if } v(i-1) < v(i), \\ \varphi(j-1) & \text{if } v(i-1) = v(i) \end{cases}$$

with $\varphi(0) := 0$. See [17, §5.6] for this definition. For a principally quasi-polarized Dieudonné module M over an arbitrary field, we denote by $ES(M)$ the elementary series of a scalar extension of M to an algebraically closed field.

The a-number of M is written as

$$a(M) = g - \varphi(g)$$

with $\varphi = ES(M)$.

Definition 2.7. For each elementary series φ as in the introduction, *the Ekedahl-Oort stratum* S_φ is the set of points of $\mathcal{A}_g$ which are associated with Dieudonné modules with the elementary series φ (see [17, §5.11]).

The stratum S_φ turns out to be a locally closed subscheme of $\mathcal{A}_g$ as shown by Oort ([17, Proposition 3.2]).

For an elementary series φ, the sequence

$$\psi : \{1, \cdots, 2g\} \to \{0, 1, \cdots, g\}$$

defined by

$$\begin{cases} \psi(i) := \varphi(i), \\ \psi(2g - i) := g + \varphi(i) - i \end{cases}$$

for all $i = 1, \cdots, g$ is called a final sequence of φ. *In this paper, for convenience we set*

$$\varphi(i) := \psi(i)$$

for all $i = g + 1, \cdots, 2g$.

By [17, Theorem 9.4], there exists a filtration refining the canonical filtration $\{N_i\}$:

$$0 = N'_0 \subset \cdots \subset N'_g \subset \cdots \subset N'_{2g} = N$$

such that $\dim_K N'_i = i$, $FN'_i = N'_{\varphi(i)}$ and $V^{-1}N'_i = N'_{g+i-\varphi(i)}$ for all $i = 1, \cdots, 2g$. In general, it is not unique (see [17, Remark 9.22]).

The following theorems are shown by Oort ([17, Theorem 9.4] and [17, §1]).

Theorem 2.8. *For two principally quasi-polarized Dieudonné modules M and M', the $K[F, V]$-modules $N = M/pM$ and $N' = M'/pM'$ are isomorphic over $\overline{K}$ if and only if $ES(M) = ES(M')$.*

Theorem 2.9. *For each elementary series φ,*

(1) the stratum S_φ is (regular as a stack) quasi-affine and the Zasiski closure $\overline{S}_\varphi$ of S_φ in $\mathcal{A}_g$ is connected;

(2) the dimension of any irreducible component of S_φ is equal to $|\varphi| = \sum_{i=1}^{g} \varphi(i)$;

(3) we have

$$\overline{S}_\varphi = \bigcup_{S_{\varphi'} \cap \overline{S}_\varphi \neq \emptyset} S_{\varphi'}.$$

There are two orderings $\prec$ and $\leq$ on the set of elementary series. For two elementary series φ and φ', we write $\varphi' \prec \varphi$ if $\varphi'(i) \leq \varphi(i)$ for all $i = 1, \cdots, g$, and write $\varphi' \leq \varphi$ if $S_{\varphi'} \cap \overline{S}_\varphi \neq \emptyset$. The second one $\leq$ becomes an order by Theorem 2.9 (3). By [17, Proposition 11.1], it follows that $\varphi' \prec \varphi$ implies $\varphi' \leq \varphi$.

3 Supersingular Loci

First of all let us investigate structure of the supersingular locus W_σ in $\mathcal{A}_g$. The number of irreducible components of W_σ and the dimension of each irreducible component of W_σ is determined in [12]. For our purpose, however, we need to analyze W_σ more explicitly.

3.1 Displays of supersingular Dieudonné modules

First let us recall the fact ([6, Lemma 3.5, 3.6]):

Proposition 3.1. *Assume K be an algebraically closed field. Then for any principally quasi-polarized supersigular Dieudonné module M over $W(K)$, there exist A-generators $v_0, \cdots, v_{g-1}$ of M such that*

$$\langle v_i, Fv_{g-1-j}\rangle = \varepsilon\delta_{ij}, \quad \langle v_i, v_j\rangle = 0, \quad (0 \leq i, j \leq g-1) \tag{2}$$

for fixed $\varepsilon = -\varepsilon^\sigma \in W(\mathbb{F}_{p^2})^\times$ and δ_{ij} Kronecker's delta, and

$$(F - V)v_i = \tau_{i,i+1}v_{i+1} + \tau_{i,i+2}v_{i+2} + \cdots + \tau_{i,g-1}v_{g-1}. \tag{3}$$

Here τ_{ij} are elements of $W(K)$, which automatically satisfy the symmetry:

$$\tau_{ij} = \tau_{g-1-j,g-1-i}. \tag{4}$$

Conversely, a principally quasi-polarized Dieudonné module which has A-generators

$$v_0, \cdots, v_{g-1}$$

satisfying the equations (2) and (3) is supersigular.

This proposition is paraphrased as follows. We introduce a new basis of M

$$\{X_1, \cdots, X_g, Y_1, \cdots, Y_g\}$$

defined by

$$X_i = v_{i-1}, \qquad Y_i = \varepsilon^{-1}Vv_{g-i}$$

for all $i = 1, 2, \cdots, g$. Then $\{X_1, \cdots, X_g, Y_1, \cdots, Y_g\}$ is a symplectic basis of M, i.e., a basis as $W(K)$-module satisfying

$$\langle X_i, Y_j\rangle = \delta_{ij}, \qquad \langle X_i, X_j\rangle = 0, \qquad \langle Y_i, Y_j\rangle = 0$$

for all $1 \leq i, j \leq g$. The display of M with respect to this basis (see [13] and [16] about displays) is written as

$$\begin{pmatrix} T & -\varepsilon^{-1}w \\ \varepsilon w & 0 \end{pmatrix} \tag{5}$$

where $w = (\delta_{i,g+1-j})$ and $T = (t_{ij})$ with

$$\begin{cases} t_{ij} = \tau_{j-1,i-1} & \text{if } 1 \leq j < i \leq g, \\ t_{ij} = 0 & \text{otherwise.} \end{cases}$$

The matrix T is strictly lower triangular and the equation (4) is equivalent to the symmetry condition:

$$Tw = {}^t(Tw). \tag{6}$$

Conversely for an arbitrary perfect field K, and for any strictly lower triangular matrix

$$T \in M_g(W(K))$$

satisfying the symmetry (6), we have a principally quasi-polarized supersingular Diendonné module, denoted by M_T, over $W(K)$ with display as (5). Hence there is a bijection from the set of strictly lower triangular matrices T satisfying the symmetry (6) with coefficients in $W(K)$ to the set of principally quasi-polarized supersingular Dieudonné modules over $W(K)$ with symplectic $W(K)$-basis $\{X_1, \cdots, X_g, Y_1, \cdots, Y_g\}$ satisfying

$$\begin{cases} (F-V)X_i = \displaystyle\sum_{j=i+1}^{g} t_{ji}X_j & \text{for some } t_{ji} \in W(K), \\ Y_i = \varepsilon^{-1}VX_{g+1-i}. \end{cases} \tag{7}$$

It follows ([6, Lemma 3.7]) that

$$a(M_T) = g - \operatorname{rk}\overline{T} \tag{8}$$

with $\overline{T} := T \bmod p$.

The semi-linear transformation for Frobenius map F on M_T is given by

$$\mathcal{F} := \begin{pmatrix} T & -p\varepsilon^{-1}w \\ \varepsilon w & 0 \end{pmatrix}.$$

Namely, we have

$$(FX_1, \cdots, FX_g, FY_1, \cdots, FY_g) = (X_1, \cdots, X_g, Y_1, \cdots, Y_g)\mathcal{F}.$$

Then F^n corresponds to the matrix $\mathcal{F}^{(n)} := \mathcal{F}\mathcal{F}^{\sigma} \cdots \mathcal{F}^{\sigma^{n-1}}$. Also the matrix for V is equal to

$$\mathcal{V} := \begin{pmatrix} 0 & -p\varepsilon^{-1}w \\ \varepsilon w & wT^{\sigma^{-1}}w \end{pmatrix}.$$

Similarly V^n is represented by $\mathcal{V}^{(n)} := \mathcal{V}\mathcal{V}^{\sigma^{-1}} \cdots \mathcal{V}^{\sigma^{-(n-1)}}$.

3.2 A certain class of supersingular Dieudonné modules

Let K be a perfect field. In this subsection, we investigate a certain class of principally quasi-polarized supersingular Dieudonné modules, i.e., we treat only those with display as in (5) satisfying

$$TT^{\sigma^i} = 0 \qquad \text{for all } i \in \mathbb{Z} \tag{9}$$

for some good choice of symplectic basis $\{X_i, Y_j\}$. The reason why we investigate a Dieudonné module with this type of T satisfying the condition (9) for some symplectic basis is that such and only such a principally quasi-polarized Dieudonné module M satisfies $S_\varphi \subset W_\sigma$ with $\varphi := ES(M)$ (the proof of this will be completed in Section 5).

Remark 3.2. For $g \geq 3$ there are principally quasi-polarized supersingular Dieudonné modules which never have displays as in (5) with condition (9).

Lemma 3.3. *Assume T satisfies the symmetry (6) and $TT^{\sigma^i} = 0$ for any $i \in \mathbb{Z}$. Then we have the following:*

(1)

$$\mathcal{F}^{(2n+1)} = \begin{pmatrix} p^n \sum_{j=0}^{n} T^{\sigma^{2j}} & -p^{n+1}\varepsilon^{-1}w \\ p^n \varepsilon w & -p^n w \sum_{j=0}^{n-1} T^{\sigma^{2j+1}} w \end{pmatrix},$$

(2)

$$\mathcal{F}^{(2n+2)} = \begin{pmatrix} p^{n+1} & p^{n+1}\varepsilon^{-1} \sum_{j=0}^{n} T^{\sigma^{2j}} w \\ p^n \varepsilon w \sum_{j=0}^{n} T^{\sigma^{2j+1}} & p^{n+1} \end{pmatrix},$$

(3)

$$\mathcal{V}^{(2n+1)} = \begin{pmatrix} -p^n \sum_{j=1}^{n} T^{\sigma^{-2j}} & -p^{n+1}\varepsilon^{-1}w \\ p^n \varepsilon w & p^n w \sum_{j=0}^{n} T^{\sigma^{-2j-1}} w \end{pmatrix},$$

(4)

$$\mathcal{V}^{(2n+2)} = \begin{pmatrix} p^{n+1} & -p^{n+1}\varepsilon^{-1} \sum_{j=1}^{n+1} T^{\sigma^{-2j}} w \\ -p^n \varepsilon w \sum_{j=0}^{n} T^{\sigma^{-2j-1}} & p^{n+1} \end{pmatrix}.$$

Proof. This lemma immediately follows by induction on n. We note that

$$p^m = \mathcal{F}^{(m)} (\mathcal{V}^{(m)})^{\sigma^m} = \mathcal{V}^{(m)} (\mathcal{F}^{(m)})^{\sigma^{-m}}$$

for every natural number m. □

Corollary 3.4. *Let M be a Dieudonné module associated with a T satisfying the symmetry (6) and $TT^{\sigma^i} = 0$ for every $i \in \mathbb{Z}$. Then for all $n \geq 0$,*

$$F^{2n+1}M \subset p^n M \quad \text{and} \quad V^{2n+1}M \subset p^n M.$$

We need a general lemma to understand $ES(M)$ for each M as in Corollary 3.4. For a $K[F,V]$-submodule S of $N = M/pM$, there uniquely exists an A-submodule $\widetilde{S}$ of M such that $pM \subset \widetilde{S} \subset M$ and $\widetilde{S}/pM = S$. Indeed the $W(K)$-module

$$\widetilde{S} := \{x \in M | x \bmod p \in S\}$$

is stable under the actions of F and V.

Lemma 3.5. *We have*

(1) $\widetilde{V^{-1}S} = \frac{1}{p}(F\widetilde{S} \cap pM)$,

(2) $\widetilde{FS} = F\widetilde{S} + pM$*. In particular if* $VM \subset \widetilde{S}$*, then* $\widetilde{FS} = F\widetilde{S}$*.*

Proof. (1) follows from the direct calculation:

$$\begin{aligned} \widetilde{V^{-1}S} &= \{x \in M | (x \bmod p) \in V^{-1}S\} = \{x \in M | (Vx \bmod p) \in S\} \\ &= V^{-1}\{Vx \in VM | (Vx \bmod p) \in S\} = V^{-1}(\tilde{S} \cap VM) = p^{-1}(F\tilde{S} \cap pM). \end{aligned}$$

Any element x of $\widetilde{FS} = \{x \in M | x \bmod p \in FS\}$ is of the form

$$x = pm + Fs$$

for some $m \in M$ and for some $s \in \tilde{S}$. That is to say, $\widetilde{FS} \subset F\tilde{S} + pM$. Conversely $F\tilde{S} + pM$ is contained in $\widetilde{FS}$ by definition. □

Proposition 3.6. *Let* M *be as in Corollary 3.4. We have*

(1) $\widetilde{(V^{-1}F)^j N} = \frac{1}{p^j}(F^{2j}M \cap pF^{2j-2}M \cap \cdots \cap p^j M)$*,*

(2) $VM \subset \widetilde{(V^{-1}F)^j N}$

for all $j \geq 0$*.*

Proof. First we show (1) implies (2). It suffices to show $p^j VM \subset p^l F^{2j-2l}M$ for all integral number $0 \leq l \leq j$. It is equivalent to $V^{2j-2l+1}M \subset p^{j-l}M$, which holds by Corollary 3.4.

We show (1) by induction on j. For $j = 0$, there is nothing to prove. Suppose that this lemma is true for $j-1$ with $j \geq 1$. Then it follows $VM \subset \widetilde{(V^{-1}F)^{j-1} N}$.

Applying the second statement of Lemma 3.5 (2) for $S = (V^{-1}F)^{j-1}N$, we have

$$F\widetilde{(V^{-1}F)^{j-1}N} = F\left\{\widetilde{(V^{-1}F)^{j-1}N}\right\} = \frac{1}{p^{j-1}}(F^{2j-1}M \cap pF^{2j-3}M \cap \cdots \cap p^{j-1}FM). \tag{10}$$

by the hypothesis of induction. Then by Lemma 3.5 (1) and the equation (10), we have

$$\begin{aligned} \widetilde{(V^{-1}F)^j N} &= p^{-1}(F\left\{F\widetilde{(V^{-1}F)^j N}\right\} \cap pM) \\ &= \frac{1}{p^j}(F^{2j}M \cap pF^{2j-2}M \cap \cdots \cap p^j M) \end{aligned}$$

as required. □

Since $N = M/pM$ is of finite length, we have a stabilizing filtration

$$\cdots \subset (V^{-1}F)^2 N \subset (V^{-1}F)N \subset N.$$

Hence $(V^{-1}F)^\infty N$ is defined. Let us introduce an invariant of N.

Definition 3.7. We set

$$c(N) := \dim N/(V^{-1}F)^\infty N.$$

Remark 3.8. By Proposition 3.6 (2), we have $c(N) = \dim FN/F(V^{-1}F)^\infty N$ under the same assumption as in Proposition 3.6.

Let $\mathcal{F}'^{(2n+2)}$ be a $g \times g$-matrix with entries in $W(K)$:

$$\mathcal{F}^{(2n+2)}/p^n = \begin{pmatrix} p & p\varepsilon^{-1}(T + T^{\sigma^2} + \cdots + T^{\sigma^{2n}})w \\ \varepsilon w(T^{\sigma} + T^{\sigma^3} + \cdots + T^{\sigma^{2n+1}}) & p \end{pmatrix}. \tag{11}$$

We denote by $\operatorname{Im} \mathcal{F}'^{(m)}$ the $W(K)$-submodule of M generated by entries of the vector

$$(X_1, \cdots, X_g, Y_1, \cdots, Y_g)\mathcal{F}'^{(m)}$$

Then immediately it follows from Proposition 3.6:

Corollary 3.9. *Let M be as in Corollary 3.4 and N be M/pM. Then we obtain*

$$(\widetilde{V^{-1}F})^j N = \frac{1}{p}\left(\operatorname{Im} \mathcal{F}'^{(2)} \cap \operatorname{Im} \mathcal{F}'^{(4)} \cap \cdots \cap \operatorname{Im} \mathcal{F}'^{(2j)} \cap pM\right) \tag{12}$$

and therefore

$$\dim (V^{-1}F)^j N = g + \dim \ker \overline{T}^{\sigma} \cap \ker \overline{T}^{\sigma^3} \cap \cdots \cap \ker \overline{T}^{\sigma^{2j-1}}$$

with $\overline{T} := T \bmod p$.

Proof. The first statement is a paraphrase of Proposition 3.6 (1). For the second, we investigate the composite ϕ of the natural inclusion and the natural projection:

$$\phi : (V^{-1}F)^j N \hookrightarrow N \to K < Y_1, \cdots, Y_g > .$$

By Proposition 3.6 (2) the map ϕ is surjective, since $Y_i \in VM$ for all $i = 1, \cdots, g$. By the equations (11) and (12), the dimension of the kernel of ϕ is calculated by using only the first g column vectors of $\mathcal{F}'^{(2i)}$ $(i = 1, \cdots, j)$. Explicitly it equals

$$\dim \ker \overline{T}^{\sigma} \cap \ker \overline{T}^{\sigma^3} \cap \cdots \cap \ker \overline{T}^{\sigma^{2j-1}}$$

as required. □

This corollary enable us to calculate the invariant $c(N)$.

Lemma 3.10. *Let c be an integer with $0 \leq c \leq [g/2]$. For a matrix $T = (t_{ij})$ satisfying $Tw = {}^t(Tw)$ with $t_{ij} = 0$ for $i \leq g - c$ or $j > c$, we consider the Dieudonné module M associated with T. Then it follows $c(N) \leq c$ with $N := M/pM$.*

Proof. For a matrix T as above, it is clear that $TT^{\sigma^i} = 0$ for all $i \in \mathbb{Z}$. Then $c(N) \leq c$ follows immediately from Corollary 3.9. □

Proposition 3.11. *Let M be as in Corollary 3.4. Set $\varphi := ES(M)$. Then it follows*

$$\begin{cases} \varphi(g - c(N)) = 0, \\ \varphi(g - c(N) + 1) = 1. \end{cases}$$

Proof. Let j be the minimal integer such that $(V^{-1}F)^j N = (V^{-1}F)^{\infty} N$. Then $(V^{-1}F)^{j+1}N = (V^{-1}F)^j N$ implies $F^2(V^{-1}F)^j N = 0$ and therefore $\varphi(g - c(N)) = 0$.

Suppose $\varphi(g - c(N) + 1) = 0$. Then taking a final filtration

$$0 = N_0 \subset \cdots \subset N_g \subset \cdots \subset N_{2g} = N,$$

we have

$$FN_{g-c(N)+1} = 0. \tag{13}$$

By the definition of j, it follows

$$N_{2g-c(N)} = (V^{-1}F)^j N \subsetneq N_{2g-c(N)+1} \subset (V^{-1}F)^{j-1}N. \tag{14}$$

Since $FN_{2g-c(N)+1} = N_{g-c(N)+1}$ by Remark 3.8, the equation (13) implies

$$(V^{-1}F)N_{2g-c(N)+1} = N_{2g-c(N)+1},$$

which contradicts (14) and $(V^{-1}F)^j N = (V^{-1}F)(V^{-1}F)^{j-1}N$. Hence $\varphi(g - c(N) + 1)$ has to be 1. $\square$

4 Moduli Space $\mathcal{T}_\mu$

In this section, we investigate the fine moduli space $\mathcal{T}_\mu$, which has already been introduced in [12, 9.11]. In particular we construct a finite étale morphism from $\mathcal{T}_\mu$ ($\mu \in \Lambda_c$) to the space $N_{g,c}$ of some matrices (see Definition 4.8 (i)). By using this morphism, we show that $\mathcal{T}_\mu$ is non-singular and irreducible (Corollary 4.10).

For each $c \leq [g/2]$, let Λ_c be the set of the equivalence classes of polarizations μ on E^g such that $\ker \mu \simeq \alpha_p^{\oplus 2c}$.

Definition 4.1. For $\mu \in \Lambda_c$ let $\mathcal{T}_\mu$ be the fine moduli scheme of isogenies

$$\rho : (E^g, \mu) \to (Y, \lambda)$$

of polarized supersingular abelian varieties such that

(i) $\mu = \rho^* \lambda$,

(ii) λ is a principal polarization.

For a given $c \leq [g/2]$, corresponding to the dual of (E^g, μ), we take a quasi-polarized superspecial Dieudonné module $(M_1, \langle\ ,\ \rangle_{M_1})$ satisfying $M_1/M_1^t \simeq K^{\oplus 2c}$, which will be shown to be unique up to isomorphism in Lemma 4.3 below.

Definition 4.2. For $(M_1, \langle\ ,\ \rangle_{M_1})$ as above, we define the moduli space $\mathcal{N}_c$ of isogenies of quasi-polarized Dieudonné modules

$$(M, \langle\ ,\ \rangle) \subset (M_1, \langle\ ,\ \rangle_{M_1})$$

satisfying

(i) $\langle\ ,\ \rangle$ is the restriction to M of $\langle\ ,\ \rangle_{M_1}$,

(ii) $\langle\ ,\ \rangle$ is a principal quasi-polarization.

For each $\mu \in \Lambda_c$, there exists a purely inseparable morphism from $\mathcal{T}_\mu$ to $\mathcal{N}_c$ over $\mathbb{F}_{p^4}$ by the same argument as [12, 7.4, 7.16], which is based on Li's theory of α-sheaves ([11, §3]).

We define a non-commutative ring

$$H := W(\mathbb{F}_{p^2})[F, V]/(F - V). \tag{15}$$

Let $\hat{M}_1$ denote the skeleton of M_1, i.e.,

$$\hat{M}_1 := \{v \in M_1 \mid (F - V)v = 0\}.$$

Then $\hat{M}_1$ is a H-module and we have $M_1 = \hat{M}_1 \otimes_{W(\mathbb{F}_{p^2})} W(K)$.

Lemma 4.3. *The quasi-polarized superspecial Dieudonné module $(M_1, \langle\ ,\ \rangle_{M_1})$ as above has A-generators $x_1, \cdots, x_g$ with $x_i \in \hat{M}_1$ such that*

$$\begin{array}{lll} \langle x_i, F^2 x_{g+1-j}\rangle = \varepsilon\delta_{ij}, & \langle x_i, F x_{g+1-j}\rangle = 0 & \text{for } 1 \le i, j \le c, \\ \langle x_i, F x_{g+1-j}\rangle = \varepsilon\delta_{ij}, & \langle x_i, x_{g+1-j}\rangle = 0 & \text{for } c < i, j \le [g/2], \\ \langle x_i, F x_{g+1-j}\rangle = 0, & \langle x_i, x_{g+1-j}\rangle = 0 & \text{otherwise.} \end{array} \tag{16}$$

In particular, such $(M_1, \langle\ ,\ \rangle_{M_1})$ are unique up to isomorphism.

Proof. Applying [12, Proposition 6.1] and then [12, Remark 6.1], we have A-generators $x_1, \cdots, x_g$ as above. □

Definition 4.4. We denote by $\tilde{\Phi}_c = \tilde{\Phi}(M_1)$ the set of $(x_1, \cdots, x_g)$ with $x_i \in \hat{M}_1$ satisfying (16). We say two elements $(x_1, \cdots, x_g)$ and $(x'_1, \cdots, x'_g)$ of $\tilde{\Phi}(M_1)$ are equivalent if $x_i \equiv x'_i \bmod pM_1$ for all $i = 1, \cdots, g$. Let $\Phi_c = \Phi(M_1)$ be a set of representatives of equivalence classes of elements of $\tilde{\Phi}(M_1)$ inductively chosen such that $(x_1, \cdots, x_g) \in \Phi(M_1)$ implies $(x_2, \cdots, x_{g-1}) \in \Phi(M'_1)$ for the principally quasi-polarized superspecial Dieudonné module M'_1 generated by $x_2, \cdots, x_{g-1}$.

Since $\tilde{M}_1/p\tilde{M}_1$ is a finite set, the set Φ_c is finite.

Definition 4.5. Let $\Theta = (x_1, \cdots, x_g)$ be an element of Φ_c. For an $\mathbb{F}_{p^4}$-algebra R, let $V^\Theta(R)$ be the subset of $\mathcal{N}_c(R)$ consisting M which has generators $X_1, \cdots, X_g$ of the form

$$\begin{cases} X_i = x_i + \sum_{j=g-c+1}^{g} \alpha_{ij} x_j & \text{for } i = 1, \cdots, c, \\ X_i = x_i & \text{for } i = c+1, \cdots, g-c, \\ X_i = F x_i^{(p)} & \text{for } i = g-c+1, \cdots, g \end{cases} \tag{17}$$

with $\alpha_{ij} \in W(R)$ and

$$\langle X_i, F X_{g+1-j}^{(p)}\rangle = \delta_{ij}, \quad \langle X_i, X_j\rangle = 0 \tag{18}$$

for all $1 \le i, j \le g$.

First we show:

Lemma 4.6. *The functor V^Θ is represented by an affine space of dimension $c(c+1)/2$.*

Proof. We show that V^Θ is represented by

$$\operatorname{Spec} \mathbb{F}_{p^4}[\xi_{ij}]/(\xi_{ij} - \xi_{g+1-j,g+1-i}) \tag{19}$$

where the ξ_{ij} are variables corresponding to $\overline{\alpha}_{ij} := \alpha_{ij} \bmod p$. Here the α_{ij} are coefficients of X_i, see (17). Any element of V^Θ is determined only by $\overline{\alpha}_{ij}$. It suffices to show for any element of V^Θ as in (17) there are only relations $\overline{\alpha}_{ij} = \overline{\alpha}_{g+1-j,g+1-i}$. The relations come only from the conditions (18). The non-trivial relations are

$$\langle X_i, X_{g+1-j}\rangle = 0$$

for $0 \leq i \leq c$ and $g+1-c \leq j \leq g$. Hence the calculation

$$\langle X_i, X_{g+1-j}\rangle = p^{-1}\left(\alpha_{ij} - \alpha_{g+1-j,g+1-i}\right)$$

ends the proof. □

Let us denote the affine scheme (19) by the same symbol V^Θ.

Lemma 4.7. *It follows that*

$$\mathcal{N}_c = \bigcup_{\Theta \in \Phi_c} V^\Theta.$$

Proof. The case of $c = 0$ is obvious because $c = 0$ implies M is a superspecial Dieudonné module. We suppose that $c \geq 1$.

We show this lemma by induction of g. Let R be an arbitrary F_{p^4}-algebra. Set $A_R = W(R)H$. Let $M \subset M_1$ be an element of $\mathcal{N}_c(R)$. Choose a $(x'_1, \cdots, x'_g) \in \Phi_c$. Without loss of generality, we may assume that there exists a surjective homomorphism

$$M \to A_R x'_1.$$

Then we have a self-dual complex

$$C^\cdot : A_R F x'_g \to M \to A_R x'_1$$

We take the cohomology $M' := H^1(C^\cdot)$ of $C^\cdot$. It is a supersingular Dieudonné module equipped with the principal quasi-polarization. Moreover if we put $M'_1 := A_R < x'_2, \cdots, x'_{g-1} >$, it follows that $(M' \subset M'_1)$ is an element of $\mathcal{N}_{c-1}(R)$ for genus $g-2$. Then by the assumption of induction, there exist $(x''_2, \cdots, x''_{g-1}) \in \Phi(M'_1)$ and generators $X'_2, \cdots, X'_{g-1}$ of M' such that

$$\begin{cases} X'_i = x''_i + \sum_{j=g-c+1}^{g-1} \alpha'_{ij} x''_j & \text{for } i = 2, \cdots, c, \\ X'_i = x''_i & \text{for } i = c+1, \cdots, g-c, \\ X'_i = F x''^{(p)}_i & \text{for } i = g-c+1, \cdots, g-1. \end{cases}$$

with $\alpha'_{ij} \in W(R)$ and

$$\langle X'_i, F X'^{(p)}_{g+1-i}\rangle = \delta_{ij}, \qquad \langle X'_i, X'_j\rangle = 0$$

for all $2 \leq i, j \leq g-1$.

By the definition of $\Phi(M_1)$, we can find an element $(x_1, \cdots, x_g)$ of $\Phi(M_1)$ satisfying $x_1 \equiv x'_1 \bmod p$, $x_g \equiv x'_g \bmod p$ and $x_i = x''_i$ $(2 \leq i \leq g-1)$.

The Dieudonné module Q generated by $x_i (i = c+1, \cdots, g-c)$ is a principally quasi-polarized Dieudonné submodule of M such that M/Q is a free Dieudonné module. We put $X_g = Fx_g^{(p)}$. For each i $(2 \leq i \leq c)$, we choose a lift X''_i of X'_i such that the coefficient of x_g is in $W(R)$, since $Fx_g^{(p)}$ is in M. Then there exists an element X''_1 of M such that X''_1 is mapped to x_1 by the map $M \to A_R x_1$, X''_1 have not terms of x_i $(i = 2, \cdots, g-c)$ and x_i-coefficients of X''_1 $(i = g-c+1, \cdots, g)$ are in $W(R)$. We can choose an element β of $W(R)$ such that $X_1 := X''_1 - \beta X_g$ satisfies $\langle X_1, FX_1^{(p)} \rangle = 0$. Set $X_i = X''_i - \langle FX_1^{(p)}, X''_i \rangle X_g$ for $i = 2, \cdots, c$. For $g-c+1 \leq i \leq g-1$, we put $X_i = Fx_i^{(p)}$. Thus we have generators $X_1, \cdots, X_g$ of M satisfying (17) and (18). $\square$

From now on, let us take generators $X_1, \cdots, X_g$ of M satisfying (17) and (18) for each point $(M \subset M_1)$ of V^Θ. Let R be an $\mathbb{F}_{p^4}$-algebra. For a point $(M \subset M_1) \in V^\Theta(R)$, let us define a $g \times g$-matrix $T = (t_{ij})$ by

$$FX_i^{(p)} - VX_i^{(p^{-1})} = \sum_{j=i+1}^{g} t_{ji} X_j. \tag{20}$$

Then $t_{ij} \in W(R)$. By the equation (18), the matrix Tw is automatically symmetric. We note that $t_{ij} = 0$ for $i \leq g-c$ or $j > c$. This means that M has a symplectic basis

$$\{X_1, \cdots, X_g, Y_1, \cdots, Y_g\}$$

for which the display of M is given by

$$\begin{pmatrix} T & -\varepsilon^{-1} w \\ \varepsilon w & 0 \end{pmatrix}$$

with T of the form

$$\begin{pmatrix} & 0 & & 0 \\ & & & \\ t_{g-c+1,1} & \cdots & t_{g-c+1,c} & \\ \vdots & & \vdots & 0 \\ t_{g,1} & \cdots & t_{g,c} & \end{pmatrix}.$$

For such a M, we can apply all results in Section 3.

Definition 4.8. (i) An affine scheme $N_{g,c}$ is defined by

$$N_{g,c}(R) := \left\{ \mathfrak{T} = (\mathsf{t}_{ij}) \in M_g(R) \;\middle|\; \mathfrak{T}w = {}^t(\mathfrak{T}w),\ \mathsf{t}_{ij} = 0\ (i \leq g-c \text{ or } j > c) \right\}$$

with $w = (\delta_{i,g+1-j})$.

(ii) Let h^Θ denote the morphism from V^Θ to $N_{g,c}$ sending $(M \subset M_1)$ to $\overline{T} = (\overline{t}_{ij})$ defined by equation (20). Here $\overline{t}_{ij} := t_{ij} \bmod p$.

Proposition 4.9. *The morphism $h^\Theta : V^\Theta \to N_{g,c}$ is finite and étale.*

Proof. With the notation in the proof of Lemma 4.6, the morphism h^Θ is given by

$$V^\Theta = \operatorname{Spec} \mathbb{F}_{p^4}[\xi_{ij}]/(\xi_{ij} - \xi_{g+1-j,g+1-i}) \to N_{g,c} = \operatorname{Spec} \mathbb{F}_{p^4}[\overline{t}_{ij}]/(\overline{t}_{ij} - \overline{t}_{g+1-j,g+1-i})$$

which on affine rings corresponds to the homomorphism sending $\overline{t}_{ij}$ to $\xi_{ij}^{p^2} - \xi_{ij}$ by the equations (20). $\square$

Corollary 4.10. *The moduli space $\mathcal{N}_c$ and therefore $\mathcal{T}_\mu$ ($\forall \mu \in \Lambda_c$) are projective non-singular geometrically integral varieties of dimension $c(c+1)/2$.*

Proof. The moduli space $\mathcal{N}_c$ is a closed subvariety of the Grassmann variety $\mathrm{Gr}_{c,2c}$, since the condition of polarization is closed. Hence $\mathcal{N}_c$ is projective.

Since $\mathcal{N}_c$ is covered by affine space V^Θ, it suffices to show $\mathcal{N}_c$ is connected. We show this by induction of g. Indeed by the proof of Lemma 4.7, $\mathcal{N}_c$ is a fiber space of $\mathcal{N}_{c-1}$ for genus $g-2$. The fiber of each point is given by the fiber of $V^\Theta \to V^{\Theta'}$ for some $\Theta = \{x_1, \cdots, x_g\}$ with $\Theta' = \{x_2, \cdots, x_{g-1}\}$. Hence each fiber is identified with the affine space $\mathbb{A}^c$ with coordinates $\overline{\alpha}_{1i}$ ($i = g-c+1, \cdots, g$), which is connected. $\square$

We will use the next lemma when we construct the morphism Ψ_c mentioned in the introduction.

Lemma 4.11. *Let c and c' be integers with $c < c' \le [g/2]$. For any element $\rho : (E^g, \mu) \to (Y, \lambda)$ of $\mathcal{T}_\mu(K)$ with $\mu \in \Lambda_c$, there exists an element $\rho' : (E^g, \mu') \to (Y, \lambda)$ of $\mathcal{T}_{\mu'}(K)$ with $\mu' \in \Lambda_{c'}$.*

Proof. It suffices to show this lemma only for $c' = c+1$. Let M be the associated quasi-polarized Dieudonné module $\mathbb{D}(Y)$. By Lemma 4.7, corresponding to $\rho^t : Y^t \to (E^g)^t$, we have an inclusion

$$M \subset M_1$$

with M_1 generated by $x_1, \cdots, x_g$ for an element $(x_1, \cdots, x_g) \in \Phi_c$.

Let us define elements x'_i ($i = 1, \cdots, g$) of $M_1 \otimes \mathrm{frac}(W(K))$ by $x'_{c'} = F^{-1} x_{c'}$ and $x'_i = x_i$ for $i \neq c'$ and denote by M'_1 the principally quasi-polarized superspecial Dieudonné module generated by $x'_1, \cdots, x'_g$. Then $M \subset M'_1$ is an element of $\mathcal{N}_{c'}$.

Let f^t be the isogeny $E^g \to E^g$ corresponding to $M_1 \subset M'_1$. Then

$$\rho \circ f : (E^g, \mu') \to (Y, \lambda)$$

with $\mu' := (\rho \circ f)^* \lambda$ is an element of $\mathcal{T}_{\mu'}(K)$. $\square$

Definition 4.12. For an elementary series φ, let $\mathcal{T}_\mu(\varphi)$ be the subspace of $\mathcal{T}_\mu$ consisting of

$$(E^g, \mu) \to (Y, \lambda)$$

with $ES(Y) = \varphi$.

In the next section, we will show that $\varphi([(g+1)/2]) = 0$ if and only if $\mathcal{T}_\mu(\varphi) \neq \emptyset$ ($\mu \in \Lambda_c$) for some $c \le [g/2]$.

5 Ekedahl-Oort stratification contained in supersingular locus

In this section, we give a lower bound of number of irreducible components of Ekedahl-Oort strata contained in supersingular locus W_σ.

5.1 Oort's criterion

In this subsection, we determine which S_φ is contained in supersingular locus W_σ.

Lemma 5.1. *For any principally quasi-polarized Dieudonné module M with $\varphi([(g+1)/2]) = 0$ with $\varphi = ES(M)$, we have*

$$F^{2n+1}M \subset p^n M \quad \text{and} \quad V^{2n+1}M \subset p^n M$$

for all $n \geq 0$. In particular M is supersingular.

Proof. Let M be a principally quasi-polarized Dieudonné module satisfying $\varphi([(g+1)/2]) = 0$ with $\varphi = ES(M)$. Put $N = M/pM$.

We show the next claim by induction of n.

Claim. For any $l \leq n$, we have $F^{2l+1}M \subset p^l M$. Moreover it follows

(n-i) $(\widetilde{V^{-1}F})^n FN = \sum_{l=0}^{n} p^{-l}F^{2l+1}M$,

(n-ii) $\dim(V^{-1}F)^n FN \leq g + [(g+1)/2]$.

Proof of Claim. It is obvious for $n = 0$. Assume it holds for n. Let us compute $(\widetilde{V^{-1}F})^{n+1}FN$. Since $\varphi([(g+1)/2]) = 0$ implies

$$\varphi(g + [(g+1)/2]) = \varphi(2g - [g/2]) = g + \varphi([g/2]) - [g/2] = [(g+1)/2],$$

from (n-(ii)) we get

$$\dim F(V^{-1}F)^n FN \leq [(g+1)/2]$$

and therefore $F^2(V^{-1}F)^n FN = 0$. This means

$$F\left\{F(\widetilde{V^{-1}F})^n FN\right\} \subset pM \tag{21}$$

with

$$F(\widetilde{V^{-1}F})^n FN = pM + \sum_{l=0}^{n} p^{-l}F^{2l+2}M$$

by Lemma 3.5 (2). In particular, we have $F^{2(n+1)+1}M \subset p^{n+1}M$. Applying Lemma 3.5 (1) for $S = F(V^{-1}F)^n FN$, we obtain

$$(\widetilde{V^{-1}F})^{n+1}FN = \sum_{l=0}^{n+1} p^{-l}F^{2l+1}M.$$

By the inclusion (21), we have

$$\dim(V^{-1}F)^{n+1}FN = g + \dim F(V^{-1}F)^n FN,$$

which is at most $g + [(g+1)/2]$. $\square$

For a Dieudonné submodule Q of M, let us denote by $\perp Q$ the Dieudonné submodule of $M \otimes \mathrm{frac}(W(K))$:

$$\{v \in M \otimes \mathrm{frac}(W(K)) | \langle v, Q\rangle \subset pM\}.$$

Then by the inclusion $F^{2n+1}M \subset p^n M$, we obtain

$$\begin{aligned} V^{2n+1}M &= p^{n-1}V^{2n+1}(p^{-n+1}M) = p^{n-1}V^{2n+1}(\perp p^n M) \\ &\subset p^{n-1}V^{2n+1}(\perp F^{2n+1}M) = p^{n-1}V^{2n+1}(V^{-(2n+1)} \perp M) \\ &= p^{n-1}(\perp M) = p^n M. \end{aligned}$$

By [2, Chapter IV §5], the property

$$F^{2n+1}M \subset p^n M \quad (\forall n = 1, 2, \cdots)$$

implies that M is supersingular. □

The following proposition is due to F. Oort[1]. Because his proof is not published, we give a proof here.

Proposition 5.2 (Oort's criterion). *We have $\varphi([(g+1)/2]) = 0$ if and only if $S_\varphi \subset W_\sigma$.*

Proof. If $\varphi([(g+1)/2]) \neq 0$, then there exists a curve C with generic point in S_φ and a special point in S_{φ_d} with $d < [(g+1)/2]$, where φ_d is the elementary series with exactly d zeros and $(g-d)$ ones. Recall the intersection of S_{φ_d} and W_σ is empty, by the classification of p-divisible groups with a-number $g-1$ in [17, §8]. Since the supersingular locus W_σ is closed, it has to follow that $S_\varphi \not\subset W_\sigma$.

Suppose $\varphi([(g+1)/2]) = 0$. Then every principally quasi-polarized Dieudonné module M with $ES(M) = \varphi$ is supersingular by Lemma 5.1. □

5.2 Construction of the morphism Ψ_c

Let us construct the morphism Ψ_c mentioned in the introduction.

Assume K is a perfect field containing $\mathbb{F}_{p^4}$.

Proposition 5.3. *Let M be a principally quasi-polarized Dieudonné module over $W(K)$ satisfying $\varphi([(g+1)/2]) = 0$ with $\varphi := ES(M)$. Then there exists a quasi-polarized superspecial Dieudonné module M_1 such that $(M \subset M_1)$ is an element of $\mathcal{N}_c(K)$ for some c (see Definition 4.2 for $\mathcal{N}_c$). Moreover we can take $S^0(M)$ as M_1.*

Proof. For any M as above, M is supersingular by Proposition 5.2. It suffices to show that $S^0(M)/S_0(M)$ is a K-vector space.

Recall that $V^{2n+1}M$ are contained in $p^n M$ for all $n \geq 0$ (Lemma 5.1). Then we have

$$F^i V^{g-1-i} M \subset F^{g-2} M \qquad 0 \leq \forall i \leq g-1. \tag{22}$$

Indeed $F^i V^{g-1-i} M \subset F^{g-2}M$ is equivalent to $V^{2g-3-2i} \subset p^{g-2-i}M$.

Since $S^0(M) = F^{1-g}(F, V)^{g-1}M$ ([11, Corollary 1.7]), we have

$$FS^0(M) \subset M,$$

by the inclusion (22). Since $S_0(M)$ is the biggest superspecial Dieudonné module contained in M, we have $FS^0(M) \subset S_0(M)$, which implies that $S^0(M)/S_0(M)$ is a K-vector space. □

[1] Private communication.

Definition 5.4. We define a map γ from the set of principally quasi-polarized supersingular Dieudonné module over $W(K)$ to $\mathbb{Z}_{\geq 0}$ by

$$\gamma(M) := \frac{1}{2}\,\mathrm{length}_K\, S^0(M)/S_0(M).$$

Proposition 5.5. *Let M be as in Proposition 5.3. Put $N = M/pM$. Then we have an equation*

$$\gamma(M) = c(N).$$

(See Definition 3.7 for $c(N)$.) In particular, $\gamma(M)$ is an invariant of N.

Proof. By Proposition 5.3, we have $(M \subset S^0(M)) \in \mathcal{N}_c(K)$ with $c = \gamma(M)$. By Lemma 4.7, we can choose A-generators $X_1, \cdots, X_g$ of M such that

(i) $X_1, \cdots, X_g, Y_1, \cdots, Y_g$ is a symplectic basis of M with $Y_i := \varepsilon^{-1} V X_{g+1-i}$,

(ii) for an element $(x_1, \cdots, x_g)$ of $\Phi(S^0(M))$ (see Definition 4.4), we have

$$\begin{cases} X_i = x_i + \sum_{j=g-c+1}^{g} \alpha_{ij} x_j & \text{for } i = 1, \cdots, c, \\ X_i = x_i & \text{for } i = c+1, \cdots, g-c, \\ X_i = F x_i & \text{for } i = g-c+1, \cdots, g. \end{cases}$$

Recall $S^0(M) = F^{1-g}(F, V)^{g-1} M$. Namely $S^0(M)$ is generated over $W(K)$ by $F^{-j} V^j X_i$ ($0 \leq j \leq g-1$ and $i = 1, \cdots, g$). Then

$$S := \{X_1, \cdots, X_{g-c}, F x_{g-c+1}, \cdots, F x_g\}$$

and

$$F^{-l} V^{l-1} (F - V) X_i \qquad (1 \leq l \leq g-1, \text{ and } i = 1, \cdots, g)$$

generate $S^0(M)$ over $W(K)$.

By using the equations

$$(F - V) X_i = \sum_{j=g-c+1}^{g} t_{ji} F x_j \qquad (1 \leq i \leq c),$$

the superspecial Dieudonné module $S^0(M)$ is generated by elements of S and

$$F^{-l} V^{l-1} (F - V) X_i = \sum_{j=g-c+1}^{g} t_{ji}^{\sigma^{-(2l-1)}} x_j$$

for all $l = 1, \cdots, g-1$. Here the matrix $T = (t_{ij})$ is of the form

$$T = \begin{pmatrix} & 0 & & 0 \\ & & & \\ t_{g-c+1,1} & \cdots & t_{g-c+1,c} & \\ \vdots & & \vdots & 0 \\ t_{g,1} & \cdots & t_{g,c} & \end{pmatrix}.$$

On the other hand, $S^0(M)$ is generated by $x_1, \cdots, x_g, Fx_1, \cdots, Fx_g$ over $W(K)$. Hence the column vectors of all of

$$\overline{T}^{\sigma^{-1}},\ \overline{T}^{\sigma^{-3}},\ \cdots,\ \overline{T}^{\sigma^{-(2g-3)}}$$

generate a K-vector space of dimension c. This is equivalent to $c(N) = c$ by Corollary 3.9 $\square$

Theorem 5.6. *For each $c \leq [g/2]$, there exists a canonical quasi-finite surjective morphism*

$$\Psi_c : \coprod_{\mu \in \Lambda_c} \mathcal{T}_\mu \to \coprod_{\varphi(g-c)=0} S_\varphi.$$

Here $\coprod_{\varphi(g-c)=0} S_\varphi$ stands for the closed subscheme of the supersingular locus W_σ whose closed points correspond to abelian varieties Y satisfying $\varphi(g-c) = 0$ with $\varphi = ES(Y)$.

Proof. It suffices to show the image of the canonical quasi-finite morphism

$$\coprod_{\mu \in \Lambda_c} \mathcal{T}_\mu \to W_\sigma$$

is precisely equal to

$$\coprod_{\varphi(g-c)=0} S_\varphi.$$

Let $E^g \to Y$ be a point of $\mathcal{T}_\mu(K)$ ($\mu \in \Lambda_c$) with a perfect field K. Set $M := \mathbb{D}(Y)$ and $N := M/pM$. By Lemma 4.7, there exists a $\Theta \in \Phi_c$ such that $(M \subset M_1) \in V^\Theta(K)$. Let $T = (t_{ij})$ be the associated matrix $h^\Theta(M \subset M_1)$ (see Definition 4.8 (ii)). By Proposition 3.11, we have $\varphi(g - c(N)) = 0$. Since $c(N) \leq c$ (Lemma 3.10), we have $\varphi(g - c) = 0$.

Let (Y, λ) be a principally polarized abelian variety satisfying $\varphi(g-c) = 0$ with $\varphi := ES(M)$ and $M := \mathbb{D}(Y)$. Then the condition $c \leq [g/2]$ implies $\varphi([(g+1)/2]) = 0$. Applying Proposition 5.3, we get an element of $\mathcal{N}_{\gamma(M)}$:

$$M \subset S^0(M).$$

By Proposition 5.5 and Proposition 3.11, we have

$$\gamma(M) = c(N) \leq c.$$

Hence (Y, λ) is in the image of $\mathcal{T}_\mu$ for some $\mu \in \Lambda_c$ by Lemma 4.11. $\square$

Lemma 5.7. *Let (Y, λ) be a geometric point of W_σ in images of two different $\mathcal{T}_\mu$ and $\mathcal{T}_{\mu'}$ with $\mu, \mu' \in \Lambda_c$. Then we have $\gamma(\mathbb{D}(Y)) < c$.*

Proof. Assume $\gamma(\mathbb{D}(Y))$ was equal to c. Let us denote by ρ_0 the minimal isogeny $E^g \to Y$, which is unique up to isomorphism. By the assumption, ρ_0 is of degree p^c. Set $\mu_0 = \rho_0^* \lambda$. Let $\rho : (E^g, \mu) \to (Y, \lambda)$ be a point of $\mathcal{T}_\mu$. Then ρ has to factor ρ_0 by the minimality of ρ_0. Since ρ_0 and ρ have the same degree, ρ and ρ_0 are the same up to automorphism of E^g, that is to say, μ is equivalent to μ_0. By the same reason, μ' is equivalent to μ_0, which contradicts. $\square$

Let us recall the definition of class numbers of the quaternion unitary group G:

$$G = \{h \in GL_g(B) \mid {}^t\overline{h}h = \lambda(h)1_g, \lambda(h) \in \mathbb{Q}\}$$

with the quaternion algebra B ramified only at p and ∞ over $\mathbb{Q}$.

Definition 5.8. For $0 \leq c \leq [g/2]$, the class number $H_{g,c}$ is defined to be

$$\sharp\ G(\mathbb{Q})\backslash G(\mathbb{A}_f)/P_{c,f}.$$

Here $P_{c,f}$ is the product of parahoric subgroups $\prod_{l:\text{prime}} P_l$ defined by

$$\delta_l^{-1} P_l \delta_l := \{h \in GL_g(\mathcal{O}_{B,l}) \mid {}^t\overline{h} f_l h = \lambda(h) f_l\}$$

where $f_l = 1_g$, $\delta_l = 1_g$ for $l \neq p$ and

$$f_p := \mathrm{diag}(\underbrace{1, \cdots, 1}_{g-2c}, \underbrace{\begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix}}_{c}) \tag{23}$$

with an element δ_p of $GL_g(\mathcal{O}_{B,p})$ satisfying ${}^t\overline{\delta}_p\delta_p = f_p$. For example, it suffices to take

$$\delta_p = \mathrm{diag}(\underbrace{1, \cdots, 1}_{g-2c}, \underbrace{\begin{pmatrix} a & bF \\ b & aF \end{pmatrix}, \cdots, \begin{pmatrix} a & bF \\ b & aF \end{pmatrix}}_{c})$$

where $a, b \in W(\mathbb{F}_{p^2})$ is defined by $a = y^{-1}$ and $b = y^{-1}x$ with a solution $(x, y) \in W(\mathbb{F}_{p^2})^{\oplus 2}$ of

$$\begin{cases} x^\sigma x = -1, \\ y^\sigma y = x^\sigma + x. \end{cases}$$

By using the class number $H_{g,c}$, we have a lower bound of the number of irreducible components of each Ekedahl-Oort stratum S_φ contained in supersingular locus W_σ.

Proposition 5.9. *Assume $\varphi(g-c) = 0$ and $\varphi(g-c+1) = 1$ with $c \leq [g/2]$. Then the number of irreducible components of S_φ is greater than or equal to the class number $H_{g,c}$.*

Proof. For each $\mu \in \Lambda_c$, there is at least one irreducible component Z of S_φ such that there is a surjective map from an irreducible component of $\mathcal{T}_\mu(\varphi)$ to Z. By Proposition 3.11, Proposition 5.5 and Lemma 5.7, there is no other $\mu' \in \Lambda_c$ such that there is a surjective map from an irreducible component of $\mathcal{T}_{\mu'}(\varphi)$ to Z. Hence the number of irreducible components of S_φ is at least $\sharp\Lambda_c$.

By the same argument as [9, §2] and [12, Chapter 8], we have a canonical bijection

$$\Lambda_c \xrightarrow{\sim} G(\mathbb{Q})\backslash G(\mathbb{A}_f)/P_{c,f}.$$

Hence the number $\sharp\Lambda_c$ is equal to the class number $H_{g,c}$. $\square$

In the next section, it will be shown that the number of irreducible components of S_φ equals the class number $H_{g,c}$.

There is an estimate of the class number $H_{g,c}$ by the mass $\mathfrak{m}_{g,c}$ of G for genus f_p at p:

$$H_{g,c} \geq 2\mathfrak{m}_{g,c}.$$

Here the mass $\mathfrak{m}_{g,c}$ is defined to be

$$\sum_{h \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/P_{c,f}} \frac{1}{\sharp(hP_{c,f}h^{-1} \cap G(\mathbb{Q}))},$$

which in fact termwise equals

$$\sum_{\mu \in \Lambda_c} \frac{1}{\sharp \operatorname{Aut}(E^g, \mu)}.$$

Furthermore we have the mass formula:

Lemma 5.10. *The mass $\mathfrak{m}_{g,c}$ is equal to*

$$\prod_{i=1}^{g} \frac{(2i-1)!\zeta(2i)}{(2\pi)^{2i}} \cdot \binom{g}{2c}_{p^2} \cdot \prod_{i=1}^{g-2c} (p^i + (-1)^i) \prod_{i=1}^{c} (p^{4i-2} - 1)$$

with the Riemann zeta function $\zeta(s)$ and the q-binomial coefficients

$$\binom{g}{r}_q := \frac{\prod_{i=1}^{g}(q^i-1)}{\prod_{i=1}^{r}(q^i-1)\prod_{i=1}^{g-r}(q^i-1)} \quad \in \mathbb{Z}[q].$$

Proof. For $g = 1$, this is none other than Deuring's mass formula [3] because c has to be 0. Suppose $g \geq 2$. Since the class number $M_g(B)$ is one for $g \geq 2$ (Eichler [4] and also see [9, Theorem 2.1]), the mass $\mathfrak{m}_{g,c}$ is equal to that associated with the group without similitude

$$G^1 = \{h \in GL_g(B) \mid {}^t\overline{h}h = 1_g\}.$$

Applying Prasad's mass formula [19] to this group G^1, we have

$$\mathfrak{m}_{g,c} = \prod_{i=1}^{g} \frac{(2i-1)!}{(2\pi)^{2i}} \cdot \prod_{l \neq p} \frac{l^{g(2g+1)}}{\sharp Sp_{2g}(\mathbb{F}_l)} \cdot \frac{p^{(\dim \overline{M}_p + g(2g+1))/2}}{\sharp \overline{M}_p(\mathbb{F}_p)} \tag{24}$$

where $\overline{M}_p(\mathbb{F}_p)$ is the Levi subgroup of $\overline{P}_p := P_p \bmod p$. Then $\overline{M}_p(\mathbb{F}_p)$ is isomorphic to the subgroup of $GL_g(\mathbb{F}_{p^2})$ consisting X satisfying

$${}^tX^\sigma f_p X = f_p$$

in $M_g(\mathbb{F}_{p^2}[F]/(Fa = a^\sigma F, a \in \mathbb{F}_{p^2}))$. Then from the direct computation by using the block expression of matrices, we have

$$\overline{M}_p(\mathbb{F}_p) = U_{g-2c}(\mathbb{F}_{p^2}) \times Sp_{2c}(\mathbb{F}_{p^2})$$

with the unitary group

$$U_m(\mathbb{F}_{p^2}) := \{A \in M_m(\mathbb{F}_{p^2}) | {}^tA^\sigma A = 1_m\}$$

in the notation of [1]. In particular we have $\dim \overline{M}_p = (g-2c)^2 + 2(2c^2+c)$. By the formulae

$$\begin{aligned} \sharp Sp_{2m}(\mathbb{F}_q) &= q^{m(2m+1)} \prod_{i=1}^{m} (1 - q^{-2i}), \\ \sharp U_m(\mathbb{F}_{q^2}) &= q^{m^2} \prod_{i=1}^{m} (1 - (-1)^i q^{-i}), \end{aligned}$$

(see [1, Chapter 1] for example), we obtain the desired equality. □

By Proposition 5.9 and Lemma 5.10, it follows:

Corollary 5.11. *For any elementary series φ satisfying $\varphi([(g+1)/2]) = 0$, the Ekedahl-Oort stratum S_φ is reducible for sufficient large p's.*

Remark 5.12. In the notation of [12, 4.6,4.7], $H_{g,0}$ is the class number $H_g(p,1)$ of G for the principal genus and $H_{g,[g/2]}$ is the class number $H_g(1,p)$ of G. Also compare Lemma 5.10 with the computation (Proposition 9 of [7] (I)) of the mass $\mathfrak{m}_{g,0}$. Although K. Hashimoto and T. Ibukiyama explicitly calculate $H_{2,0}$ and $H_{2,1}$ in [7], it seems difficult to get the explicit formula of class numbers $H_{g,c}$ for higher g's. See [9], [10] and [8] for closer investigations for $g = 2$. However for any elementary series φ satisfying $\varphi(g-c) = 0$ and $\varphi(g-c+1) = 1$, the number of irreducible components of S_φ *as a stack* is equal to the mass $\mathfrak{m}_{g,c}$ (by using Theorem 6.19 below). In other words, for a natural number n such that $(n,p) = 1$ and $n \geq 3$, the number of irreducible components of a variant $S_{\varphi,n}$ with level n-structure is equal to $\sharp Sp_{2g}(\mathbb{Z}/n\mathbb{Z}) \cdot \mathfrak{m}_{g,c}$.

5.3 Examples

In this subsection, we give some examples of S_φ contained in W_σ. By using such examples, we can give a geometric proof of Proposition 3.11, which was used only in the proof of Theorem 5.6.

Lemma 5.13. *For a natural number r less than or equal to $[g/2]$, let M be a supersingular Dieudonné module associated with a $g \times g$-matrix $T = (t_{ij})$ of rank r with $t_{ij} = 0$ for $i \leq g - r$ or for $j > r$. Then we have*

$$ES(M) = \varphi_r^{\mathrm{top}}$$

with

$$\varphi_r^{\mathrm{top}} := (0, \cdots, 0, 1, 2, \cdots, r).$$

Furthermore we have $\gamma(M) = r$.

Proof. By $\mathrm{rk}\, T = r$ and Corollary 3.9, we see that $F(V^{-1}F)(M/pM)$ is generated by

$$FX_{r+1}, \cdots, FX_g$$

and therefore

$$\begin{aligned}
&\dim F(V^{-1}F)(M/pM) = g - r \\
&F^2(V^{-1}F)(M/pM) = 0.
\end{aligned}$$

Since T is of rank r, we have $\varphi(g) = r$ and therefore $ES(M)$ has to be $(0, \cdots, 0, 1, 2, \cdots, r)$. $\square$

Corollary 5.14. *For the generic point $(E^g \to \mathcal{Y})$ of $\mathcal{T}_\mu$ $(\mu \in \Lambda_c)$, we obtain*

$$ES(\mathcal{Y}) = \varphi_c^{\mathrm{top}}.$$

Next we investigate, for $c \leq [g/2]$, the Ekedahl-Oort stratum with elementary series

$$\varphi_c^{\mathrm{bot}} := (0, \cdots, 0, \underbrace{1, \cdots, 1}_{c}). \tag{25}$$

Let $T(t_1, \cdots, t_c)$ be $g \times g$-matrix of rank 1 for $t_1, \cdots, t_c \in W(K)$ with $t_1 \neq 0$:

$$T(t_1, \cdots, t_c) := \begin{pmatrix} & 0 & & 0 \\ & & & \\ t_c & \cdots & t_1 & \\ \vdots & & \vdots & 0 \\ t_c^2/t_1 & \cdots & t_c & \end{pmatrix}. \tag{26}$$

Also we introduce a polynomial $J(\bar{t}_1, \cdots, \bar{t}_c)$ in $\bar{t}_1, \cdots, \bar{t}_c$ with $\bar{t}_i := t_i \bmod p$:

$$J(\bar{t}_1, \cdots, \bar{t}_c) := \det \begin{pmatrix} \bar{t}_1 & \bar{t}_2 & \cdots & \bar{t}_c \\ \bar{t}_1^{\sigma^2} & \bar{t}_2^{\sigma^2} & \cdots & \bar{t}_c^{\sigma^2} \\ \vdots & & & \vdots \\ \bar{t}_1^{\sigma^{2c-2}} & \bar{t}_2^{\sigma^{2c-2}} & \cdots & \bar{t}_c^{\sigma^{2c-2}} \end{pmatrix}. \tag{27}$$

Lemma 5.15. *Assume $M = M_T$ for $T = T(t_1, \cdots, t_c)$ and $J(\bar{t}_1, \cdots, \bar{t}_c) \neq 0$. Then*

(1) $\dim(V^{-1}F)^j N = 2g - j$ *for* $j = 0, 1, \cdots, c$,

(2) $\dim F(V^{-1}F)^j N = g - j$ *for* $j = 0, 1, \cdots, c$,

(3) $\dim F^2(V^{-1}F)^j N = 1$ *for* $j = 0, 1, \cdots, c - 1$,

(4) $F^2(V^{-1}F)^c N = 0$.

Proof. (1) By Corollary 3.9 and the assumption $J(\bar{t}_1, \cdots, \bar{t}_c) \neq 0$, it follows

$$\begin{aligned}
\dim(V^{-1}F)^j N &= g + \dim\ker \overline{T}^{\sigma} \cap \ker \overline{T}^{\sigma^3} \cap \cdots \cap \ker \overline{T}^{\sigma^{2j-1}} \\
&= 2g - j.
\end{aligned}$$

(2) immediately follows from (1) and Proposition 3.6 (2).

(3) Since we have

$$\dim F^2(V^{-1}F)^{c-1} N \leq \dim F^2(V^{-1}F)^j N \leq \dim F^2 N = 1$$

for all $0 \leq j \leq c-1$, it suffices to show $F^2(V^{-1}F)^{c-1}N \neq 0$. By Lemma 3.5 (2) and Proposition 3.6 (2), we see

$$\{F^2(\widetilde{V^{-1}F})^{c-1}N\} = \operatorname{Im} \mathcal{F}'^{(2)} \cap \operatorname{Im} \mathcal{F}'^{(4)} \cap \cdots \cap \operatorname{Im} \mathcal{F}'^{(2c)} + pM.$$

This is not contained in pM by the assumption $J(\bar{t}_1, \cdots, \bar{t}_c) \neq 0$.

(4) Since $F(V^{-1}F)^cN$ is generated by $FX_{c+1}, \cdots, FX_g$ by Corollary 3.9 and

$$FX_i = VX_i$$

for all $i = c+1, \cdots, g$, we have $F(V^{-1}F)^cN = V(V^{-1}F)^cN$ and therefore $F^2(V^{-1}F)^cN = 0$. □

By the lemma above and Definition 2.6, we obtain:

Proposition 5.16. *Let $T = T(t_1, \cdots, t_c)$ and M_T be the associated Dieudonné module. If $J(\bar{t}_1, \cdots, \bar{t}_c) \neq 0$, then it follows that*

$$ES(M_T) = \varphi_c^{\mathrm{bot}}$$

We also have $\gamma(M_T) = c$.

Corollary 5.17. *There exists a quasi-finite surjective morphism from T'_{2c} defined in the proof of [12, Proposition 9.11] to each connected component of $S_{\varphi_c^{\mathrm{bot}}}$. Moreover we have a finite étale morphism from T'_{2c} to*

$$\operatorname{Spec} \mathbb{F}_{p^4}\left[x_1, x_2, \cdots, x_c, \frac{1}{J(x_1, x_2, \cdots, x_c)}\right].$$

Let us re-prove Proposition 3.11.

Lemma 5.18. *For an integer $n \geq 3$ with $(n, p) = 1$, let $W_{\sigma,n}$ be the supersingular locus in $\mathcal{A}_{g,1,n}$. We denote by Ω_n the subvariety of $W_{\sigma,n}$ consisting abelian varieties with elementary series φ satisfying $\varphi([(g+1)/2]) = 0$. Let f be a morphism from an $\mathbb{F}_p$-scheme S to Ω_n and $\mathcal{X} \to S$ be the corresponding family of principally polarized supersingular abelian varieties with level n structure. Then the map γ from S to $\mathbb{Z}_{\geq 0}$ sending $s \in S$ to*

$$\gamma(s) := \gamma(\mathbb{D}(\mathcal{X}_{\bar{s}})) = \frac{1}{2}\deg(S^0(\mathcal{X}_{\bar{s}}) \to S_0(\mathcal{X}_{\bar{s}}))$$

is lower semi-continuous. Here $\mathcal{X}_{\bar{s}}$ is the abelian variety $\mathcal{X}_s \otimes_{k(s)} \overline{k(s)}$.

Proof. By a version with level structure of Proposition 5.3, there is a proper surjective morphism

$$\coprod_{\mu \in \Lambda_{[g/2],n}} \mathcal{T}_\mu \to \Omega_n,$$

where $\Lambda_{[g/2],n}$ is the set

$$\{\text{polarizations } \mu \text{ on } E^g \mid \ker(\mu) \simeq \alpha_p^{2c}\} / \operatorname{Aut}(E^g, \theta)$$

with level n-structure θ. Then it suffices to show that only for families on $\mathcal{T}_\mu$.

By definition, closed points of an affine open subvariety V^Θ of $\mathcal{N}_{[g/2]}$ correspond to principally quasi-polarized supersingular Dieudonné modules generated by

$$X_i = \sum_{j \geq i} \alpha_{ij} F^e x_j \qquad (e = 0 \text{ or } 1)$$

with $\alpha_{ij} \in A$ and $\alpha_{ii} = 1$ (see the equations (17)).

For any quasi-polarized superspecial Dieudonné module N, the condition

$$M \subset N$$

is equivalent to

$$X_i \in N \qquad \text{for all } i = 0, \cdots, g-1,$$

which is a closed condition in the parameter space of α_{ij}. Since $M \subset N$ implies $\gamma(M) \leq \operatorname{length} N/M$ and for each non-negative integer m there are only finitely many superspecial Dieudonné modules N satisfying $M \subset N \subset M \otimes \operatorname{frac} W(K)$ and $\operatorname{length} N/M = m$, we have the semi-continuity of γ. □

Lemma 5.19. *For each elementary series φ satisfying*

$$\varphi(g-c) = 0, \qquad \varphi(g-c+1) = 1, \tag{28}$$

any generic point η of S_φ satisfies

$$\gamma(\eta) = c.$$

Proof. Let φ satisfy the equations (28). It follows

$$\varphi_c^{\text{bot}} \prec \varphi \prec \varphi_c^{\text{top}}.$$

Then by [17, Proposition 11.1], the Zariski closure of any irreducible component of S_φ contains an irreducible component of $S_{\varphi_c^{\text{bot}}}$ and the Zariski closure of any irreducible component of $S_{\varphi_c^{\text{top}}}$ contains an irreducible component of S_φ. Using the last statements in Lemma 5.13 and Proposition 5.16, at the generic point η of each irreducible component of S_φ, we have $\gamma(\eta) = c$, by Lemma 5.18. □

Proposition 5.20. *For any point $s = (Y, \lambda)$ of S_φ with φ satisfying (28), we have $\gamma(s) = c$.*

Proof. Proposition 5.5 means the invariant $\gamma(s) = \gamma(\mathbb{D}(Y \otimes_K \overline{K}))$ of a principally polarized abelian variety Y over K is determined by the isomorphism class of $Y[p]$. Then by Proposition 5.19, it follows

$$\gamma(s) = \gamma(\eta) = c$$

with η in Proposition 5.19. □

6 The number of irreducible components of S_φ

In this section, we prove the main theorem (Theorem 6.19) mentioned in the introduction.

Given an elementary series φ, we have to determine what kind of matrix T gives M_T such that $\varphi = \varphi(M_T)$. It seems difficult to write down explicitly which T is associated with M_T with $\varphi = \varphi(M_T)$. (It was possible for the two cases: $\varphi = \varphi_c^{\mathrm{bot}}$ and φ_c^{top}. See Lemma 5.13 and Proposition 5.16.) However it is relatively easy to determine the form of the matrix giving the Dieudonné module associated with each generic point of S_φ. This is done by introducing the subspace L_s of $N_{g,c}$. (Here s is a combinatorial data determined by φ.) The main theorem follows from a close investigation of the structure of L_s, for example the a-number stratification on L_s.

We will use three sets:

Definition 6.1. (i) $I_g = \{\varphi : \text{elementary series } \mid \varphi([(g+1)/2]) = 0\}$.

(ii) $J_g = \{(r; s_1, \cdots, s_r) \mid \sum_{i=1}^r s_i \le [g/2],\ s_i \in \mathbb{Z}_{\ge 1}\ (\forall i = 1, \cdots, r)\}$. For an element of $s = (r; s_1, \cdots, s_r)$ of J_g, we put $s_0 = g - \sum_{j=1}^r s_j$.

(iii) Let P_g be the set of monotonically increasing functions π satisfying $\pi(1) = 0$, $\pi(g) - \pi(g-1) \le [g/2]$ and for all $a = 2, \cdots, g-1$

$$\pi(a+1) - \pi(a) > \pi(a) - \pi(a-1)$$

unless $\pi(a) = 0$.

There are canonical bijections $\nu : I_g \to J_g$ and $\beta : J_g \to P_g$ defined as:

Definition 6.2. (i) Let ν denote the map from I_g to J_g sending φ to

$$\nu(\varphi) := (r(\varphi); s_1(\varphi), \cdots, s_r(\varphi))$$

defined by

$$\begin{cases} r(\varphi) := \varphi(g), \\ s_i(\varphi) := \sharp\{j \in \{1, \cdots, g\} \mid \varphi(j) = i\}. \end{cases}$$

(ii) Let β be the map from J_g to P_g sending $s = (r; s_1, \cdots, s_r)$ to β_s defined by

$$\begin{cases} \beta_s(a) := 0 & \text{for } a \le g - r, \\ \beta_s(a) := \displaystyle\sum_{i=0}^{a-g+r-1} (a - g + r - i) s_{r-i} & \text{for } a > g - r. \end{cases}$$

Let us introduce an invariant of elementary series φ.

Definition 6.3. We define a map

$$\alpha : I_g \to \mathrm{Map}(\{1, \cdots, g\},\ \mathbb{Z}_{\ge 0})$$

in the following way. For an element φ of I_g, we associate α_φ defined by

$$\alpha_\varphi(a) := \mathrm{codim}_{\overline{S}_\varphi} \overline{S}_\varphi(a)$$

with a-number loci $\overline{S}_\varphi(a) = \overline{S}_\varphi \cap T_a$ on $\overline{S}_\varphi$. Here T_a is the closed subvariety of $\mathcal{A}_g$ consisting of principally polarized abelian variety X with $a(X) \ge a$ (see [5] for the structure of T_a).

In this section, we will show that $\alpha = \beta \circ \nu$. First we see:

Lemma 6.4. *For any $\varphi \in I_g$, we have*

$$\alpha_\varphi(a) \leq \beta_{\nu(\varphi)}(a) \tag{29}$$

for all a.

Proof. By [17, Proposition 11.1], $\overline{S}_\varphi(a)$ contains $\overline{S}_{\varphi'}$ with elementary series φ' defined by

$$\begin{cases} \varphi'(j) = \varphi(j) & \text{if } \varphi(j) \leq g - a \\ \varphi'(j) = g - a & \text{if } \varphi(j) > g - a. \end{cases}$$

Moreover we have $\operatorname{codim}_{\overline{S}_\varphi} \overline{S}_{\varphi'} = \beta_{\nu(\varphi)}(a)$ by loc. cit. $\square$

Definition 6.5. Let V be an $\mathbb{F}_p$-vector space of dimension g. Fix a flag

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_g = V$$

with $\dim V_l = l$ and a basis $X_{g+1-l}, \cdots, X_g$ of V_l for each $l = 0, \cdots, g$. For any $\mathbb{F}_p$-algebra R, we denote by V_R and $V_{l,R}$ by $V \otimes R$ and $V_l \otimes R$ respectively.

(i) For $s = (r; s_1, \cdots, s_r) \in J_g$, we define the closed subscheme L_s of $N_{g,c}$ as follows. For any $\mathbb{F}_p$-algebra R, the set $L_s(R)$ of R-valued points consists of $T \in N_{g,c}(R)$ satisfying

$$\dim_{R/m}(V_{g-c_i,R}T \bmod mV) \leq i - 1$$

for any $i = 1, \cdots, r+1$ and for any maximal ideal m of R with $c := c_1$ and

$$\begin{cases} c_i := s_i + s_{i+1} + \cdots + s_r & (i = 1, 2, \cdots, r), \\ c_{r+1} := 0 \end{cases}$$

where $V_{l,R}T$ stands for the image of $V_{l,R}$ by the right multiplication of the matrix $T = (t_{ij})$:

$$X_i \cdot T := \sum_{j=1}^{g} t_{ji} X_j.$$

(ii) Let L_s^0 denote subvariety of L_s whose R-valued points correspond to matrices

$$T = \sum_{i=1}^{r} T(a_{i1}, a_{i2}, \cdots, a_{i,c_i})$$

with $a_{i1} \in R^\times$ and $a_{ij} \in R$ for all i, j. Here $T(t_1, t_2, \cdots, t_c)$ is the matrix of rank 1 defined as the equation (26).

Remark 6.6. In general, for each elementary series φ, the action of $T \in L_s^0$ $(s = \nu(\varphi))$ on

$$0 = V_0 \subset V_1 \subset \cdots \subset V_g$$

does not coincide with the action of F on the lower half of a final filtration (see §2.3) for M with $ES(M) = \varphi$:

$$0 = N'_0 \subset N'_1 \subset \cdots \subset N'_g.$$

All that we can say is

$$\dim V_l T = \max\{\, i \mid s_0 + s_1 + \cdots + s_i \leq l \,\} = \varphi(l) = \dim FN'_l.$$

Proposition 6.7. *(1) L_s^0 is dense in L_s.*

(2) For any $s = (r; s_1, \cdots, s_r) \in J_g$, the variety L_s is irreducible and of dimension $\beta_s(g)$.

(3) Let $L_s(a)$ be the subvariety of L_s consisting of elements T of rank $g - a$. Then we have

$$\operatorname{codim}_{L_s} L_s(a) = \beta_s(a).$$

Proof. By definition, L_s^0 is isomorphic to

$$\prod_{i=1}^{r} (\mathbb{G}_m \times \mathbb{A}^{c_i - 1}).$$

Hence (2) immediately follows from (1), since we have in general

$$\sum_{i=g-a+1}^{r} c_i = \beta_s(a).$$

We show (1) by induction of r. For an element T of L_s^0, let l be the maximal integer such that $X_l T \neq 0$. There are two cases.
Case 1: $X_l T = \sum_{j=g-l+1}^{g} t_{g+1-j} X_j$ with $t_l \neq 0$. Put

$$T' := T - T(t_l, t_{l-1}, \cdots, t_1).$$

Then T' is in $L_{s'}$ with $s' = (r-1; s_2, \cdots, s_r)$. By the hypothesis of induction, T' has a generalization $\mathcal{T}'_x$ to $L_{s'}^0$ $(x \in \overline{K})$. Namely $\mathcal{T}'_0 = T'$ and $\mathcal{T}'_x \in L_{s'}^0$ for $x \neq 0$.

We construct a generalization $\mathcal{T}''_x$ of $T(t_l, t_{l-1}, \cdots, t_1)$ to $L_{(1;c_1)}^0$ by adding x^2 to the $(g - c_1 + 1, c_1)$-th entry,

$${}^t(0, \cdots, 0, \sqrt{t_l}x, \frac{t_{l-1}}{\sqrt{t_l}}x, \cdots, \frac{t_1}{\sqrt{t_l}}x)$$

to the c_1-th column vector and

$$(\frac{t_1}{\sqrt{t_l}}x, \cdots, \frac{t_{l-1}}{\sqrt{t_l}}x, \sqrt{t_l}x, 0, \cdots, 0)$$

to the $(g - c_1 + 1)$-th row vector.

Then we have a generalization $\mathcal{T}_x := \mathcal{T}'_x + \mathcal{T}''_x$ of T to L_s^0.
Case 2: $X_l T = \sum_{j=g-l'+1}^{g} t_{g+1-j} X_j$ with $t'_l \neq 0$ and $l' < l$. Then l' has to be at most c_2. Let

$$T(t_1, \cdots, t_{l'} \ ; \ t_{l'+1}, \cdots, t_{l'+l-1})$$

be the matrix of rank 2 with the form:

$$\begin{pmatrix} & 0 & & & & 0 \\ t_1 & \cdots & t_{l'} & & & \\ & & \vdots & & & \\ t_{l'+l-1} & \cdots & t_l & \cdots & t_{l'} & \\ & & \vdots & & \vdots & 0 \\ & & t_{l'+l-1} & & t_1 & \end{pmatrix}.$$

Set $T' = T - T(t_1, \cdots, t_{l'} \ ; \ t_{l'+1}, \cdots, t_{l'+l-1})$.

As a generalization of $T(t_1, \cdots, t_{l'} \ ; \ t_{l'+1}, \cdots, t_{l'+l-1})$, we can take matrix $\mathcal{T}''_x$ of rank 2 with the same (i,j)-th entries at $i \leq g - l' + 1$ or $j \geq l'$ as

$$T(t_1, \cdots, t_{l'} \ ; \ t_{l'+1}, \cdots, t_{l'+l-1}) + xT(t_{l'}, t_{l'+1}, \cdots, t_{l-1}, \underbrace{0, \cdots, 0}_{l'}).$$

Then we have a generalization $\mathcal{T}_x := T' + \mathcal{T}''_x$ of T to matrices of Case 1.

Let us prove (3). For any $s = (r; s_1, \cdots, s_r) \in J_g$, by the proof of (1), any matrix T of rank $g - a$ in L_s has a generalization to a family of elements of $L^0_{s'}$ with

$$s' := (g - a; s_1, \cdots, s_{g-a-1}, s'_{g-a}), \qquad s'_{g-a} := \sum_{j=g-a}^{r} s_j.$$

Hence

$$\mathrm{codim}_{L_s} L_s(a) = \dim L_s - \dim L_{s'} = \beta_s(a).$$

$\square$

Definition 6.8. For an element $s = (r; s_1, \cdots, s_r)$ of J_g, we define an elementary series $\mathcal{ES}(s)$ as the elementary series of the generic point of L_s. Then we have a well-defined map

$$\mathcal{ES} : J_g \to I_g.$$

We will use the important fact:

Proposition 6.9. *Let $\varphi' = \mathcal{ES}(s)$ for $s \in J_g$. Then all Dieudonné modules with elementary series φ' have displays*

$$\begin{pmatrix} \tilde{T} & -\varepsilon^{-1} w \\ \varepsilon w & 0 \end{pmatrix}$$

for some $T \in L^0_s$ where $\tilde{T}$ is a lift of T. Moreover the set of such T's is dense in L^0_s.

In order to show this proposition, we need two lemmas:

Lemma 6.10. *Let $(M \subset M_1)$ be an element of $V^{\Theta}(K)$ which is mapped to $T \in N_{g,c}(K)$ through the map in Definition 4.8 (ii). With the notation of Definition 4.5, the first cohomology $M' := H^1(C^{\cdot})$ of the self-dual complex*

$$C^{\cdot} : AFx_g \to M \to Ax_1,$$

which is a principally quasi-polarized supersingular Dieudonné module of genus $g-2$, is associated with the matrix $\tilde{T}'$ which is obtained by removing the first and the last column vectors and the top and the bottom row vectors from $\tilde{T}$.

Proof of Lemma 6.10. It obviously follows from the definition of the morphism $h^{\Theta} : V^{\Theta} \to N_{g,c}$ (Definition 4.8). $\square$

In the next lemma, for a principally quasi-polarized supersingular Dieudonné module M, if we say

$$C^{\cdot} : Ay \to M \to Ax$$

is a self-dual complex, it is supposed to satisfy, in addition to the self-duality,

(i) $(F - V)x = 0$ and $(F - V)y = 0$,

(ii) $M \to Ax$ is surjective and $Ay \to M$ is injective,

(iii) $H^1(C^{\cdot})$ is a free Dieudonné module.

Lemma 6.11. *For given elementary series φ of length g and φ' of length $g - 2$, assume there exist a principally quasi-polarized supersingular Dieudonné module M and a self-dual complex*

$$C^{\cdot} : \quad Ay \to M \to Ax$$

such that we have

$$\begin{cases} ES(M) = \varphi, \\ ES(H^1(C^{\cdot})) = \varphi'. \end{cases}$$

Then for any principally quasi-polarized supersingular Dieudonné module M_0 with $ES(M_0) = \varphi$, there exists a self-dual complex

$$C_0^{\cdot} : \quad Ay_0 \to M_0 \to Ax_0$$

such that

$$ES(H^1(C_0^{\cdot})) = \varphi'.$$

Proof of Lemma 6.11. For any M_0 as above, since $ES(M_0) = \varphi$, we have an isomorphism

$$M_0/pM_0 \simeq M/pM.$$

By the existence of complex $C^{\cdot} : Ay \to M \to Ax$, we have a self-dual complex

$$C_{0,p}^{\cdot} : \quad Ay/pAy \to M_0/pM_0 \to Ax/pAx.$$

Taking a lift $M_0 \to Ax_0$ of $M_0/pM_0 \to Ax/pAx$ and combining its dual, say $Ay_0 \to M_0$, we have a self-dual complex

$$C_0^{\cdot} : \quad Ay_0 \to M_0 \to Ax_0,$$

which is a lift of $C_{0,p}^{\cdot}$. Since $H^1(C_0^{\cdot}) \bmod p = H^1(C_{0,p}^{\cdot})$, we have $ES(H^1(C_0^{\cdot})) = \varphi'$. $\square$

Proof. Let us show Proposition 6.9 by induction of g. For the generic point of L_s^0, the associated Dieudonné module $\mathcal{M}$ has the elementary series $\varphi' = \mathcal{ES}(s)$ by definition. We have a self-dual complex

$$C^{\cdot} : \quad AFx_g \to \mathcal{M} \to Ax_1$$

for some $(x_1, \cdots, x_g) \in \Phi_{c_1}$ and the matrix of $\mathcal{M}' := H^1(C^{\cdot})$ gives the generic point of $L^0_{s'}$ with

$$\begin{cases} s' := (r; s_1, \cdots, s_{r-1}, s_r - 1) & \text{if } s_r > 1, \\ s' := (r-1; s_1, \cdots, s_{r-1}) & \text{if } s_r = 1, \end{cases}$$

where $r = ES(\mathcal{M})(g)$, by Lemma 6.10. By the hypothesis of induction, all principally quasi-polarized supersingular Dieudonné modules M' with the same elementary series as $\mathcal{M}'$ correspond to elements of $L^0_{s'}$.

By Lemma 6.11, for any principally quasi-polarized supersingular Dieudonné module M with the same elementary series as $\mathcal{M}$, there exists a self-dual complex

$$C^{\cdot}_0 : Ay \to M \to Ax$$

such that $H^1(C^{\cdot}_0)$ is associated with an element of $L^0_{s'}$.

Then, by Lemma 6.10 and the fact that $ES(M_{\tilde{T}})(g) = \operatorname{rk} T$ for any T, the Dieudonné module M has to be associated with a matrix in L^0_s for a certain element $(x_1, \cdots, x_g) \in \Phi_{c_1}$ with $x_1 = x$ and $Fx_g = y$, $\square$

The task we should do is to show that $\nu(\varphi') = s$ in Proposition 6.9, that is, $\nu \circ \mathcal{ES} = \mathrm{id}_{J_g}$. The next is the key for this purpose.

Proposition 6.12. *We have an equality: $\beta = \alpha \circ \mathcal{ES}$.*

Proof. For $s \in J_g$, let $\varphi' := \mathcal{ES}(s)$. By Theorem 5.6 and Definition 4.12, there is a quasi-finite surjective morphism

$$\coprod_{\mu \in \Lambda_{c'}} \mathcal{T}_\mu(\varphi') \to S_{\varphi'}$$

with c' satisfying $\varphi'(g - c') = 0$ and $\varphi'(g - c' + 1) = 1$, and a finite étale morphism

$$V^{\Theta} \to N_{g,c'}$$

for an affine covering $\mathcal{N}_{c'} := \cup_{\Theta \in \Phi_{c'}} V^{\Theta}$. Let $V^{\Theta}_{\varphi'}$ be the inverse image of the closed subscheme L_s of $N_{g,c'}$. Corresponding to $V^{\Theta}_{\varphi'}$, we have a subscheme $U^{\Theta}_{\mu,\varphi'}$ of $\mathcal{T}_\mu$ for each $\mu \in \Lambda_{c'}$. By Proposition 6.9, we have a quasi-finite surjective morphism

$$\coprod_{\mu \in \Lambda_{c'}} \coprod_{\Theta \in \Phi_{c'}} U^{\Theta}_{\mu,\varphi'} \to \overline{S}_{\varphi'}.$$

Hence in order to see the codimension $\alpha_{\varphi'}(a)$ of $\overline{S}_{\varphi'}(a)$ in $\overline{S}_{\varphi'}$, it suffices to investigate that of the a-number locus $V^{\Theta}_{\varphi'}(a)$ in $V^{\Theta}_{\varphi'}$ and therefore that of $L_s(a)$ in L_s. It has already been calculated in Proposition 6.7 (3). Namely it follows

$$\alpha_{\varphi'}(a) = \operatorname{codim}_{L_s} L_s(a) = \beta_s(a)$$

for all $a = 1, \cdots, g$. $\square$

Corollary 6.13. *(1) The map α is injective and the image of α is P_g.*

(2) $\mathcal{ES}$ is a bijection.

By Corollary 6.13, we obtain a commutative diagram of bijections:

$$\begin{tikzcd} J_g \arrow[r, "\mathcal{ES}"] & I_g \arrow[r, "\alpha"] \arrow[d, "\nu"'] & P_g \arrow[d, "\xi"] \\ & J_g \arrow[r, "\beta"'] & P_g \end{tikzcd} \tag{30}$$

where ξ is defined to be $\beta \circ \nu \circ \alpha^{-1}$.

In order to prove that ξ is the identity map, we introduce a partial order on P_g.

Definition 6.14. For two element π_1, π_2 of P_g, we write $\pi_1 \leq \pi_2$ if and only if $\pi_1(a) \leq \pi_2(a)$ for all $a = 1, \cdots, g$.

Lemma 6.15. *We have $\xi(\pi) \geq \pi$ for all $\pi \in P_g$.*

Proof. This is none other than Lemma 6.4, i.e., $\alpha_\varphi \leq \beta_{\nu(\varphi)}$. □

The next obvious lemma says that ξ is the identity map and therefore $\alpha_\varphi = \beta_{\nu(\varphi)}$.

Lemma 6.16. *Let $(P, <)$ be a finite partial ordered set. Any bijective map*

$$\xi : P \to P$$

satisfying $\xi(\pi) \geq \pi$ for all $\pi \in P$, is the identity map.

Proposition 6.17. $\nu \circ \mathcal{ES} = \mathrm{id}_{J_g}$.

Proof. It immediately follows from Proposition 6.12 and the commutative diagram (30), since ξ is the identity map. □

Corollary 6.18. *For a non-negative integer c at most $[g/2]$, let φ be an elementary series satisfying $\varphi(g-c) = 0$ and $\varphi(g-c+1) = 1$. Any principally quasi-polarized supersingular Dieudonné module M with $ES(M) = \varphi$ has a display*

$$\begin{pmatrix} \tilde{T} & -\varepsilon^{-1} w \\ \varepsilon w & 0 \end{pmatrix}$$

such that $\tilde{T}$ is a lift of

$$T \in L_s^0, \qquad s = \nu(\varphi)$$

for a certain symplectic basis. Moreover the set of such T is dense in L_s^0.

We denote by L_s^{gen} the dense subscheme of L_s^0 consisting of such T as in Corollary 6.18.

Theorem 6.19. *Let c be a non-negative integer at most $[g/2]$. For any elementary series φ satisfying $\varphi(g-c) = 0$ and $\varphi(g-c+1) = 1$, the number of irreducible component of S_φ is equal to the class number $H_{g,c}$.*

Proof. For such an elementary series φ, we set $s := \nu(\varphi) \in J_g$. Let us denote by $V_\varphi^{\Theta,\mathrm{gen}}$ an irreducible component of the inverse image of L_s^{gen} by the morphism $h^\Theta : V^\Theta \to N_{g,c}$, and by $U_{\mu,\varphi}^{\Theta,\mathrm{gen}}$ the associated affine subscheme of $\mathcal{T}_\mu$ for each $\mu \in \Lambda_c$.

For any generic point η of S_φ associated with $\mu \in \Lambda_c$, its display is of the same form by Corollary 6.18. This means that they are isomorphic as principally quasi-polarized Dieudonné modules. Then for any η as above, by using the same affine subscheme $U_{\mu,\varphi}^{\Theta,\mathrm{gen}}$ of $\mathcal{T}_\mu$, we have the diagram:

$$\begin{tikzcd} & U_{\mu,\varphi}^{\Theta,\mathrm{gen}} \arrow[r] \arrow[d, "f"] & L_s^{\mathrm{gen}} \\ \eta \arrow[r] & S_\varphi & \end{tikzcd}$$

where the image of f is corresponding to η by the second statement of Corollary 6.18. Hence there is only one η associated with $\mu \in \Lambda_c$. $\square$

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