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Matrix coefficients of representations of $SU(2,2)$: — the case of P_J-principal series —

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Matrix coefficients of representations of $SU(2,2)$:
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Introduction

Our concern in this paper is to have an explicit integral expression of the radial part of the matrix coefficients with particular K-type of certain generalized principal series representations, i.e. P_J-principal series, of the real unitary group $SU(2,2)$ of signature $(2+,2-)$.

The non-compact Lie group $G = SU(2,2)$ of dimension 15 has two types of the standard maximal parabolic subgroups. One, which corresponds to the long root in the restricted root system (C_2-type), that has non-abelian unipotent radical, is denoted by $P_J = M_J A_J N_J$.

Given a discrete series representation σ of M_J and a complex linear form ν on the Lie algebra $\mathfrak{a}_J = \mathrm{Lie}(A_J)$, the parabolic induction $\pi = \mathrm{Ind}_{P_J}^G(\sigma \otimes e^{\nu+\rho_J} \otimes 1_{N_J})$ is a P_J-principal series representation of G. Here ρ_J denotes the half-sum of roots in the unipotent part N_J, $e^{\nu+\rho_J} : A_J \to \mathbf{C}^*$ the quasi-character with derivation $\nu + \rho_J : \mathfrak{a}_J \to \mathbf{C}$, and $\sigma \otimes e^{\nu+\rho_J}$ is the exterior tensor representation of the Levi part $M_J A_J$ of P_J.

Consider the subset consisting of the highest weights of the K-types occurring in $\pi|K$ in the weight lattice of the maximal compact subgroup K of G. Then we can find that it is a "translation" of the similar set of the highest weights of the K-types of a large discrete series representation of G, here "large" is in the sense of Kostant-Vogan. Therefore corresponding to the minimal K-type of a large discrete series, we can consider the "corner" K-type of our π, which is legitimately defined by Hayata [5], [6] in this case.

Let τ be the irreducible finite-dimensional representation of K such that its dual τ^* is the corner K-type (*cf.* Definition (2.1)). Fix a basis $\{v_i\}_{0 \le i \le d}$ of the representation space W_τ of τ, and let $\{v_j^*\}$ be its dual basis in W_{τ^*}. Then we can form $(d+1) \times (d+1)$-matrix

$$\Phi(g) := \left((\pi(g)v_i^*, v_j^*)_{0 \le i,j \le d}\right) = \left((c_{i,j}(g))_{0 \le i,j \le d}\right).$$

Here $(*,*)$ is the given inner product on the representation space H_π of a Hilbert space representation π. The value of Φ is determined by its restriction $\Phi|A$ to the radical part $A \cong \mathbf{R}^2_{>0}$, because of the Cartan decomposition $G = KAK$.

Our main result is to show that the radial part of each entry of the matrix Φ satifies a holonomic system which is equivalent to Appell's hypergeometric system of type F_2, and to have an integral expression of it in terms of Gaussian hypergeometric series (Theorem (5.4)).

Let us explain the outline of the method of proof. Set $M = Z_A(K)$ the centralizer of A in K. Then $\Phi(mam^{-1}) = \Phi(a)$ for any $a \in A$ and $m \in M$. This $M-$compatibility implies that many entries of the matrix $\Phi(a)$ vanishes. Let C be the Casimir operator in $Z(\mathfrak{g})$. Then, since π is quasi-simple, the $A-$radial part $\rho_A(C)$ of C acts on $\Phi|A$ as a scalar multiple of $\chi_\pi(C)$, the value of the infinitesimal character χ_π of π at C:

$$\rho_A(C)(\Phi|A) = \chi_\pi(C)(\Phi|A).$$

Moreover we consider a gradient-type operator called Schmid operator ∇, which were also utlized in other papers [13], [8], [5], [6], [12]. The A-radial part of this gives the Euler-Darboux operator which plays a crucial role in the classical theory of hypergeometric functions of two variables. Note that our result is quite similar to that of Iida [8], for both $Sp(2, \mathbf{R})$ and $SU(2,2)$ has the same restricted root system.

Let us review the outline of the contents of this paper. In §1 we prepare fundamental facts on the structure of the group $SU(2,2)$ and its subgroups, and their associated Lie algebras. Moreover some facts on the represenations of the maximal compact subgroup $K = S(U(2) \times U(2))$ are recalled. In §2, we define the P_J-principal series represenations and its corner K-types. Some more facts about the representations of K are prepared.

In §3 we define two type of the annihilators of the corner K-types. One is the composition of the gradient operator and the projectors in the Clebsh-Gordan decomposition of the tensor products of irreducible K-modules. This kind of operators are used to characterize the minimal K-type of the discrete series representation by Schmid [14]. We call these operators *Schimid operators*. Another annihilator comes from the Casmir operator. We compute the A-radial parts of these operators.

In §4, we compute these eqautions of §3 in terms of the coefficients of the spherical functions. The result is a system of differential-difference equations apparently rather complicated.

In §5 we reduce the equations in §4 firstly to simple equations for each single coefficients. After a simple change of unknown functions (dependent variables) and a simple change of (indepent) variables, we find that the finial equation is no other than a modified system of Appell's F_2. To have an integral expression is a simple problem (*cf.* Theorem (5.4)).

Our formula thus obtained has rather limited domain of convergence, different from the famous fomula by Harish-Chandra on the spherical functions representing the matrix coefficients of the principal series. However, we have a connection with F_2 in one part. And also our formula has very natural affinity with the result of Akhiezer and Gindikin [1], because the limitation of convergence is caused by the 'hidden' singularities on a complexification of G/K.

Now let us comment on the relation between this paper and other papers in the literature. Firstly this work is motivated by the desire to know the 'complexity' of the matrix coefficients of representations which should reflect the largeness of the Gelfand-Kirillov dimension of representations, as mensioned in the introduction of the previous paper [7]. However, chronologically speaking the result of this paper is obtained much earlier than that of [7].

Secondly from the view point of spherical functions on Lie groups, we have to mention that in the case of the compact dual $G^c = SU(4)$ of $G = SU(2,2)$, the spherical functions on the compact symmetric space $G^c/K = SU(4)/S(U(2) \times U(2))$, i.e., the orthogonal polynomials in two variables of the BC_2 type, are investigated by Koornwinder [10], and Deviard-Gaveaux [2], [3]. Note that this kind of results are later generalized further by Heckemann-Opdam to arbitrary symmetric spaces.

Thirdly we have to note that the "confluent versions" of our spherical functions are alreday found in Hayata [6] (Whittaker functions, after utilizing the holonomic systems similar to those of Yamashita [17]) and in Gon [4] for the Siegel-Whittaker functions.

The group $Sp(2, \mathbf{R})$ has the same restricted root system as $SU(2,2)$, therefore their spherical functions resemble to those of $SU(2,2)$. As we already remarked, the case of matrix coefficients, which shoulde be the counter part to our paper for $Sp(2, \mathbf{R})$, is Iida [8]. The Whittaker functions and the Siegel-Whittaker functions of the P_J-principal series are discussed in Miyazaki-Oda [12] and in Miyazaki [11], respectively.

Finally we have to mention the important relation with Appell's F_2. The reader can find there the same equations (2.1) of N. Takayama [15] as the formulae of Proposition (5.3) in this paper, which is called the *modified* F_2.

1 Preliminaries.

1.1 Basic terminology for $SU(2,2)$

We fix some notations in this section.

1.1.1 The structure of the Lie group $SU(2,2)$ and its Lie algebra

Our Lie group is the special unitary group of signature $(2+,2-)$ with real dimension 15

$$G = SU(2,2) = \{g \in SL_4(\mathbf{C}) | {}^t\bar{g} I_{2,2} g = I_{2,2}\};\ I_{2,2} = \mathrm{diag}(1,1,-1,-1),$$

and its associated Lie algebra is

$$\mathfrak{g} = \mathrm{Lie}(G) = \{\begin{pmatrix} X & Z \\ {}^t\bar{Z} & Y \end{pmatrix} \in \mathfrak{sl}_4(\mathbf{C}) | X, Y, Z \in M_2(\mathbf{C}), {}^t\bar{X} + X = {}^t\bar{Y} + Y = 0\}.$$

We fix a maximal compact subgroup K of G and its Lie algebra by

$$\begin{aligned} K &= \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in G | A, B \in U(2), \det A \det B = 1\} \\ &\cong S(U(2) \times U(2)), \\ \mathfrak{k} &= \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{sl}_4(\mathbf{C}) | {}^t\bar{A} + A = {}^t\bar{B} + B = 0\}. \end{aligned}$$

These are 7-dimensional. We fix a compact Cartan subgroup T of G in K and its associated Lie algebra :

$$\begin{aligned} T &= \{\text{diagonal matrices} \in K\}, \\ \mathfrak{t} &= \{\text{diagonal matrices} \in \mathfrak{k}\}. \end{aligned}$$

The complexifications of $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$ are denoted by $\mathfrak{g}_{\mathbf{C}}, \mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}$.

We denote by X_{ij} the matrix unit with 1 at (i,j)-component. The complement space $\mathfrak{p}$ of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is given by

$$\mathfrak{p} = \{\begin{pmatrix} 0 & Z \\ {}^t\bar{Z} & 0 \end{pmatrix} | Z \in M_2(\mathbf{C})\}.$$

An maximal abelian subspace in $\mathfrak{p}$ is given by

$$\mathfrak{a} = \mathbf{R}H_1 \oplus \mathbf{R}H_2 \quad \text{with} \quad H_1 = X_{13} + X_{31}, \quad H_2 = X_{24} + X_{42}.$$

Moreover we set

$$A = \exp(\mathfrak{a}), \quad M = Z_K(A), \quad \mathfrak{m} = \mathfrak{z}_{\mathfrak{K}}(\mathfrak{a}).$$

Then

$$M = \{\exp(\theta\sqrt{-1}I_0)\gamma^j | \theta \in \mathbf{R}, j = 1,2\}, \quad \mathfrak{m} = \mathbf{R}\sqrt{-1}I_0$$

with matrices

$$\gamma = I_0 = \mathrm{diag}(1,-1,1,-1).$$

Define matrices $h_i, e_{i,\pm}$ $(i = 1,2)$ by

$$h_1 = \mathrm{diag}(1,-1,0,0), h_2 = \mathrm{diag}(0,0,1,-1), e_{1,+} = X_{12}, e_{1,-} = X_{21}, e_{2,+} = X_{34}, e_{2,-} = X_{43}.$$

Let $\mathfrak{z}(\mathfrak{k}_{\mathbf{C}}) = \mathbf{C}I_{2,2}$ be the center of $\mathfrak{k}_{\mathbf{C}}$. Then $\{h_i, e_{i,\pm}\ (i = 1,2)\}$ is a basis of the commutator algebra $[\mathfrak{k},\mathfrak{k}]$, and $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}_{\mathbf{C}}) \oplus [\mathfrak{k},\mathfrak{k}]$. The set $\{h_1, h_2, I_{2,2}\}$ is a basis of the Cartan subalgebra $\mathfrak{t}_{\mathbf{C}}$.

It is convenient to set $Z_{ij} = X_{ii} - X_{jj}$ for $(i,j) = (1,2)$ or $(3,4)$. Then

$$Z_{13} = \frac{1}{2}(I_{2,2} + h_1 - h_2), \quad Z_{24} = \frac{1}{2}(I_{2,2} - h_1 + h_2).$$

Obviously $Z_{13}, Z_{24} \notin [\mathfrak{k},\mathfrak{k}]$.

1.1.2 Restricted root system

Define two linear forms $\lambda_i \in \mathfrak{a}^*$ $(i = 1,2)$ on $\mathfrak{a}$ defined by $\lambda_i(H_j) = \delta_{ij}$. Then the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$, which is of type C_2, is given by

$$\Delta = \Delta(\mathfrak{g},\mathfrak{a}) = \{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}.$$

For a fixed positive system $\Delta_+ = \{\lambda_1 \pm \lambda_2, 2\lambda_1, 2\lambda_2\}$, the associated root space decomposition is given as

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{m} + \sum_{\lambda\in\Delta} \mathfrak{g}_\lambda$$

with

$$\mathfrak{g}_{2\lambda_1} = \mathbf{R}E_1, \mathfrak{g}_{2\lambda_2} = \mathbf{R}E_2, \mathfrak{g}_{\lambda_1+\lambda_2} = \mathbf{R}E_3 + \mathbf{R}E_4, \mathfrak{g}_{\lambda_1-\lambda_2} = \mathbf{R}E_5 + \mathbf{R}E_6,$$

and

$$\mathfrak{g}_{-\mu} =^t \bar{\mathfrak{g}}_\mu = \{{}^t\bar{X} | X \in \mathfrak{g}_\mu\}.$$

Here

$$E_1 = \frac{\sqrt{-1}}{2}\begin{pmatrix} 1 & & -1 & \\ & 0 & & 0 \\ 1 & & -1 & \\ & 0 & & 0 \end{pmatrix}, \quad E_2 = \frac{\sqrt{-1}}{2}\begin{pmatrix} 0 & & 0 & \\ & 1 & & -1 \\ 0 & & 0 & \\ & 1 & & -1 \end{pmatrix},$$

$$E_3 = \frac{1}{2}\begin{pmatrix} & 1 & & -1 \\ -1 & & 1 & \\ & 1 & & -1 \\ -1 & & 1 & \end{pmatrix}, \quad E_4 = \frac{\sqrt{-1}}{2}\begin{pmatrix} & 1 & & -1 \\ 1 & & -1 & \\ & 1 & & -1 \\ 1 & & -1 & \end{pmatrix},$$

$$E_5 = \frac{1}{2}\begin{pmatrix} & 1 & & 1 \\ -1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \end{pmatrix}, \quad E_6 = \frac{\sqrt{-1}}{2}\begin{pmatrix} & 1 & & 1 \\ 1 & & -1 & \\ & 1 & & 1 \\ -1 & & 1 & \end{pmatrix}.$$

1.1.3 Hyperbolic trigonometric functions

We identify $a \in \mathbf{R}_+^\times$ and $t \in \mathbf{R}$ by $a = e^t = \exp(t)$. Hence we set

$$sh(a) = sh(t) = \frac{1}{2}(a - a^{-1}), ch(a) = ch(t) = \frac{1}{2}(a + a^{-1}),$$
$$th(a) = sh(a)/ch(a), ct(a) = ch(a)/sh(a),$$

and define a function in two variables

$$D = D(a_1, a_2) = ch(a_1)^2 - ch(a_2)^2 = sh(a_1)^2 - sh(a_2)^2 = \frac{1}{4}(a_1/a_2 - a_2/a_1)(a_1a_2 - a_1^{-1}a_2^{-1}).$$

Added to the basic relations

$$ch(a)^2 - sh(a)^2 = 1,\ sh(a^2) = 2sh(a)ch(a),\ ch(a^2) = ch(a)^2 + sh(a)^2 = 1 + 2sh(a)^2,$$

in this paper we particularly use

$$sh(a_1/a_2)^2 + sh(a_1a_2)^2 = ch(a_1^2)ch(a_2^2) - 1 = 2\{sh(a_1)^2 + sh(a_2)^2 + 2sh(a_1)^2sh(a_2)^2\},$$

and

$$ch(a_1a_2)sh(a_1/a_2)^2 + ch(a_1/a_2)sh(a_1a_2)^2 = 2ch(a_1)ch(a_2)\{sh(a_1)^2 + sh(a_2)^2\}.$$

Moreover for Euler operators $\partial = a\frac{\partial}{\partial a}, \partial_i = a_i\frac{\partial}{\partial a_i}$ $(i = 1, 2)$, we have

$$\partial(sh(a)) = ch(a), \quad \partial(ch(a)) = sh(a),$$

and

$$\frac{ch(a_1)}{D} \cdot \partial_1 \cdot \frac{D}{ch(a_1)} = \frac{sh(a_1^2)}{D} - ct(a_1^2) + sh(a_1^2)^{-1},$$
$$\frac{ch(a_2)}{D} \cdot \partial_2 \cdot \frac{D}{ch(a_2)} = -\frac{sh(a_2^2)}{D} - ct(a_2^2) + sh(a_2^2)^{-1}.$$

1.2 K-module, projectors, and (τ_1, τ_2)-spherical functions

1.2.1 Parametrization and realization of irreducible representations of K

The unitary dual $\hat{K}$ of the connected compact group $K = S(U(2) \times U(2))$ or its complexified Lie algebra $\mathfrak{k}_{\mathbf{C}} = \mathfrak{s}(\mathfrak{gl}_2(\mathbf{C}) \otimes \mathfrak{gl}(\mathbf{C}))$ is parametrized as follows.

Definition (1.1) The element $\tau_{[r,s;u]}$ in the set

$$\hat{K} = \hat{\mathfrak{k}_{\mathbf{C}}} = \{\tau_{[r,s;u]} : r, s \in \mathbf{Z}_{\geq 0}, u \in \mathbf{Z}, r + s + u \in 2\mathbf{Z}\}$$

with parameter $[r, s; u]$ is given by

$$\begin{aligned}
&\tau_{[r,s;u]}(\begin{pmatrix} zg_1 & \\ & z^{-1}g_2 \end{pmatrix}) &&= \mathrm{Sym}^r(g_1) \otimes \mathrm{Sym}^s(g_2) \otimes z^u && (g_j \in SU(2), z \in U(1)),\\
&\tau_{[r,s;u]}(\begin{pmatrix} Y_1 & \\ & Y_2 \end{pmatrix}) &&= \mathrm{Sym}^r(Y_1) \otimes Id + Id \otimes \mathrm{Sym}^s(Y_2) && (Y_j \in \mathfrak{sl}_2(\mathbf{C})),\\
&\tau_{[r,,s;u]}(I_{2,2}) &&= u \cdot Id.
\end{aligned}$$

Here Sym^d means the symmetric tensor representation of degree d of $SU(2, 2)$ or its Lie algebra. More explicitly, we specify the standard basis in the representation space $V_\tau = V_{r,s}$ in the same way as in [5], §§3.5 or §§1.3 of [7]:

$$f_{kl} = f_k \otimes f_l; \quad f_k \leftrightarrow x^ky^{r-k}, \quad f_l \leftrightarrow x^ly^{s-l} \quad (0 \leq k \leq r, 0 \leq l \leq s).$$

Then we have the following.

Lemma (1.1) *Write* $\tau = \tau_{[r,s,;u]}$. *Then*

$$\begin{array}{rclrcl} \tau(h^1)f_{kl} & = & (2k-r)f_{kl}, & \tau(h^2)f_{kl} & = & (2l-s)f_{kl}, \\ \tau(e^1_+)f_{kl} & = & (r-k)f_{k+1,l}, & \tau(e^2_+)f_{kl} & = & (s-l)f_{k,l+1}, \\ \tau(e^1_-)f_{kl} & = & kf_{k-1,l}, & \tau(e^2_-)f_{kl} & = & lf_{k,l-1}, \\ \tau(I_{2,2})f_{kl} & = & uf_{kl}. & & & \end{array}$$

Moreover we have

$$\tau(Z_{13})f_{kl} = \tfrac{1}{2}(-r+s+u+2k-2l)f_{kl}, \quad \tau(Z_{24})f_{kl} = \tfrac{1}{2}(r-s+u-2k+2l)f_{kl}.$$

See Lemma (3.8) of [5].

The contragradient represenation τ^* of $\tau = \tau_{[r,s,;u]}$ has the parameter $\tau^* = \tau_{[r,s;-u]}$.

1.2.2 Projectors from $V_\tau \otimes \mathfrak{p}_\pm$ to irreducible factors

Given an irreducible representation $\tau = \tau_{[r,s;u]}$, we want to decompose the tensor product $V_\tau \otimes \mathfrak{p}_\pm$ into irreducible components. The adoint representations $Ad_\pm$ of K on $\mathfrak{p}_\pm$ are irreducible and thier parameters are given by

$$Ad_+ = \tau_{[1,1;2]}, \quad Ad_- = \tau_{[1,1;-2]}.$$

Therefore

$$\tau \otimes Ad_\pm = \tau_{[r+1,s+1;u\pm 2]} \otimes \tau_{[r+1,s-1;u\pm 2]} \otimes \tau_{[r-1,s+1;u\pm 2]} \otimes \tau_{[r-1,s-1;u\pm 2]}.$$

Here the factor including the parameter $r-1$ (resp. $s-1$) is dropped if $r=0$ (resp. $s=0$). We can define the projectors corresponding to this decomposition:

$$P^{(e,f)} : V_{r,s} \otimes \mathfrak{p}_+ \to V_{r+e,s+f}, \quad \bar{P}^{(e,f)} : V_{r,s} \otimes \mathfrak{p}_- \to V_{r+e,s+f}, \ (e,f=\pm).$$

Lemma (1.2) *For each projector given above, up to scalar multiple the basis of* $V_\tau \otimes \mathfrak{p}_\pm$ *are mapped as follows :*

$$\begin{array}{rcrcr} P^{(-,-)}(f_{kl}\otimes X_{13}) & = & -\bar{P}^{(-,-)}(f_{kl}\otimes X_{42}) & = & (k-r)lf_{k,l-1} \\ P^{(-,-)}(f_{kl}\otimes X_{24}) & = & -\bar{P}^{(-,-)}(f_{kl}\otimes X_{31}) & = & k(s-l)f_{k-1,l} \\ P^{(-,-)}(f_{kl}\otimes X_{23}) & = & \bar{P}^{(-,-)}(f_{kl}\otimes X_{41}) & = & klf_{k-1,l-1} \\ P^{(-,-)}(f_{kl}\otimes X_{14}) & = & \bar{P}^{(-,-)}(f_{kl}\otimes X_{32}) & = & (k-r)(s-l)f_{k,l} \\ P^{(+,-)}(f_{kl}\otimes X_{13}) & = & -\bar{P}^{(+,-)}(f_{kl}\otimes X_{42}) & = & (-l)f_{k+1,l-1} \\ P^{(+,-)}(f_{kl}\otimes X_{24}) & = & -\bar{P}^{(+,-)}(f_{kl}\otimes X_{31}) & = & (l-s)f_{k,l} \\ P^{(+,-)}(f_{kl}\otimes X_{23}) & = & \bar{P}^{(+,-)}(f_{kl}\otimes X_{41}) & = & (-l)f_{k,l-1} \\ P^{(+,-)}(f_{kl}\otimes X_{14}) & = & \bar{P}^{(+,-)}(f_{kl}\otimes X_{32}) & = & (l-s)f_{k,l} \\ P^{(-,+)}(f_{kl}\otimes X_{13}) & = & -\bar{P}^{(-,+)}(f_{kl}\otimes X_{42}) & = & (r-k)f_{k,l} \\ P^{(-,+)}(f_{kl}\otimes X_{24}) & = & -\bar{P}^{(-,+)}(f_{kl}\otimes X_{31}) & = & kf_{k-1,l+1} \\ P^{(-,+)}(f_{kl}\otimes X_{23}) & = & \bar{P}^{(-,+)}(f_{kl}\otimes X_{41}) & = & (-k)f_{k-1,l} \\ P^{(-,+)}(f_{kl}\otimes X_{14}) & = & \bar{P}^{(-,+)}(f_{kl}\otimes X_{32}) & = & (k-r)f_{k,l+1} \end{array}$$

See Lemma (3.12) of [5]. Because we do not use $P^{(+,+)}, \bar{P}^{(+,+)}$, they are omitted.

1.2.3 (τ_1, τ_2)-spherical functions

Let (τ_1, V_{τ_1}) and (τ_2, V_{τ_2}) be two irreducible representations of K with parameters $d_1 = [r_1, s_1; u_1]$ and $d_2 = [r_2, s_2; u_2]$. Similarly as in the previous paper [7], we consider the space

$$C^{\infty}_{d_1,d_2}(K\backslash G/K)$$

$$:= \{c : G \to V_{d_1} \otimes V_{d_2} | c(k_1 g k_2) = \tau_1(k_1) \otimes \tau_2(k_2^{-1}) \cdot c(g),\ k_1, k_2 \in K\}.$$

Let

$$c(g) = \sum_{M=(k_1,l_1;k_2,l_2)} c_M(g) f_{k_1,l_1} \otimes f_{k_2,l_2}$$

be the expression of c in terms of standard basis. Then the redundancy of the double coset decomposition $G = KAK$ implies the following.

Lemma (1.3) (M-compatibility) *The restriction of any coefficient c_{k_1,l_1,k_2,l_2} to A vanishes, unless*

$$k_1 + l_1 + k_2 + l_2 = \frac{1}{2}(r_1 + s_1 + r_2 + s_2)$$

and

$$k_1 - l_1 + k_2 - l_2 \equiv \frac{1}{2}(r_1 - s_1 + u_1 + r_2 - s_2 + u_2) \mod 2.$$

Remark We note here that under the first condition, the second condition is equivalent to $2s_1 + 2s_2 \equiv u_1 + u_2 \mod 4$ (or to $2r_1 + 2r_2 \equiv u_1 + u_2 \mod 4$).

Proof of Lemma. The first condition is already proved in [7], Lemma (3.1) in a more special situation. The same proof is applicale to our case. We have to show the second condition.

Take an element m of M of the form

$$m = \begin{pmatrix} zg_1 & 0 \\ 0 & z^{-1}g_2 \end{pmatrix}, \quad g_1 = \mathrm{diag}(uz^{-1}, \bar{u}z),\ g_2 = \mathrm{diag}(uz, \bar{u}z^{-1})$$

with $u \in \mathbb{C}^{\times}, |u| = 1$, and $z = \sqrt{-1}$. Then $g_1, g_2 \in SU(2)$ and for $a \in A$ we get

$$\begin{aligned} c(a) &= c(mam^{-1}) = \tau_1(m) \otimes \tau_2(m) \cdot c(a) \\ &= \sum_{k_1,l_1,k_2,l_2} (uz^{-1})^{k_1+k_2} (\bar{u}z)^{r_1-k_1+r_2-k_2} (uz)^{l_1+l_2} (\bar{u}z^{-1})^{s_1-l_1+s_2-l_2} \\ &\quad \cdot z^{u_1+u_2} c_{k_1,l_1,k_2,l_2}(a) f_{k_1,l_1} \otimes f_{k_2,l_2}. \end{aligned}$$

Comparing the exponents in u and z of the coefficients, we have our lemma.

2 The P_J-principal series representarions and their corner K-types

2.1 The P_J-principal series of $SU(2,2)$

Recall the generalized principal series representation obtained by parabolic induction with respect to the parabolic subgroup P_J. This group is the unique maximal cuspidal parabolic subgroup in G. For a Langlands decomposition $P_J = M_J A_J N_J$, its split component $A_J = \exp(\mathfrak{a}_J)$ is given by

$$\mathfrak{a}_J = \mathfrak{a}_{2\lambda_2} = \mathbf{R}H_1,$$

its unipotent radical $N_J = \exp(\mathfrak{n}_J)$ by

$$\mathfrak{n}_J = \mathfrak{g}_{2\lambda_1} + \mathfrak{g}_{\lambda_1+\lambda_2} + \mathfrak{g}_{\lambda_1-\lambda_2},$$

and the component $M_J = \exp(\mathfrak{m}_J)$ with

$$\mathfrak{m}_J = \mathbf{R}H_2 \oplus \mathbf{R}E_2 \oplus \mathbf{R}\sqrt{-1}I_0 \oplus \mathbf{R}\sqrt{-1}Z_{24},$$

where $Z_{24} = \frac{1}{2}(I_{2,2} - H_1' + H_2')$. Put

$$T = \{\exp(\sqrt{-1}\theta I_0) | \theta \in \mathbf{R}\} \cong \mathbf{C}^{(1)},$$

and

$$G_0 = \left\{ \begin{pmatrix} 1 & & & \\ & \alpha & & \beta \\ & & 1 & \\ & \bar{\beta} & & \bar{\alpha} \end{pmatrix} | \theta \in \mathbf{R}, \alpha, \beta \in \mathbf{C}, |\alpha|^2 - |\beta|^2 = 1 \right\} \cong SU(1,1).$$

Then $M_J = T \cdot G_0$. Note that here a maximal compact subgroup K_0 of G_0 is given by

$$K_0 = \{\mathrm{diag}(1, \alpha, 1, \bar{\alpha}) | \alpha \in \mathbf{C}^{(1)}\} \cong \mathbf{C}^{(1)}.$$

The essentially discrete series representations of M_J is given as the composition $\sigma = (\chi_m, D_k^{\pm})$ of the characters

$$\chi_m(e^{\sqrt{-1}\theta}) = e^{m\sqrt{-1}\theta} \quad (e^{\sqrt{-1}\theta} \in \mathbf{C}^{(1)}, i.e., \theta \in \mathbf{R}; m \in \mathbf{Z})$$

of $\mathbf{C}^{(1)}$ and the discrete series representations $D_k^{\pm}$ of $G_0 = SU(1,1)$ with Blatter parameter $\pm k$ (*cf.* [5] §§3.1).

For a complex-valued linear form $\nu \in \mathfrak{a}_{J,\mathbf{C}} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_J, \mathbf{C})$ on $\mathfrak{a}_J$, we denote by e^ν the character of A_J defined by $e^\nu(a_1) = e^{\nu(\log a_1)}$ $(a_1 \in A_J)$.

Now we define the generalized principal series associated with P_J by

$$\pi_{J,\sigma,\nu} = \pi_J(m; \pm k; \nu) = \mathrm{Ind}_{P_J}^G(\sigma \otimes e^{\nu+\rho_J} \otimes 1_{N_J}).$$

Here $\rho_J = 2\lambda_1$ and the group G acts by right translation.

2.2 The corner K-types of P_J principal series

We construct some elements in $U(\mathfrak{g})$ which annihilate the vectors in a P_J-principal series represenation with the corner K-type.

We refer Hayata [5] in this subsection. The K-type of the representation $\pi_{J,\sigma,\nu}$ is described in Proposition (3.3) of [5]. The necessary data for our purpose is to specifty the *corner K-type*.

Definition (2.1) The corner K-type τ_d in $\pi = \pi_{J,\sigma,\nu}$ is characterized by the properties:
1) its dimension is minimal among the K-types in π;
2) the restriction $\tau_d|K_0$ has the minimal K_0-type of $D_k^{\pm}$;
3) When $m \neq 0$, $\tau_{d+\delta}$ does not occur in π for some non-compact root δ with respect to $(\mathfrak{g}, \mathfrak{h})$. See §§4.3 of [5].

Notation (2.1) In order to describe the corner K-type, it is convenient to introduce the notation

$$sgn(\sigma) = (sgn(m); \varepsilon) \text{ for } \sigma = (\chi_m, D_k^{\varepsilon}).$$

Here $sgn(m) = +1, 0, -1$ according as $m > 0, m = 0, m < 0$, respectively, and $\varepsilon = +1$ for D_k^+ and $\varepsilon = -1$ for D_k^-.

Here is the list of the corner K-types.

Proposition (2.1) *The parameter $[r, s; u]$ for the contragredient representation of the corner K-type is*

$$[0, |m|; -\varepsilon 2k + m] \text{ if } \varepsilon sgn(m) \geq 0,$$

$$[|m|, 0; -\varepsilon 2k - m] \text{ if } \varepsilon sgn(m) \leq 0.$$

We refer the table before Remark 4.3 in [6] for this proposition.

Throughout the rest of this paper, $\tau_2 = \tau_{[r,s,;u_2]}$ stands for the contragredient of the corner K-type of $\pi = \pi_{J,\sigma,\nu}$.

Proposition (2.2) *Let $[r, s; u_2]$ be as above. Then the constituents in $\pi|K$ with parameter of the form $[r, s; u_1]$ occurs if and only if*

$$\varepsilon(u_1 + u_2) \geq 0, \quad u_1 + u_2 \equiv \text{mod } 4.$$

Moreover, if it is the case, the K-type with parameter $[r, s; u_1]$ has multiplicity one in $\pi|K$.

This is deduced immediately from Propostion (3.3) in [6]. From now on, $\tau_1 = \tau_{[r,s;u_1]}$ stands for these K-types satisfying the above conditions of multiplicity one.

2.3 The behavior of the standard basis of the corner K-type $V_{\tau_1} \otimes V_{\tau_2}$

The standard basis of $V_{\tau_1} \otimes V_{\tau_2}$ is the collection of

$$f_M = f^L_{k_1,\ell_1} \otimes f^R_{k_2,\ell_2}, \text{ with } M = (k_1, \ell_1, k_2, \ell_2),$$

where $\{f^L_{k_1,\ell_1}\}$ (resp. $\{f^R_{k_2,\ell_2}\}$) is the tandard basis of V_{τ_1} (resp. V_{τ_2}) given in Sect.1.2. By the compatibility condition (Lemma (1.3)) M runs through under the following condition:

$$\begin{array}{llll} k_1 = k_2 = 0, & 0 \leq l_i \leq r = |m|, & l_1 + l_2 = |m| & \text{if } \varepsilon m \geq 0, \\ l_1 = l_2 = 0, & 0 \leq k_i \leq s = |m|, & k_1 + k_2 = |m| & \text{if } \varepsilon m < 0. \end{array}$$

In order to give a unified description to the behavior of the standird basis we use some linear functions of $M = (k_1, \ell_1, k_2, \ell_2)$. Firstly we define $t_i = t_i(M)$ by the following table:

	$m \geq 0, \varepsilon > 0$	$m < 0, \varepsilon < 0$	$m < 0, \varepsilon > 0$	$m \geq 0, \varepsilon < 0$
$(t_1, t_2) =$	(l_2, l_1),	(l_1, l_2),	(k_1, k_2),	(k_2, k_1)

We also define $A_i = A_i(M)$, $i = 1, 2$, and $B = B(M)$ by

$$A_1 = \varepsilon \frac{1}{4}(u_2 - u_1) + \frac{1}{2}(t_2 - t_1), \quad A_2 = \varepsilon \frac{1}{4}(u_2 - u_1) - \frac{1}{2}(t_2 - t_1),$$

$$B = \varepsilon\frac{1}{4}(u_1 + u_2).$$

The action of $\mathfrak{k}_{\mathbf{C}}$ on the standard basis is written in the following way.

Lemma (2.2) *With $A_i = A_i(M)$, $B = B(M)$, one has*

$$\begin{aligned}
\tau_2(I_0)f_M &= sgn(m)(t_1 - t_2)f_M,\\
\tau_1(Z_{13})f_M &= \varepsilon(-A_1 + B)f_M, \quad \tau_2(Z_{13})f_M = \varepsilon(A_1 + B)f_M,\\
\tau_1(Z_{24})f_M &= \varepsilon(-A_2 + B)f_M, \quad \tau_2(Z_{24})f_M = \varepsilon(A_2 + B)f_M.
\end{aligned}$$

Lemma (2.3) *The compact root vectors act on f_M, $M = (k_1, \ell_1, k_2, \ell_2)$, as follows: If $\varepsilon m \geq 0$ then $\tau_j(e^1_\pm)f_M = 0$, and if $\varepsilon m < 0$ then*

$$\tau_1(e^1_+)f_M = (r - k_1)f^L_{k_1+1,l_1} \otimes f^R_{k_2,l_2}, \ \tau_2(e^1_+)f_M = (r - k_2)f^L_{k_1,l_1} \otimes f^R_{k_2+1,l_2},$$
$$\tau_1(e^1_-)f_M = k_1 f^L_{k_1-1,\ell_1} \otimes f^R_{k_2,l_2}, \ \tau_2(e^1_-)f_M = k_2 f^L_{k_1,l_1} \otimes f^R_{k_2-1,l_2}.$$

If $\varepsilon m < 0$ then $\tau_j(e^2_\pm)f_M = 0$, and if $\varepsilon m \geq 0$ then

$$\tau_1(e^2_+)f_M = (s - \ell_1)f^L_{k_1,l_1+1} \otimes f^R_{k_2,l_2}, \ \tau_2(e^2_+)f_M = (s - l_2)f^L_{k_1,l_1} \otimes f^R_{k_2,l_2+1},$$
$$\tau_1(e^2_-)f_M = l_1 f^L_{k_1,l_1-1} \otimes f^R_{k_2,l_2}, \ \tau_2(e^2_-)f_M = l_2 f^L_{k_1,l_1} \otimes f^R_{k_2,l_2-1}.$$

2.4 Differential equations for the corner K-type

Let τ_1, τ_2 be those irreducible representations of K specified in (2.2). Let $\{f^{L*}_{k_1,l_1}\}$ and $\{f^{R*}_{k_2,l_2}\}$ be the dual basis of $\{f^L_{k_1,l_1}\}$ in τ_1 and $\{f^R_{k_2,l_2}\}$ in τ_2, respectively. Let $\iota_i : \tau_i \to \pi$ $(i = 1, 2)$ be the injective homomorphisms of K-modules, unique up to scalar multiple, since τ_i occurs with multiplicity one in π and by Schur's Lemma. Now we put

$$\varphi(g) = \sum_{(k_1,l_1)} \sum_{(k_2,l_2)} (\pi(g)\iota_2(f^{R*}_{k_2,l_2}), \iota_1(f^{L*}_{k_1,l_1}))f^L_{k_1,l_1} \otimes f^R_{k_2,l_2}.$$

This function belongs to the space $C^\infty_{\tau_1,\tau_2}(K\backslash G/K)$, defined in (1.2.3).

2.4.1 Gradient operators

We want to recall the gradient operators for the space $C^\infty_{d_1,d_2}(K\backslash G/K)$. Up to scalar constant, the Killing form determines the bilinear form

$$\mathrm{tr}(\begin{pmatrix} 0 & Z \\ {}^t\bar{Z} & \end{pmatrix}\begin{pmatrix} 0 & W \\ {}^t\bar{W} & \end{pmatrix}) = \mathrm{tr}_{\mathbf{C}/\mathbf{R}}(\sum_{i=1}^4 z_i\bar{w}_i) \text{ for } Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix},\ W = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}.$$

Hence up to scalar multiple, an orthogonal basis with respect to this is given by

$$X_{13} + X_{31}, \sqrt{-1}(X_{13} - X_{31}), X_{14} + X_{41}, \sqrt{-1}(X_{14} - X_{41}),$$
$$X_{23} + X_{32}, \sqrt{-1}(X_{23} - X_{32}), X_{24} + X_{42}, \sqrt{-1}(X_{24} - X_{42}).$$

Let R_X, L_X be the right derivation and the left derivation with respect to $X \in \mathfrak{g}$. Then we can define the right gradient operator by

$$\begin{aligned}
\nabla^R_{d_1,d_2}\phi(g) :&= \sum_{i=3,4}\{R_{X_{1i}+X{i1}}\phi(g) \otimes (X_{1i} + X_{i1}) + R_{\sqrt{-1}(X_{1i}-X_{i1}}\phi(g) \otimes \sqrt{-1}(X_{1i} - X_{i1})\}\\
&= 2\sum_{i=3,4}\{R_{X_{1i}}\phi(g) \otimes X_{i1} + R_{X_{i1}}\phi(g) \otimes X_{1i}\}.
\end{aligned}$$

The left gradient operator $\nabla^L_{d_1,d_2}$ is obtained similarly by replacing the right derivations R_X's by the left derivations L_R's.

2.4.2 Schmid operators

Proposition (2.4) *For the function* φ *defined above, we have*

$$\begin{array}{rcl} P^{(+,-)} \cdot \nabla^R_+ \varphi &=& 0, \text{ if } \varepsilon = +1 \text{ and } m > 0; \\ \bar{P}^{(-,+)} \cdot \nabla^R_- \varphi &=& 0, \text{ if } \varepsilon = -1 \text{ and } m > 0; \\ P^{(-,+)} \cdot \nabla^R_+ \varphi &=& 0, \text{ if } \varepsilon = +1 \text{ and } m < 0; \\ \bar{P}^{(+,-)} \cdot \nabla^R_- \varphi &=& 0, \text{ if } \varepsilon = -1 \text{ and } m < 0; \end{array}$$

$$\begin{array}{rcl} P^{(-,-)} \cdot \nabla^R_+ \cdot P^{(+,+)} \cdot \nabla^R_+ \varphi &=& 0, \text{ if } \varepsilon = +1 \text{ and } m = 0; \\ \bar{P}^{(-,-)} \cdot \nabla^R_- \cdot \bar{P}^{(+,+)} \cdot \nabla^R_- \varphi &=& 0, \text{ if } \varepsilon = -1 \text{ and } m = 0. \end{array}$$

Remark (2.1) The compostions $P^{(*,*)} \cdot \nabla^R_\pm$ are called Schmid operator. These are used to characterize the minimal K-types of the discrete series representations [14].

Proof of Proposition. This is derived from the nature of the corner K-type. The proof of Proposition (4.4) of [6] is applicable in our case.

2.4.3 The Casimir operator

Proposition (2.5) *The spherical fucntion* φ *belonging to* $\pi J(m; \pm k; \nu)$ *with the corner* K*-type from the right hand side satisfyies the Casimir equation*:

$$\chi_{\pi_{J,\sigma,\nu}}(\Omega)\varphi = \{\nu^2 + (k-1)^2 + \frac{1}{2}m^2 - 10\}\varphi.$$

with respect to the Casimir operator given by

$$\Omega = H_1^2 + H_2^2 + \frac{1}{2}I_0^2 + 2\sum_{j=1}^{2}(E_j^t \bar{E}_j +^t \bar{E}_j E_j) + \sum_{j=3}^{6}(E_j^t \bar{E}_j +^t \bar{E}_j E_j).$$

See \$§3.4 of [6] and §§5.1 [5].

3 The A-radial part of the annihilators

3.1 Cartan decompotion of basis of $\mathfrak{p}_\pm$

There are two kinds of Cartan decompositions:

$$\mathfrak{g} = Ad(a^{-1})\mathfrak{k} + \mathfrak{a} + \mathfrak{k}; \quad a \in A^+ \text{ (the right decompostion)},$$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + Ad(a)\mathfrak{k}; \quad a \in A^+ \text{ (the left decomposition)},$$

where the adjoint representation Ad is defined by

$$Ad(g)(X) = gXg^{-1} \quad (g \in G, X \in \mathfrak{g}_{\mathbf{C}}).$$

Here note that the sums are not direct sums, for the subspace $\mathfrak{m}$ in $\mathfrak{k}$ is $Ad(A)$-invariant.

Let $a = \exp(t_1H_1 + t_2H_2)$, then the entries of the matrix $a = (a_{ij})_{1\le i,j\le 4}$ is given as follows:

$$a_{11} = a_{33} = ch(a_1),\ a_{22} = a_{44} = ch(a_2),\ a_{13} = a_{31} = sh(a_1),\ a_{24} = a_{42} = sh(a_2)$$

with $a_i = \exp(t_i)$ $(i = 1, 2)$.

The Cartan decompositions of the standard generators (basis にする？) of $\mathfrak{p}_\pm$ is calculated as follows.

Lemma (3.1) (The right Cartan decomposition)

$$\begin{aligned}
X_{14} &= \frac{1}{D}\{-ch(a_1)sh(a_2)Ad(a^{-1})(e_+^1) - sh(a_1)ch(a_2)Ad(a^{-1})(e_+^2) \\
&\quad +sh(a_2)ch(a_2)e_+^1 + sh(a_1)ch(a_1)e_+^2\}, \\
X_{41} &= \frac{1}{D}\{ch(a_1)sh(a_2)Ad(a^{-1})(e_-^1) + sh(a_1)ch(a_2)Ad(a^{-1})(e_-^2) \\
&\quad -sh(a_2)ch(a_2)e_-^1 - sh(a_1)ch(a_1)e_-^2\}, \\
X_{23} &= \frac{1}{D}\{sh(a_1)ch(a_2)Ad(a^{-1})(e_-^1) + ch(a_1)sh(a_2)Ad(a^{-1})(e_-^2) \\
&\quad -sh(a_1)ch(a_1)e_-^1 - sh(a_2)ch(a_2)e_-^2\}, \\
X_{32} &= \frac{1}{D}\{-sh(a_1)ch(a_2)Ad(a^{-1})(e_+^1) - ch(a_1)sh(a_2)Ad(a^{-1})(e_+^2) \\
&\quad +sh(a_1)ch(a_1)e_+^1 + sh(a_2)ch(a_2)e_+^2\}, \\
X_{13} &= \frac{1}{2}\{sh(a_1^2)^{-1}Ad(a^{-1})(Z_{13}) + H_1 - ct(a_1^2)(Z_{13})\}, \\
X_{31} &= \frac{1}{2}\{-sh(a_1^2)^{-1}Ad(a^{-1})(Z_{13}) + H_1 + ct(a_1^2)(Z_{13})\}, \\
X_{24} &= \frac{1}{2}\{sh(a_2^2)^{-1}Ad(a^{-1})(Z_{24}) + H_2 - ct(a_2^2)(Z_{24})\}, \\
X_{42} &= \frac{1}{2}\{-sh(a_2^2)^{-1}Ad(a^{-1})(Z_{24}) + H_2 + ct(a_2^2)(Z_{24})\},
\end{aligned}$$

3.2 A-radial part of the annihilators

3.2.1 A-radial part of the gradient operators

Given two finite dimensional irreducible representations (τ_1, V_1), (τ_2, V_2) of K, we already defined the gradient operators $\nabla_\pm^*$ $(* = R, L)$:

$$\nabla_\pm^R : C_{\tau_1,\tau_2}^\infty(K\backslash G/K) \to C_{\tau_1,\tau_2\otimes\mathrm{Ad}_\pm}^\infty(K\backslash G/K)$$

$$\nabla^L_\pm : C^\infty_{\tau_1,\tau_2}(K\backslash G/K) \to C^\infty_{\tau_1\otimes\pm,\tau_2}(K\backslash G/K).$$

Here we compute their A-radial parts of these operators:

$$\rho_A(\nabla^R_\pm) : C^\infty_{\tau_1,\tau_2}(A, V_1\otimes V_2) \to C^\infty_{\tau_1,\tau_2\otimes \mathrm{Ad}_\pm}(A, V_1\otimes V_2)$$

$$\rho_A(\nabla^L_\pm) : C^\infty_{\tau_1,\tau_2}(A, V_1\otimes V_2) \to C^\infty_{\tau_1\otimes\pm,\tau_2}(A, V_1\otimes V_2).$$

Let us denote the Euler operator with respect to a_i by $\partial_i = a_i \cdot \frac{\partial}{\partial a_i}$. Then, for example, in the case of $\rho_A(\nabla^R_+)$, by Cartan decomposition we have

$$\rho_A(\nabla^R_+) = R_{X_{31}}\varphi\otimes X_{13} + R_{X_{41}}\varphi\otimes X_{14} + R_{X_{32}}\varphi\otimes X_{23} + R_{X_{42}}\varphi\otimes X_{24}$$

$$\begin{aligned}
&= \{\frac{1}{2}\partial_1 - \frac{1}{2}sh(a_1^2)^{-1}\tau_1(Z_{13}) - \frac{1}{2}ct(a_1^2)\tau_2(Z_{13})\}\varphi\otimes X_{13} \\
&\quad +\frac{1}{D}\{ch(a_1)sh(a_2)\tau_1(e^1_-) + sh(a_2)ch(a_2)\tau_1(e^2_-) \\
&\quad +sh(a_2)ch(a_2)\tau_2(e^1_-) + sh(a_1)ch(a_2)\tau_2(e^2_-)\}\varphi\otimes X_{14} \\
&\quad -\frac{1}{D}\{sh(a_1)ch(a_2)\tau_1(e^1_+) + ch(a_1)ch(a_2)\tau_1(e^2_+) \\
&\quad +sh(a_1)ch(a_1)\tau_2(e^1_+) + sh(a_2)ch(a_2)\tau_2(e^2_+)\}\varphi\otimes X_{23} \\
&\quad +\{\frac{1}{2}\partial_2 - \frac{1}{2}sh(a_2^2)^{-1}\tau_1(Z_{24}) - \frac{1}{2}ct(a_2^2)\tau_2(Z_{24})\}\varphi\otimes X_{24}
\end{aligned}$$

for $\varphi \in C^\infty(A, V_1\otimes V_2)$.

Here we rewrite $\tau_2(Z_{13})\varphi\otimes X_{13}$ etc. by

$$(\tau_2\otimes \mathrm{Ad}_+)(Z_{13})(\varphi\otimes X_{13}) - \varphi\otimes[Z_{13}, X_{13}] = (\tau_2\otimes \mathrm{Ad}_+)(Z_{13})(\varphi\otimes X_{13}) - 2\varphi\otimes X_{13}.$$

If we rewrite other operators $\rho_A(\nabla^R_-)$ similarly, we obtain the following.

Lemma (3.2) (the A-radial part of the gradient operators)

$$\begin{aligned}
\rho_A(\nabla^R_+)\varphi &= \frac{1}{2}\{\partial_1 - sh(a_1^2)^{-1}\tau_1(Z_{13}) - ct(a_1^2)(\tau_2\otimes \mathrm{Ad}_+)(Z_{13}) \\
&\quad +2ct(a_1^2) + \frac{2}{D}sh(a_1^2)\}(\varphi\otimes X_{13}) \\
&\quad +\frac{1}{2}\{\partial_2 - sh(a_2^2)^{-1}\tau_1(Z_{24}) - ct(a_2^2)(\tau_2\otimes \mathrm{Ad}_+)(Z_{24}) \\
&\quad +2ct(a_2^2) - \frac{2}{D}sh(a_2^2)\}(\varphi\otimes X_{24}) \\
&\quad +\frac{1}{D}\{ch(a_1)sh(a_2)\tau_1(e^1_-) + sh(a_1)ch(a_2)\tau_1(e^2_-) \\
&\quad +sh(a_2)ch(a_2)(\tau_2\otimes \mathrm{Ad}_+)(e^1_-) + sh(a_1)ch(a_1)(\tau_2\otimes \mathrm{Ad}_+)(e^2_-)\}(\varphi\otimes X_{14}) \\
&\quad -\frac{1}{D}\{sh(a_1)ch(a_2)\tau_1(e^1_+) + ch(a_1)sh(a_2)\tau_1(e^2_+) \\
&\quad +sh(a_1)ch(a_1)(\tau_2\otimes \mathrm{Ad}_+)(e^1_+) + sh(a_2)ch(a_2)(\tau_2\otimes \mathrm{Ad}_+)(e^2_+)\}(\varphi\otimes X_{23})
\end{aligned}$$

$$\begin{aligned}
\rho_A(\nabla^R_-)\varphi &= \frac{1}{2}\{\partial_1 + sh(a_1^2)^{-1}\tau_1(Z_{13}) + ct(a_1^2)(\tau_2\otimes \mathrm{Ad}_-)(Z_{13}) \\
&\quad +2ct(a_1^2) + \frac{2}{D}sh(a_1^2)\}(\varphi\otimes X_{31}) \\
&\quad +\frac{1}{2}\{\partial_2 + sh(a_2^2)^{-1}\tau_1(Z_{24}) + ct(a_2^2)(\tau_2\otimes \mathrm{Ad}_-)(Z_{24})
\end{aligned}$$

$$+2ct(a_2^2) - \frac{2}{D}sh(a_2^2)\}(\varphi \otimes X_{42})$$
$$-\frac{1}{D}\{ch(a_1)sh(a_2)\tau_1(e_+^1) + sh(a_1)ch(a_2)\tau_1(e_+^2)$$
$$+sh(a_2)ch(a_2)(\tau_2 \otimes \mathrm{Ad}_+)(e_+^1) + sh(a_1)ch(a_1)(\tau_2 \otimes \mathrm{Ad}_+)(e_+^2)\}(\varphi \otimes X_{41})$$
$$+\frac{1}{D}\{sh(a_1)ch(a_2)\tau_1(e_-^1) + ch(a_1)sh(a_2)\tau_1(e_-^2)$$
$$+sh(a_1)ch(a_1)(\tau_2 \otimes \mathrm{Ad}_-)(e_-^1) + sh(a_2)ch(a_2)(\tau_2 \otimes \mathrm{Ad}_-)(e_-^2)\}(\varphi \otimes X_{32})$$

3.3 The A-radial part of the Casimir equation

In this subsection, we compute the $A-$radial part $\rho_A(\Omega)$ of Casimir operator Ω for $\varphi \in C^\infty_{\tau_1,\tau_2}(K\backslash G/K)$.

We have to rewrite the Casimir element Ω of $Z(\mathfrak{g}_{\mathbf{C}})$:

$$\Omega = H_1^2 + H_2^2 + \frac{1}{2}I_0^2 + 2\sum_{j=1}^{2}(E_j {}^t\overline{E}_j + {}^t\,\overline{E}_j E_j) + \sum_{j=3}^{6}(E_j {}^t\overline{E}_j + {}^t\,\overline{E}_j E_j)$$

(*cf.* Proposition (2.5)), utilizing the right Cartan decomposition $\mathfrak{g} = Ad(a^{-1})\mathfrak{k} + \mathfrak{a} + \mathfrak{k}$.

For each $\mu \in \Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ we denote by E_μ a root vector corresponding to μ and put $F_\mu = E_\mu - E_{-\mu}$, where we may assume $E_{-\mu} = {}^t\,\overline{E}_\mu$. Then we have $F_\mu = -F_{-\mu} \in \mathfrak{k}_{\mathbf{C}}$ and for each $a \in A^+$ we can write

$$Ad(a^{-1})F_\mu = a^{-\mu}E_\mu - a^\mu E_{-\mu} = a^{-\mu}F_\mu - (a^\mu - a^{-\mu})E_{-\mu}.$$

Hence each root vector has a right Cartan decomposition

$$E_{\pm\mu} = -\frac{1}{a^\mu - a^{-\mu}}Ad(a^{-1})F_\mu + \frac{a^{\pm\mu}}{a^\mu - a^{-\mu}}F_\mu.$$

Moreover noting the commutation relation

$$[F_\mu, Ad(a^{-1})F_\mu] = -(a^\mu - a^{-\mu})H_\mu,$$

where $H_\mu \in \mathfrak{a}$ is defined by $H_\mu = [E_\mu, E_{-\mu}]$, we get

$$\begin{aligned} E_\mu E_{-\mu} &= \frac{a^\mu}{a^\mu - a^{-\mu}}H_\mu + \frac{1}{(a^\mu - a^{-\mu})^2}\{(Ad(a^{-1})F_\mu)^2 + F_\mu\} - \frac{a^\mu + a^{-\mu}}{(a^\mu - a^{-\mu})^2}(Ad(a^{-1})F_\mu)F_\mu, \\ E_{-\mu}E_\mu &= \frac{a^{-\mu}}{a^\mu - a^{-\mu}}H_\mu + \frac{1}{(a^\mu - a^{-\mu})^2}\{(Ad(a^{-1})F_\mu)^2 + F_\mu\} - \frac{a^\mu + a^{-\mu}}{(a^\mu - a^{-\mu})^2}(Ad(a^{-1})F_\mu)F_\mu. \end{aligned}$$

In view of Section 1.1.2, for the positive roots $\{2\lambda_1, 2\lambda_2, \lambda_1 \pm \lambda_2\}$ in Δ, we here use the substitutions:

$$E_{2\lambda_1} = E_1, \quad E_{2\lambda_2} = E_2, \quad E_{\lambda_1+\lambda_2} = E_3 \quad \text{or} \quad E_4, \quad E_{\lambda_1-\lambda_2} = E_5 \quad \text{or} \quad E_6,$$

to have

$$\begin{aligned} &H_{2\lambda_1} = H_1, \quad H_{2\lambda_2} = H_2, \quad H_{\lambda_1+\lambda_2} = H_1 + H_2, \quad H_{\lambda_1-\lambda_2} = H_1 - H_2, \\ &F_{2\lambda_1} = \sqrt{-1}Z_{13}, \quad F_{2\lambda_2} = \sqrt{-1}Z_{24}, \end{aligned}$$

$$\begin{aligned} F_{\lambda_1+\lambda_2} &= (e_+^1 - e_-^1) - (e_+^2 - e_-^2) & &\text{if} E_{\lambda_1+\lambda_2} = E_3, \\ F_{\lambda_1+\lambda_2} &= \sqrt{-1}(e_+^1 + e_-^1) - \sqrt{-1}(e_+^2 + e_-^2) & &\text{if } E_{\lambda_1+\lambda_2} = E_4, \\ F_{\lambda_1-\lambda_2} &= (e_+^1 - e_-^1) + (e_+^2 - e_-^2) & &\text{if } E_{\lambda_1-\lambda_2} = E_5, \\ F_{\lambda_1-\lambda_2} &= \sqrt{-1}(e_+^1 + e_-^1) + \sqrt{-1}(e_+^2 + e_-^2) & &\text{if } E_{\lambda_1-\lambda_2} = E_6. \end{aligned}$$

From the above results we see that the radial part $\rho_A(\Omega)$ with respect to the right Cartan decomposition $\mathfrak{g} = Ad(a^{-1})\mathfrak{k} + \mathfrak{a} + \mathfrak{k}$ has the form, for each $a \in A^+$,

$$\rho_A(\Omega) = H_1^2 + H_2^2 + f_1(a)H_1 + f_2(a)H_2 + F(a)$$

with $f_i(a) \in \mathbf{R}$ and $F(a) \in Ad(a^{-1})\mathfrak{k} + \mathfrak{k}$. Here each $f_i(a)$ is given by

$$f_i(a) = 2\frac{a_1^2 + a_1^{-2}}{a_1^2 - a_1^{-2}} + 2\frac{a_1a_2 + a_1^{-1}a_2^{-1}}{a_1a_2 - a_1^{-1}a_2^{-1}} + (-1)^{i-1}2\frac{a_1a_2^{-1} + a_1^{-1}a_2}{a_1a_2^{-1} - a_1^{-1}a_2}.$$

To get the explicit form of $F(a)$ we note that the equalities

$$(F_\mu\varphi)(a) = \{-\tau_2(F_\mu)\varphi\}(a), \quad \{Ad(a^{-1})F_\mu\varphi\}(a) = \{\tau_1(F_\mu)\varphi\}(a)$$

hold for any $\varphi \in C^\infty_{\tau_1,\tau_2}(K\backslash G/K)$. Then the partial sum in $F(a)$ corresponding to the positive roots $2\lambda_1, 2\lambda_2$ is

$$\begin{aligned}
&-\frac{4}{(a_1^2 - a_1^{-2})^2}\{\tau_1(Z_{13})^2 + \tau_2(Z_{13})^2\} \quad - \quad \frac{4(a_1^2 + a_1^{-2})}{(a_1^2 - a_1^{-2})^2}\tau_1(Z_{13})\tau_2(Z_{13}) \\
&-\frac{4}{(a_2^2 - a_2^{-2})^2}\{\tau_1(Z_{24})^2 + \tau_2(Z_{24})^2\} \quad - \quad \frac{4(a_2^2 + a_2^{-2})}{(a_2^2 - a_2^{-2})^2}\tau_1(Z_{24})\tau_2(Z_{24}).
\end{aligned}$$

The partial sum in $F(a)$ corresponding to the positive roots $\lambda_1 \pm \lambda_2$ is

$$\begin{aligned}
&\frac{2}{(a_1a_2 - a_1^{-1}a_2^{-1})^2}\{\tau_1(e_+^1 - e_-^1 - e_+^2 + e_-^2)^2 + \tau_2(e_+^1 - e_-^1 - e_+^2 + e_-^2)^2 \\
&\qquad\qquad -\tau_1(e_+^1 + e_-^1 - e_+^2 - e_-^2)^2 - \tau_2(e_+^1 + e_-^1 - e_+^2 - e_-^2)^2\} \\
&+\frac{2(a_1a_2 + a_1^{-1}a_2^{-1})}{(a_1a_2 - a_1^{-1}a_2^{-1})^2}\{\tau_1(e_+^1 - e_-^1 - e_+^2 + e_-^2)) \cdot \tau_2(e_+^1 - e_-^1 - e_+^2 + e_-^2) \\
&\qquad\qquad -\tau_1(e_+^1 + e_-^1 - e_+^2 - e_-^2) \cdot \tau_2(e_+^1 + e_-^1 - e_+^2 - e_-^2)\} \\
&+\frac{2}{(a_1a_2^{-1} - a_1^{-1}a_2)^2}\{\tau_1(e_+^1 - e_-^1 + e_+^2 - e_-^2)^2 + \tau_2(e_+^1 - e_-^1 + e_+^2 - e_-^2)^2 \\
&\qquad\qquad -\tau_1(e_+^1 + e_-^1 + e_+^2 + e_-^2)^2 - \tau_2(e_+^1 + e_-^1 + e_+^2 + e_-^2)^2\} \\
&+\frac{2(a_1a_2^{-1} + a_1^{-1}a_2)}{(a_1a_2^{-1} - a_1^{-1}a_2)^2}\{\tau_1(e_+^1 - e_-^1 + e_+^2 - e_-^2) \cdot \tau_2(e_+^1 - e_-^1 + e_+^2 - e_-^2) \\
&\qquad\qquad -\tau_1(e_+^1 + e_-^1 + e_+^2 + e_-^2)) \cdot \tau_2(e_+^1 + e_-^1 + e_+^2 + e_-^2))\} \\
&= \frac{4}{(a_1a_2 - a_1^{-1}a_2^{-1})^2}\sum_{i=1}^{2}\tau_i(e_+^1e_-^2 + e_-^2e_+^1 + e_-^1e_+^2 + e_+^2e_-^1 - e_+^1 - e1_- - e_-^1e_+^1 - e_+^2e_-^2 - e_-^2e_+^2) \\
&+\frac{4(a_1a_2 + a_1^{-1}a_2^{-1})}{(a_1a_2 - a_1^{-1}a_2^{-1})^2}\{\tau_1(e_+^1)\tau_2(e_-^2) + \tau_1(e_-^2)\tau_2(e_+^1) + \tau_1(e_-^1)\tau_2(e_+^2) + \tau_1(e_+^2)\tau_2(e_-^1) \\
&\qquad\qquad -\tau_1(e_+^1)\tau_2(e_-^1) - \tau_1(e_-^1)\tau_2(e_+^1) - \tau_1(e_+^2)\tau_2(e_-^2) - \tau_1(e_-^2)\tau_2(e_+^2)\} \\
&-\frac{4}{(a_1a_2^{-1} - a_1^{-1}a_2)^2}\sum_{i=1}^{2}\tau_i(e_+^1e_-^2 + e_-^2e_+^1 + e_-^1e_+^2 + e_+^2e_-^1 + e_+^1e_-^1 + e_-^1e_+^1 + e_+^2e_-^2 + e_-^2e_+^2) \\
&-\frac{4(a_1a_2^{-1} + a_1^{-1}a_2)}{(a_1a_2^{-1} - a_1^{-1}a_2)^2}\{\tau_1(e_+^1)\tau_2(e_-^2) + \tau_1(e_-^2)\tau_2(e_+^1) + \tau_1(e_-^1)\tau_2(e_+^2) + \tau_1(e_+^2)\tau_2(e_-^1) \\
&\qquad\qquad +\tau_1(e_+^1)\tau_2(e_-^1) + \tau_1(e_-^1)\tau_2(e_+^1) + \tau_1(e_+^2)\tau_2(e_-^2) + \tau_1(e_-^2)\tau_2(e_+^2)\}.
\end{aligned}$$

Now using the equalities

$$
\begin{aligned}
&\frac{a_1a_2+a_1^{-1}a_2^{-1}}{a_1a_2-a_1^{-1}a_2^{-1}}+\frac{a_1a_2^{-1}+a_1^{-1}a_2}{a_1a_2^{-1}-a_1^{-1}a_2}=\frac{1}{D}sh(a_1^2),\\
&\frac{a_1a_2+a_1^{-1}a_2^{-1}}{a_1a_2-a_1^{-1}a_2^{-1}}-\frac{a_1a_2^{-1}+a_1^{-1}a_2}{a_1a_2^{-1}-a_1^{-1}a_2}=-\frac{1}{D}sh(a_2^2),\\
&\frac{4}{(a_1a_2-a_1^{-1}a_2^{-1})^2}+\frac{4}{(a_1a_2^{-1}-a_1^{-1}a_2)^2}=\frac{1}{D^2}\{ch(a_1^2)ch(a_2^2)-1\},\\
&\frac{4}{(a_1a_2-a_1^{-1}a_2^{-1})^2}-\frac{4}{(a_1a_2^{-1}-a_1^{-1}a_2)^2}=-\frac{1}{D^2}sh(a_1^2)sh(a_2^2),\\
&\frac{4(a_1a_2+a_1^{-1}a_2^{-1})}{(a_1a_2-a_1^{-1}a_2^{-1})^2}+\frac{4(a_1a_2^{-1}+a_1^{-1}a_2)}{(a_1a_2^{-1}-a_1^{-1}a_2)^2}=\frac{4}{D^2}ch(a_1)ch(a_2)\{sh(a_1)^2+sh(a_2)^2\},\\
&\frac{4(a_1a_2+a_1^{-1}a_2^{-1})}{(a_1a_2-a_1^{-1}a_2^{-1})^2}-\frac{4(a_1a_2^{-1}+a_1^{-1}a_2)}{(a_1a_2^{-1}-a_1^{-1}a_2)^2}=-\frac{4}{D^2}sh(a_1)sh(a_2)\{ch(a_1)^2+ch(a_2)^2\}
\end{aligned}
$$

we get the following formula for $\rho_A(\Omega)$.

Lemma (3.3) *For the Casimir operator Ω acting on $C^\infty_{\tau_1,\tau_2}(K\backslash G/K)$, its radial part $\rho_A(\Omega)$ is given follows*:

$$
\begin{aligned}
\rho_A(\Omega) &= \partial_1^2+\partial_2^2+2\{cth(a_1^2)+\frac{1}{D}sh(a_1^2)\}\partial_1+2\{cth(a_2^2)-\frac{1}{D}sh(a_2^2)\}\partial_2\\
&\quad+\frac{1}{2}\tau_2(I_0)^2+T_1+T_2+T_3+T_4+T_5+T_6.
\end{aligned}
$$

Here

$$
\begin{aligned}
T_1 &= -sh(a_1^2)^{-2}\{\tau_1(Z_{13})^2+\tau_2(Z_{13})^2\}-sh(a_2^2)^{-2}\{\tau_1(Z_{24})^2+\tau_2(Z_{24})^2\},\\
T_2 &= -2ch(a_1^2)sh(a_1^2)^{-2}\tau_1(Z_{13})\tau_2(Z_{13})-2ch(a_2^2)sh(a_2^2)^{-2}\tau_2(Z_{24})\tau_2(Z_{24}),\\
T_3 &= -\frac{1}{D^2}\{ch(a_1^2)ch(a_2^2)-1\}\sum_{i=1}^{2}\tau_i(e^1_+e^1_-+e^1_-e^1_++e^2_+e^2_-+e^2_-e^2_+),\\
T_4 &= -\frac{1}{D^2}sh(a_1^2)sh(a_2^2)\sum_{i=1}^{2}\tau_i(e^1_+e^2_-+e^2_-e^1_++e^1_-e^2_++e^2_+e^1_-),\\
T_5 &= -\frac{4}{D^2}ch(a_1)ch(a_2)\{sh(a_1)^2+sh(a_2)^2\}\\
&\quad\{\tau_1(e^1_+)\tau_2(e^1_-)+\tau_1(e^1_-)\tau_2(e^1_+)+\tau_1(e^2_+)\tau_2(e^2_-)+\tau_1(e^2_-)\tau_2(e^2_+)\},\\
T_6 &= -\frac{4}{D^2}sh(a_1)sh(a_2)\{ch(a_1)^2+ch(a_2)^2\}\\
&\quad\{\tau_1(e^1_+)\tau_2(e^2_-)+\tau_1(e^2_-)\tau_2(e^1_+)+\tau_1(e^1_-)\tau_2(e^2_+)+\tau_1(e^2_+)\tau_2(e^1_-)\}.
\end{aligned}
$$

4 Explict differential equations for each coefficient

The A-radial part $\rho_A(\Omega)$ of Casimir operator is a second order differential operator for the corner K-type, namely it is a second order operator whose values are matrices of size $(|m|+1)^2$. And we have yet another second order differential operator comming from the Schmid operator.

In this section, we have the explicit fomulae of these operators, in terms of coefficients of the spherical functions.

4.1 Differential equation obtained by Schmid operators

From the equation for the spherical function $\varphi = \sum c_M f_M$ obtained by Schmid operators, we deduce a second order differential equation for each coefficien c_M.

Let us firstly suppose $m \neq 0$. Then we have

$$[P^{(+,-)} \circ \rho_A(\nabla_+^R)]\varphi = 0, \quad [\overline{P}^{(+,-)} \circ \rho_A(\nabla_-^R)]\varphi = 0,$$
$$[P^{(-,+)} \circ \rho_A(\nabla_+^R)]\varphi = 0, \quad [\overline{P}^{(-,+)} \circ \rho_A(\nabla_-^R)]\varphi = 0$$

in the cases $(sgn(m), \varepsilon) = (+1,+1), (-1,-1), (-1,+1), (+1,-1)$, respectively. By Lemma(3.2) and the projection formula (1.2) we see that the left hand sides of the above equations are written uniformly as $-sgn(m)\sum_M \Gamma_M$ where

$$\begin{aligned}\Gamma_M \;=\;& \tfrac{t_1}{2}\{\partial_1 + (A_1 - B)sh(a_1^2)^{-1} - (A_1 + B)cth(a_1^2) + \tfrac{2}{D}sh(a_1^2)\}c_M f_{M+\delta_1}\\ &+\tfrac{t_2}{2}\{\partial_2 + (A_2 - B)sh(a_2^2)^{-1} - (A_2 + B)cth(a_2^2) - \tfrac{2}{D}sh(a_2^2)\}c_M f_{M+\delta_2}\\ &+t_2 c_M \tfrac{1}{D}\{t_2 sh(a_1)ch(a_2) f_{M+\delta_3} + t_1 sh(a_1)ch(a_1) f_{M+\delta_4} + sh(a_2)ch(a_2) f_{M+\delta_5}\}\\ &-t_1 c_M \tfrac{1}{D}\{t_1 ch(a_1)sh(a_2) f_{M+\delta_6} + t_2 sh(a_2)ch(a_2) f_{M+\delta_7} + sh(a_1)ch(a_1) f_{M+\delta_8}\}.\end{aligned}$$

Here $A_i = A_i(M), B = B(M)$ and the shift parameters $\{\delta_i\}$ are given as follows:

	$m>0, \varepsilon>0$	$m<0, \varepsilon>0$	$m<0, \varepsilon>0$	$m>0, \varepsilon<0$
δ_1	$(0,0,1,-1)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,-1,1)$
δ_2	$(0,0,0,0)$	$(0,0,1,-1)$	$(0,0,-1,1)$	$(0,0,0,0)$
δ_3	$(0,-1,1,0)$	$(0,1,0,-1)$	$(1,0,-1,0)$	$(-1,0,0,1)$
δ_4	$(0,0,1,-1)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,-1,1)$
δ_5	$(0,0,0,0)$	$(0,0,1,-1)$	$(0,0,-1,1)$	$(0,0,0,0)$
δ_6	$(0,1,0,-1)$	$(0,-1,1,0)$	$(-1,0,0,1)$	$(1,0,-1,0)$
δ_7	$(0,0,0,0)$	$(0,0,1,-1)$	$(0,0,-1,1)$	$(0,0,0,0)$
δ_8	$(0,0,1,-1)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,-1,1)$

Throughout the following δ denotes the element of $\mathbf{Z}^4$ defined by

	$m \geq 0, \varepsilon>0$	$m<0, \varepsilon>0$	$m<0, \varepsilon>0$	$m \geq 0, \varepsilon<0$
$\delta =$	$(0,1,0,-1)$,	$(0,-1,0,1)$,	$(-1,0,1,0)$,	$(1,0,-1,0)$.

From the above we see that four functions

- the coefficient of $f_{M+(0,0,1,-1)}$ in $\sum_M \Gamma_M$ in the case $m>0$, $\varepsilon>0$,
- the coefficient of f_M in $\sum_M \Gamma_M$ in the case $m<0$, $\varepsilon<0$,
- the coefficient of f_M in $\sum_M \Gamma_M$ in the case $m<0$, $\varepsilon>0$,
- the coefficient of $f_{M+(0,0,1,-1)}$ in $\sum_M \Gamma_M$ in the case $m<0$, $\varepsilon<0$

have the same form

$$\begin{aligned}&\tfrac{t_1}{2}\{\partial_1+(A_1-B)sh(a_1^2)^{-1}-(A_1+B)cth(a_1^2)+\tfrac{2}{D}sh(a_1^2)\}c_M\\&+(t_2+1)^2\tfrac{1}{D}sh(a_1)ch(a_2)c_{M+\delta}+t_1t_2\tfrac{1}{D}sh(a_1)ch(a_1)c_M-t_1\tfrac{1}{D}sh(a_1)ch(a_1)c_M.\end{aligned}$$

Likewise, four functions

- the coefficient of f_M in $\sum_M \Gamma_M$ in the case $m>0$, $\varepsilon>0$,
- the coefficient of $f_{M+(0,0,1,-1)}$ in $\sum_M \Gamma_M$ in the case $m<0$, $\varepsilon<0$,
- the coefficient of $f_{M+(0,0,-1,1)}$ in $\sum_M \Gamma_M$ in the case $m<0$, $\varepsilon>0$,
- the coefficient of f_M in $\sum_M \Gamma_M$ in the case $m<0$, $\varepsilon<0$

are uniformly written as

$$\begin{aligned}&\tfrac{t_1}{2}\{\partial_2+(A_2-B)sh(a_2^2)^{-1}-(A_2+B)cth(a_2^2)-\tfrac{2}{D}sh(a_2^2)\}c_M\\&+t_2\tfrac{1}{D}sh(a_2)ch(a_2)c_M-(t_1+1)^2\tfrac{1}{D}ch(a_1)sh(a_2)c_{M-\delta}-t_1t_2\tfrac{1}{D}sh(a_2)ch(a_2)c_M.\end{aligned}$$

Hence we get the following

Lemma (4.1) *If $m\neq 0$ one has*

$$\begin{aligned}&(t_2+1)^2\tfrac{1}{D}sh(a_1)ch(a_2)c_{M+\delta}\\&=-\tfrac{t_1}{2}\{\partial_1+(A_1-B)sh(a_1^2)^{-1}-(A_1+B)cth(a_1^2)+(t_2+1)\tfrac{1}{D}sh(a_1^2)\}c_M,\\&(t_1+1)^2\tfrac{1}{D}ch(a_1)sh(a_2)c_{M-\delta}\\&=\tfrac{t_2}{2}\{\partial_2+(A_2-B)sh(a_2^2)^{-1}-(A_2+B)cth(a_2^2)-(t_1+1)\tfrac{1}{D}sh(a_2^2)\}c_M\end{aligned}$$

with $t_i=t_i(M)$, $A_i=A_i(M)$, $B=B(M)$.

From the above lemma we get a defferential equation satisfied by each coefficient:

$$\begin{aligned}&\{\partial_1+(A_1-B)sh(a_1^2)^{-1}-(A_1+B)cth(a_1^2)+(t_2+1)\tfrac{1}{D}sh(a_1^2)\}\\&\{\partial_2+(A_2-B)sh(a_2^2)^{-1}-(A_2+B)cth(a_2^2)-(t_1+1)\tfrac{1}{D}sh(a_2^2)\}c_M\\&+(t_1+1)t_2\tfrac{1}{D^2}sh(a_1^2)sh(a_2^2)c_M=0.\end{aligned}$$

We now treat the case $m=0$. Thus the space $V_{\tau_1}\otimes V_{\tau_2}$ is one-dimensional with the base f_O, $O=(0,0,0,0)$. Suppose $\varepsilon>0$ and consider the equation

$$[P^{(-,-)}\circ\rho_A(\nabla_+^R)\circ P^{(+,+)}\circ\rho_A(\nabla_+^R)]\varphi=0,\quad \varphi=c_Of_O.$$

Since $\tau_2\otimes Ad_+$ is irreducible with parameter $[1,1;u_2+2]$, our equation reduces to

$$[P^{(-,-)}\circ\rho_A(\nabla_+^R)\circ\rho_A(\nabla_+^R)]\varphi=0.$$

In $\mathfrak{p}_+$ we have

$$X_{13}=f_{10},\quad X_{24}=-f_{01},\quad X_{14}=-f_{11},\quad X_{23}=f_{00}$$

up to a common scalar. Then using Lemma(3.2) and the projection formula (1.2), we get

$$\begin{aligned}\rho_A(\nabla_+^R)\varphi&=\frac{1}{2}\{\partial_1-\frac{1}{2}u_1sh(a_1^2)^{-1}-\frac{1}{2}u_2cth(a_1^2)\}c_Of_{(0,0,1,0)}\\&-\frac{1}{2}\{\partial_2-\frac{1}{2}u_1sh(a_2^2)^{-1}-\frac{1}{2}u_2cth(a_2^2)\}c_Of_{(0,0,0,1)}.\end{aligned}$$

Write the right hand side as $q_1 f_{(0,0,1,0)} + q_2 f_{(0,0,1,0)}$. Then noting that $P^{(-,-)} \circ (\tau_2 \otimes Ad_+ \otimes Ad_+)$ has parameter $[0, 0; u_2 + 4]$ we get

$$\begin{aligned}
&[P^{(-,-)} \circ \rho_A(\nabla_+^R) \circ \rho_A(\nabla_+^R)]\varphi \\
&= -\tfrac{1}{2}\{\partial_1 - \tfrac{1}{2}u_1 sh(a_1^2)^{-1} - \tfrac{1}{2}u_2 cth(a_1^2) + \tfrac{2}{D} sh(a_1^2)\} q_2 f_O \\
&\quad + \tfrac{1}{2}\{\partial_2 - \tfrac{1}{2}u_1 sh(a_2^2)^{-1} - \tfrac{1}{2}u_2 cth(a_2^2) - \tfrac{2}{D} sh(a_2^2)\} q_1 f_O \\
&= \tfrac{1}{4}[\{\partial_1 - \tfrac{1}{2}u_1 sh(a_1^2)^{-1} - \tfrac{1}{2}u_2 cth(a_1^2) + \tfrac{2}{D} sh(a_1^2)\}\{\partial_2 - \tfrac{1}{2}u_1 sh(a_2^2)^{-1} - \tfrac{1}{2}u_2 cth(a_2^2)\} \\
&\quad + \{\partial_2 - \tfrac{1}{2}u_1 sh(a_2^2)^{-1} - \tfrac{1}{2}u_2 cth(a_2^2) - \tfrac{2}{D} sh(a_2^2)\}\{\partial_2 - \tfrac{1}{2}u_1 sh(a_2^2)^{-1} - \tfrac{1}{2}u_2 cth(a_2^2)\}] \\
&\quad \cdot c_O f_O.
\end{aligned}$$

Now it is easy to show that the coefficient of f_O in the last expression is equal to $\frac{1}{4}$ times left hand side of (1), if we put $t_1 = t_2 = 0$ in (1). We get the same result in the case $m = 0, \varepsilon < 0$ by a similar calculation. Summing up, we get

Proposition (4.2) *For any (m, ε) each coefficient c_M is annihilated by the differential operator*

$$\begin{aligned}
E = \; & \{\partial_1 + (A_1 - B) sh(a_1^2)^{-1} - (A_1 + B) cth(a_1^2) + (t_2 + 1)\tfrac{1}{D} sh(a_1^2)\} \\
& \{\partial_2 + (A_2 - B) sh(a_2^2)^{-1} - (A_2 + B) cth(a_2^2) - (t_1 + 1)\tfrac{1}{D} sh(a_2^2)\} \\
& + (t_1 + 1) t_2 \tfrac{1}{D^2} sh(a_1^2) sh(a_2^2)
\end{aligned}$$

with $t_i = t_i(M)$, $A_i = A_i(M)$, $B = B(M)$.

4.2 Explicit differential equation obtained by the Casimir operator

We start from Lemma (3.3). Let us consider $\varphi(a_1, a_2) = c_M(a_1, a_2) f_M \in C^\infty_{\tau_1,\tau_2}(K\backslash G/K)$. Recall that we have $k_1 = k_2 = 0$ or $\ell_1 = \ell_2 = 0$ according as $\varepsilon m \geq 0$ or $\varepsilon m < 0$. We need to compute the action on φ of $\frac{1}{2}\tau_2(I_0)^2$ and $T_1, \cdots, T_6$ in Lemma (3.3). By Lemmas (2.2), (2.3), we have

$$\begin{aligned}
&\tfrac{1}{2}\tau_2(I_0)^2\varphi = \tfrac{1}{2}(t_1 - t_2)^2\varphi, \\
&T_1\varphi = -2\{(A_1^2 + B^2) sh(a_1^2)^{-2} + (A_2^2 + B^2) sh(a_2^2)^{-2}\}\varphi, \\
&T_2\varphi = 2\{(A_1^2 - B^2) ch(a_1^2) sh(a_1^2)^{-2} + (A_2^2 - B^2) ch(a_2^2) sh(a_2^2)^{-2}\}\varphi, \\
&T_3\varphi = -2(2t_1 t_2 + |m|)\tfrac{1}{D^2}\{ch(a_1^2) ch(a_2^2) - 1\}\varphi, \\
&T_4\varphi = T_6\varphi = 0.
\end{aligned}$$

As for the action of T_5 we set

$$\begin{aligned}
&T_5 = -\tfrac{4}{D^2} ch(a_1) ch(a_2)\{sh(a_1)^2 + sh(a_2)^2\}\Lambda, \text{ with} \\
&\Lambda := \tau_1(e_+^1)\tau_2(e_-^1) + \tau_1(e_-^1)\tau_2(e_+^1) + \tau_1(e_+^2)\tau_2(e_-^2) + \tau_1(e_+^2)\tau_2(e_-^2).
\end{aligned}$$

By Lemma (2.3) we have, in the case $\varepsilon m > 0$

$$\begin{aligned}
\Lambda\varphi &= \textstyle\sum_{l_1 + l_2 = s}\{(l_1 + 1)^2 c_{0,l_1+1,0,l_2-1} + (l_2 + 1)^2 c_{0,l_1-1,0,l_2+1}\} f^L_{0,l_1} \otimes f^R_{0,l_2} \\
&= \textstyle\sum_M\{(t_2 + 1)^2 c_{M+\delta} + (t_1 + 1)^2 c_{M-\delta}\} f_M
\end{aligned}$$

and in the case $\varepsilon m < 0$

$$\begin{aligned}
\Lambda\varphi &= \textstyle\sum_{k_1 + k_2 = r}\{(k_1 + 1)^2 c_{k_1+1,0,k_2-1,0} + (k_2 + 1)^2 c_{k_1-1,0,k_2+1,0}\} f^L_{k_1,0} \otimes f^R_{k_2,0} \\
&= \textstyle\sum_M\{(t_1 + 1)^2 c_{M-\delta} + (t_2 + 1)^2 c_{M+\delta}\} f_M,
\end{aligned}$$

where $t_i = t_i(M)$. But Lemma(4.1) implies in both cases

$$\begin{aligned}
&(t_2 + 1)^2 c_{M+\delta} + (t_1 + 1)^2 c_{M-\delta} \\
&= -\tfrac{t_1}{2} D sh(a_1)^{-1} ch(a_2)^{-1} \\
&\quad \{\partial_1 + (A_1 - B) sh(a_1^2)^{-1} - (A_1 + B) cth(a_1^2) + (t_2 + 1)\tfrac{1}{D} sh(a_1^2)\} c_M \\
&\quad + \tfrac{t_2}{2} D ch(a_1)^{-1} sh(a_2)^{-1} \\
&\quad \{\partial_2 + (A_2 - B) sh(a_2^2)^{-1} - (A_2 + B) cth(a_2^2) - (t_1 + 1)\tfrac{1}{D} sh(a_2^2)\} c_M.
\end{aligned}$$

This computation covers the case $m = 0$ if we put $t_1 = t_2 = 0$.

From these results we see that each coefficient c_M is annihilated by the second order differential operator

$$\begin{aligned} P = \; & \partial_1^2 + \partial_2^2 \\ & +2\{cth(a_1^2) + \tfrac{1}{D} sh(a_1^2) + \tfrac{t_1}{D}(sh(a_1)^2 + sh(a_2)^2) cth(a_1)\}\partial_1 \\ & +2\{cth(a_2^2) - \tfrac{1}{D} sh(a_2^2) - \tfrac{t_2}{D}(sh(a_1)^2 + sh(a_2)^2) cth(a_2)\}\partial_2 \\ & +\tfrac{1}{2}(t_1 - t_2)^2 \\ & -2\{(A_1^2 + B^2) sh(a_1^2)^{-2} + (A_2^2 + B^2) sh(a_2^2)^{-2}\} \\ & +2\{(A_1^2 - B^2) ch(a_1^2) sh(a_1^2)^{-2} + (A_2^2 - B^2) ch(a_2^2) sh(a_2^2)^{-2}\} \\ & -2(2t_1t_2 + |m|)\tfrac{1}{D^2}(ch(a_1^2)ch(a_2^2) - 1) \\ & +\tfrac{2t_1}{D}(sh(a_1)^2 + sh(a_2)^2) cth(a_1) \\ & \quad \{(A_1 - B) sh(a_1^2)^{-1} - (A_1 + B) cth(a_1^2) + (t_2 + 1)\tfrac{1}{D} sh(a_1^2)\} \\ & -\tfrac{2t_2}{D}(sh(a_1)^2 + sh(a_2)^2) cth(a_2) \\ & \quad \{(A_2 - B) sh(a_2^2)^{-1} - (A_2 + B) cth(a_2^2) - (t_1 + 1)\tfrac{1}{D} sh(a_2^2)\}. \end{aligned}$$

In this expression of P, the sum of the terms with denominator D^2 has the numerator

$$\begin{aligned} & -2(2t_1t_2 + |m|)(ch(a_1^2)ch(a_2^2) - 1) + 4t_1(t_2 + 1)(sh(a_1)^2 + sh(a_2)^2) ch(a_1)^2 \\ & +4t_2(t_1 + 1)(sh(a_1)^2 + sh(a_2)^2) ch(a_2)^2 \\ = \; & -4(2t_1t_2 + |m|)(2sh(a_1)^2 sh(a_2)^2 + sh(a_1)^2 + sh(a_2)^2) \\ & +4(sh(a_1)^2 + sh(a_2)^2)\{t_1(t_2 + 1)(1 + sh(a_1)^2) + t_2(t_1 + 1)(1 + sh(a_2)^2)\} \\ = \; & 4D\{t_1(t_2 + 1) sh(a_1)^2 - t_2(t_1 + 1) sh(a_2)^2\}. \end{aligned}$$

Moreover we have $\frac{1}{2}(t_1 - t_2)^2 = \frac{1}{2}m^2 - 2t_1t_2$ in the constant term. Summung up, we get the following

Proposition (4.3) *Each coefficient c_M is annihilated by the differential operator*

$$\begin{aligned} P = \; & \partial_1^2 + \partial_2^2 \\ & +2\{cth(a_1^2) + \tfrac{1}{D} sh(a_1^2) + \tfrac{t_1}{D}(sh(a_1)^2 + sh(a_2)^2) cth(a_1)\}\partial_1 \\ & +2\{cth(a_2^2) - \tfrac{1}{D} sh(a_2^2) - \tfrac{t_2}{D}(sh(a_1)^2 + sh(a_2)^2) cth(a_2)\}\partial_2 \\ & -2\{(A_1^2 + B^2) sh(a_1^2)^{-2} + (A_2^2 + B^2) sh(a_2^2)^{-2}\} \\ & +2\{(A_1^2 - B^2) ch(a_1^2) sh(a_1^2)^{-2} + (A_2^2 - B^2) ch(a_2^2) sh(a_2^2)^{-2}\} \\ & +\tfrac{4}{D}\{t_1(t_2 + 1) sh(a_1)^2 - t_2(t_1 + 1) sh(a_2)^2\} \\ & +\tfrac{2t_1}{D}(sh(a_1)^2 + sh(a_2)^2) cth(a_1)\{(A_1 - B) sh(a_1^2)^{-1} - (A_1 + B) cth(a_1^2)\} \\ & -\tfrac{2t_2}{D}(sh(a_1)^2 + sh(a_2)^2) cth(a_2)\{(A_2 - B) sh(a_2^2)^{-1} - (A_2 + B) cth(a_2^2)\} \\ & -\{\nu^2 + (k - 1)^2 - 10 + 2t_1t_2\} \end{aligned}$$

with $t_i = t_i(M)$, $A_i = A_i(M)$, $B = B(M)$.

5 The holonomic system and its solutions

5.1 Change of functions

For each coefficient function $c_M(a_1, a_2)$ we introduce a new function $h_M(a_1, a_2)$ by

$$c_M(a_1, a_2) = f(a_1, a_2) h_M(a_1, a_2)$$

with multiplier

$$f(a_1, a_2) = ch(a_1)^{A_1} sh(a_1)^B ch(a_2)^{A_2} sh(a_2)^B; \quad A_i = A_i(M), B = B(M).$$

We rewrite the differential equations in Propositions (4.2) and (4.3) to differential equations for h_M.

We start with the following formulas:

$$\begin{aligned} f^{-1}\partial_i f &= \partial_i - (A_i - B)sh(a_i^2)^{-1} + (A_i + B)cth(a_i^2) \\ &= \partial_i + 2A_i cth(a_i^2) - (A_i - B)cth(a_i^2), \\ f^{-1}\partial_i^2 f &= \partial_i^2 + 2(2A_i cth(a_i^2) - (A_i - B)cth(a_i^2))\partial_i + ((A_i - B)^2 - 2(A_i + B))sh(a_i^2)^{-2} \\ &\quad -2(A_i + B - 1)(A_i - B)ch(a_i^2)sh(a_i^2)^{-2} + (A_i + B)^2 cth(a_i^2)^2. \end{aligned}$$

Firstly the operator E given in Proposition(4.2) is transformed to $\mathcal{E} := f^{-1}Ef$. By the formulas above, we have

$$\begin{aligned} \mathcal{E} &= f^{-1}\{\partial_1 + (A_1 - B)sh(a_1^2)^{-1} - (A_1 + B)cth(a_1^2) + \frac{t_2+1}{D}sh(a_1^2)\}f \\ &\quad \cdot f^{-1}\{\partial_2 + (A_2 - B)sh(a_2^2)^{-1} - (A_2 + B)cth(a_2^2) - \frac{t_1+1}{D}sh(a_2^2)\}f \\ &\quad +(t_1+1)t_2\frac{1}{D^2}sh(a_1^2)sh(a_2^2) \\ &= \{\partial_1 + \frac{t_2+1}{D}sh(a_1^2)\}\{\partial_2 - \frac{t_1+1}{D}sh(a_2^2)\} + (t_1+1)t_2\frac{1}{D^2}sh(a_1^2)sh(a_2^2), \end{aligned}$$

from which we immediately get the following

Proposition (5.1) *The function h_M is annihilated by the differential operator*

$$\mathcal{E} = \partial_1\partial_2 - \frac{t_1+1}{D}sh(a_2^2)\partial_1 + \frac{t_2+1}{D}sh(a_1^2)\partial_2$$

with $t_i = t_i(M)$.

We now consider the operator P given in Proposition (4.3). The function h_M is annihilated by $\mathcal{P} := f^{-1}Pf$:

$$\begin{aligned} \mathcal{P} &= \partial_1^2 + \partial_2^2 \\ &\quad +2[(2A_1+1)cth(a_1^2) + \frac{1}{D}sh(a_1^2) + \{\frac{t_1}{D}(sh(a_1)^2 + sh(a_2)^2) - A_1 + B\}cth(a_1)]\partial_1 \\ &\quad +2[(2A_2+1)cth(a_2^2) - \frac{1}{D}sh(a_2^2) + \{-\frac{t_2}{D}(sh(a_1)^2 + sh(a_2)^2) - A_2 + B\}cth(a_2)]\partial_2 \\ &\quad +U - (\nu^2 + (k-1)^2 - 10 + 2t_1t_2), \end{aligned}$$

where the term U is given by

$$\begin{aligned} U &= \sum_{i=1}^{2}\{((A_i - B)^2 - 2(A_i + B))sh(a_i^2)^{-2} + G_i + (A_i + B)^2 cth(a_i^2)^2\}\} \\ &\quad +2\{cth(a_1^2) + \frac{1}{D}sh(a_1^2) + H_1\}\{I_1 + (A_1 + B)cth(a_1^2)\} \\ &\quad +2\{cth(a_2^2) - \frac{1}{D}sh(a_2^2) - H_2\}\{I_2 + (A_2 + B)cth(a_2^2)\} \\ &\quad -2\{(A_1^2 + B^2)sh(a_1^2)^{-2} + (A_2^2 + B^2)sh(a_2^2)^{-2}\} \\ &\quad +2\{(A_1^2 - B^2)ch(a_1^2)sh(a_1^2)^{-2} + (A_2^2 - B^2)ch(a_2^2)sh(a_2^2)^{-2}\} \\ &\quad +\frac{4}{D}\{t_1(t_2+1)sh(a_1)^2 - t_2(t_1+1)sh(a_2)^2\} \\ &\quad +J_1 - J_2 \end{aligned}$$

with

$$\begin{aligned}
G_i &= -2(A_i+B-1)(A_i-B)ch(a_i^2)sh(a_i^2)^{-2},\\
H_i &= \frac{t_i}{D}(sh(a_1)^2+sh(a_2)^2)cth(a_i),\\
I_i &= -(A_i-B)sh(a_i^2)^{-1},\\
J_i &= \frac{2t_i}{D}(sh(a_1)^2+sh(a_2)^2)cth(a_i)\{(A_i-B)sh(a_i^2)^{-1}-(A_i+B)cth(a_i^2)\}.
\end{aligned}$$

In this expression for $\mathcal{P}$, the coefficients of ∂_1 and ∂_2 are simplified as

$$\begin{gathered}
2\{(2A_1+1)cth(a_1^2)+\frac{t_1+1}{D}sh(a_1^2)-(A_1-B+t_1)cth(a_1)\},\\
2\{(2A_2+1)cth(a_2^2)-\frac{t_2+1}{D}sh(a_2^2)-(A_2-B+t_2)cth(a_2)\},
\end{gathered}$$

respectively. We need to simplfy the expression for U. Noting the cancellations

$$\begin{gathered}
G_i+2cth(a_i^2)I_i+2(A_i^2-B^2)ch(a_i^2)sh(a_i^2)^{-2}=0,\\
2H_i\{I_i+(A_i+B)cth(a_i^2)\}+J_i=0,
\end{gathered}$$

we have

$$\begin{aligned}
U &= \sum_{i=1}^{2}\{((A_i-B)^2-2(A_i+B))sh(a_i^2)^{-2}+(A_i+B)^2cth(a_i^2)^2+2(A_i+B)cth(a_i^2)^2\\
&\qquad -2(A_i^2+B^2)sh(a_i^2)^{-2}\}\\
&\quad +\frac{2}{D}[sh(a_1^2)\{I_1+(A_1+B)cth(a_1^2)\}-sh(a_2^2)\{I_2+(A_2+B)cth(a_2^2)\}\\
&\quad +2t_1sh(a_1)^2-2t_2sh(a_2)^2]+4t_1t_2\\
&= \sum_{i=1}^{2}((A_i+B)^2+2(A_i+B))(cth(a_i^2)^2-sh(a_i^2)^{-2})\\
&\quad +\frac{4}{D}[(A_1+B+t_1)sh(a_1)^2-(A_2+B+t_2)sh(a_2)^2]+4t_1t_2.
\end{aligned}$$

Then using $cth(a)^2-sh(a)^{-2}=1$ and $2(A_1+B+t_1)=2(A_2+B+t_2)=|m|+\varepsilon u_2$ we get

$$\begin{aligned}
U &= \sum_{i=1}^{2}((A_i+B)^2+2(A_i+B))+2(|m|+\varepsilon u_2)+4t_1t_2\\
&= \frac{1}{2}m^2+2|m|+\frac{1}{2}u_2(u_2+8\varepsilon)+2t_1t_2.
\end{aligned}$$

Hence we have established the following

Proposition (5.2) *The function h_M is annihilated by the differential operator*

$$\begin{aligned}
\mathcal{P} &= \partial_1^2+\partial_2^2\\
&\quad +2\{(2A_1+1)cth(a_1^2)+\frac{t_1+1}{D}sh(a_1^2)-(A_1-B+t_1)cth(a_1)\}\partial_1\\
&\quad +2\{(2A_2+1)cth(a_2^2)-\frac{t_2+1}{D}sh(a_2^2)-(A_2-B+t_2)cth(a_2)\}\partial_2\\
&\quad +\frac{1}{2}m^2+2|m|-(k-1)^2-\nu^2+10+\frac{1}{2}u_2(u_2+8\varepsilon)
\end{aligned}$$

with $t_i=t_i(M)$, $A_i=A_i(M)$, $B=B(M)$.

5.2 Change of variables

Now we introduce new variables y_1 and y_2 by

$$y_i = -sh(a_i)^2 \ \ (i = 1, 2).$$

Then we have

$$\begin{aligned} &ch(a_i)^2 = 1 - y_i, \quad ch(a_i^2) = 1 - 2y_i, \quad sh(a_i^2)^2 = 4y_i(y_i - 1), \quad D = y_2 - y_1, \\ &\partial_i = -sh(a_i^2)\frac{\partial}{\partial y_i}. \end{aligned}$$

Putting $h_M(a_1, a_2) = h_M^*(y_1, y_2)$ we transform the differential equations for h_M given in Propositions (5.1) and (5.2) into the equation for h_M^*. Now the equation $\mathcal{P}h_M = 0$ reads

$$\begin{aligned} &\sum_{i=1}^{2} 4y_i(y_i - 1)\frac{\partial^2}{\partial y_i^2}h_M^* \\ &+\sum_{i=1}^{2}\{4(A_i + B - t_i + 2)y_i + 4(-B + t_i - 1) + (-1)^{i-1}8(t_i + 1)\frac{y_i(y_i - 1)}{y_1 - y_2}\}\frac{\partial}{\partial y_i}h_M^* \\ &+\{\frac{1}{2}m^2 + 2|m| - (k-1)^2 - \nu^2 + 10 + \frac{1}{2}u_2(u_2 + 8\varepsilon)\}h_M^* = 0, \end{aligned}$$

and the other equation is easier to handle. Recalling the definition of A_i and B we get

Proposition (5.3) *The function h_M^* satisfies the system*

$$[\frac{\partial^2}{\partial y_1 \partial y_2} - \frac{t_1 + 1}{y_1 - y_2}\cdot\frac{\partial}{\partial y_1} + \frac{t_2 + 1}{y_1 - y_2}\cdot\frac{\partial}{\partial y_2}]h_M^* = 0,$$

$$\begin{aligned} &\sum_{i=1}^{2} y_i(y_i - 1)\frac{\partial^2}{\partial y_i^2}h_M^* \\ &+\sum_{i=1}^{2}[\{\frac{1}{2}(|m| + \varepsilon u_2) - 2t_i + 2\}y_i - \varepsilon\frac{1}{4}(u_1 + u_2) + t_i - 1 + (-1)^{i-1}2(t_i + 1)\frac{y_i(y_i - 1)}{y_1 - y_2}]\frac{\partial}{\partial y_i}h_M^* \\ &+\frac{1}{4}[\frac{1}{2}m^2 + 2|m| - (k-1)^2 - \nu^2 + 10 + \frac{1}{2}u_2(u_2 + 8\varepsilon)]h_M^* = 0 \end{aligned}$$

with $t_i = t_i(M)$.

The first equation in the above system is often called *Euler-Darboux equation* from historical reason. The second, we call *Poisson equation* in this paper.

This system is the same as those in [2], Theorem 4,Theorem 7, [8] §8. Also it is the modified F_2 of [15] §2 Formula (2.1), p.211. We discuss the solutions in the next subsection.

5.3 Integral expression of the solutions

Our system is

$$[\frac{\partial^2}{\partial y_1 \partial y_2} - \frac{b_2}{y_1 - y_2}\cdot\frac{\partial}{\partial y_1} + \frac{b_1}{y_1 - y_2}\cdot\frac{\partial}{\partial y_2}]f = 0, \tag{1}$$

$$[\quad \sum_{i=1}^{2} y_i(y_i - 1)\frac{\partial^2}{\partial y_i^2} + \{(a + b_1 - b_2 + 1)y_1 + b_2 - c + 2b_2\frac{y_1(y_1 - 1)}{y_1 - y_2}\}\frac{\partial}{\partial y_1}$$

$$+ \quad \{(a-b_1+b_2+1)y_2+b_1-c-2b_1\frac{y_1(y_2-1)}{y_1-y_2}\}\frac{\partial}{\partial y_2}-\lambda]f=0 \tag{2}$$

where $\mathrm{Re}(b_i)>0$. To recover the system given in Proposition (5.3), the parameters should be specified by

$$b_1=t_2+1,\ b_2=t_1+1,\ a=-k+1,\ c=\frac{\varepsilon}{4}(u_1+u_2)+2=\frac{1}{4}(|m|-2k+\varepsilon u_1+8),$$

$$\lambda=-\frac{1}{4}\{\frac{1}{2}m^2+2|m|-(k-1)^2-\nu^2+10\frac{1}{2}u_2(u_2+8\varepsilon)\}=-\frac{1}{2}\{(|m|-k+3)^3-\nu^2\}.$$

In order to investigate the behaviour of the solutions of the Euler-Darboux equation, we want to determine the characteristic indices along the divisor of singularity $D=\{(y_1,y_2)|y_1=y_2\}$ of the equation. Let

$$f(y_1,y_2)=\sum_{n=0}^{\infty}\varphi_n(y_2)t^{\rho+n} \qquad (t=y_1-y_2)$$

be a formal power series solution at a generic point (y_2,y_2) of D. Since the Euler-Darboux equation in the new sytem of variables (t,y_2) reads

$$[(-\frac{\partial}{\partial t}+\frac{\partial}{\partial y_2})t\frac{\partial}{\partial t}-(b_1+b_2)\frac{\partial}{\partial t}+b_1\frac{\partial}{\partial y_2}]f=0,$$

the characteristic equation is $\rho^2+(b_1+b_2)\rho=0$. But the value $\rho=-(b_1+b_2)=-(t_2+t_1+2)<0$ should be discarded, because our solution is regular at the generic point of D. Hence we have

$$\rho=0, \quad \text{and} \quad \varphi_0(y_2)=f(y_1,y_2)|_{t=0}=f(y_2,y_2).$$

The reccurence relation among $\{\varphi_n\}_{n=0,1,\cdots}$, implies an integral expression

$$f(y_1,y_2)=B(b_1,b_2)^{-1}\int_0^1\varphi_0(y_2+(y_1-y_2)s)s^{b_1-1}(1-s)^{b_2-1}ds. \tag{3}$$

valid around D.

Now recall that the solution (3) of the Euler-Darboux equation satisfies the Poisson equation (2) if and only if $\varphi_0(z)$ satisfies Gaussian hypergeometric equation:

$$[z(1-z)\frac{d^2}{dz^2}+\{\gamma-(\alpha+\beta+1)z\}\frac{d}{dz}-\alpha\beta]\varphi_0(z)=0,$$

(*cf.* [8], Lemma (8.6)) with

$$\alpha+\beta=a+b_1+b_2, \alpha\cdot\beta=-\lambda, \gamma=c.$$

By Proposition (2.2), $\gamma\geq 2$ is a positive integer. Therefore the other solution of the Gaussian hypergeotric equation has the characteristic index $1-\gamma=-w$ wih $w=\frac{\varepsilon}{4}(u_1+u_2)+1=B+1\geq 1$, i.e., it has pole of order at least $-B-1$. Meanwhile the regularity of the coefficient

$$c_M(a,a)=ch(a)^{A_1+A_2}\{sh^2(a)\}^B\varphi_0(-sh^2(a))$$

at $a=1$ implies that the pole of $\varphi_0(z)$ at $z=0$ should be at most of order B. Hence $\varphi_0(z)$ is a constant multiple of the Gaussian hypergeometric series.

Summing up these facts, we have the following

Theorem (5.4) *The function $h^*_M(y_1, y_2)$, normalized by $h^*_M(0,0) = 1$, is expressed as*

$$h^*_M(y_1, y_2) = B(t_2+1, t_1+1)^{-1} \int_0^1 {}_2F_1(\alpha, \beta, \gamma; sy_1 + (1-s)y_2) s^{t_2} (1-s)^{t_1} ds.$$

with $t_i = t_i(M)$ and α, β, γ as above.
In particular the coefficient

$$c_M(a_1, a_2) = ch(a_1)^{A_1} ch(a_2)^{A_2} sh(a_1)^B sh(a_2)^B h^*_M(-sh^2(a_1), -sh^2(a_2))$$

has zeros of order $B = \frac{\varepsilon}{4}(u_1 + u_2)$ at $a_i = 1$ in each variable a_i $(i = 1, 2)$.

Remark The integral expression of the above theorem converges only for (y_1, y_2) such that $|y_i| = |sh(a_i)|^2 < 1$ $(i = 1, 2)$. This seems to be strange, because the matrix coefficients are defined everywhere on A. The point is our holonomic system has singularities over the complexification $A_{\mathbf{C}}$ at $y_1 = 1$ or at $y_2 = 1$, but not on A. These hidden singlarities show up in the complexification of G/K as seen in the paper of Akhiezer and Gindikin [1], as remarked in the Introduction.

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