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UNIQUE DETERMINATION OF INHOMOGENEITY IN A STATIONARY ISOTROPIC LAMÉ SYSTEM WITH VARIBLE COEFFICIENTS

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Abstract. We consider a two dimensional stationary Lamé system with variable coefficients. We prove the uniqueness in the inverse source problem of determining polygonal supports of distributing force by observations of surface displacement and stress. Our method is based on the regularity property of solutions to the Poisson equation in a polygonal domain.

1. Introduction

Let Ω be a bounded domain with C^2-boundary $\partial\Omega$. We assume that Ω is occupied by a nonhomogeneous isotropic elastic medium and consider a two dimensional stationary isotropic Lamé system with variable coefficients

$$\begin{aligned}(L\mathbf{u})(x) \ &\equiv \mu(x)\Delta\mathbf{u}(x) + (\lambda(x)+\mu(x))\nabla(\nabla\cdot\mathbf{u}(x)) \\ &+(\nabla\cdot\mathbf{u}(x))\nabla\lambda(x) + (\nabla\mathbf{u}(x)+(\nabla\mathbf{u}(x))^T)\nabla\mu(x) \\ &= \mathbf{F}(x), \qquad\qquad x\in\Omega\end{aligned} \tag{1.1}$$

with the boundary condition

$$\mathbf{u}(x) = \mathbf{f}(x), \qquad x\in\partial\Omega. \tag{1.2}$$

Here and henceforth,

$$x = (x_1, x_2)\in\mathbb{R}^2$$

$$\mathbf{u}(x) = (u_1(x), u_2(x))^T$$

$\cdot^T$: the transpose of a vector or a matrix under consideration

$$\nabla\cdot\mathbf{u}(x) = \frac{\partial u_1}{\partial x_1}(x) + \frac{\partial u_2}{\partial x_2}(x)$$

$$\nabla \mathbf{u}(x) = \left(\frac{\partial u_k}{\partial x_l} \right)_{1 \le k,l \le 2} \ : \ 2 \times 2 - \ \text{matrix}$$

$$\lambda, \mu \in C^2(\overline{\Omega}) : \ \text{the Lamé coefficients depending on } x.$$

We assume that

$$\mu(x) > 0, \quad \lambda(x) + 2\mu(x) > 0, \qquad x \in \overline{\Omega}. \tag{1.3}$$

System (1.1) with boundary input (1.2) describes the elastic displacement by an exterior force $\mathbf{F}(x) = (F_1(x), F_2(x))^T$. In practise, we often need determine the acting distributing force by the resulting surface stress $\sigma(\mathbf{u})\nu$ on a subboundary $\Gamma \subset \Omega$. Here and henceforth, the surface stress $\sigma(\mathbf{u})\nu$ is defined as follows: let $\nu = \nu(x)$ be the unit outward normal vector to $\partial\Omega$ at x, and we define a 2×2 matrix $\sigma(\mathbf{u})$ by

$$\sigma(\mathbf{u}) = \lambda(x)(\nabla \cdot \mathbf{u}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu(x)(\nabla \mathbf{u}(x) + (\nabla \mathbf{u}(x))^T) \tag{1.4}$$

for $\mathbf{u}(x) = (u_1(x), u_2(x))^T$.

In particular, we consider a force $\mathbf{F}$ in the form

$$\mathbf{F}(x) = q(x)(\chi_D(x), \chi_E(x))^T, \tag{1.5}$$

where q is a given and χ_D denotes the characteristic function of a set $D \subset \Omega$. Force (1.5) describes that the x_1- and x_2-components of the force distribute only in D and E respectively with the strength $q(x)$. In this paper, we will discuss

Inverse source problem. Let Γ be an arbitrary relatively open subset of $\partial\Omega$, $\mathbf{f}$ in (1.2) be fixed and $q \in C^2(\overline{\Omega})$ be given such that $q > 0$ on $\overline{\Omega}$. Then determine D and E in (1.1) with (1.5) by $\sigma(\mathbf{u})\nu$ on Γ.

Our main concern is the uniqueness: is the correspondence

$$\sigma(\mathbf{u})\nu|_\Gamma \longleftrightarrow (D, E)$$

one to one?

A similar problem for the Laplacian is called an inverse gravimetry problem: In $-\Delta u(x) = q(x)\chi_D(x)$ in $\mathbb{R}^3$ and $\lim_{|x|\to\infty} u(x) = 0$, determine D by $\nabla u|_{\partial\Omega}$ with a bounded domain Ω.

As for the inverse gravimetry, there are many papers and we can consult Anger [2], Isakov [5], [6]. However there seems no papers treating the determination of supports of right hand sides in elliptic systems with variable coefficients. Kim and Yamamoto [8] proposed a proof for the uniqueness within polygonal D's and we apply the argument in [7], [8] to prove the uniqueness in determining (D, E) in (1.1) and (1.5) by $\mathbf{f}$ on $\partial\Omega$ and $\sigma(\mathbf{u})\nu$ on Γ. Our argument is applicable to the higher dimensional cases and to other systems such as the stationary Maxwell's equations, but for conciseness, we restrict ourselves to the two dimensional stationary isotropic Lamé system.

The paper is organized as follows:
Section 2: Main results
Section 3: Preliminaries
Section 4: Proof of the main results.

2. Main results

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2 boundary $\partial\Omega$ and let D_j, E_j, $j = 1$ or 2, be sums of a finite number of polygons such that $\overline{D_j}, \overline{E_j} \subset \Omega$:

$$D_j = \cup_{n=1}^{M_j} D_{n,j} \quad \text{and} \quad E_j = \cup_{n=1}^{N_j} E_{n,j},$$

where $D_{n,j}$, $E_{n,j} \neq \emptyset$ are open polygons and the $\overline{D_{n,j}}, \overline{E_{n,j}} \subset \Omega$, $\overline{D_{n,j}} \cap \overline{D_{m,j}} = \overline{E_{n,j}} \cap \overline{E_{m,j}} = \emptyset$ if $n \neq m$.

Let $\mathbf{u_j} = (u_{j,1}, u_{j,2}) \in H^1(\Omega) \times H^1(\Omega)$, $j = 1$ or 2, be the weak solution to the Lamé system

$$(L\mathbf{u_j})(x) = q(x)(\chi_{D_j}(x), \chi_{E_j}(x))^T, \qquad x \in \Omega \tag{2.1}$$

with the Dirichlet boundary data

$$\mathbf{u_j}(x) = \mathbf{f}(x), \qquad x \in \Gamma \subset \partial\Omega. \tag{2.2}$$

Here $\mathbf{f} \in H^{\frac{3}{2}}(\Omega) \times H^{\frac{3}{2}}(\Omega)$, $q \in C^2(\overline{\Omega})$ and $q > 0$ on $\overline{\Omega}$.

Then it is well-known that

$$\mathbf{u_j} \in H^2(\Omega) \times H^2(\Omega). \tag{2.3}$$

Henceforth, for $D \subset \mathbb{R}^2$, we denote the convex hull (i,e., the smallest convex set containing D) by $co(D)$. Now we are ready to state our main results. First we consider the case where $D_1 = E_1$ and $D_2 = E_2$. That is, let $\mathbf{u_j}$, $j = 1$ or 2, satisfy the Lamé system

$$(L\mathbf{u_j})(x) = q(x)(\chi_{D_j}(x), \chi_{D_j}(x))^T, \qquad x \in \Omega. \tag{2.4}$$

Theorem 2.1. *We assume that $D_1 = E_1$ and $D_2 = E_2$. Then*

$$\sigma(\mathbf{u_1}(x))\nu(x) = \sigma(\mathbf{u_2}(x))\nu(x), \qquad x \in \Gamma \tag{2.5}$$

implies that $co(D_1 \cup E_1) = co(D_2 \cup E_2)$, where $\nu = \nu(x) = (\nu_1(x), \nu_2(x))^T$ is the outward unit normal vector to $\partial\Omega$.

Corollary 2.2. *Under the conditions in Theorem 2.1, we further assume that D_1, D_2 are convex. Then* (2.5) *implies $D_1 \cup E_1 = D_2 \cup E_2$.*

In the case where $D_1 \neq E_1$ or $D_2 \neq E_2$, we do not know whether $co(D_1 \cup E_1) = co(D_2 \cup E_2)$, in general. However, if $D_1 \cap E_1 = D_2 \cap E_2 = \emptyset$, we can obtain some global uniqueness result through the same argument as one of the proof of Theorem 2.1.

Theorem 2.3. *We assume that $D_1 \cap E_1 = D_2 \cap E_2 = \emptyset$. Then*

$$\sigma(\mathbf{u_1}(x))\nu(x) = \sigma(\mathbf{u_2}(x))\nu(x), \qquad x \in \Gamma \tag{2.6}$$

implies that $co(D_1 \cup E_1) = co(D_2 \cup E_2)$, where $\nu = \nu(x) = (\nu_1(x), \nu_2(x))^T$ is the outward unit normal vector to $\partial\Omega$.

In a succeeding paper, we will consider a more general case.

3. Preliminaries

In this section, we will give preliminary results for the proof of our main theorems. Henceforth let $\overline{AB}$ denote the line segment including A and B. The first lemma shows the regularity of an H^1-solution to an elliptic equation, which plays an important role in proving our main theorem. The proof is essentially based on [4]. For completeness, we will give the proof.

Lemma 3.1. *Let $\triangle P_1P_2P_3$ be the interior of a triangle which has three vertices $P_j \in \mathbb{R}^2$, $j = 1, 2, 3$. Assume that $f \in L^\mu(\triangle P_1P_2P_3)$ for some $\mu > 2$. If $v \in H^1(\triangle P_1P_2P_3)$ is the solution to a Dirichlet problem for the Laplace equation*

$$\begin{cases} \Delta v = f & \text{in } \triangle P_1P_2P_3 \\ v = 0 & \text{on } \overline{P_1P_2} \cup \overline{P_2P_3} \cup \overline{P_3P_1}, \end{cases} \tag{3.1}$$

then there exists a number $p > 2$ such that

$$v \in W^{2,p}(\triangle P_1P_2P_3). \tag{3.2}$$

Proof. Let θ_1, θ_2 and θ_3 be the angles $\measuredangle P_3P_1P_2$, $\measuredangle P_1P_2P_3$ and $\measuredangle P_2P_3P_1$, respectively. Since $0 < \theta_j < \pi$ for any $j = 1, 2, 3$, we can take a real number $q_0 \in (1, 2)$ so that

$$\frac{2}{q_0} < \min\left\{\frac{\pi}{\theta_1}, \frac{\pi}{\theta_2}, \frac{\pi}{\theta_3}\right\}. \tag{3.3}$$

Let $p := \min\left\{\frac{q_0}{q_0-1}, \mu\right\}$. Clearly the number p is greater than 2. We claim that

$$v \in W^{2,p}(\triangle P_1P_2P_3). \tag{3.4}$$

Let $q := \frac{p}{p-1}$. Then by (3.3) we have

$$\frac{2}{q} = \frac{2(p-1)}{p} \le \frac{2}{q_0} < \min\left\{\frac{\pi}{\theta_1}, \frac{\pi}{\theta_2}, \frac{\pi}{\theta_3}\right\},$$

which implies that the number $\frac{2\theta_j}{q\pi}$ is not an integer for any $j = 1, 2, 3$. Since $p \le \mu$ and $f \in L^\mu(\triangle P_1P_2P_3)$, we see that $f \in L^p(\triangle P_1P_2P_3)$. Therefore it follows from Theorem 4.4.4.13 in [4] that there exist real numbers $c_{j,m}$ and a function w such that

$$w - \sum_{\substack{1 \le j \le 3 \\ -\frac{2}{q} < \lambda_{j,m} < 0 \\ \lambda_{j,m} \ne -1}} c_{j,m} S_{j,m} \in W^{2,p}(\triangle P_1P_2P_3)$$

and

$$\begin{cases} \Delta w = f & \text{in } \triangle P_1P_2P_3 \\ w = 0 & \text{on } \overline{P_1P_2} \cup \overline{P_2P_3} \cup \overline{P_3P_1}, \end{cases}$$

where m is a negative integer, $\lambda_{j,m} = \frac{m\pi}{\theta_j}$, and the functions $S_{j,m}$ are defined in equation $(4, 4, 3, 7)$ in [4]. We note that $S_{j,m}$ does not necessarily belong to

$W^{2,p}(\triangle P_1P_2P_3)$. The uniqueness of the Dirichlet problem yields

$$w = v.$$

Furthermore our choice of constants p, q implies that there are not negative integers m such that

$$-\frac{2}{q} < \lambda_{j,m} = \frac{m\pi}{\theta_j} < 0.$$

Hence we can conclude that

$$v \in W^{2,p}(\triangle P_1P_2P_3).$$

$\square$

Applying the above lemma and the Sobolev imbedding theorem (e.g., [1]), we can prove that an H^1-solution to a Cauchy problem of the Laplace equation is of C^2-class in a neighbourhood of a corner of a triangular domain. This proposition plays an essential role in proving our theorems.

Lemma 3.2. *Let $\triangle P_1P_2P_3$ be the interior of a triangle which has three vertices $P_j \in \mathbb{R}^2$, $j = 1, 2, 3$. Let $G \in W^{1,\mu}(\triangle P_1P_2P_3)$ for some $\mu > 2$ and let $y \in H^1(\triangle P_1P_2P_3)$ be the solution to the Laplace equation*

$$\begin{cases} \Delta y = G & \text{in} \quad \triangle P_1P_2P_3 \\ y = |\nabla y| = 0 & \text{on} \quad \overline{P_1P_2} \cup \overline{P_1P_3}. \end{cases} \tag{3.5}$$

Then there exists a neighbourhood U of P_1 such that the solution y belongs to $C^2(\overline{U \cap \triangle P_1P_2P_3})$.

The proof is done similarly to Proposition 2.2 in [7] and the details are omitted here.

4. Proof of Theorem 2.1

Assume contrarily that $co(D_1) \neq co(D_2)$. Then, since $co(D_1)$ and $co(D_2)$ are convex polygons, there exists a vertex O of $co(D_1)$ such that $O \in \Omega \setminus \overline{co(D_2)}$ or a vertex O of $co(D_2)$ such that $O \in \Omega \setminus \overline{co(D_1)}$. Without loss of generality, we may assume the former case. Then there exists a polygon $D_{n,1}$ for some integer

$1 \leq n \leq M_1$ such that O is a vertex of $D_{n,1}$. Since $O \in \overline{D_{n,1}} \setminus \overline{co(D_2)}$, we can take a small triangle $\triangle OAB$ such that

$$\overline{OA} \cup \overline{OB} \subset \partial D_{n,1} \quad \text{and} \quad \triangle OAB \subset D_{n,1} \setminus \overline{co(D_2)}. \tag{4.1}$$

Here and henceforth $\triangle OAB$ means the interior of the triangle with the vertices O, A and B.

Let us define $\mathbf{v} = (v_1, v_2) := \mathbf{u_1} - \mathbf{u_2}$ in Ω. Then it follows from (2.2) and (2.4) that the function $\mathbf{v}$ satisfies

$$\begin{aligned}(L\mathbf{v}) &= \mu\Delta\mathbf{v} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + (\nabla\cdot\mathbf{v})\nabla\lambda + (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\nabla\mu \\ &= q(x)((\chi_{D_1} - \chi_{D_2}), (\chi_{D_1} - \chi_{D_2}))^T \qquad \text{in} \quad \Omega\end{aligned} \tag{4.2}$$

and

$$\mathbf{v} = \sigma(\mathbf{v})\nu = 0 \qquad \text{on} \quad \Gamma. \tag{4.3}$$

Let us denote by $\mathfrak{D}$ the connected component of $\Omega \setminus (\overline{D_1 \cup D_2})$ with $\partial\Omega$ as its boundary portion. Therefore, the unique continuation (e.g., [3]) implies that

$$\mathbf{v} \equiv 0 \qquad \text{on} \quad \overline{\mathfrak{D}}. \tag{4.4}$$

Since $\overline{OA} \cup \overline{OB} \subset \overline{\mathfrak{D}}$, we have

$$\mathbf{v} = |\nabla\mathbf{v}| = 0 \qquad \text{on} \quad \overline{OA} \cup \overline{OB}. \tag{4.5}$$

For simplicity, we define for $j = 1, 2$

$$\mathbf{H_j} = (H_{j,1}, H_{j,2}) := (\nabla\cdot\mathbf{u_j})\nabla\lambda + (\nabla\mathbf{u_j} + (\nabla\mathbf{u_j})^T)\nabla\mu$$

and

$$G_j := -\sum_{i=1}^{2}\{(\partial_{x_i}\mu)(\Delta u_{j,i}) + [\partial_{x_i}(\lambda+\mu)][\partial_{x_i}(\nabla\cdot\mathbf{u_j})] + \partial_{x_i}H_{j,i}\}.$$

Then by (2.3), (4.1), (4.2) and (4.5), we obtain that the function $\nabla\cdot\mathbf{v}$ is an H^1-solution to the following elliptic equation

$$\begin{cases}\Delta(\nabla\cdot\mathbf{v}) = \frac{1}{\lambda+2\mu}[(G_1 - G_2) + \partial_{x_1}q + \partial_{x_2}q] \qquad \text{in} \quad \triangle OAB \\ \nabla\cdot\mathbf{v} = 0 \qquad \text{on} \quad \overline{OA} \cup \overline{OB}.\end{cases} \tag{4.6}$$

Since μ, λ, q_1, $q_2 \in C^2(\overline{\Omega})$ and $\lambda + 2\mu > 0$ on $\overline{\Omega}$, the definition of $\mathbf{H_j}$ and G_j yields

$$\frac{1}{\lambda+2\mu}[(G_1 - G_2) + \partial_{x_1}q + \partial_{x_2}q] \in L^2(\triangle OAB). \tag{4.7}$$

By using an appropriate cut-off function, we can easily deduce from (4.6) and (4.7) that

$$\nabla \cdot \mathbf{v} \in H^2(\frac{1}{2}\triangle OAB). \tag{4.8}$$

Here and henceforth, for any $s > 0$, $\frac{1}{s}\triangle OAB$ means the interior of the triangle with the vertices O, $\frac{1}{s}(A - O) + O$ and $\frac{1}{s}(B - O) + O$.

On the other hand, we have from (4.1) and (4.2) that for $i = 1, 2$

$$\mu\Delta v_i = -(\lambda + \mu)[\partial_{x_i}(\nabla \cdot \mathbf{v})] - (H_{1,i} - H_{2,i}) + q \qquad \text{in} \quad \triangle OAB. \tag{4.9}$$

Since $\mu \in C^2(\overline{\Omega})$ and $\mu > 0$ on $\overline{\Omega}$, (4.8) yields

$$-\frac{\lambda + \mu}{\mu}[\partial_{x_i}(\nabla \cdot \mathbf{v})] - \frac{1}{\mu}(H_{1,i} - H_{2,i}) + \frac{1}{\mu}q \in H^1(\frac{1}{2}\triangle OAB). \tag{4.10}$$

Then (4.5), (4.8) and the Sobolev imbedding theorem implies that

$$\begin{cases} \Delta v_i \in L^\ell(\frac{1}{2}\triangle OAB) & \text{for any } \ell \geq 2 \\ v_i = 0 \quad \text{on} & \overline{OA} \cup \overline{OB}. \end{cases} \tag{4.11}$$

By using an appropriate cut-off function and Lemma 3.1, we obtain that there exists a number $\eta > 2$ such that

$$v_i \in W^{2,\eta}(\frac{1}{4}\triangle OAB). \tag{4.12}$$

Then (4.12) and the definition of $\mathbf{H_j}$ and G_j imply that

$$G_1 - G_2 \in L^\eta(\frac{1}{4}\triangle OAB), \tag{4.13}$$

and hence

$$\frac{1}{\lambda + 2\mu}[(G_1 - G_2) + \partial_{x_1}q + \partial_{x_2}q] \in L^\eta(\frac{1}{4}\triangle OAB). \tag{4.14}$$

Applying again Lemma 3.1 to (4.6) and using an appropriate cut-off function, we obtain that there exists a number $\rho > 2$ such that

$$\nabla \cdot \mathbf{v} \in W^{2,\rho}(\frac{1}{8}\triangle OAB). \tag{4.15}$$

Let $p = \min\{\eta, \rho\}$. It is clear that $p > 2$. Hence, it follows from (4.5), (4.9) and (4.15) that the function v_i is an H^2-function satisfying

$$\begin{cases} \Delta v_i = -\frac{\lambda+\mu}{\mu}[\partial_{x_i}(\nabla \cdot \mathbf{v})] - \frac{1}{\mu}(H_{1,i} - H_{2,i}) + \frac{1}{\mu}q \in W^{1,p}(\frac{1}{8}\triangle OAB) \\ v_i = |\nabla v_i| = 0 \quad \text{on} \quad \overline{OA} \cup \overline{OB}. \end{cases} \tag{4.16}$$

Then Lemma 3.2 yield that there exists a large number $S > 0$ such that

$$v_i \in C^2(\overline{\frac{1}{S}\triangle OAB}), \tag{4.17}$$

and hence by (4.5), for $i = 1, 2$, we have $\Delta v_i(O) = 0$ and

$$-\frac{\lambda(O)+\mu(O)}{\mu(O)}[\partial_{x_i}(\nabla \cdot \mathbf{v}(O))] - \frac{1}{\mu(O)}(H_{1,i}(O) - H_{2,i}(O)) = 0.$$

Since $q > 0$ on $\overline{\Omega}$, this contradicts (4.16). Therefore the proof of Theorem 2.1 is complete.

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