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Microsupport of Whitney solutions to systems with regular singularities and its applications

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Abstract. For systems of holomorphic linear differential equation with regular singularities in the sense of Kashiwara-Oshima, it is obtained that the bound to microsupport of the solution complex of the formal cohomology associated with constructible sheaf due to Kashiwara-Schapira. As applications, hyperbolic Cauchy and boundary value problems are considered for Whitney functions.

Introduction.

In algebraic analysis, a system of holomorphic linear differential equations on a complex manifold X is nothing but a (left) coherent Module $\mathscr{M}$ over the Ring $\mathscr{D}_X$ of *holomorphic linear differential operators* (in this paper, we shall write *Module* or *Ring* with capital letters, instead of *sheaf of modules* or *sheaf of rings*). Let F be a complex of sheaves on X with $\mathbb{R}$-constructible cohomologies (we fix the field $\mathbb{C}$ of complex numbers as a base ring). Then the complex of *generalized functions* associated with F is given by $\boldsymbol{R}\mathscr{H}om_{\mathbb{C}_X}(F, \mathscr{O}_X)$, and corresponding *solution sheaf complex* is $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \boldsymbol{R}\mathscr{H}om_{\mathbb{C}_X}(F, \mathscr{O}_X))$. Let us denote by $\mathrm{SS}(F)$ the *microsupport* of F due to Kashiwara-Schapira (see [K-S 2]). Then it is known that

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \boldsymbol{R}\mathscr{H}om_{\mathbb{C}_X}(F, \mathscr{O}_X))\big) \subset \mathrm{char}(\mathscr{M}) \mathbin{\widehat{+}} \mathrm{SS}(F)^a$$

(see § 1 for the notation) and various results can be obtained from this estimate. Next, we replace $\boldsymbol{R}\mathscr{H}om_{\mathbb{C}_X}(F, \mathscr{O}_X)$ by $T\mathscr{H}om(F, \mathscr{O}_X)$ of the *moderate cohomology* or $F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X$ of the *formal cohomology* ([K-S 3]). Then, the estimate above does not hold in general. However in a recent paper [MF-K-S], Monteiro Fernandes-Kashiwara-Schapira showed that if $\mathscr{M}$ has regular singularities along a regular involutory complex subbundle V of T^*X in the sense of Kashiwara-Oshima [K-O], then it follows that

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, T\mathscr{H}om(F, \mathscr{O}_X))\big) \subset V \mathbin{\widehat{+}} \mathrm{SS}(F)^a.$$

In this paper, we shall show that by the same methods and conditions as in [MF-K-S]

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\big) \subset V \mathbin{\widehat{+}} \mathrm{SS}(F)$$

holds. Moreover as applications, we shall show unique solvability theorems for *Cauchy* and *boundary value problems* for Whitney functions under a kind of hyperbolicity condition.

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We remark that the results in this paper may be generalized by the theory of *ind-sheaves* recently developed by Kashiwara-Schapira (cf. [K-S 4]).

1. Review and Preliminaries.

In this section, we shall fix the notation and recall results used in later sections. General references are made to Kashiwara-Schapira [K-S 2].

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, of real numbers and of complex numbers respectively. Further we set $\mathbb{N} := \{n \in \mathbb{Z};\, n \geqslant 1\} \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_{>0} := \{t \in \mathbb{R};\, t > 0\} \subset \mathbb{R}_{\geqslant 0} := \{t \in \mathbb{R};\, t \geqslant 0\}$ and $\mathbb{C}^{\times} := \mathbb{C} \setminus \{0\}$.

For a topological space S and a subset $A \subset S$, we denote by $\operatorname{Cl} A$ and $\operatorname{Int} A$ the *closure* and *interior* of A respectively.

In this paper, all the manifold are assumed to be paracompact. If $\tau\colon E \to Z$ is a vector bundle over a manifold Z, then we set $\dot{E} := E \setminus Z$ (the zero-section removed) and $\dot{\tau} := \tau\big|_{\dot{E}}$. Let $\pi\colon E^* \to Z$ be a dual bundle to E. We set

$$P^+ := \{(v, \xi) \in E \underset{Z}{\times} E^*;\, \langle v, \xi\rangle > 0\}.$$

Let $p_1^+\colon P^+ \to E$ and $p_2^+\colon P^+ \to E^*$ be the canonical projections. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(E)$ the full subcategory of $\mathbf{D}^{\mathrm{b}}(E) := \mathbf{D}^{\mathrm{b}}(\mathbb{C}_E)$ consisting of conic objects. Then the following proposition is used to define boundary value morphisms:

1.1. Proposition ([Ud, Corollary A.2], [S-K-K, Chapter I])**.** *For any $F \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(E)$, there exists the following distinguished triangle*:

$$F \to \tau^! \boldsymbol{R}\tau_! \, F \to \boldsymbol{R}p^+_{1*}\, p_2^{+\,!}\, F^\wedge \xrightarrow{+1}.$$

Here $F^\wedge$ denotes the Fourier-Sato transform of F.

Let X be a complex manifold, $\tau\colon TX \to X$ and $\pi\colon T^*X \to X$ the *tangent* and the *cotangent bundles* respectively. For conic subsets A, $B \subset T^*X$, we set:

$$\begin{aligned} A + B &:= \{(z; \zeta_1 + \zeta_2) \in T^*X;\, (z; \zeta_1) \in A,\, (z; \zeta_2) \in B\},\\ A^a &:= \{(z; \zeta) \in T^*X;\, (z; -\zeta) \in A\},\\ A^\circ &:= \bigcap_{(z;\zeta)\in A} \{(z; v) \in TX;\, \operatorname{Re}\langle v, \zeta\rangle \geqslant 0\}. \end{aligned}$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle\colon T_zX \times T_z^*X \to \mathbb{C}$ is the inner product. For conic subsets A, $B \subset TX$, we shall define $A + B$, $A^a \subset TX$ and $A^\circ \subset T^*X$ as same manners.

Normal and Conormal Bundles. Let M be a closed real analytic submanifold of X, $\tau_M\colon T_MX \to X$ and $\pi_M\colon T^*_MX \to X$ the *normal* and the *conormal bundles* to M in X respectively. Let $(x) = (x', x'')$ be local coordinates of X such that M is given by $x'' = 0$.

We also use $(x';x'')$ as local coordinates of T_MX. Let $(x;\xi)$ be local coordinates of T^*X associated with (x). Then the Hamiltonian isomorphism induces isomorphisms:

$$
\begin{array}{ccccc}
T^*T_MX & \xrightarrow{\sim} & T^*T^*_MX & \xrightarrow{\sim} & T_{T^*_MX}T^*X \\
\Cup & & \Cup & & \Cup \\
(x',x'';\xi',\xi'') & \longleftrightarrow & (x',\xi'';\xi',-x'') & \longleftrightarrow & (x',\xi'';x'',\xi').
\end{array}
\tag{1.1}
$$

We obtain a natural embedding $T^*M \hookrightarrow T_{T^*_MX}T^*X$ by:

$$
T^*M \ni (x';\xi') \mapsto (x',0;0,\xi') \in T_{T^*_MX}T^*X. \tag{1.2}
$$

For a subset $S \subset X$, we denote by $C_M(S)$ the *normal cone* which is a closed conic subset of T_MX given as follows: $(x'_0;x''_0) \in C_M(S)$ if and only if there exists a sequence $\{(x'_n,x''_n;c_n)\}_{n\in\mathbb{N}} \subset S \times \mathbb{R}_{>0}$ such that

$$
(x'_n,x''_n) \underset{n}{\to} (x'_0,0), \quad c_nx''_n \underset{n}{\to} x''_0. \tag{1.3}
$$

Let $i\colon M \hookrightarrow X$ be the natural embedding and $A \subset T^*X$ a conic subset. Then by (1.2) we set $i^\sharp(A) := T^*M \cap C_{T^*_MX}(A) \subset T^*M$. Note that $(x'_0;\xi'_0) \in i^\sharp(A)$ if and only if there exists a sequence $\{(x'_n,x''_n;\xi'_n,\xi''_n)\}_{n\in\mathbb{N}} \subset A$ such that

$$
(x'_n,x''_n;\xi'_n) \underset{n}{\to} (x'_0,0;\xi'_0), \quad |x''_n|\,|\xi''_n| \underset{n}{\to} 0. \tag{1.4}
$$

Diagonal Embedding Case. We denote by $\varDelta \subset X \times X$ the diagonal set and identify with X. Further, we identify TX with $T_\varDelta(X \times X)$ by the first projection. Similarly, TT^*X is identified with $T_{T^*_\varDelta(X\times X)}T^*(X\times X)$. Let $(x,\widetilde{x})$ be local coordinates of $X \times X$. Then $X = \varDelta$ is defined by $y := x - \widetilde{x} = 0$:

$$
X = \{(x,y);\, y = 0\} \subset X \times X = \{(x,y)\}. \tag{1.5}
$$

Let $(x,y;\xi,\eta)$ be local coordinates of $T^*(X\times X)$ associated with (x,y). Then isomorphisms of (1.1) are read as

$$
\begin{array}{ccccc}
T^*TX & \xrightarrow{\sim} & T^*T^*X & \xrightarrow{\sim} & TT^*X \\
\Cup & & \Cup & & \Cup \\
(x,y;\xi,\eta) & \longleftrightarrow & (x,\eta;\xi,-y) & \longleftrightarrow & (x,\eta;y,\xi).
\end{array}
\tag{1.6}
$$

In view of (1.1) and (1.2), we have the inclusion:

$$
T^*X \subset T_{T^*_\varDelta(X\times X)}T^*(X\times X) = TT^*X, \tag{1.7}
$$

which is given by $(x;\xi) \to (x,0;0,\xi)$. For any subsets S_1, $S_2 \subset X$, we set $C(S_1,S_2) := C_\varDelta(S_1 \times S_2) \subset TT^*X$. Further, we set

$$
A \mathbin{\widehat{+}} B := T^*X \cap C(A,B^a) \subset T^*X.
$$

By the definition, $A + B \subset A \mathbin{\widehat{+}} B = B \mathbin{\widehat{+}} A$ hold, and $(x_0;\xi_0) \in A \mathbin{\widehat{+}} B$ if and only if there exist sequences $\{(x_n;\xi_n)\}_{n\in\mathbb{N}} \subset A$ and $\{(y_n;\eta_n)\}_{n\in\mathbb{N}} \subset B$ such that

$$
x_n,\, y_n \underset{n}{\to} x_0, \quad \xi_n + \eta_n \underset{n}{\to} \xi_0, \quad |x_n - y_n|\,|\xi_n| \underset{n}{\to} 0. \tag{1.8}
$$

Microsupport. For any object F of $\mathbf{D}^{\mathrm{b}}(X)$, we denote by $\mathrm{SS}(F)$ the *microsupport* of F which is a closed conic subset of T^*X and described as follows:

Let (x) be local coordinates of X and $(x_0;\xi_0)$ a point of T^*X. Then $(x_0;\xi_0) \notin \mathrm{SS}(F)$ if and only if the following condition holds: There exist an open neighborhood U of x_0 in X and a proper convex (subanalytic) closed cone $\gamma \subset X$ satisfying $\xi_0 \in \mathrm{Int}\,\gamma^{\circ a} \cup \{0\}$ such that

$$\boldsymbol{R}\Gamma(H \cap (x+\gamma); F) \xrightarrow{\sim} \boldsymbol{R}\Gamma(L \cap (x+\gamma); F) \tag{1.9}$$

holds for any $x \in U$ and any sufficiently small $\varepsilon > 0$. Here

$$L := \{y \in X;\, \mathrm{Re}\,\langle y - x_0, \xi_0\rangle = -\varepsilon\} \subset H := \{y \in X;\, \mathrm{Re}\,\langle y - x_0, \xi_0\rangle \geqslant -\varepsilon\}.$$

Note that $\mathrm{SS}(F) \cap T^*_X X = \mathrm{supp}\, F$. Since $H \cap (x+\gamma)$ and $L \cap (x+\gamma)$ are compact, if we set

$$Z(x,\varepsilon) := (H \setminus L) \cap (x+\gamma) = \{y \in X;\, \mathrm{Re}\,\langle y - x_0, \xi_0\rangle > -\varepsilon\} \cap (x+\gamma), \tag{1.10}$$

then (1.9) is equivalent to

$$\boldsymbol{R}\Gamma_{\mathrm{c}}(Z(x,\varepsilon); F) = 0. \tag{1.11}$$

2. Systems with Regular Singularities.

From now on, M denotes an n-dimensional real analytic manifold, X a complexification of M, and $i\colon M \hookrightarrow X$ the natural embedding. We denote by $\mathscr{O}_X$ the sheaf *of holomorphic functions*, and by $\mathscr{D}_X$ the *Ring of holomorphic linear differential operators* on X respectively. Let $\mathscr{E}_X$ be the *Ring of microdifferential operators* on T^*X and $\{\mathscr{E}_X^{(m)}\}_{m\in\mathbb{Z}}$ the usual *order filtration* on $\mathscr{E}_X$ (see [S-K-K] or [Sc]). Let V be a $\mathbb{C}^\times$-conic involutory closed subset of $\dot{T}^*X$. Then we set

$$\mathscr{I}_V := \{P \in \mathscr{E}_X^{(1)};\, \sigma_1(P)\big|_V \equiv 0\}, \quad \mathscr{E}_V := \bigcup_{m\in\mathbb{N}_0} \mathscr{I}_V^{\,m}.$$

Here $\sigma_m(P)$ denotes the *principal symbol* of $P \in \mathscr{E}_X^{(m)}$. Namely, $\mathscr{E}_V \subset \mathscr{E}_X$ is a sheaf of subring generated by $\mathscr{I}_V$. By the definition, $\mathscr{E}_X^{(0)} \subset \mathscr{E}_V$ holds. Further Kashiwara-Oshima [K-O] proved that $\mathscr{E}_V$ is a Noetherian Ring, and that every coherent $\mathscr{E}_X$-Module is pseudocoherent as an $\mathscr{E}_V$-Module.

2.1. Definition ([K-O])**.** Let V be a $\mathbb{C}^\times$-conic involutory closed subset of $\dot{T}^*X$ and $\mathfrak{M}$ a coherent $\mathscr{E}_X$-Module defined in an open set of $\dot{T}^*X$. Then we say that $\mathfrak{M}$ *has regular singularities along* V if there exists locally a sheaf of $\mathscr{E}_V$-submodule $\mathfrak{L} \subset \mathfrak{M}$ such that $\mathfrak{L}$ is $\mathscr{E}_X^{(0)}$ coherent and that $\mathscr{E}_X\mathfrak{L} = \mathfrak{M}$.

If $\mathfrak{M}$ has regular singularities along V, then $\mathrm{supp}\,\mathfrak{M} \subset V$ ([K-K, Lemma 1.13]).

2.2. Definition. Let V be a $\mathbb{C}^\times$-conic involutory closed subset of T^*X and $\mathscr{M}$ a coherent $\mathscr{D}_X$-Module. Then we say that *$\mathscr{M}$ has regular singularities along V* if a coherent $\mathscr{E}_X$-Module $\mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \pi^{-1}\mathscr{M}$ has regular singularities along $V \cap \dot{T}^*X$ and the characteristic variety $\operatorname{char}(\mathscr{M}) := \operatorname{supp}(\mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \pi^{-1}\mathscr{M})$ is contained in V.

The notion of regular singularities is closely related to the Levi condition (see [D'A-T]).

If $V \subset T^*X$ is a regular involutory complex vector subbundle, then there exists locally a smooth morphism $f\colon X \to Z$ of complex manifolds such that

$$V = X \underset{Z}{\times} T^*Z. \tag{2.1}$$

2.3. Proposition ([MF-K-S], see also [H])**.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module which has regular singularities along $X \underset{Z}{\times} T^*Z$. Then there exists locally on X a finite free resolution of $\mathscr{M}$:*

$$0 \to \mathscr{D}_{X\to Z}^{\oplus N_r} \to \mathscr{D}_{X\to Z}^{\oplus N_{r-1}} \to \cdots \to \mathscr{D}_{X\to Z}^{\oplus N_1} \to \mathscr{D}_{X\to Z}^{\oplus N_0} \to \mathscr{M} \to 0.$$

3. The Functor of Formal Cohomology.

Let us briefly recall the functor of formal cohomology due to Kashiwara-Schapira [K-S 3]. We inherit the notation form the preceding section. Since the base ring is fixed to $\mathbb{C}$, we simply write $\mathscr{H}om(*,*) = \mathscr{H}om_{\mathbb{C}_X}(*,*)$, $* \otimes * = * \underset{\mathbb{C}_X}{\otimes} *$ and so on. We set $\mathscr{D}_M^A := i^{-1}\mathscr{D}_X$ to avoid the confusion. Let $\mathscr{B}_M$ and C_M^∞ be the sheaves on M *of Sato hyperfunctions* and *of complex valued C^∞ functions* respectively. We denote by $\mathbb{R}\text{-}\mathfrak{Cons}(M)$ and $\mathfrak{Mod}(\mathscr{D}_M^A)$ the Abelian categories *of $\mathbb{R}$-constructible sheaves on M* and *of (left) $\mathscr{D}_M^A$-Modules* respectively. Let $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(M)$ and $\mathbf{D}^{\mathrm{b}}(\mathscr{D}_M^A)$ be the bounded derived categories of $\mathbb{R}\text{-}\mathfrak{Cons}(M)$ and $\mathfrak{Mod}(\mathscr{D}_M^A)$ respectively. We denote by

$$* \overset{\mathrm{w}}{\otimes} C_M^\infty \colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(M) \to \mathbf{D}^{\mathrm{b}}(\mathscr{D}_M^A) \tag{3.1}$$

the *Whitney functor* due to Kashiwara-Schapira [K-S 3]. We recall:

3.1. Theorem ([K-S 3])**.** (1) $* \overset{\mathrm{w}}{\otimes} C_M^\infty \colon \mathbb{R}\text{-}\mathfrak{Cons}(M) \to \mathfrak{Mod}(\mathscr{D}_M^A)$ *is an exact functor.*

(2) *If $U \subset M$ is a subanalytic open subset, then $\mathbb{C}_U \overset{\mathrm{w}}{\otimes} C_M^\infty = \mathscr{I}^\infty_{M,M\setminus U} \subset C_M^\infty$ is the subsheaf consisting of sections vanishing at infinite order on $M \setminus U$.*

(3) *If $Z \subset M$ is a subanalytic closed subset, then $\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} C_M^\infty = \mathscr{W}^\infty_{M,Z} := C_M^\infty \big/ \mathscr{I}^\infty_{M,Z}$ is the sheaf of Whitney functions on Z.*

Let $X^{\mathbb{R}}$ be the real underlying manifold of X, and $\overline{X}$ the complex conjugate manifold of X. The *functor of formal cohomology* is defined by

$$* \overset{\mathrm{w}}{\otimes} \mathscr{O}_X := \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}}}(\mathscr{O}_{\overline{X}}, * \overset{\mathrm{w}}{\otimes} C^\infty_{X^{\mathbb{R}}}) \colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X) \to \mathbf{D}^{\mathrm{b}}(\mathscr{D}_X).$$

3.2. Theorem ([K-S 3]). (1) *For any* $F \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(M)$, *it follows that*

$$\boldsymbol{R}i_* F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X = \boldsymbol{R}i_*(F \overset{\mathrm{w}}{\otimes} C^\infty_M).$$

In particular, $\mathbb{C}_M \overset{\mathrm{w}}{\otimes} \mathscr{O}_X = C^\infty_M$ *holds.*

(2) *If* $Z \subset X$ *is a closed analytic subset, then* $\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{O}_X$ *is the formal completion of* $\mathscr{O}_X$ *along* Z.

(3) *There exists the following chain of morphisms*:

$$F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X \to T\mathscr{H}\!om(\mathrm{D}'_X F, \mathscr{O}_X) \to \boldsymbol{R}\mathscr{H}\!om(\mathrm{D}'_X F, \mathscr{O}_X).$$

Here $T\mathscr{H}\!om(*, \mathscr{O}_X)$ *denotes the functor of moderate cohomology due to Kashiwara* [K], *and* $\mathrm{D}'_X F := \boldsymbol{R}\mathscr{H}\!om(F, \mathbb{C}_X)$.

Let N be a real analytic closed submanifold of M. Let Y be a complexification of N in X, and $f_N \colon N \hookrightarrow M$ the canonical embedding with a complexification $f \colon Y \hookrightarrow X$:

$$\begin{tikzcd} N \arrow[r, hook, "f_N"] \arrow[d, hook] & M \arrow[d, hook, "i"] \\ Y \arrow[r, hook, "f"] & X. \end{tikzcd} \tag{3.2}$$

Let $\tau_N \colon T_N M \to N$ and $\pi_N \colon T^*_N M \to N$ be the normal and the conormal bundles to N in M respectively. We denote by $\nu_N(*) \colon \mathbf{D}^{\mathrm{b}}(M) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T_N M)$ the *specialization functor*, and by

$$\mathrm{W}\text{-}\nu_N(* \otimes C^\infty_M) = w\nu_N(*, C^\infty_M) \colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(M) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T_N M),$$
$$\mathrm{W}\text{-}\mu_N(* \otimes C^\infty_M) := \mathrm{W}\text{-}\nu_N(* \otimes C^\infty_M)^\wedge \colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(M) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T^*_N M),$$

the *Whitney specialization functor* and its Fourier-Sato transform due to Colin [C 1], [C 3]. In particular, we set:

$$\mathrm{W}\text{-}\nu_N(C^\infty_M) := \mathrm{W}\text{-}\nu_N(\mathbb{C}_M \otimes C^\infty_M), \quad \mathrm{W}\text{-}\mu_N(C^\infty_M) := \mathrm{W}\text{-}\mu_N(\mathbb{C}_M \otimes C^\infty_M).$$

Then we recall:

3.3. Theorem ([C 1]). (1) $\mathrm{W}\text{-}\nu_N(C^\infty_M)$ *and* $\mathrm{W}\text{-}\mu_N(C^\infty_M)$ *are concentrated in degree zero, and there exist the following natural monomorphisms of sheaves*:

$$\mathrm{W}\text{-}\nu_N(C^\infty_M) \rightarrowtail \nu_N(\mathscr{B}_M), \quad \mathrm{W}\text{-}\mu_N(C^\infty_M) \rightarrowtail \mu_N(\mathscr{B}_M).$$

(2) $\boldsymbol{R}\tau_{N!}\mathrm{W}\text{-}\nu_N(C^\infty_M) = (\mathbb{C}_N \overset{\mathrm{w}}{\otimes} C^\infty_M) \otimes \omega_{N/M} = \mathscr{W}^\infty_{M,N} \otimes \omega_{N/M}$ *and* $\boldsymbol{R}\tau_{N*}\mathrm{W}\text{-}\nu_N(C^\infty_M) = f_N^{-1} C^\infty_M$ *hold. Here* $\omega_{N/M}$ *is the relative dualizing complex.*

Taking $F = \mathrm{W}\text{-}\nu_N(C^\infty_M)$, $\nu_N(C^\infty_M)$ or $\nu_N(\mathscr{B}_M)$ in Proposition 1.1, we obtain:

3.4. Proposition. *There exists the following morphism of distinguished triangles*:

$$\begin{array}{ccccc}
\mathrm{W}\text{-}\nu_N(C^\infty_M) & \longrightarrow & \tau_N^{-1}\mathscr{W}^\infty_{M,N} & \longrightarrow & \boldsymbol{R}p^+_{1*}\,p_2^{+\,-1}\mathrm{W}\text{-}\mu_N(C^\infty_M)\otimes\omega^{\otimes-1}_{N/M}\xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow \\
\nu_N(C^\infty_M) & \longrightarrow & \tau_N^{-1}\boldsymbol{R}\varGamma_N(C^\infty_M)\otimes\omega^{\otimes-1}_{N/M} & \longrightarrow & \boldsymbol{R}p^+_{1*}\,p_2^{+\,-1}\mu_N(C^\infty_M)\otimes\omega^{\otimes-1}_{N/M}\xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow \\
\nu_N(\mathscr{B}_M) & \longrightarrow & \tau_N^{-1}\varGamma_N(\mathscr{B}_M)\otimes\omega^{\otimes-1}_{N/M} & \longrightarrow & \boldsymbol{R}p^+_{1*}\,p_2^{+\,-1}\mu_N(\mathscr{B}_M)\otimes\omega^{\otimes-1}_{N/M}\xrightarrow{+1}\,.
\end{array}$$

Note that applying the functor $\boldsymbol{R}\pi_{N*}$ to the distinguished triangles in Proposition 3.4 (or using Sato's fundamental distinguished triangle), we obtain the following morphisms of distinguished triangles:

$$\begin{array}{ccccc}
f_N^{-1}C^\infty_M & \longrightarrow & \mathscr{W}^\infty_{M,N} & \longrightarrow & \boldsymbol{R}\dot\pi_{N*}\mathrm{W}\text{-}\mu_N(C^\infty_M)\otimes\omega^{\otimes-1}_{N/M}\xrightarrow{+1} \\
\| & & \downarrow & & \downarrow \\
f_N^{-1}C^\infty_M & \longrightarrow & \boldsymbol{R}\varGamma_N(C^\infty_M)\otimes\omega^{\otimes-1}_{N/M} & \longrightarrow & \boldsymbol{R}\dot\pi_{N*}\,\mu_N(C^\infty_M)\otimes\omega^{\otimes-1}_{N/M}\xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow \\
f_N^{-1}\mathscr{B}_M & \longrightarrow & \varGamma_N(\mathscr{B}_M)\otimes\omega^{\otimes-1}_{N/M} & \longrightarrow & \boldsymbol{R}\dot\pi_{N*}\,\mu_N(\mathscr{B}_M)\otimes\omega^{\otimes-1}_{N/M}\xrightarrow{+1}\,.
\end{array}\tag{3.3}$$

For any coherent $\mathscr{D}_X$-Module $\mathscr{M}$, we denote by $\boldsymbol{D}f^*\mathscr{M} := \mathscr{O}_Y\overset{\boldsymbol{L}}{\underset{f^{-1}\mathscr{O}_X}{\otimes}} f^{-1}\mathscr{M}$ the *induced system* of $\mathscr{M}$. Assume that Y is non-characteristic for $\mathscr{M}$; that is, $\mathrm{char}(\mathscr{M})\cap\dot T^*_YX=\emptyset$. Then, it is known that $\boldsymbol{D}f^*\mathscr{M}$ is identified with $\mathscr{M}_Y := \mathcal{H}^0\boldsymbol{D}f^*\mathscr{M}$ which is a coherent $\mathscr{D}_Y$-Module. By [K-S 2, Exercise XI.11] and [K-S 3, Theorem 7.2], we have

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathscr{W}^\infty_{M,N}) & \longrightarrow & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\varGamma_N(\mathscr{B}_M))\otimes\omega^{\otimes-1}_{N/M} \\
\downarrow\wr & & \downarrow\wr \\
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y}(\mathscr{M}_Y,C^\infty_N) & \longrightarrow & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{B}_N).
\end{array}\tag{3.4}$$

In particular, by Proposition 3.4 and (3.4), we have a morphism $\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\nu_N(\mathscr{B}_M))\to \tau_N^{-1}\mathscr{H}\!om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{B}_N)$ which is nothing but the non-characteristic boundary value morphism (hence a monomorphism) (see for example [O-Y, Theorem 5.3]). Therefore, by Proposition 3.4 and (3.4), we obtain the following:

3.5. Proposition. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module for which Y is non-characteristic. Then the diagram below is commutative*:

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathrm{W}\text{-}\nu_N(C^\infty_M)) & \xrightarrow{\mathrm{W}\text{-}\gamma} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y}(\mathscr{M}_Y,C^\infty_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\nu_N(C^\infty_M)) & \longrightarrow & \tau_N^{-1}\boldsymbol{R}\varGamma_N\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},C^\infty_M)\otimes\omega^{\otimes-1}_{N/M} \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\nu_N(\mathscr{B}_M)) & \xrightarrow{\gamma} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{B}_N).
\end{array}$$

Further following all the morphisms are monomorphisms:

$$\begin{array}{ccccc}
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \text{W-}\nu_N(C^\infty_M)) & \rightarrowtail & \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(C^\infty_M)) & \rightarrowtail & \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{B}_M)) \\
\big\downarrow{\scriptstyle \text{W-}\gamma^0} & & & & \big\downarrow{\scriptstyle \gamma^0} \\
\tau_N^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, C^\infty_N) & & \longrightarrow & & \tau_N^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).
\end{array}$$

Let Z be a closed real analytic submanifold of X. We denote by

$$\begin{aligned}
\text{W-}\nu_Z(*\otimes\mathscr{O}_X) &:= \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{\overline{X}}}(\mathscr{O}_{\overline{X}}, \text{W-}\nu_Z(*\otimes C^\infty_{X^\mathbb{R}}))\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T_ZX),\\
\text{W-}\mu_Z(*\otimes\mathscr{O}_X) &:= \text{W-}\nu_Z(*\otimes\mathscr{O}_X)^\wedge \ : \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X) \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T^*_ZX),
\end{aligned}$$

the *formal specialization functor along* Z and its Fourier-Sato transform due to Colin [C 1], [C 3]. Note that as in Proposition 3.4, there exists the distinguished triangle below:

$$\text{W-}\nu_Z(F\otimes\mathscr{O}_X) \to \tau_Z^{-1}(F_Z \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \to \boldsymbol{R}p^+_{1*}\,p_2^{+\,-1}\text{W-}\mu_Z(F\otimes\mathscr{O}_X)\otimes\omega^{\otimes -1}_{Z/X} \xrightarrow{+1} . \tag{3.5}$$

4. Formal Microlocalization and Estimate of Microsupports.

We inherit the notation from the preceding sections. First, we impose the following:

4.1. Condition. V is a regular involutory complex subbundle of T^*X, and $\mathscr{M}$ is a coherent $\mathscr{D}_X$-Module and has regular singularities along V.

The following theorem is the first main result in this paper:

4.2. Theorem (cf. [MF-K-S, Theorem 2.1])**.** *Let* V *and* $\mathscr{M}$ *satisfy Condition* 4.1*. Then for any* $F \in \operatorname{Ob}\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X)$*, it follows that*

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F\overset{\mathrm{w}}{\otimes}\mathscr{O}_X)\big) \subset V \mathbin{\widehat{+}} \mathrm{SS}(F).$$

Proof. Since

$$\begin{aligned}
\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F\overset{\mathrm{w}}{\otimes}\mathscr{O}_X)\big)\cap T^*_XX &= \operatorname{supp}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F\overset{\mathrm{w}}{\otimes}\mathscr{O}_X)\big)\\
&\subset \operatorname{supp}\mathscr{M}\cap\operatorname{supp}F \subset \big(V\mathbin{\widehat{+}}\mathrm{SS}(F)\big)\cap T^*_XX,
\end{aligned}$$

we shall consider on $\dot{T}^*X$. The method of proof is same as in [MF-K-S]. Since the problem is local, we may assume that $X = Y\times Z$, $f\colon Y\times Z\to Z$ is a canonical projection, and $V = X\underset{Z}{\times}T^*Z = Y\times T^*Z$. Hence by Proposition 2.3 and a standard argument, we may assume that $\mathscr{M} = \mathscr{D}_{X\to Z}$.

Let $(x_0;\xi_0)$ be a point of $\dot{T}^*X$. Assume that $(x_0;\xi_0)\notin V\mathbin{\widehat{+}}\mathrm{SS}(F)$. We take a neighborhood U of x_0 and a proper convex subanalytic closed cone $\gamma\subset X$ such that $\xi_0\in\operatorname{Int}\gamma^{\circ a}$ and $(U\times\gamma^{\circ a})\cap\big(V\mathbin{\widehat{+}}\mathrm{SS}(F)\big)\subset T^*_XX$. Set for short:

$$\mathcal{H}(F) := \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Z}, F\overset{\mathrm{w}}{\otimes}\mathscr{O}_X).$$

Let $Z(x,\varepsilon)$ be as (1.10). By (1.11), we may show:

$$\boldsymbol{R}\varGamma_{\mathrm{c}}(Z(x,\varepsilon);\mathcal{H}(F)) = 0, \tag{4.1}$$

Take $v \in \operatorname{Int}\gamma$ and set $Z_\delta := Z(x-\delta v, \varepsilon-\delta)$ for $0<\delta \ll \varepsilon$. We may assume that $x-\delta v \in U$. Then, for any $j \in \mathbb{Z}$ we have

$$H^j_{\mathrm{c}}(Z(x,\varepsilon);\mathcal{H}(F)) = \varinjlim_{\delta>0} H^j_{\mathrm{c}}(X;\mathbb{C}_{Z_\delta} \overset{\boldsymbol{L}}{\otimes} \mathcal{H}(F)).$$

By [K-S 3, Proposition 2.8], we have a natural morphism:

$$\mathbb{C}_{Z_\delta} \overset{\boldsymbol{L}}{\otimes} (F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \to (\mathbb{C}_{Z_\delta} \overset{\boldsymbol{L}}{\otimes} F) \overset{\mathrm{w}}{\otimes} \mathscr{O}_X. \tag{4.2}$$

Next we set

$$Z'_\delta := (x-\delta v + \operatorname{Int}\gamma) \cap \{y \in X;\, \operatorname{Re}\langle y-x_0,\xi\rangle \geqslant \delta-\varepsilon\}.$$

Note that $Z_\delta \cap Z'_\delta = \operatorname{Int} Z_\delta$. Since

$$\begin{aligned}\boldsymbol{R}\mathscr{H}om(\mathbb{C}_{Z_\delta}, \mathrm{D}'_X\mathbb{C}_{Z'_\delta}) &= \boldsymbol{R}\mathscr{H}om(\mathbb{C}_{Z_\delta}, \boldsymbol{R}\mathscr{H}om(\mathbb{C}_{Z'_\delta}, \mathbb{C}_X))\\ &= \boldsymbol{R}\mathscr{H}om(\mathbb{C}_{Z_\delta} \overset{\boldsymbol{L}}{\otimes} \mathbb{C}_{Z'_\delta}, \mathbb{C}_X) = \boldsymbol{R}\mathscr{H}om(\mathbb{C}_{\operatorname{Int} Z_\delta}, \mathbb{C}_X),\end{aligned}$$

a natural morphism $\mathbb{C}_{\operatorname{Int} Z_\delta} \to \mathbb{C}_X$ induces a morphism

$$\mathbb{C}_{Z_\delta} \to \mathrm{D}'_X\mathbb{C}_{Z'_\delta}. \tag{4.3}$$

Since $\mathrm{D}'_X\mathrm{D}'_X F = F$, we have

$$\begin{aligned}\mathrm{D}'_X\mathbb{C}_{Z'_\delta} \overset{\boldsymbol{L}}{\otimes} F &= \boldsymbol{R}\mathscr{H}om(\mathbb{C}_{Z'_\delta}, \mathbb{C}_X) \overset{\boldsymbol{L}}{\otimes} F \to \boldsymbol{R}\mathscr{H}om(\mathbb{C}_{Z'_\delta}, F)\\ &= \boldsymbol{R}\mathscr{H}om(\mathbb{C}_{Z'_\delta}, \boldsymbol{R}\mathscr{H}om(\mathrm{D}'_X F, \mathbb{C}_X))\\ &\simeq \boldsymbol{R}\mathscr{H}om(\mathbb{C}_{Z'_\delta} \overset{\boldsymbol{L}}{\otimes} \mathrm{D}'_X F, \mathbb{C}_X) = \mathrm{D}'_X(\mathbb{C}_{Z'_\delta} \overset{\boldsymbol{L}}{\otimes} \mathrm{D}'_X F).\end{aligned} \tag{4.4}$$

For $0<\delta'<\delta$, we set

$$W := (x-\delta v + \operatorname{Int}\gamma) \cap \{y \in X;\, \operatorname{Re}\langle y-x_0,\xi\rangle > \delta'-\varepsilon\}.$$

Then W is an open subset of X and both Z'_δ and $Z_{\delta'}$ are closed subsets of W. Hence, there exists the following chain of morphisms:

$$\boldsymbol{R}\Gamma_{Z'_\delta}(F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \to \boldsymbol{R}\Gamma_W(F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \to \boldsymbol{R}\Gamma_W\big((F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)_{Z_{\delta'}}\big) = (F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)_{Z_{\delta'}}. \tag{4.5}$$

Therefore by (4.3), (4.4), [K-S 3, Proposiotion 2.8] and (4.5), we have the following chain of natural morphisms:

$$\begin{aligned}(\mathbb{C}_{Z_\delta} \overset{\boldsymbol{L}}{\otimes} F) \overset{\mathrm{w}}{\otimes} \mathscr{O}_X &\to (\mathrm{D}'_X\mathbb{C}_{Z'_\delta} \overset{\boldsymbol{L}}{\otimes} F) \overset{\mathrm{w}}{\otimes} \mathscr{O}_X \to \big(\mathrm{D}'_X(\mathbb{C}_{Z'_\delta} \overset{\boldsymbol{L}}{\otimes} \mathrm{D}'_X F)\big) \overset{\mathrm{w}}{\otimes} \mathscr{O}_X\\ &\to \boldsymbol{R}\mathscr{H}om(\mathbb{C}_{Z'_\delta}, \mathrm{D}'_X\mathrm{D}'_X F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \simeq \boldsymbol{R}\Gamma_{Z'_\delta}(F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\\ &\to (F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)_{Z_{\delta'}} = \mathbb{C}_{Z_{\delta'}} \overset{\boldsymbol{L}}{\otimes} (F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X).\end{aligned} \tag{4.6}$$

Thus by (4.2) and (4.6), we have a chain of natural morphisms:

$$\mathbb{C}_{Z_\delta} \overset{\boldsymbol{L}}{\otimes} (F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \to (\mathbb{C}_{Z_\delta} \overset{\boldsymbol{L}}{\otimes} F) \overset{\mathrm{w}}{\otimes} \mathscr{O}_X \to \mathbb{C}_{Z_{\delta'}} \overset{\boldsymbol{L}}{\otimes} (F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \to (\mathbb{C}_{Z_{\delta'}} \overset{\boldsymbol{L}}{\otimes} F) \overset{\mathrm{w}}{\otimes} \mathscr{O}_X.$$

Hence taking inductive limits, we have

$$H^j_{\mathrm{c}}(Z(x,\varepsilon);\mathcal{H}(F)) = \varinjlim_{\delta>0} H^j_{\mathrm{c}}(X;\mathbb{C}_{Z_\delta}\overset{\boldsymbol{L}}{\otimes}\mathcal{H}(F)) = \varinjlim_{\delta>0} H^j_{\mathrm{c}}(X;\mathcal{H}(\mathbb{C}_{Z_\delta}\overset{\boldsymbol{L}}{\otimes} F)).$$

Since f is proper over $\operatorname{supp}\mathbb{C}_{Z_\delta}$, we have by [K-S 3, Theorem 7.2]:

$$\boldsymbol{R}f_!\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Z},(\mathbb{C}_{Z_\delta}\overset{\boldsymbol{L}}{\otimes} F)\overset{\mathrm{w}}{\otimes}\mathscr{O}_X) \simeq \boldsymbol{R}f_!(\mathbb{C}_{Z_\delta}\overset{\boldsymbol{L}}{\otimes} F)\overset{\mathrm{w}}{\otimes}\mathscr{O}_Z\,. \tag{4.7}$$

Hence applying the functor $\boldsymbol{R}\mathit{\Gamma}_{\mathrm{c}}(Y;*)$ to (4.7), we have

$$\varinjlim_{\delta>0} H^j_{\mathrm{c}}(X;\mathcal{H}(\mathbb{C}_{Z_\delta}\overset{\boldsymbol{L}}{\otimes} F)) \simeq \varinjlim_{\delta>0} H^j_{\mathrm{c}}(Y;\boldsymbol{R}f_!(\mathbb{C}_{Z_\delta}\overset{\boldsymbol{L}}{\otimes} F)\overset{\mathrm{w}}{\otimes}\mathscr{O}_Z).$$

Hence the proof of (4.1) is reduced to show

$$\boldsymbol{R}f_!(\mathbb{C}_{Z_\delta}\overset{\boldsymbol{L}}{\otimes} F) = 0. \tag{4.8}$$

Set $\varphi(y) := \operatorname{Re}\langle x_0 - y, \xi_0\rangle$ and $X_t := \{y\in X;\, \varphi(y) < t\}$. Then $Z_\delta = (x-\delta v+\gamma)\cap X_{\delta-\varepsilon}$. If we prove that for any $y\in U$

$$-d\varphi(y) = (y;\xi_0) \notin \big(\mathrm{SS}(\mathbb{C}_{(x-\delta v+\gamma)}\overset{\boldsymbol{L}}{\otimes} F) + V\big) \tag{4.9}$$

holds, then we have by [K-S 2, Proposition 5.4.17 (c)], for any t with $X_t\cap U\neq\emptyset$,

$$\boldsymbol{R}f_!(\mathbb{C}_{Z_\delta}\overset{\boldsymbol{L}}{\otimes} F) = \boldsymbol{R}f_!(\mathbb{C}_{(x-\delta v+\gamma)\cap X_t}\overset{\boldsymbol{L}}{\otimes} F)$$

holds. Hence choosing $t<0$ as $(x-\delta v+\gamma)\cap X_t = \emptyset$, we can obtain (4.8).

Now we prove (4.9). Since

$$(U\times\gamma^{\circ a})\cap\mathrm{SS}(F) \subset (U\times\gamma^{\circ a})\cap\big(V\mathbin{\widehat{+}}\mathrm{SS}(F)\big)\subset T^*_X X,$$

we have

$$\mathrm{SS}(\mathbb{C}_{(x-\delta v+\gamma)}\otimes F)\subset (U\times\gamma)\mathbin{\widehat{+}}\mathrm{SS}(F) = (U\times\gamma)+\mathrm{SS}(F).$$

On the other hand, since

$$(U\times\gamma^{\circ a})\cap\big(V+\mathrm{SS}(F)\big)\subset (U\times\gamma^{\circ a})\cap\big(V\mathbin{\widehat{+}}\mathrm{SS}(F)\big)\subset T^*_X X,$$

we have:

$$(U\times\gamma^{\circ a})\cap\big(\mathrm{SS}(\mathbb{C}_{(x-\delta v+\gamma)}\otimes F)+V\big)\subset (U\times\gamma^{\circ a})\cap\big(V+(U\times\gamma)+\mathrm{SS}(F)\big)\subset T^*_X X.$$

Thus we obtain

$$(U\times\operatorname{Int}\gamma^{\circ a})\cap\big(\mathrm{SS}(\mathbb{C}_{(x-\delta v+\gamma)}\overset{\boldsymbol{L}}{\otimes} F)+V\big) = \emptyset.$$

This proves (4.9) since $\xi_0\in\operatorname{Int}\gamma^{\circ a}$. $\square$

We denote by $p_j\colon X\times X\to X$ the j-th projection, and by $\varDelta\simeq X$ the diagonal set of $X\times X$. Then the *formal specialization* of $F\in\operatorname{Ob}\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X)$ is defined by

$$F\overset{\mathrm{w}}{\underset{\nu}{\otimes}}\mathscr{O}_X:=\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_{X\times X}}(\mathscr{D}_{X\times X\underset{p_1}{\longrightarrow}X},\mathrm{W}\text{-}\nu_{\varDelta}(p_2^{-1}F\otimes\mathscr{O}_{X\times X}))\otimes\omega_{TX/X}\,.$$

We give another expression of $F\overset{\mathrm{w}}{\underset{\nu}{\otimes}}\mathscr{O}_X$: Let $\widetilde{X}^{\mathbb{C}}$ be the *complex normal deformation* of $\varDelta$ in $X\times X$ and $t\colon\widetilde{X}^{\mathbb{C}}\to\mathbb{C}$ the canonical mapping. Set $\varOmega:=t^{-1}(\mathbb{R}_{>0})\subset\widetilde{X}^{\mathbb{C}}$ and consider the commutative diagram below:

$$\begin{tikzcd}
TX \arrow[r, hook, "\sigma"] \arrow[d] & \widetilde{X}^{\mathbb{C}} \arrow[d, "p"] & \varOmega \arrow[l, hook', "j"'] \arrow[dl, "\widetilde{p}"] \\
X \arrow[r, hook] & X\times X.
\end{tikzcd}$$

Set $\rho_j:=p_j\circ p\colon\widetilde{X}^{\mathbb{C}}\to X$. Then we have

$$F\overset{\mathrm{w}}{\underset{\nu}{\otimes}}\mathscr{O}_X=\sigma^{-1}\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_{\widetilde{X}^{\mathbb{C}}}}(\mathscr{D}_{\widetilde{X}^{\mathbb{C}}\underset{\rho_1}{\longrightarrow}X},(\rho_2^{-1}F\overset{\boldsymbol{L}}{\otimes}\mathbb{C}_{\mathrm{Cl}\,\varOmega})\overset{\mathrm{w}}{\otimes}\mathscr{O}_{\widetilde{X}^{\mathbb{C}}})\otimes\omega_{TX/X}\,.$$

The *formal microlocalization* is the Fourier-Sato transform of $F\overset{\mathrm{w}}{\underset{\nu}{\otimes}}\mathscr{O}_X$:

$$F\overset{\mathrm{w}}{\underset{\mu}{\otimes}}\mathscr{O}_X:=(F\overset{\mathrm{w}}{\underset{\nu}{\otimes}}\mathscr{O}_X)^{\wedge}\,.$$

Note that the original definition in [C 2] is $F\overset{\mathrm{w}}{\underset{\mu}{\otimes}}\mathscr{O}_X=(F\overset{\mathrm{w}}{\underset{\nu}{\otimes}}\mathscr{O}_X)^{\wedge a}$. However, in view of Theorem 4.3 (3) below, we slightly changed the definition.

We recall the fundamental properties of the formal microlocalization functor

$$*\overset{\mathrm{w}}{\underset{\mu}{\otimes}}\mathscr{O}_X\colon\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X)\to\mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T^*X).$$

4.3. Theorem ([C 1], [C 2]). (1) *$F\overset{\mathrm{w}}{\underset{\mu}{\otimes}}\mathscr{O}_X|_X=F\overset{\mathrm{w}}{\otimes}\mathscr{O}_X$ and there exists the following distinguished triangle*:

$$F\overset{\boldsymbol{L}}{\otimes}\mathscr{O}_X\to F\overset{\mathrm{w}}{\otimes}\mathscr{O}_X\to\boldsymbol{R}\dot{\pi}_*(F\overset{\mathrm{w}}{\underset{\mu}{\otimes}}\mathscr{O}_X)\xrightarrow{+1}.$$

(2) *Each cohomology $\mathcal{H}^j(F\overset{\mathrm{w}}{\underset{\mu}{\otimes}}\mathscr{O}_X)$ is an $\mathscr{E}_X$-Module for any $j\in\mathbb{Z}$.*

(2) $\operatorname{supp}(F\overset{\mathrm{w}}{\underset{\mu}{\otimes}}\mathscr{O}_X)\subset\mathrm{SS}(F)^a$ *and there exists the following chain of morphisms*:

$$F\overset{\mathrm{w}}{\underset{\mu}{\otimes}}\mathscr{O}_X\to T\text{-}\mu hom(\mathrm{D}'_XF,\mathscr{O}_X)\to\mu hom(\mathrm{D}'_XF,\mathscr{O}_X),$$

where T-$\mu hom(*,\mathscr{O}_X)$ is the temperate μhom functor due to Andronikof [A].

Note that both $\operatorname{supp}\bigl(T\text{-}\mu\mathscr{hom}(\mathrm{D}'_X F, \mathscr{O}_X)\bigr)$ and $\operatorname{supp}\bigl(\mu\mathscr{hom}(\mathrm{D}'_X F, \mathscr{O}_X)\bigr)$ are contained in $\mathrm{SS}(\mathrm{D}'_X F) = \mathrm{SS}(F)^a$.

Further we can show that every quantized contact transformation acts on $F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X$ as an isomorphism. Precisely, let X and Y be complex manifolds with same dimension n and $(p_X, p_Y) \in \dot{T}^*X \times \dot{T}^*Y$. Let $\chi\colon (T^*Y)_{p_Y} \to (T^*X)_{p_X}$ be a germ of complex canonical transformation and $\Lambda \subset T^*(X \times Y)$ the Lagrangian submanifold associated with χ. Let K be an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(X \times Y; p_X, p_Y^a)$ such that $\mathrm{SS}(K) = \Lambda$ and K is simple with shift zero along Λ (for the notation and terminology, see [A] and [K-S 2]). We denote by q_j the j-th projection on $X \times Y$. For every $G \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(Y; p_Y^a)$, we set $\Phi_{K[n]}(G) := \boldsymbol{R}q_{1\,!}(K[n] \overset{\boldsymbol{L}}{\otimes} q_2^{-1}G) \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X; p_X)$.

4.4. Theorem. *Under the notation above, the quantized contact transformation induces the following isomorphisms at p_X:*

$$\begin{array}{ccccc}
\chi_*(\mathrm{D}'_Y G \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_Y) & \longrightarrow & \chi_* T\text{-}\mu\mathscr{hom}(G, \mathscr{O}_Y) & \longrightarrow & \chi_* \mu\mathscr{hom}(G, \mathscr{O}_Y) \\
\big\downarrow \wr & & \big\downarrow \wr & & \big\downarrow \wr \\
\mathrm{D}'_X \Phi_{K[n]}(G) \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X & \longrightarrow & T\text{-}\mu\mathscr{hom}(\Phi_{K[n]}(G), \mathscr{O}_X) & \longrightarrow & \mu\mathscr{hom}(\Phi_{K[n]}(G), \mathscr{O}_X).
\end{array}$$

Let N be a closed real analytic submanifold of M, and $\pi_N\colon T^*_N X \to N$ the canonical projection. We see that $\operatorname{supp}(\mathbb{C}_N \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X) \subset T^*_N X$, and by [C 1], for any $p \in \dot{T}^*_N X$ we have

$$\mathscr{H}^k(\mathbb{C}_N \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X)_p \simeq \varinjlim_{U,V} H^{\operatorname{codim}_X N + k}(X; \mathbb{C}_{V \cap \operatorname{Cl} U} \overset{\mathrm{w}}{\otimes} \mathscr{O}_X). \tag{4.10}$$

Here U ranges through the family of subanalytic open neighborhoods of $\pi_N(p)$ in X, and V ranges through the family of subanalytic open sets of X such that $C_N(V)_{\pi_N(p)} \subset \operatorname{Int}\{p\}^\circ$. Since the problem is local, we may assume that $X = \mathbb{C}^n$ and both U and V can be chosen as bounded convex sets. On the other hand, by the proof of [Be, Theorem 4.4], for any relatively compact Stein open subset $V \Subset X$, it follows that $\boldsymbol{R}\Gamma(X; \mathbb{C}_V \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)$ is concentrated in degree $\dim_{\mathbb{C}} X = \dim M$. Hence choosing $V \Subset U$ in (4.10), we obtain the following:

4.5. Proposition. *Let M be a real analytic manifold, N a closed real analytic submanifold of M. Then $\mathbb{C}_N \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X\big|_{\dot{T}^*_N X}$ is concentrated in degree $-\operatorname{codim}_M N$.*

4.6. Remark. Under the same notation in Proposition 4.5, $\mathbb{C}_N \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X\big|_N = \mathbb{C}_N \overset{\mathrm{w}}{\otimes} C^\infty_M = \mathscr{W}^\infty_{M,N}$ is concentrated in degree zero. Hence in general, the complex $\mathbb{C}_N \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X$ is not concentrated in a single degree in T^*X.

Let $f\colon Y \to X$ be a morphism of manifolds. We set natural mappings associated with f as follows:

$$T^*Y \xleftarrow{f_d} Y \underset{X}{\times} T^*X \xrightarrow{f_\pi} T^*X.$$

We extend Theorem 4.2 to the formal microlocalization functor:

4.7. Theorem (cf. [MF-K-S, Theorem 2.3])**.** *Let V be a closed $\mathbb{C}^\times$-conic regular involutory submanifold of T^*X and F an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X)$. Suppose one of the following conditions*:

(1) *$\mathscr{M}$ is a coherent $\mathscr{D}_X$-Module such that V and $\mathscr{M}$ satisfy Condition* 4.1*.*

(2) *$\mathscr{M}$ is a coherent $\mathscr{E}_X$-Module defined on an open subset of $\dot{T}^*X$ and has regular singularities along V, and $F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X\big|_U$ is concentrated in a single degree.*

Then it follows that

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X)\big) \subset C(V, \mathrm{SS}(F)^a).$$

Proof. Since the problem is local, in Case (1) we may assume that $X = Y \times Z$, $V = X \underset{Z}{\times} T^*Z = Y \times T^*Z$ and $\mathscr{M} = \mathscr{D}_{X\to Z}$ by Proposition 2.3. In Case (2), we may assume that $X = Y \times Z$ and $V = X \underset{Z}{\times} T^*Z = Y \times T^*Z$ by a suitable contact transformation. By [K-O, Theorem 1.9], we can find an exact sequence

$$0 \to \mathscr{N} \to \mathscr{E}_{X\to Z}^{\oplus N_0} \to \mathscr{M} \to 0,$$

and $\mathscr{N}$ has also regular singularities along V. Hence by a standard argument, the proof can be reduced to the case where $\mathscr{M} = \mathscr{E}_{X\to Z} = \mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \pi^{-1}\mathscr{D}_{X\to Z}$. Therefore in both cases, the proof is reduced to the estimation of

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Z}, F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X)\big).$$

Let $f\colon X = Y \times Z \to Z$ be the canonical projection. We work on the space T^*TX under the identifications of (1.6), and by [K-S 2, Theorem 5.5.5] we see

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Z}, F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X)\big) = \mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Z}, F \overset{\mathrm{w}}{\underset{\nu}{\otimes}} \mathscr{O}_X)\big).$$

Then setting $h := f \circ \rho_1\colon \widetilde{X}^{\mathbb{C}} \to Z$, we have

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Z}, F \overset{\mathrm{w}}{\underset{\nu}{\otimes}} \mathscr{O}_X) = \sigma^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{\widetilde{X}^{\mathbb{C}}}}(\mathscr{D}_{\widetilde{X}^{\mathbb{C}}\to Z}, (\rho_2^{-1}F \otimes \mathbb{C}_{\mathrm{Cl}\,\Omega}) \overset{\mathrm{w}}{\otimes} \mathscr{O}_{\widetilde{X}^{\mathbb{C}}}).$$

Let $X = \{(x,y);\, y = 0\} \subset X \times X = \{(x,y)\}$ be local coordinates of (1.5). Then the coordinates of $\widetilde{X}^{\mathbb{C}}$ are $\{(x,y,t);\, t \in \mathbb{C},\, (x, x-ty) \in X \times X\}$ and

$$p(x,y,t) = (x, x-ty), \qquad \rho_1(x,y,t) = x, \qquad \rho_2(x,y,t) = x - ty.$$

Let $(x, y, t; \xi, \eta, \tau)$ be the coordinates of $T^*\widetilde{X}^{\mathbb{C}}$ associated with (x, y, t). Since $h\colon \widetilde{X}^{\mathbb{C}} \to Z$ is smooth, we see

$$\operatorname{char}(\mathscr{D}_{\widetilde{X}^{\mathbb{C}}\to Z}) = \widetilde{V} := \rho_d(\widetilde{X}^{\mathbb{C}} \underset{Z}{\times} T^*Z) = \{(x, y, t; \xi, 0, 0) \in T^*\widetilde{X}^{\mathbb{C}};\ (x; \xi) \in Y \times T^*Z\}.$$

By [K-S 2, Corolllary 6.4.4], we have

$$\begin{aligned}
\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Z}, F \underset{\nu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X)\big) &= \mathrm{SS}\big(\sigma^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{\widetilde{X}^{\mathbb{C}}}}(\mathscr{D}_{\widetilde{X}^{\mathbb{C}}\to Z}, (\rho_2^{-1}F \otimes \mathbb{C}_{\mathrm{Cl}\,\Omega}) \overset{\mathrm{w}}{\otimes} \mathscr{O}_{\widetilde{X}^{\mathbb{C}}})\big) \\
&\subset \sigma^\sharp\big(\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{\widetilde{X}^{\mathbb{C}}}}(\mathscr{D}_{\widetilde{X}^{\mathbb{C}}\to Z}, (\rho_2^{-1}F \otimes \mathbb{C}_{\mathrm{Cl}\,\Omega}) \overset{\mathrm{w}}{\otimes} \mathscr{O}_{\widetilde{X}^{\mathbb{C}}})\big)\big) \\
&\subset \sigma^\sharp\big(\widetilde{V} \mathbin{\widehat{+}} \mathrm{SS}(\rho_2^{-1}F \otimes \mathbb{C}_{\mathrm{Cl}\,\Omega})\big) \subset \sigma^\sharp\big(\widetilde{V} \mathbin{\widehat{+}} \big(\mathrm{SS}(\rho_2^{-1}F) \mathbin{\widehat{+}} \mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega})\big)\big).
\end{aligned}$$

Hence we may show

$$\sigma^\sharp\big(\widetilde{V} \mathbin{\widehat{+}} \big(\mathrm{SS}(\rho_2^{-1}F) \mathbin{\widehat{+}} \mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega})\big)\big) \subset C(\widetilde{V}, \mathrm{SS}(F)^a).$$

Since ρ_2 is smooth, by [K-S 2, Proposition 5.4.5], we have

$$\mathrm{SS}(\rho_2^{-1}F) = \rho_{2\,d}\,\rho_{2\,\pi}^{-1}\,\mathrm{SS}(F) = \{(x, y, t; \xi, -t\xi, -\langle y, \xi\rangle) \in T^*\widetilde{X}^{\mathbb{C}};\ (x - ty; \xi) \in \mathrm{SS}(F)\}.$$

Hence it follows that $\mathrm{SS}(\rho_2^{-1}F) \cap \mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega})^a \subset T^*_{\widetilde{X}^{\mathbb{C}}}\widetilde{X}^{\mathbb{C}}$ since

$$\mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega}) = \{(x, y, t; 0, 0, \tau) \in T^*\widetilde{X}^{\mathbb{C}};\ \operatorname{Im} t = \operatorname{Re} t \operatorname{Re}\tau = 0,\ \operatorname{Re} t \geqslant 0,\ \operatorname{Re}\tau \geqslant 0\}.$$

Thus we have (see [K-S 2, Remark 6.2.6])

$$\mathrm{SS}(\rho_2^{-1}F) \mathbin{\widehat{+}} \mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega}) = \mathrm{SS}(\rho_2^{-1}F) + \mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega}).$$

Let $(x_0, y_0; \xi_0, \eta_0)$ be a point of T^*TX. Assume that

$$(x_0, y_0; \xi_0, \eta_0) \in \sigma^\sharp\big(\widetilde{V} \mathbin{\widehat{+}} \big(\mathrm{SS}(\rho_2^{-1}F) + \mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega})\big)\big).$$

Then by (1.4) there exists a sequence

$$\{(x_n, y_n, t_n; \xi_n, \eta_n, \tau_n)\}_{n\in\mathbb{N}} \subset \widetilde{V} \mathbin{\widehat{+}} \big(\mathrm{SS}(\rho_2^{-1}F) + \mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega})\big)$$

such that $(x_n, y_n, t_n; \xi_n, \eta_n) \underset{n}{\to} (x_0, y_0, 0; \xi_0, \eta_0)$ and $|t_n|\,|\tau_n| \underset{n}{\to} 0$. Thus by (1.8) there exist sequences

$$\begin{cases}
\{(x_{n,j}, y_{n,j}, t_{n,j}; \xi_{n,j}, 0, 0)\}_{j,n\in\mathbb{N}} \subset \widetilde{V}, \\
\{(x'_{n,j}, y'_{n,j}, t'_{n,j}; \xi'_{n,j}, -t'_{n,j}\xi'_{n,j}, -\langle y'_{n,j}, \xi'_{n,j}\rangle)\}_{j,n\in\mathbb{N}} \subset \mathrm{SS}(\rho_2^{-1}F), \\
\{(x'_{n,j}, y'_{n,j}, t'_{n,j}; 0, 0, \tau''_{n,j})\}_{j,n\in\mathbb{N}} \subset \mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega}),
\end{cases}$$

such that

$$\begin{cases}
(x_{n,j}, y_{n,j}, t_{n,j}),\ (x'_{n,j}, y'_{n,j}, t'_{n,j}) \underset{j}{\to} (x_n, y_n, t_n), \\
(\xi_{n,j} + \xi'_{n,j}, -t'_{n,j}\xi'_{n,j}, \tau''_{n,j} - \langle y'_{n,j}, \xi'_{n,j}\rangle) \underset{j}{\to} (\xi_n, \eta_n, \tau_n), \\
|(x_{n,j} - x'_{n,j}, y_{n,j} - y'_{n,j}, t_{n,j} - t'_{n,j})|\,|\xi_{n,j}| \underset{j}{\to} 0,
\end{cases}$$

hold. Hence by extracting subsequences, we may assume that there exist sequences

$$\begin{cases} \{(x_n, y_n, t_n; \xi_n, 0, 0)\}_{n\in\mathbb{N}} \subset \widetilde{V}, \\ \{(x'_n, y'_n, t'_n; \xi'_n, -t'_n\xi'_n, -\langle y'_n, \xi'_n\rangle)\}_{n\in\mathbb{N}} \subset \mathrm{SS}(\rho_2^{-1}F), \\ \{(x'_n, y'_n, t'_n; 0, 0, \tau''_n)\}_{n\in\mathbb{N}} \subset \mathrm{SS}(\mathbb{C}_{\mathrm{Cl}\,\Omega}), \end{cases}$$

such that

$$\begin{cases} (x_n, y_n, t_n),\ (x'_n, y'_n, t'_n) \underset{n}{\to} (x_0, y_0, 0), \\ (\xi_n + \xi'_n, -t'_n\xi'_n) \underset{n}{\to} (\xi_0, \eta_0), \end{cases}$$

hold. In particular, we have $\operatorname{Re} t'_n \geqslant 0$ and $\operatorname{Im} t'_n = 0$. Since $t'_n \underset{n}{\to} 0$, we see $t'_n\xi_n + t'_n\xi'_n \underset{n}{\to} 0$, and we have $t'_n\xi_n \underset{n}{\to} \eta_0$ since $-t'_n\xi'_n \underset{n}{\to} \eta_0$. Thus we can find $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_{>0}$ such that $c_n \underset{n}{\to} 0$ and $c_n\xi_n$, $-c_n\xi'_n \underset{n}{\to} \eta_0$. Consider sequences

$$\begin{cases} \{(x'_n + (c_n - t'_n)y'_n; c_n\xi_n)\}_{n\in\mathbb{N}} \subset V, \\ \{(x'_n - t'_ny'_n; -c_n\xi'_n)\}_{n\in\mathbb{N}} \subset \mathrm{SS}(F)^a. \end{cases}$$

Then $(x'_n + (c_n - t'_n)y'_n; c_n\xi_n)$, $(x'_n - t'_ny'_n; -c_n\xi'_n) \underset{n}{\to} (x_0; \eta_0)$ and

$$\frac{1}{c_n}\big((x'_n + (c_n - t'_n)y'_n; c_n\xi_n) - (x'_n - t'_ny'_n; -c_n\xi'_n)\big) = (y'_n; \xi_n + \xi'_n) \underset{n}{\to} (y_0, \xi_0).$$

Therefore by (1.3) we have

$$(x_0, \eta_0; y_0, \xi_0) \in C(V, \mathrm{SS}(F)^a).$$

The proof is complete. □

Next we introduce the following condition:

4.8. Condition. $V \subset T^*X$ is a closed $\mathbb{C}^\times$-conic regular involutory submanifold, and $\mathscr{M}$ is a coherent $\mathscr{D}_X$-Module and has regular singularities along V.

4.9. Theorem. *Let V be a closed $\mathbb{C}^\times$-conic regular involutory submanifold of T^*X, $\mathscr{M}$ a coherent $\mathscr{D}_X$-Module, and F an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X)$. Suppose one of the following conditions:*

(1) V and $\mathscr{M}$ satisfy Condition 4.1.

(2) V and $\mathscr{M}$ satisfy Condition 4.8, and $F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X|_{\dot{T}^*X}$ is concentrated in a single degree.

Then it follows that

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\big) \subset V \mathbin{\widehat{+}} \mathrm{SS}(F).$$

Proof. Consider the distinguished triangle below:

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F \overset{\boldsymbol{L}}{\otimes} \mathscr{O}_X) \to \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \to \boldsymbol{R}\dot{\pi}_*\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X) \xrightarrow{+1}.$$

Since $\mathscr{M}$ is coherent, we have

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F \overset{\boldsymbol{L}}{\otimes} \mathscr{O}_X) = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \overset{\boldsymbol{L}}{\otimes} F.$$

By [K-S 2, Theorem 11.3.3], we see

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)\big) = \mathrm{char}(\mathscr{M}) \subset V.$$

Thus by virtue of [K-S 2, Corollary 6.4.5], we have

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F \overset{\boldsymbol{L}}{\otimes} \mathscr{O}_X)\big) \subset V \mathbin{\widehat{+}} \mathrm{SS}(F).$$

On the other hand, by [K-S 2, Propositions 5.5.4, 6.2.4] and Theorem 4.7, we have

$$\mathrm{SS}\big(\boldsymbol{R}\dot{\pi}_* \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X)\big) \subset V \underset{\infty}{\widehat{+}} \mathrm{SS}(F) \subset V \mathbin{\widehat{+}} \mathrm{SS}(F).$$

Therefore by [K-S 2, Proposition 5.1.3], we obtain the desired result. $\square$

4.10. Example (cf. [Bo])**.** Let $\pi_M : T^*_M X \to M$ be the natural projection and $k: T^*_M X \hookrightarrow T^*X$ the canonical embedding. We set

$$\mathscr{C}^d_M := k^{-1}(\mathbb{C}_M \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X).$$

Note that $\mathrm{supp}(\mathbb{C}_M \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X) \subset T^*_M X$. Let us set $\mathscr{A}_M := \mathscr{O}_X\big|_M$. Then we have

$$0 \to \mathscr{A}_M \to C^\infty_M \to \dot{\pi}_{M*}\mathscr{C}^d_M \to 0, \qquad R^j\dot{\pi}_{M*}\mathscr{C}^d_M = 0 \quad (j \neq 0),$$

and there exist natural monomorphisms

$$\mathscr{C}^d_M \rightarrowtail \mathscr{C}^f_M := T\text{-}\mu hom(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X) \rightarrowtail \mathscr{C}_M = \mu hom(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X).$$

$\mathscr{C}^d_M\big|_{\dot{T}^*_M X}$ is concentrated in degree zero by Proposition 4.5, and

$$\boldsymbol{R}\pi_{M*}\mathscr{C}^d_M = \mathscr{C}^d_M\big|_M = \mathbb{C}_M \overset{\mathrm{w}}{\otimes} \mathscr{O}_X\big|_M = C^\infty_M$$

is also concentrated in degree zero. Therefore $\mathscr{C}^d_M$ is a conic sheaf of $T^*_M X$, and in particular defined as an object of $\mathbf{D}^{\mathrm{b}}(\mathscr{E}_X)$. Let $p: TT^*X = T^*T^*X \to T^*X$ be the canonical projection. By (1.1), (1.2) and (1.7), we have:

$$\begin{tikzcd}
T^*X \arrow[d, tail] & M \underset{X}{\times} T^*X \arrow[l, tail, "i_\pi"'] \arrow[r, two heads, "i_d"] \arrow[d, tail] & T^*M \arrow[d, tail] \\
TT^*X \arrow[d, equal] & T^*_M X \underset{T^*X}{\times} TT^*X \arrow[l, tail] \arrow[r, two heads] \arrow[d, equal] & T_{T^*_M X}T^*X \arrow[d, equal] \\
T^*T^*X & T^*_M X \underset{T^*X}{\times} T^*T^*X \arrow[l, tail, "k_\pi"'] \arrow[r, two heads, "k_d"] & T^*T^*_M X.
\end{tikzcd}$$

Let V and $\mathscr{M}$ satisfy Condition 4.8. Then, we see $C(V, T^*_MX) = k_d^{-1}C_{T^*_MX}(V)$ (cf. [K-S 2, Proposition 4.4.2]). Hence, we have by [K-S 2, Proposition 5.4.4]

$$\begin{aligned}
k_d^{-1}\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^d_M)\big) \cap p^{-1}(\dot{T}^*X) &= k_\pi k_d^{-1}\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^d_M)\big) \cap p^{-1}(\dot{T}^*X) \\
&= \mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \boldsymbol{R}k_*\mathscr{C}^d_M)\big) \cap p^{-1}(\dot{T}^*X) \\
&= \mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_M \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X)\big) \cap p^{-1}(\dot{T}^*X) \\
&\subset C(V, \mathrm{SS}(\mathbb{C}_M)^a) = C(V, T^*_MX) = k_d^{-1}C_{T^*_MX}(V).
\end{aligned}$$

Therefore we have:

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^d_M)\big) \cap p^{-1}(\dot{T}^*_MX) \subset C_{T^*_MX}(V).$$

For the same reason, we obtain by Theorem 4.9:

$$\begin{aligned}
i_d^{-1}\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, C^\infty_M)\big) &= i_\pi i_d^{-1}\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, C^\infty_M)\big) \\
&= \mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \boldsymbol{R}i_*C^\infty_M)\big) = \mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_M \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\big) \\
&\subset V \mathbin{\widehat{+}} \mathrm{SS}(\mathbb{C}_M) = V \mathbin{\widehat{+}} T^*_MX = (M \underset{X}{\times} T^*X) \cap k_d^{-1}C_{T^*_MX}(V).
\end{aligned}$$

Thus we have

$$\mathrm{SS}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, C^\infty_M)) \subset i_d\big((M \underset{X}{\times} T^*X) \cap k_d^{-1}C_{T^*_MX}(V)\big) = i^\sharp(V). \tag{4.11}$$

5. Hyperbolic Boundary Value Problem for Whitney Functions.

In this section, we consider a hyperbolic boundary value problem for Whitney functions. First we shall prove the following:

5.1. Proposition (cf. [MF-K-S, Theorem 2.2]). *Let V and $\mathscr{M}$ satisfy Condition* 4.1, *F an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(X)$. Let Y be a real analytic closed submanifold of X and $f\colon Y \hookrightarrow X$ the embedding. Assume*:

$$T^*_YX \cap \big(V \mathbin{\widehat{+}} \mathrm{SS}(F)\big) \subset T^*_XX. \tag{5.1}$$

Then, the distinguished triangle (3.5) *induces the following isomorphism*:

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{W}\text{-}\nu_Y(F \otimes \mathscr{O}_X)) \simeq \tau_Y^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, F_Y \overset{\mathrm{w}}{\otimes} \mathscr{O}_Y).$$

Proof. By [K-S 3, Theorem 7.2], we have

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F_Y \overset{\mathrm{w}}{\otimes} \mathscr{O}_X) \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, F_Y \overset{\mathrm{w}}{\otimes} \mathscr{O}_Y).$$

Hence by (3.5), we may prove

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{W}\text{-}\mu_Y(F \otimes \mathscr{O}_X))\big|_{\dot{T}^*_YX} = 0. \tag{5.2}$$

Let d be the codimension of Y. Since the problem is local, we may assume that $X = \mathbb{C}^n = Y \times Z$, $f\colon Y \ni y \mapsto (y,0) \in X$, and that V is of the form (2.1). By the stalk formula ([C 1]), for any $x^* \in \dot{T}^*_Y X$ and $j \in \mathbb{Z}$ we have

$$\mathscr{H}^j \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{W}\text{-}\mu_Z(F \otimes \mathscr{O}_X))_{x^*} = \varinjlim_{U} \mathscr{H}^j \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)_{\pi(x^*)}\,.$$

Here U ranges through the family of subanalytic open sets of X such that $C_Y(\operatorname{Cl} U)_{\pi(x^*)} \subset \operatorname{Int}\{x^*\}^\circ$. We may assume that $\pi(x^*) = 0$ and that U has a form $Y \times \Gamma$. Here Γ is a proper convex subanalytic open cone of Z. By Theorem 4.2, we have

$$\mathrm{SS}\bigl(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\bigr) \subset V \mathbin{\widehat{+}} \mathrm{SS}(F_U).$$

Set $W := \{(y,z) \in Y \times Z;\ z \in \operatorname{Cl}\Gamma\}$. We shall show

$$N^*_0(\mathbb{C}_W) \cap \mathrm{SS}\bigl(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\bigr) \subset \{0\}. \tag{5.3}$$

Here $N^*_0(\mathbb{C}_W) \subset T^*X \cap \pi^{-1}(0)$ denotes the *conormal cone.* Note that

$$\mathrm{SS}(\mathbb{C}_U) \cap \pi^{-1}(0) \subset \{(0;-\zeta);\ \zeta \in \Gamma^\circ\}, \quad N^*_0(\mathbb{C}_W) = \{(0;\zeta);\ \zeta \in \Gamma^\circ\}.$$

Assume that $(0;\zeta_0) \in N^*_0(\mathbb{C}_W) \cap (V \mathbin{\widehat{+}} \mathrm{SS}(F_U))$ and $\zeta_0 \neq 0$. Then, since $\mathrm{SS}(F_U) \subset \mathrm{SS}(F) \mathbin{\widehat{+}} \mathrm{SS}(\mathbb{C}_U)$ by [K-S 2, Corollary 6.4.5], we have

$$(0;\zeta_0) \in V \mathbin{\widehat{+}} \bigl(\mathrm{SS}(\mathbb{C}_U) \mathbin{\widehat{+}} \mathrm{SS}(F)\bigr). \tag{5.4}$$

Thus by (1.8) there exist sequences $\{(z_n;\zeta_n)\}_{n\in\mathbb{N}} \subset V$, $\{(z'_n;\zeta'_n)\}_{n\in\mathbb{N}} \subset \mathrm{SS}(\mathbb{C}_U) \mathbin{\widehat{+}} \mathrm{SS}(F)$ such that

$$z_n,\ z'_n \underset{n}{\to} 0, \quad \zeta_n + \zeta'_n \underset{n}{\to} \zeta_0.$$

Using (1.8) again, we can find sequences $\{(z'_{n,j};-\zeta'_{n,j})\}_{n,j\in\mathbb{N}} \subset \mathrm{SS}(\mathbb{C}_U)$, $\{(z''_{n,j};\zeta''_{n,j})\}_{n,j\in\mathbb{N}} \subset \mathrm{SS}(F)$ such that

$$z'_{n,j},\ z''_{n,j} \underset{j}{\to} z'_n, \quad -\zeta'_{n,j} + \zeta''_{n,j} \underset{j}{\to} \zeta'_n, \quad |z'_{n,j} - z''_{n,j}|\,|\zeta'_{n,j}| \underset{j}{\to} 0.$$

By extracting subsequences, we may assume that there exist sequences $\{(z_n;\zeta_n)\}_{n\in\mathbb{N}} \subset V$, $\{(z'_n;-\zeta'_n)\}_{n\in\mathbb{N}} \subset \mathrm{SS}(\mathbb{C}_U)$ and $\{(z''_n;\zeta''_n)\}_{n\in\mathbb{N}} \subset \mathrm{SS}(F)$ such that

$$z_n,\ z'_n,\ z''_n \underset{n}{\to} 0, \quad \zeta_n - \zeta'_n + \zeta''_n \underset{n}{\to} \zeta_0.$$

Then the sequence $\{|\zeta_n + \zeta''_n|\}_{j=1}^\infty$ does not converge to zero. Indeed, assume that $|\zeta_n + \zeta''_n| \underset{n}{\to} 0$. Then we see $\Gamma^{\circ a} \ni -\zeta'_n \underset{n}{\to} \zeta_0 \in \Gamma^\circ$. Since Γ° is a proper convex closed cone, we have $\zeta_0 \in \Gamma^\circ \cap \Gamma^{\circ a} = \{0\}$, which is a contradiction. Hence extracting subsequence, setting $c_n := 1/|\zeta_n + \zeta''_n| > 0$, we may assume that $\{c_n\}_{j\in\mathbb{N}}$ and $\{c_n(\zeta_n + \zeta''_n)\}_{j\in\mathbb{N}}$ converge to some $c \in \mathbb{R}_{\geqslant 0}$ and $\theta_0 \neq 0$ respectively. Hence we have $c_n(\zeta_n - \zeta'_n + \zeta''_n) \underset{n}{\to} c\zeta_0$. In particular, $\{c_j(0,\zeta'_n)\}_{j\in\mathbb{N}} \subset \{0\} \times \Gamma^\circ$ converges to $\theta_0 - c\zeta_0$. Since $\{0\} \times \Gamma^\circ$ is closed, we have $\theta_0 - c\zeta_0 \in \{0\} \times \Gamma^\circ$. Thus we have

$$\theta_0 = \theta_0 - c\zeta_0 + c\zeta_0 \in \{0\} \times \Gamma^\circ + \{0\} \times \Gamma^\circ \subset \{0\} \times \Gamma^\circ \subset T^*_Y X.$$

Therefore, $\{(z''_n; c_n\zeta_n)\}_{n\in\mathbb{N}} \subset V$ and $\{(z''_n; c_n\zeta''_n)\}_{n\in\mathbb{N}} \subset \mathrm{SS}(F)$ satisfy:

$$z''_n \underset{n}{\to} z_0, \quad c_n(\zeta_n + \zeta''_n) \underset{n}{\to} \theta_0, \quad |z''_n - z''_n|\,|c_n\zeta_n| = 0.$$

This implies $(0;\theta_0) \in T^*_Y X \cap (V \widehat{+} \mathrm{SS}(F))$, which contradicts (5.1). This proves (5.3). Further, by $\mathrm{supp}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\big) \subset W$ and [K-S 2, Corollary 5.4.9] we have

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)_0 = \boldsymbol{R}\Gamma_W \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, F_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)_0 = 0.$$

Therefore we obtain (5.2). $\square$

Let $g\colon L \to M$ be a morphism of manifolds, and $W \subset T^*X$ a conic subset. Recall that *g is hyperbolic for W* if:

$$\dot{T}^*_L M \cap C_{T^*_M X}(W) = \emptyset.$$

We denote by N a d-codimensional closed real analytic submanifold of M, and use the notation in (3.2). We shall show the following:

5.2. Theorem. *Let V and $\mathscr{M}$ satisfy Condition 4.8. Suppose that $f_N\colon N \hookrightarrow M$ is hyperbolic for V. Then there exist the following isomorphisms:*

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{W}\text{-}\nu_N(C^\infty_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, C^\infty_N) \\
\downarrow\wr & & \downarrow\wr \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(C^\infty_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\Gamma_N \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, C^\infty_M) \otimes \omega^{\otimes -1}_{N/M}.
\end{array}$$

Note that the hyperbolicity condition implies that Y is non-characteristic for $\mathscr{M}$ in a neighborhood of N.

Proof. We show:

$$(5.5) \qquad \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{W}\text{-}\mu_N(C^\infty_M))\big|_{\dot{T}^*_N M} = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_N(C^\infty_M))\big|_{\dot{T}^*_N M} = 0.$$

First, consider $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_N(C^\infty_M))$. By [K-S 2, Corollary 5.4.10], (4.11) and the hyperbolicity condition, we have:

$$\begin{aligned}
\mathrm{supp}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_N(C^\infty_M))\big) \cap \dot{T}^*_N M &\subset \dot{T}^*_N M \cap \mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, C^\infty_M)\big) \\
&\subset \dot{T}^*_N M \cap C_{T^*_M X}(V) = \emptyset.
\end{aligned}$$

Next, consider $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{W}\text{-}\mu_N(C^\infty_M))$. By the stalk formula ([C 1]), for any $x^* \in \dot{T}^*_N M$ and $j \in \mathbb{Z}$ we have

$$\mathscr{H}^j \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{W}\text{-}\mu_N(C^\infty_M))_{x^*} = \varinjlim_{U} \mathscr{H}^j \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} C^\infty_M)_{\pi(x^*)}\,.$$

Here U ranges through the family of subanalytic open sets of M such that $C_N(\mathrm{Cl}\,U)_{\pi(x^*)} \subset \mathrm{Int}\{x^*\}^\circ$. Since the problem is local, we may assume that $M = \mathbb{R}^{n-d}_{x'} \times \mathbb{R}^d_{x''} \supset N = \{x \in M;\, x'' = 0\} = \mathbb{R}^{n-d}_{x'} \times \{0\}$, $\pi(x^*) = 0$ and that U has a form $\mathbb{R}^{n-d} \times \Gamma$. Here $\Gamma = \bigcap\limits_{j=1}^{m} \Gamma_j \subset$

$\mathbb{R}^d$ is a proper convex subanalytic open cone with $\Gamma_j = \{x'' \in \mathbb{R}^d;\ \langle x'', \xi''_j\rangle > 0\}$ for some $\xi''_j \in \mathbb{R}^d$. By Theorem 3.2, we have

$$\varinjlim_U \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} C^\infty_M)_0 = \varinjlim_U \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)_0\,.$$

5.3. Lemma. *$\mathbb{C}_U \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X|_{\dot{T}^*X}$ is concentrated in degree zero.*

Proof. We set

$$W_+^{p,q} := \{x \in \mathbb{R}^n;\ \langle x'', \xi''_1\rangle = \cdots = \langle x'', \xi''_p\rangle = 0,\ \langle x'', \xi''_{p+1}\rangle > 0, \ldots, \langle x'', \xi''_{p+q}\rangle > 0\}$$

and shall prove that $\mathbb{C}_{W_+^{p,q}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X|_{\dot{T}^*X}$ in concentrated in degree $-p$.

Assume that $p+q=1$. If $(p,q)=(1,0)$, then by Proposition 4.5, $\mathbb{C}_{W_+^{1,0}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X|_{\dot{T}^*X}$ is concentrated in degree -1. If $(p,q)=(0,1)$, then setting $W_-^{0,1} := \{x \in \mathbb{R}^n;\ \langle x'', \xi''_1\rangle < 0\}$, we have $\mathbb{C}_{W_+^{0,1}} \oplus \mathbb{C}_{W_-^{0,1}} \to \mathbb{C}_M \to \mathbb{C}_{W_+^{1,0}} \xrightarrow{+1}$. Hence we have

$$(\mathbb{C}_{W_+^{0,1}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X) \oplus (\mathbb{C}_{W_-^{0,1}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X) \to \mathbb{C}_M \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X \to \mathbb{C}_{W_+^{1,0}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X \xrightarrow{+1}.$$

Therefore $\mathcal{H}^j(\mathbb{C}_{W_\pm^{0,1}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X)|_{\dot{T}^*X} = 0$ for $j \neq 0$ and the sequence

$$0 \to \mathcal{H}^{-1}(\mathbb{C}_{W_+^{1,0}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X)|_{\dot{T}^*X} \to \big(\mathcal{H}^0(\mathbb{C}_{W_+^{0,1}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X) \oplus \mathcal{H}^0(\mathbb{C}_{W_-^{0,1}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X)\big)|_{\dot{T}^*X} \to \mathscr{C}^d_M|_{\dot{T}^*X} \to 0$$

is exact.

Next assume that we have proved the desired result for $p+q=\nu-1$. $\mathbb{C}_{W_+^{\nu,0}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X|_{\dot{T}^*X}$ is concentrated in degree $-\nu$ by Proposition 4.5. Assume that $\mathbb{C}_{W_+^{p+1,\nu-p-1}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X|_{\dot{T}^*X}$ is concentrated in degree $-p-1$. Then setting

$$W_-^{p,\nu-p} := \{x \in \mathbb{R}^n;\ \langle x'', \xi''_1\rangle = \cdots = \langle x'', \xi''_p\rangle = 0,\ \langle x'', \xi''_{p+1}\rangle < 0, \ldots, \langle x'', \xi''_{p+q}\rangle > 0\},$$
$$W'^{p,\nu-p-1}_+ := \{x \in \mathbb{R}^n;\ \langle x'', \xi''_1\rangle = \cdots = \langle x'', \xi''_p\rangle = 0,\ \langle x'', \xi''_{p+2}\rangle > 0, \ldots, \langle x'', \xi''_{p+q}\rangle > 0\},$$

we have $\mathbb{C}_{W_+^{p,\nu-p}} \oplus \mathbb{C}_{W_-^{p,\nu-p}} \to \mathbb{C}_{W'^{p,\nu-p-1}_+} \to \mathbb{C}_{W_+^{p+1,\nu-p-1}} \xrightarrow{+1}$. Hence we have:

$$(\mathbb{C}_{W_+^{p,\nu-p}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X) \oplus (\mathbb{C}_{W_-^{p,\nu-p}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X) \to \mathbb{C}_{W'^{p,\nu-p-1}_+} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X \to \mathbb{C}_{W_+^{p+1,\nu-p-1}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X \xrightarrow{+1}.$$

By the induction hypothesis, we see that $\big(\mathbb{C}_{W'^{p,\nu-p-1}_+} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X\big)|_{\dot{T}^*X}$ is concentrated in degree $-p$. Therefore $\mathcal{H}^j(\mathbb{C}_{W_\pm^{p,\nu-p}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X)|_{\dot{T}^*X} = 0$ for $j \neq -p$ and the sequence

$$\begin{aligned}0 \to \mathcal{H}^{-p-1}(\mathbb{C}_{W_+^{p+1,\nu-p-1}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X)|_{\dot{T}^*X} &\to \big(\mathcal{H}^{-p}(\mathbb{C}_{W_+^{p,\nu-p}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X) \oplus \mathcal{H}^{-p}(\mathbb{C}_{W_-^{p,\nu-p}} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X)\big)|_{\dot{T}^*X} \\ &\to \mathcal{H}^{-p}(\mathbb{C}_{W'^{p,\nu-p-1}_+} \underset{\mu}{\overset{\mathrm{w}}{\otimes}} \mathscr{O}_X)|_{\dot{T}^*X} \to 0\end{aligned}$$

is exact. Therefore the induction proceeds.

In particular, $\mathbb{C}_U \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X\big|_{\dot{T}^*X} = \mathbb{C}_{W_+^{0,m}} \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathscr{O}_X\big|_{\dot{T}^*X}$ is concentrated in degree zero. □

By the preceding lemma and Theorem 4.9, we have

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\big) \subset V \mathbin{\widehat{+}} \mathrm{SS}(\mathbb{C}_U).$$

Hence we can apply the same method as in Proposition 5.1 to prove (5.5). Set $W := \{z \in \mathbb{C}^n;\ \mathrm{Re}\, z'' \in \mathrm{Cl}\,\Gamma\}$. We shall show

$$N_0^*(\mathbb{C}_W) \cap \mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\big) \subset \{0\}. \tag{5.6}$$

Note that

$$\mathrm{SS}(\mathbb{C}_U) \cap \pi^{-1}(0) \subset \{(0; \sqrt{-1}\,\eta', -\xi'' + \sqrt{-1}\,\eta'');\ \xi'' \in \Gamma^\circ\}, \quad N_0^*(\mathbb{C}_W) = \{(0; \xi'');\ \xi'' \in \Gamma^\circ\}.$$

Assume that $(0; \xi_0'') \in N_0^*(\mathbb{C}_W) \cap (V \mathbin{\widehat{+}} \mathrm{SS}(\mathbb{C}_U))$ and $\xi_0'' \neq 0$. Then by (1.5) there exist sequences $\{(z_j; \zeta_j)\}_{j\in\mathbb{N}} \subset V$, $\{(\widetilde{x}_j; \sqrt{-1}\,\widetilde{\eta}_j', -\widetilde{\xi}_j'' + \sqrt{-1}\,\widetilde{\eta}_j'')\}_{j\in\mathbb{N}} \subset \mathrm{SS}(\mathbb{C}_U)$ such that

$$z_j,\ \widetilde{x}_j \underset{j}{\to} 0, \quad \zeta_j + (\sqrt{-1}\,\widetilde{\eta}_j', -\widetilde{\xi}_j'' + \sqrt{-1}\,\widetilde{\eta}_j'') \underset{j}{\to} (0, \xi_0''), \quad |z_j - \widetilde{x}_j|\,|\zeta_j| \underset{j}{\to} 0.$$

In particular, we have

$$\xi_j - (0, \widetilde{\xi}_j'') \underset{j}{\to} (0, \xi_0''), \quad |y_j|\,|\eta_j| \leqslant |z_j - \widetilde{x}_j|\,|\zeta_j| \underset{j}{\to} 0. \tag{5.7}$$

Then the sequence $\{|\xi_j|\}_{j=1}^\infty$ does not converge to zero. Indeed, assume that $|\xi_j| \underset{j}{\to} 0$. Then by (5.7), we see $\Gamma^{\circ a} \ni -\widetilde{\xi}_j'' \underset{j}{\to} \xi_0'' \in \Gamma^\circ$. Since Γ° is a proper convex closed cone, we have $\xi_0'' \in \Gamma^\circ \cap \Gamma^{\circ a} = \{0\}$, which is a contradiction. Hence extracting subsequence, setting $c_j := 1/|\xi_j| > 0$, we may assume that $\{c_j\}_{j\in\mathbb{N}}$ and $\{c_j\xi_j\}_{j\in\mathbb{N}}$ converge to some $c \in \mathbb{R}_{\geqslant 0}$ and $\theta_0 \in \mathbb{R}^n \setminus \{0\}$ respectively. Hence we have $c_j(\xi_j - (0, \widetilde{\xi}_j'')) \underset{j}{\to} (0, c\xi_0'')$. In particular, $\{c_j(0, \widetilde{\xi}_j'')\}_{j\in\mathbb{N}} \subset \{0\} \times \Gamma^\circ$ converges to $\theta_0 - (0, c\xi_0'')$. Since $\{0\} \times \Gamma^\circ$ is closed, we have $\theta_0 - (0, c\xi_0'') \in \{0\} \times \Gamma^\circ$. Thus we have

$$\theta_0 = \theta_0 - (0, c\xi_0'') + (0, c\xi_0'') \in \{0\} \times \Gamma^\circ + \{0\} \times \Gamma^\circ \subset \{0\} \times \Gamma^\circ.$$

Thus we write $\theta_0 = (0, \theta_0'') \neq 0$. By virtue of (5.7), the sequence $\{(z_j; c_j\zeta_j)\}_{j\in\mathbb{N}} \subset V$ satisfies

$$(x_j; c_j\xi_j) \underset{j}{\to} (0; 0, \theta_0''), \quad |y_j|\,|c_j\eta_j| \underset{j}{\to} 0.$$

By (1.4), this implies that $(x_0; \theta_0'') \in \dot{T}_N^* M \cap C_{T_M^*X}(V)$, which contradicts the hyperbolicity condition. Hence we obtain (5.6). Further by $\mathrm{supp}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)\big) \subset W$ and [K-S 2, Corollary 5.4.9], we have

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)_0 = \boldsymbol{R}\Gamma_W \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)_0 = 0.$$

Therefore we obtain (5.5).

In view of Propositions 3.4 and 3.5, we have by (5.5)

$$\begin{array}{ccc} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\text{W-}\nu_N(C^\infty_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,C^\infty_N) \\ \downarrow & & \downarrow \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\nu_N(C^\infty_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\Gamma_N\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},C^\infty_M)\otimes\omega^{\otimes -1}_{N/M}. \end{array}$$

Further by (3.3), (3.4) and (5.5), we have

$$\begin{array}{ccccc} f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},C^\infty_M) & \xrightarrow{\sim} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{W}^\infty_{M,N}) & \xrightarrow{\sim} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,C^\infty_N) \\ \| & & \downarrow\wr & & \\ f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},C^\infty_M) & \xrightarrow{\sim} & \boldsymbol{R}\Gamma_N\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},C^\infty_M)\otimes\omega^{\otimes -1}_{N/M}. & & \end{array}$$

The proof is complete. □

5.4. Remark. Let V, $\mathscr{M}$ and $f_N\colon N\hookrightarrow M$ be as in Theorem 5.2. Then by (3.3) and proof of Theorem 5.2, there exists the following commutative diagram (cf. [K-S 1]):

$$\begin{array}{ccccc} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\text{W-}\nu_N(C^\infty_M)) & \xrightarrow{\sim} & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\nu_N(C^\infty_M)) & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\nu_N(\mathscr{B}_M)) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,C^\infty_N) & = & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,C^\infty_N) & \longrightarrow & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{B}_N). \end{array}$$

6. Hyperbolic Cauchy Problem for C^∞ Functions.

In this section, we consider a hyperbolic Cauchy problem for C^∞ functions.

6.1. Theorem. *Let $f_N\colon N\to M$ be a morphism of real analytic manifolds and $f\colon Y\to X$ a complexification. Let V and $\mathscr{M}$ satisfy Condition* 4.8*. Suppose that f_N is hyperbolic for V. Then there exists the following isomorphism*:

$$f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},C^\infty_M)\simeq\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M},C^\infty_N).$$

Proof. (i) Suppose that f is smooth. Then, by [K-S 3, Theorem 3.3], we have

$$\begin{aligned} f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},C^\infty_M) &= \boldsymbol{R}\mathscr{H}om_{f_N^{-1}i^{-1}\mathscr{D}_X}(f_N^{-1}i^{-1}\mathscr{M},\boldsymbol{R}\mathscr{H}om_{\mathscr{D}^A_N}(\mathscr{D}^A_{N\to M},C^\infty_N)) \\ &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}^A_N}(\mathscr{D}^A_{N\to M}\overset{\boldsymbol{L}}{\underset{f_N^{-1}\mathscr{D}^A_M}{\otimes}}f_N^{-1}i^{-1}\mathscr{M},C^\infty_N)\simeq\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M},C^\infty_N). \end{aligned}$$

(ii) Suppose that f is an embedding of a closed submanifold. Restricting isomorphisms of Theorem 5.2 to the zero-section N, we obtain:

$$f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},C^\infty_M)\xrightarrow{\sim}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,C^\infty_N)=\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M},C^\infty_N). \tag{6.1}$$

(iii) In general, we decompose f by the graph embedding:

$$Y\xrightarrow{g}Z:=Y\times X\xrightarrow{h}X,\qquad f=h\circ g.$$

Here $g\colon Y \ni y \mapsto (y, f(y)) \in Y \times X$ and h is the canonical projection. We identify Y with $g(Y)$. Set $L := N \times M \subset Z$. Then $\boldsymbol{D}h^*\mathscr{M}$ has regular singularities along $\widetilde{V} := h_d\, h_\pi^{-1}(V)$, and we easily see that

$$\dot{T}^*_N L \cap C_{T^*_L Z}(\widetilde{V}) = \emptyset.$$

Thus by (i) and (ii) we have

$$\begin{aligned} f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, C^\infty_M) &\simeq g_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Z}(\boldsymbol{D}h^*\mathscr{M}, C^\infty_L) \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}g^*\boldsymbol{D}h^*\mathscr{M}, C^\infty_N) \\ &= \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, C^\infty_N). \end{aligned}$$

The proof is complete. □

7. Remark on One-Codimensional Case.

In this section, we assume that N is a one-codimensional closed submanifold of M in (3.2). Let $\mathscr{M}$ be a coherent $f^{-1}\mathscr{D}_X$-Module. Assume that Y is non-characteristic for $\mathscr{M}$. We consider:

7.1. Condition. $\mathscr{M}$ satisfies:

$$f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, C^\infty_M) \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, C^\infty_N).$$

7.2. Theorem. *Assume Condition* 7.1. *Then there exist the following isomorphisms*:

$$\begin{array}{ccc} \tau_N^{-1}f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, C^\infty_M) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, C^\infty_N) \\ \downarrow\wr & & \| \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \text{W-}\nu_N(C^\infty_M)) & \xrightarrow{\sim} & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, C^\infty_N). \end{array}$$

Proof. Since the problem is local, we assume that $X = \mathbb{C}^n_z \times \mathbb{C}_\tau \supset Y = \{(z,\tau) \in X;\, \tau = 0\}$ and so on. Hence $f(z,\tau) = \tau$. We set for short, $v := (0; 1\, d/dt) \in \dot{T}_N M$, $p := (0; 1\, dt)$, $p^a := (0; -1\, dt) \in \dot{T}^*_N M$. We identify $\omega_{N/M}[-1]$ with $\mathbb{Z}_N$, and we may prove:

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \text{W-}\nu_N(C^\infty_M))_v \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, C^\infty_N)_0\,.$$

By (3.3), (3.4) and Condition 7.1, we have

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \text{W-}\mu_N(C^\infty_M))_p \oplus \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \text{W-}\mu_N(C^\infty_M))_{p^a} = 0. \tag{7.1}$$

Hence by Proposition 3.4, we obtain

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \text{W-}\nu_N(C^\infty_M))_v \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{W}^\infty_{M,N})_0 \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, C^\infty_N)_0\,.$$

The proof is complete. □

7.3. Example. Let $P(z,\tau,\partial_z,\partial_\tau)$ be a differential operator of order m on X, and set $\mathscr{M} := \mathscr{D}_X \big/ \mathscr{D}_X P$. Assume that P has the following form:

$$P(z,\tau,\partial_z,\partial_\tau) = \partial_z^m + \sum_{j=0}^{m-1} P_j(z,\tau,\partial_z)\,\partial_\tau^{\,j}\,.$$

Note that Y is non-characteristic for $\mathscr{M}$. We impose the following:

7.4. Condition. (1) There exist holomorphic functions $\eta - \lambda_j(z,\tau;\zeta)$ $(1 \leqslant j \leqslant m)$ such that each $\lambda_j(z,\tau;\zeta)$ is a polynomial with respect to ζ of degree one, and that

$$\sigma_m(P)(z,\tau;\zeta,\eta) = \prod_{j=1}^{m} (\eta - \lambda_j(z,\tau;\zeta)).$$

(2) If $(x,t) \in M$ and $\xi \in \mathbb{R}^n$, then $\lambda_j(x,t;\xi) \in \mathbb{R}$.

For P satisfying Condition 7.4 (1), Uchikoshi [Uk] defined a rational number $\operatorname{Irr} P \in [1,m]$. We briefly recall the definition. Set $\Lambda_j(z,\tau,\partial_z,\partial_\tau) := \partial_\tau - \lambda_j(z,\tau,\partial_z) \in \Gamma(X;\mathscr{D}_X^{(1)})$. For $1 \leqslant q \leqslant m$, set $\mathfrak{S}^q := \{\mu = (\mu_1,\dots,\mu_q) \in \mathbb{N}^q;\, 1 \leqslant \mu_i \leqslant m,\, i \neq j \Rightarrow \mu_i \neq \mu_j\}$, $\mathfrak{S}^0 := \{0\}$ and $\mathfrak{S}' := \bigcup_{q=0}^{m-1} \mathfrak{S}^q$. For $\mu = (\mu_1,\dots,\mu_q) \in \mathfrak{S}^q$, we set $|\mu| := q$ (with convention $|0| := 0$) and $\Lambda^\mu(z,\tau,\partial_z,\partial_\tau) := \Lambda_{\mu_q}(z,\tau,\partial_z,\partial_\tau)\cdots\Lambda_{\mu_1}(z,\tau,\partial_z,\partial_\tau) \in \Gamma(X;\mathscr{D}_X^{(|\mu|)})$ with convention $\Lambda^0 := 1$. Then for any $\sigma \in \mathfrak{S}^m$, we can write

$$P(z,\tau,\partial_z,\partial_\tau) = \Lambda^\sigma(z,\tau,\partial_z,\partial_\tau) + \sum_{\mu\in\mathfrak{S}'} (\tau^{|\mu|-m} a_\mu^\sigma(z,\tau) + b_\mu^\sigma(z,\tau,\partial_z))\Lambda^\mu(z,\tau,\partial_z,\partial_\tau),$$

where $\operatorname{ord} b_\mu^\sigma \leqslant m - |\mu| - 1$. This expression is referred as a *Lascar decomposition subordinate to* σ. For each Lascar decomposition, we set

$$\kappa_\sigma := \max\{1, \max_{\mu\in\mathfrak{S}'}\{\frac{m-|\mu|}{m-|\mu|-\operatorname{ord} b_\mu^\sigma}\}\}.$$

Then, setting $\operatorname{irr}_\sigma P := \min\{\kappa_\sigma;$ Lascar decompositions subordinate to $\sigma\}$, we define:

$$\operatorname{Irr} P := \max\{\operatorname{irr}_\sigma P;\, \sigma \in \mathfrak{S}^m\}.$$

Then Uchikoshi proved:

7.5. Theorem ([Uk]). *If P satisfies Condition* 7.4 *and* $\operatorname{Irr} P = 1$, *then Condition* 7.1 *is satisfied*:

$$f_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, C_M^\infty) \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, C_N^\infty) = (C_N^\infty)^{\oplus m}.$$

Hence in this case, Theorem 7.2 holds.

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