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New curvilinear integrals along paths of Feynman path integral

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New Curvilinear Integrals along Paths of Feynman Path Integral

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Abstract

We define new curvilinear integrals along paths of Feynman path integral. In order to define curvilinear integrals along paths on a path space, Itô integrals use initial points of line segments of broken line paths, and Stratonovich integrals use middle points of them. However, our new curvilinear integrals use classical curvilinear integrals along broken line paths. Therefore, the classical fundamental theorem of calculus holds in Feynman path integral.

1 Introduction

In this paper, using the theory of the time slicing approximation in [4], we give a mathematically rigorous definition of new curvilinear integrals along paths of Feynman path integral [2]. It is a mathematical problem how to define curvilinear integrals along paths on a path space. Itô integrals [1] and Stratonovich [5] integrals are enormously successful in stochastic analysis.

Fix $x_0 \in \mathbf{R}^d$ and $x \in \mathbf{R}^d$. Let $\Delta_{T,0}$ is an arbitrary division of the interval $[0,T]$ into subintervals, i.e.,

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0\,. \tag{1.1}$$

Set $x_{J+1} = x$. Let x_J, x_{J-1}, $\cdots$, x_1 be arbitrary ponits of $\mathbf{R}^d$. Let $\gamma_{\Delta_{T,0}}$ be the broken line path which connects (T_j, x_j) and (T_{j-1}, x_{j-1}) by a line segment for any $j = 1, 2, \ldots, J+1$, i.e., $\gamma_{\Delta_{T,0}}(T_j) = x_j$. Let $0 \le T' \le T'' \le T$.

Roughly speaking, when the Brownian motion $B(\tau)$ is equal to the broken line path $\gamma_{\Delta_{T,0}}(\tau)$, Itô integrals are defined by initial points of line segments of the broken line path $\gamma_{\Delta_{T,0}}$, i.e.,

$$\int_{T'}^{T''} f(\tau, B(\tau))dB(\tau) \approx \sum_j f(T_{j-1}, x_{j-1})(x_j - x_{j-1})\,, \tag{1.2}$$

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and Stratonovich integrals are defined by middle points of line segments of the broken line path $\gamma_{\Delta_{T,0}}$, i.e.,

$$\int_{T'}^{T''} f(\tau, B(\tau)) \circ dB(\tau) \approx \sum_j f\left(\frac{T_j + T_{j-1}}{2}, \frac{x_j + x_{j-1}}{2}\right)(x_j - x_{j-1})\,. \tag{1.3}$$

However, our new curvilinear integrals are the classical curvilinear integrals themselves along the broken line path $\gamma_{\Delta_{T,0}}$, i.e.,

$$\int_{T'}^{T''} f(\tau, \gamma_{\Delta_{T,0}}(\tau)) d\gamma_{\Delta_{T,0}}(\tau)\,. \tag{1.4}$$

Therefore, the classical fundamental theorem of calculus holds in Feynman path integral $\int e^{\frac{i}{\hbar}S[\gamma]} \sim \mathcal{D}[\gamma]$ in [4].

2 Results

In order to state our results, we recall the notations of Chapter 2 in [4].

For a path $\gamma : [0,T] \to \mathbf{R}^d$ which starts from $x_0 \in \mathbf{R}^d$ at time 0 and reaches $x \in \mathbf{R}^d$ at time T, i.e. $\gamma(0) = x_0$ and $\gamma(T) = x$, we define the action $S[\gamma]$ along the path γ by

$$S[\gamma] = \int_0^T \frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 - V(t,\gamma)dt\,. \tag{2.1}$$

Let $F[\gamma]$ be a functional on the path space $C([0,T] \to \mathbf{R}^d)$ and $0 < \hbar < 1$ be a parameter.

Let $\Delta_{T,0}$ be an arbitrary division of the interval $[0,T]$ into subintervals, i.e.,

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0\,. \tag{2.2}$$

Set $x_{J+1} = x$. Let x_J, x_{J-1}, $\ldots$, x_1 be arbitrary points of $\mathbf{R}^d$. Let $\gamma_{\Delta_{T,0}}$ be the broken line path which connects (T_j, x_j) and (T_{j-1}, x_{j-1}) by a line segment for any $j = 1, 2, \ldots, J+1$, i.e., $\gamma_{\Delta_{T,0}}(T_j) = x_j$. Set $t_j = T_j - T_{j-1}$. Let $|\Delta_{T,0}|$ be the size of the division defined by $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j$.

We define the Feynman path integral by

$$\begin{aligned}&\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\\ =\;& \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]}F[\gamma_{\Delta_{T,0}}]\prod_{j=1}^{J} dx_j\,.\end{aligned} \tag{2.3}$$

Definition 1 *We say that a functional $F[\gamma]$ is path integrable, if and only if the right-hand side of* (2.3) *converges.*

Our assumption for the potential $V(t,x)$ is the following:

Assumption 1 *$V(t,x)$ is a real-valued function of $(t,x) \in \mathbf{R}\times\mathbf{R}^d$, and for any multi-index α, $\partial_x^\alpha V(t,x)$ is continuous in $\mathbf{R}\times\mathbf{R}^d$. For any integer $k \ge 2$, there exists a positive constant A_k such that*

$$|\partial_x^\alpha V(t,x)| \le A_k\,, \quad (|\alpha| = k)\,. \tag{2.4}$$

Using the assumption which was first found by D. Fujiwara [3], we defined Fujiwara's class $\mathcal{F}$ of functionals, and proved the following result. (See Theorem 2 in [4].)

Proposition 1 *Under Assumption 1, assume that T is sufficiently small. Then, for any $F[\gamma]$ which belongs to Fujiwara's $\mathcal{F}$, the right-hand side of (2.3) really converges uniformly on any compact set of the configuration space $(x, x_0) \in \mathbf{R}^{2d}$, i.e., $F[\gamma]$ is path integrable.*

Now we are ready to state theorems of this paper. First, we state a sufficient condition to define our new curvilinear integrals along paths of Feynman path integral.

Assumption 2 *Let m be non-negative integer. $Z(t,x)$ is a vector-valued function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$ into $\mathbf{R}^d$. For any multi-index α, $\partial_x^\alpha Z(t,x)$ and $\partial_x^\alpha \partial_t Z(t,x)$ are continuous on $[0,T] \times \mathbf{R}^d$, and there exists a positive constant C_α such that*

$$|\partial_x^\alpha Z(t,x)| + |\partial_x^\alpha \partial_t Z(t,x)| \leq C_\alpha (1+|x|)^m \,. \tag{2.5}$$

Furthermore, $\partial_x Z(t,x)$ is a symmetric matrix, i.e.,

$${}^t(\partial_x Z) = \partial_x Z \,. \tag{2.6}$$

Theorem 1 *Let $0 \leq T' \leq T'' \leq T$. Under Assumption 1 and Assumption 2, the curvilinear integral along paths of Feynman path integral*

$$F[\gamma] = \int_{T'}^{T''} Z(\tau, \gamma(\tau)) \cdot d\gamma(\tau) \,, \tag{2.7}$$

belongs to $\mathcal{F}$. Therefore, if T is sufficiently small, $F[\gamma]$ is path integrable.
Here $Z \cdot d\gamma$ is the inner product of Z and $d\gamma$ in $\mathbf{R}^d$.

Remark 1 *The domain of the functional $F[\gamma]$ defined by (2.7) contains all of broken line paths at least.*

Next, we state a fundamental theorem of calculus for our new curvilinear integrals in Feynman path integral. This theorem is different from Itô's formula in stochastic analysis. However this theorem is the same as the classical fundamental theorem of calculus.

Assumption 3 *Let m be non-negative integer. $f(t,x)$ is a function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$. For any multi-index α, $\partial_x^\alpha f(t,x)$ and $\partial_x^\alpha \partial_t f(t,x)$ are continuous on $[0,T] \times \mathbf{R}^d$, and there exists a positive constant C_α such that*

$$|\partial_x^\alpha f(t,x)| + |\partial_x^\alpha \partial_t f(t,x)| \leq C_\alpha (1+|x|)^m \,. \tag{2.8}$$

Theorem 2 *Let $0 \leq T' \leq T'' \leq T$. Under Assumption 1 and Assumption 3, the curvilinear integral along paths of Feynman path integral*

$$F[\gamma] = \int_{T'}^{T''} (\partial_x f)(\tau, \gamma(\tau)) \cdot d\gamma(\tau) \,, \tag{2.9}$$

belongs to $\mathcal{F}$. Therefore, if T is sufficiently small, $F[\gamma]$ is path integrable.

Furthermore, the classical fundamental theorem of calculus

$$\begin{aligned}
&f\left(T'',\gamma(T'')\right)-f\left(T',\gamma(T')\right)\\
&=\int_{T'}^{T''}(\partial_x f)(\tau,\gamma(\tau))\cdot d\gamma(\tau)+\int_{T'}^{T''}(\partial_t f)(\tau,\gamma(\tau))d\tau\,. \qquad (2.10)
\end{aligned}$$

holds in the Feynman path integral $\int e^{\frac{i}{\hbar}S[\gamma]}\sim\mathcal{D}[\gamma]$.

Remark 2 *If* $g=g(t,x)$ *also satisfies Assumption 3, then the classical formula of integration by parts*

$$\begin{aligned}
&\Big[f(\tau,\gamma)g(\tau,\gamma)\Big]_{T'}^{T''}-\int_{T'}^{T''}f(\tau,\gamma)(\partial_x g)(\tau,\gamma)\cdot d\gamma-\int_{T'}^{T''}f(\tau,\gamma)(\partial_t g)(\tau,\gamma)d\tau\\
&=\int_{T'}^{T''}g(\tau,\gamma)(\partial_x f)(\tau,\gamma)\cdot d\gamma+\int_{T'}^{T''}g(\tau,\gamma)(\partial_t f)(\tau,\gamma)d\tau\,, \qquad (2.11)
\end{aligned}$$

holds in the Feynman path integral $\int e^{\frac{i}{\hbar}S[\gamma]}\sim\mathcal{D}[\gamma]$.

3 Proofs

Proof of Theorem 1. For simplicity, we prove the case when $0=T'<T''<T$.
(1) Let $\Delta_{T,0}$ be an arbitrary division of the interval $[0,T]$ into subintervals, i.e.,

$$\Delta_{T,0}:T=T_{J+1}>T_J>\cdots>T_1>T_0=0\,. \qquad (3.1)$$

Let $\gamma_{\Delta_{T,0}}$ be the broken line path which connects (T_j,x_j) and (T_{j-1},x_{j-1}) by a line segment for any $j=1,2,\ldots,J+1$, i.e., $\gamma_{\Delta_{T,0}}(T_j)=x_j$. Set $t_j=T_j-T_{j-1}$. Then there exist an integer M and $0<\vartheta\le 1$ such that

$$T''=\vartheta t_M+T_{M-1}\,. \qquad (3.2)$$

Therefore we can write

$$\begin{aligned}
F[\gamma_{\Delta_{T,0}}]&=\int_0^{T''}Z(\tau,\gamma_{\Delta_{T,0}}(\tau))\cdot d\gamma_{\Delta_{T,0}}(\tau)\\
&=\sum_{j=1}^{M-1}\int_0^1 Z(\theta t_j+T_{j-1},\theta x_j+(1-\theta)x_{j-1})\cdot(x_j-x_{j-1})d\theta\\
&+\int_0^{\vartheta}Z(\theta t_M+T_{M-1},\theta x_M+(1-\theta)x_{M-1})\cdot(x_M-x_{M-1})d\theta\,. \qquad (3.3)
\end{aligned}$$

We set

$$\begin{aligned}
b_j(x,y)&=\int_0^1 Z(\theta t_j+T_{j-1},\theta x+(1-\theta)y)\cdot(x-y)d\theta \quad (1\le j\le M-1)\,,\\
b_M(x,y)&=\int_0^{\vartheta}Z(\theta t_M+T_{M-1},\theta x+(1-\theta)y)\cdot(x-y)d\theta\,,\\
b_j(x,y)&=0 \quad (j>M)\,. \qquad (3.4)
\end{aligned}$$

(2) We consider $\partial_y b_M(x,y)$.

$$\partial_y b_M(x,y) = -\int_0^{\vartheta} Z(\theta t_M + T_{M-1}, \theta x + (1-\theta)y)d\theta$$
$$+\int_0^{\vartheta} {}^t(\partial_x Z)(\theta t_M + T_{M-1}, \theta x + (1-\theta)y)(1-\theta)(x-y)d\theta\,. \tag{3.5}$$

Since ${}^t(\partial_x Z) = (\partial_x Z)$, we have

$$\begin{aligned}\partial_y b_M(x,y) &= -\int_0^{\vartheta} Z(\theta t_M + T_{M-1}, \theta x + (1-\theta)y)d\theta \\ &\quad +\int_0^{\vartheta} \partial_\theta\Big(Z(\theta t_M + T_{M-1}, \theta x + (1-\theta)y)\Big)(1-\theta)d\theta \\ &\quad -t_M\int_0^{\vartheta} (\partial_t Z)(\theta t_M + T_{M-1}, \theta x + (1-\theta)y)(1-\theta)d\theta\,. \end{aligned}\tag{3.6}$$

Integrating by parts, we get

$$\begin{aligned}\partial_y b_M(x,y) &= Z(\vartheta t_M + T_{M-1}, \vartheta x + (1-\vartheta)y)(1-\vartheta) - Z(T_{M-1}, y) \\ &\quad -t_M\int_0^{\vartheta} (\partial_t Z)(\theta t_M + T_{M-1}, \theta x + (1-\theta)y)(1-\theta)d\theta\,. \end{aligned}\tag{3.7}$$

(3) We consider $\partial_y b_M(y,z)$. In a similar way to (2), we get

$$\begin{aligned}\partial_y b_M(y,z) &= Z(\vartheta t_M + T_{M-1}, \vartheta y + (1-\vartheta)z)\vartheta \\ &\quad -t_M\int_0^{\vartheta} (\partial_t Z)(\theta t_M + T_{M-1}, \theta y + (1-\theta)z)\theta d\theta\,. \end{aligned}\tag{3.8}$$

(4) We consider $\partial_y b_j(x,y)$ for $1 \le j \le M-1$. In a similar way to (2), we get

$$\begin{aligned}\partial_y b_j(x,y) &= -Z(T_{j-1}, y) \\ &\quad -t_j\int_0^{1} (\partial_t Z)(\theta t_j + T_{j-1}, \theta x + (1-\theta)y)(1-\theta)d\theta\,. \end{aligned}\tag{3.9}$$

(5) We consider $\partial_y b_j(y,z)$ for $1 \le j \le M-1$. In a similar way to (3), we get

$$\begin{aligned}\partial_y b_j(y,z) &= Z(T_j, y) \\ &\quad -t_j\int_0^{1} (\partial_t Z)(\theta t_j + T_{j-1}, \theta y + (1-\theta)z)\theta d\theta\,. \end{aligned}\tag{3.10}$$

(6) By (2) – (5), we have the following.
If $1 \le j \le M-2$,

$$\begin{aligned}&\partial_y\Big(b_{j+1}(x,y) + b_j(y,z)\Big) \\ &= -t_{j+1}\int_0^{1} (\partial_t Z)(\theta t_{j+1} + T_j, \theta x + (1-\theta)y)(1-\theta)d\theta \\ &\quad -t_j\int_0^{1} (\partial_t Z)(\theta t_j + T_{j-1}, \theta y + (1-\theta)z)\theta d\theta\,. \end{aligned}\tag{3.11}$$

If $j = M-1$,

$$\begin{aligned}
&\partial_y\Big(b_M(x,y) + b_{M-1}(y,z)\Big) \\
&= Z(\vartheta t_M + T_{M-1}, \vartheta y + (1-\vartheta)z)(1-\vartheta) \\
&\quad -t_M \int_0^{\vartheta} (\partial_t Z)(\theta t_M + T_{M-1}, \theta x + (1-\theta)y)(1-\theta)d\theta \\
&\quad -t_{M-1} \int_0^1 (\partial_t Z)(\theta t_{M-1} + T_{M-2}, \theta y + (1-\theta)z)\theta d\theta\,.
\end{aligned} \tag{3.12}$$

If $j = M$,

$$\begin{aligned}
&\partial_y\Big(b_{M+1}(x,y) + b_M(y,z)\Big) \\
&= Z(\vartheta t_M + T_{M-1}, \vartheta y + (1-\theta)z)\vartheta \\
&\quad -t_M \int_0^{\vartheta} (\partial_t Z)(\theta t_j + T_{j-1}, \theta y + (1-\theta)z)\theta d\theta\,.
\end{aligned} \tag{3.13}$$

If $j > M$,

$$\partial_y\Big(b_{j+1}(x,y) + b_j(y,z)\Big) = 0\,. \tag{3.14}$$

We set $u_j = t_j$ ($1 \le j \le M-1$), $u_M = 1 + t_M$ and $u_j = 0$ ($j > M$). Then we have

$$\sum_{j=1}^{J+1} u_j \le T + 2\,. \tag{3.15}$$

In a similar way to Chapter 10 in [4], we can prove that $F[\gamma]$ satisfies Assumption 5 in [4]. Therefore $F[\gamma] \in \mathcal{F}$. □

Proof of Theorem 2. By ${}^t(\partial_x^2 f) = (\partial_x^2 f)$ and Theorem 1, we have

$$F[\gamma] = \int_{T'}^{T''} (\partial_x f)(\tau, \gamma(\tau)) \cdot d\gamma(\tau) \in \mathcal{F}\,. \tag{3.16}$$

We set

$$F_1[\gamma] \;=\; f\left(T'', \gamma(T'')\right) - f\left(T', \gamma(T')\right)\,, \tag{3.17}$$

$$F_2[\gamma] \;=\; \int_{T'}^{T''} (\partial_x f)(\tau, \gamma(\tau)) \cdot d\gamma(\tau) + \int_{T'}^{T''} (\partial_t f)(\tau, \gamma(\tau)) d\tau\,. \tag{3.18}$$

By Theorem 3 in [4], we have

$$f(\tau, \gamma(\tau)) \in \mathcal{F}\,, \quad \int_{T'}^{T''} (\partial_t f)(\tau, \gamma(\tau)) d\tau \in \mathcal{F}\,. \tag{3.19}$$

By Theorem 1 in [4], we have

$$F_1[\gamma] \in \mathcal{F}\,, \quad F_2[\gamma] \in \mathcal{F}\,. \tag{3.20}$$

Now, using the classical fundamental theorem of calculus, we have

$$F_1[\gamma_{\Delta_{T,0}}] = F_2[\gamma_{\Delta_{T,0}}]\,, \tag{3.21}$$

for any broken line path $\gamma_{\Delta_{T,0}}$. Therefore we get

$$
\begin{aligned}
& \int e^{\frac{i}{\hbar}S[\gamma]}F_1[\gamma]\mathcal{D}[\gamma] \\
= \; & \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} F_1[\gamma_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j \\
= \; & \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} F_2[\gamma_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j \\
= \; & \int e^{\frac{i}{\hbar}S[\gamma]}F_2[\gamma]\mathcal{D}[\gamma] \, .
\end{aligned} \tag{3.22}
$$

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