UTMS 2003–47 December 22, 2003

# On the defining equations of abelian surfaces and modular forms

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## 0. Introduction

It is a well-known fact that Hesse's cubic curve in $\mathbb{P}^2$:

$$X^3 + Y^3 + Z^3 = 3\mu XYZ$$

gives a defining equation of an elliptic curve $E$ with level 3 structure whenever $\mu \neq \infty, 1, \omega$ and $\omega^2$ ($\omega = e^{2\pi i/3}$). Write $E = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ ($\tau \in \mathbb{H}_1$), then the coefficient $\mu$ can be regarded as a modular function in $\tau$. More precisely we define two functions:

$$\vartheta(\tau) = \sum_{l\in\mathbb{Z}^2} \exp \pi i (\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} [l]\tau),$$

$$\chi(\tau) = \sum_{l\in\mathbb{Z}^2} \exp \pi i (\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} [l + \frac{1}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix}]\tau)$$

in the space $M_1(\Gamma(3))$ of modular forms of weight 1. Moreover $\vartheta$ and $\chi$ generate the graded ring of modular forms of level 3

$$\bigoplus_{k=0}^{\infty} M_k(\Gamma(3)) = \mathbb{C}[\vartheta, \chi]$$

(cf. [G, Theorem 1.1]). Then we can show

$$\mu(\tau) = \frac{\vartheta(\tau)}{\chi(\tau)}.$$

In this paper, we consider the case of abelian surfaces with level 3 structure. The cubic theta relations are already given by Ch. Birkenhake and H. Lange ([BL]), and it is possible to consider their coefficients as functions on Siegel upper half space. One might expect that all the coefficients of the defining equation are Siegel modular forms of level 3, but the naive hope is not true. However we find some of the coefficients are modular forms of level 3.

We have two main results in this paper. One is the generalization of Hesse's cubic curve:

**Theorem 0.1** (Theorem 4.1)**.** *We give an explicit form of the degree* 3 *part of defining equations for a principally polarized abelian surface.*

The other result corresponds to the coefficients of the defining equation, which are considered as generalizations of $\mu$.

**Theorem 0.2** (Theorem 3.2)**.** *We specify the coefficients of defining equations which are written by theta functions of quadratic forms, that belong to the space of Siegel modular forms of degree* 2*, level* 3*.*

In the previous paper ([G]), the author gives the dimensions and the generators of the space of Siegel modular forms of degree 2 and level 3 for low weights. The generators are all given by theta functions, and we see that they appear in the coefficients of the defining equations. Therefore one can regard this paper as an enhancement of the previous paper [G].

**Acknowledgement**: The author thanks Professor Takayuki Oda for his constant support and much advice to this work.

**Notations**: We write the space of holomorphic Siegel modular forms of weight $k$ associated to the congruence subgroup $\Gamma \subset Sp(g,\mathbb{Z})$ by $M_k(\Gamma)$. For a holomorphic function $f$ on Siegel upper half space $\mathbb{H}_g$ and $\gamma \in Sp(g,\mathbb{Z})$, we put $f|_k\gamma(\tau) = \det(C\tau + D)^{-k} f(\gamma\langle\tau\rangle)$, here $C$ (resp. $D$) is the lower-left (resp. lower-right) $g \times g$ block of $\gamma$, and $\gamma\langle\tau\rangle$ is a standard action.

For matrices $X$ and $Y$, we write $Y[X] = {}^tXYX$. We put $\omega = e^{2\pi i/3}$.

## 1. Defining equations of abelian varieties

Let $A = V/\Lambda$ be a $g$-dimensional abelian variety over $\mathbb{C}$, that is, $V$ is a complex vector space of dimension $g$ and $\Lambda$ is a $\mathbb{Z}$-lattice of rank $2g$. Let $L_0$ be an ample line bundle on $A$. Then by the theorem of Koizumi([Ko, Corollary 4.7]), $L = L_0^3$ is normally generated, i.e. the natural map

$$\operatorname{Sym}^n H^0(A,L) \longrightarrow H^0(A,L^n)$$

is surjective for each $n \geq 1$. In particular $L$ is very ample.

**Theorem 1.1** ([Ke, Theorem 2]). *Let $L = L_0^3$ be an ample line bundle on $A$. Then the kernel of the natural map*

$$\bigoplus_{n=0}^{\infty} \operatorname{Sym}^n H^0(A,L) \longrightarrow \bigoplus_{n=0}^{\infty} H^0(A,L^n)$$

*is generated by the elements of degree* 2 *and* 3.

All the relations of theta functions of degree 3 is already given by Birkenhake and Lange ([BL]). Now we explain their theorem.

Let $L = L(H,\alpha)$ be an ample line bundle on $A = V/\Lambda$. Here $H$ is a positive definite Hermitian form on $V$, such that for $E = \operatorname{Im} H$, $E(\Lambda,\Lambda) \subset \mathbb{Z}$; $\alpha$ is a semicharacter of $\Lambda$, that is, a map from $\Lambda$ to the group $\mathbb{C}_1^\times$ of complex number of absolute value 1 which satisfies

$$\alpha(\lambda_1 + \lambda_2) = \alpha(\lambda_1)\alpha(\lambda_2)\exp \pi i E(\lambda_1,\lambda_2), \quad \lambda_1,\lambda_2 \in \Lambda.$$

Then $L(H,\alpha)$ is given by the quotient of the trivial line bundle $\mathbb{C} \times V$ by $\Lambda$ according to the action

$$\Lambda \times (\mathbb{C} \times V) \ni (\lambda,(x,v)) \longmapsto (e_\lambda(v)x, v+\lambda) \in \mathbb{C} \times V,$$
$$e_\lambda(v) = \alpha(\lambda)\exp(\pi H(v,\lambda) + \frac{\pi}{2}H(\lambda,\lambda)).$$

Since $H$ is positive definite, $E$ is a non-degenerate alternating form on $\Lambda$. Then we can take Frobenius base of $\Lambda$ $e_1,\dots,e_g,f_1,\dots,f_g$, that is, $E$ is represented by this basis as

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \quad D = \operatorname{diag}(d_1,\dots,d_g), \quad d_i > 0, \quad d_i | d_{i+1}.$$

We fix such basis and decompose $\Lambda = \Lambda_1 \oplus \Lambda_2$, further $V = V_1 \oplus V_2$ as real vector spaces.

For $\lambda = \lambda_1 + \lambda_2$ as above decomposition, we define semicharacter $\alpha_0 : \Lambda \to \mathbb{C}_1^\times$ by

$$\alpha_0(\lambda) = \exp \pi i E(\lambda_1,\lambda_2).$$

We call $L = L(H,\alpha_0)$ a line bundle of characteristic zero. Then $L$ is symmetric i.e. $(-1)^*L \cong L$.

Next for $x \in A$, let $T_x : A \to A$, $y \mapsto x+y$ be a translation of $x$. We put

$$\begin{aligned} K(L) &= \ker\{\phi_L : A \longrightarrow \widehat{A} = \operatorname{Pic}^0(A), \quad x \longmapsto T_x^*L \otimes L^{-1}\} \\ &= \Lambda(L)/\Lambda, \quad \Lambda(L) = \{v \in V \mid E(v,\Lambda) \subset \mathbb{Z}\}. \end{aligned}$$

We decompose $K(L) = K(L)_1 \oplus K(L)_2$ according to the above decomposition. Then by Riemann-Roch theorem,

$$\sharp K(L)_1 = \sharp K(L)_2 = \dim H^0(A,L).$$

Now since the $\mathbb{Z}$-basis of $\Lambda_2$, $f_1,\dots,f_g$ generates $V$ over $\mathbb{C}$, we define a symmetric form $B$ on $V$ by $\mathbb{C}$-linear extension of $H$ restricted on $V_2$. We put for $x \in K(L)_1$,

$$\vartheta_x^L(v) = \exp\left(\frac{\pi}{2}B(v,v) - \frac{\pi}{2}(H-B)(x+2v,x)\right) \sum_{\lambda \in \Lambda_1} \exp\left(\pi(H-B)(x+v,\lambda) - \frac{\pi}{2}(H-B)(\lambda,\lambda)\right), \tag{1}$$

then $\{\vartheta^L_x\}_{x\in K(L)_1}$ form a basis of $H^0(A,L)$ of characteristic zero.

We assume that $L = L_0^3$ for some line bundle $L_0$. Let $A_6$ be the set of 6 divisible point of $A$ and $Z_6 = A_6 \cap K(L^2)_1$ (notice $K(L^2) \supset A_6$).

For $\rho \in \widehat{Z}_6 = \mathrm{Hom}(Z_6, \mathbb{C}_1^\times)$, $y_1 \in K(L^6)_1$ and $y_2 \in K(L^2)_1$ we define

$$\theta_{(y_1,y_2),\rho}(v) = \sum_{a\in Z_6} \rho(a)\vartheta^{L^6}_{y_1-a}(v)\vartheta^{L^2}_{y_2-3a}(v). \tag{2}$$

Now we state the theorem of Birkenhake and Lange.

**Theorem 1.2** (Cubic theta relations [BL, Theorem 3.3])**.** *Let $L$ be an ample line bundle on $A$ and assume $L$ is a third power. Then all the cubic theta relations are given as follows.*

$$\begin{aligned}&\theta_{(y_1,y_2),\rho}(0)\sum_{b\in Z_6}\vartheta^L_{y_1'+y_2'+y_3+2b}\vartheta^L_{y_1'-y_2'+y_3+2b}\vartheta^L_{-2y_1'+y_3+2b}\\ &=\theta_{(y_1',y_2'),\rho}(0)\sum_{b\in Z_6}\vartheta^L_{y_1+y_2+y_3+2b}\vartheta^L_{y_1-y_2+y_3+2b}\vartheta^L_{-2y_1+y_3+2b}.\end{aligned}$$

*Here $\rho\in\widehat{Z}_6$, $y_1, y_1' \in K(L^6)_1$, $y_2, y_2' \in K(L^2)_1$ and $y_3 \in K(L^3)_1$ such that*

$$\begin{cases} y_1+y_2+y_3, \quad y_1-y_2+y_3, \quad -2y_1+y_3,\\ y_1'+y_2'+y_3, \quad y_1'-y_2'+y_3, \quad -2y_1'+y_3\end{cases}$$

*all belong to $K(L)_1$.*

## 2. Theta constants as modular forms

Let $A = V/\Lambda$ be an abelian variety of dimension $g$ with principal polarization $H$. Let $L_0 = L(H,\alpha_0)$ be a line bundle of characteristic zero. We decompose $\Lambda = \Lambda_1 \oplus \Lambda_2$ according to Frobenius basis $e_1,\dots,e_g,f_1,\dots,f_g$. Since $f_1,\dots,f_g$ generates $V$ over $\mathbb{C}$, we identify $V=\mathbb{C}^g$ according to this base. Then $\Lambda = \tau\mathbb{Z}^g + \mathbb{Z}^g$ for $\tau\in\mathbb{H}_g$, and the Hermitian form $H$ is given by $(\operatorname{Im}\tau)^{-1}$ (cf. [LB, Chapter 8]).

Now for $v,w\in V$, we write

$$v = v_1\tau + v_2, \quad w = w_1\tau + w_2, \quad v_1,v_2,w_1,w_2\in\mathbb{R}^g.$$

Then by definition we have

$$(H-B)(v,w) = {}^tv(\operatorname{Im}\tau)^{-1}(\bar w - w) = {}^tv(\operatorname{Im}\tau)^{-1}(-2i(\operatorname{Im}\tau)w_1) = -2i^t(\tau v_1+v_2)w_1. \tag{3}$$

Now let $L = L_0^3$. Since $L_0$ is principal, $K(L_0^n) = A_n$. We write,

$$K(L^6)_1 = K(L_0^{18})_1 \ni y_1 = \frac{1}{18}\tau\eta_1, \quad K(L^2)_1 = K(L_0^9)_1 \ni y_2 = \frac{1}{6}\tau\eta_2, \quad Z_6 \ni b = \frac{1}{6}\tau\beta,$$

for $\eta_1,\eta_2,\beta\in\mathbb{Z}^g$. Then using (3), theta functions (1) can be rewritten as follows.

$$\begin{aligned}\vartheta^{L^6}_{y_1-b}(0) &= \sum_{m\in\mathbb{Z}^g}\exp 18\pi i(\tau[m+\frac{1}{6}\beta-\frac{1}{18}\eta_1])\\ \vartheta^{L^2}_{y_2-3b}(0) &= \sum_{m\in\mathbb{Z}^g}\exp 6\pi i(\tau[m+\frac{1}{2}\beta-\frac{1}{6}\eta_2])\end{aligned} \tag{4}$$

Now we use the following.

**Proposition 2.1.** *Let $Q\in M_m(\mathbb{Z})$ be a symmetric positive definite matrix with even diagonal entries, and let $q$ be a level of $Q$, that is, the minimum positive integer such that $qQ^{-1}$ is also integral with even diagonal entries. We put $T^g(Q) = \{T\in M_{m,g}(\mathbb{Z}) |\ QT\equiv 0 \mod q\}$.*

*We define for $\tau\in\mathbb{H}_g$ and $T\in T^g(Q)$,*

$$\theta^g(\tau,Q|T) = \sum_{N\in M_{m,g}(\mathbb{Z})}\exp\pi i\operatorname{Tr}(Q[N+\frac{1}{q}T]\tau).$$

*Then $\theta^g(\tau,Q|T)\in M_{m/2}(\Gamma^g(q))$, with principal congruence subgroup $\Gamma^g(q)\subset Sp(g,\mathbb{Z})$. Moreover the following properties hold.*

$$\theta^g(\left(\begin{smallmatrix} V & 0 \\ 0 & {}^tV^{-1}\end{smallmatrix}\right)\langle\tau\rangle, Q|T) = \theta^g(\tau, Q|TV) \quad \textit{for } V \in GL_g(\mathbb{Z}).$$
$$\theta^g(\left(\begin{smallmatrix} 1 & S \\ 0 & 1\end{smallmatrix}\right)\langle\tau\rangle, Q|T) = \exp\pi i\,\mathrm{Tr}(\frac{1}{q^2}{}^tTQTS)\theta^g(\tau, Q|T).$$
$$\theta^g(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)\langle\tau\rangle, Q|T)$$
$$= (\det Q)^{-g/2}(\det(-i\tau))^{m/2}\sum_{\substack{T'\in T^n(Q)\\ \bmod q}}\exp 2\pi i\,\mathrm{Tr}(\frac{1}{q^2}{}^tTQT')\theta^g(\tau, Q|T').$$

For the proof, see [A, Proposition 1.3.14, Exercise 2.2.3]. By this Proposition, we see that above functions (4) belong to the space of Siegel modular forms of weight 1/2, $M_{1/2}(\Gamma^g(36))$ and $M_{1/2}(\Gamma^g(12))$ respectively.

Thus the coefficients of the defining equations (2) are given as

$$\begin{aligned}\theta_{(y_1,y_2),\rho}(0) &= \sum_{\beta\in(\mathbb{Z}/6\mathbb{Z})^g}\rho(\beta)\sum_{m,n\in\mathbb{Z}^g}\exp\pi i(18\tau[m+\frac{1}{6}\beta-\frac{1}{18}\eta_1]+6\tau[n+\frac{1}{2}\beta-\frac{1}{6}\eta_2]) \\ &= \sum_{\beta\in(\mathbb{Z}/6\mathbb{Z})^g}\rho(\beta)\sum_{N\in M_{g,2}(\mathbb{Z})}\exp\pi i\,\mathrm{Tr}(\begin{pmatrix}18 & 0\\ 0 & 6\end{pmatrix}\left[N+\frac{1}{36}\begin{pmatrix}6{}^t\beta-2{}^t\eta_1\\ 18{}^t\beta-6{}^t\eta_2\end{pmatrix}\right]\tau).\end{aligned} \tag{5}$$

By Proposition 2.1, we have $\theta_{(y_1,y_2),\rho}\in M_1(\Gamma^g(36))$.

**Example** ([BL, §4])

We consider the case $g=1$. Since $\dim \mathrm{Sym}^3 H^0(A,L)=10$ and $\dim H^0(A,L^3)=9$, there is only one nontrivial equation. We fix the isomorphism of $K(L^6)_1$ to $\mathbb{Z}/18\mathbb{Z}$, and denote its elements by $\{0,1,\dots,17\}$. The groups $K(L)_1$, $K(L^2)_1 = Z_6$, $K(L^3)_1$ are embedded into this group, and we write $K(L)_1=\{0,6,12\}$, etc. We write the coordinates of $\mathbb{P}_2$ by $X_0, X_6, X_{12}$.

Using this notation, the defining equation of elliptic curve is given as

$$X_0^3+X_6^3+X_{12}^3 = 3\frac{\theta_{(0,0),1}(0)}{\theta_{(0,6),1}(0)}X_0X_6X_{12}.$$

By (5), we have

$$\theta_{(0,0),1}(0)=\sum_{b\in\mathbb{Z}/6\mathbb{Z}}\sum_{l\in\mathbb{Z}^2}\exp\pi i(\begin{pmatrix}18 & 0\\ 0 & 6\end{pmatrix}\left[l+\frac{1}{36}\begin{pmatrix}6b\\ 18b\end{pmatrix}\right]\tau),$$
$$\theta_{(0,6),1}(0)=\sum_{b\in\mathbb{Z}/6\mathbb{Z}}\sum_{l\in\mathbb{Z}^2}\exp\pi i(\begin{pmatrix}18 & 0\\ 0 & 6\end{pmatrix}\left[l+\frac{1}{36}\begin{pmatrix}6b\\ 18b-12\end{pmatrix}\right]\tau),$$

as functions on $\tau$. Moreover we can show $\theta_{(0,0),1}=\vartheta$ and $\theta_{(0,6),1}=\chi$ as in the introduction, by comparing sufficiently many Fourier coefficients, or by the similar method we shall give bellow in the case of $g=2$.

## 3. Modular forms in the coefficients of defining equations of abelian surfaces

Now we shall consider the case of an abelian surface. We use the similar notation as in the example of elliptic curves in the previous section, that is, $K(L^6)_1={}^t(a,b)$, $a,b\in\{0,1,\dots,17\}$ etc. In the defining equation of Theorem 1.2, we put $y_3={}^t(0,0)$ and $\rho=1$. As in [BL, §4], we only consider the case $y_1,y_1'\in\{0,1,\dots,5\}^2$, $y_2,y_2'\in\{0,3,\dots,15\}^2/\{\pm1\}$. Then all the elements $y_1, y_2(y_1',y_2')$ such that $y_1+y_2+$

$y_3, y_1-y_2+y_3, -2y_1+y_3 \in K(L)_1$ are as follows.

$$\begin{aligned}\{y_1,y_2\} = \quad & \left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}6\\0\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\6\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}6\\6\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}6\\12\end{pmatrix}\right\},\\ & \left\{\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}3\\6\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}3\\12\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}9\\0\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}9\\6\end{pmatrix}\right\},\\ & \left\{\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}0\\9\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}6\\3\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}6\\9\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}6\\15\end{pmatrix}\right\},\\ & \left\{\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}3\\9\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}3\\15\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}9\\3\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}9\\9\end{pmatrix}\right\}.\end{aligned}$$

Let $X_{0,0}, X_{0,6}, X_{0,12}, X_{6,0}, X_{6,6}, X_{6,12}, X_{12,0}, X_{12,6}$ and $X_{12,12}$ be homogenous coordinates of $\mathbb{P}^8$. We write

$$P_{(y_1,y_2,y_3)}(\mathbf{X}) = \sum_{b\in Z_6} X_{y_1+y_2+y_3+2b}X_{y_1-y_2+y_3+2b}X_{-2y_1+y_3+2b}.$$

We only consider the first row of the above list, since for example we can show

$$P_{\{\binom{3}{0},\binom{3}{0},\binom{0}{0}\}}(\mathbf{X}) = P_{\{\binom{0}{0},\binom{6}{0},\binom{0}{0}\}}(\mathbf{X}), \quad \theta_{\{\binom{3}{0},\binom{3}{0}\},1} = \theta_{\{\binom{0}{0},\binom{6}{0}\},1}.$$

Now we shall write

$$\Theta\begin{bmatrix}a & b\\ c & d\end{bmatrix}(\tau) := \sum_{N\in M_2(\mathbb{Z})} \exp \pi i(\begin{pmatrix}18 & 0\\ 0 & 6\end{pmatrix}\left[N+\frac{1}{36}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right]\tau). \tag{6}$$

And we set

$$\begin{aligned}\Theta_1(\tau) &= \theta_{\{\binom{0}{0},\binom{0}{0}\},1} = \sum_{a,b\in\mathbb{Z}/6\mathbb{Z}} \Theta\begin{bmatrix}6a & 6b\\ 18a & 18b\end{bmatrix}(\tau);\\ \Theta_2(\tau) &= \theta_{\{\binom{0}{0},\binom{6}{0}\},1} = \sum_{a,b\in\mathbb{Z}/6\mathbb{Z}} \Theta\begin{bmatrix}6a & 6b\\ 18a-12 & 18b\end{bmatrix}(\tau);\\ \Theta_3(\tau) &= \theta_{\{\binom{0}{0},\binom{0}{6}\},1} = \sum_{a,b\in\mathbb{Z}/6\mathbb{Z}} \Theta\begin{bmatrix}6a & 6b\\ 18a & 18b-12\end{bmatrix}(\tau);\\ \Theta_4(\tau) &= \theta_{\{\binom{0}{0},\binom{6}{6}\},1} = \sum_{a,b\in\mathbb{Z}/6\mathbb{Z}} \Theta\begin{bmatrix}6a & 6b\\ 18a-12 & 18b-12\end{bmatrix}(\tau);\\ \Theta_5(\tau) &= \theta_{\{\binom{0}{0},\binom{6}{12}\},1} = \sum_{a,b\in\mathbb{Z}/6\mathbb{Z}} \Theta\begin{bmatrix}6a & 6b\\ 18a-12 & 18b+12\end{bmatrix}(\tau).\end{aligned}$$

We shall write:

$$\begin{aligned}P_1(\mathbf{X}) &= X_{0,0}^3+X_{0,6}^3+X_{0,12}^3+X_{6,0}^3+X_{6,6}^3+X_{6,12}^3+X_{12,0}^3+X_{12,6}^3+X_{12,12}^3;\\ P_2(\mathbf{X}) &= 3(X_{0,0}X_{6,0}X_{12,0}+X_{0,6}X_{6,6}X_{12,6}+X_{0,12}X_{6,12}X_{12,12});\\ P_3(\mathbf{X}) &= 3(X_{0,0}X_{0,6}X_{0,12}+X_{6,0}X_{6,6}X_{6,12}+X_{12,0}X_{12,6}X_{12,12});\\ P_4(\mathbf{X}) &= 3(X_{0,0}X_{6,6}X_{12,12}+X_{0,6}X_{6,12}X_{12,0}+X_{0,12}X_{12,6}X_{6,0});\\ P_5(\mathbf{X}) &= 3(X_{0,0}X_{6,12}X_{12,6}+X_{6,6}X_{0,12}X_{12,0}+X_{12,12}X_{0,6}X_{6,0}).\end{aligned}$$

Then we have 4 independent equations: $\Theta_i P_1(\mathbf{X}) = \Theta_1 P_i(\mathbf{X})$, $(2 \le i \le 5)$.

The aim of this paper is to show that $\Theta_i$ belongs to the space of Siegel modular forms $M_1(\Gamma^2(3))$ of weight 1 and level 3, for each $i$. We use the following fact ([G, Lemma 5.2]).

**Lemma 3.1.** *We define 5 functions:*

$$t_j(\tau) = \sum_{N \in M_2(\mathbb{Z})} \exp \pi i (\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} [N + \frac{1}{9} T_j] \tau),$$

$$T_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

*Then $t_1, \dots, t_5$ form a basis of $M_1(\Gamma^2(3))$. The action of $Sp(2, \mathbb{Z})$ on $M_1(\Gamma^2(3))$ is irreducible.*

Our main theorem is as follows.

**Theorem 3.2.** *We have $\Theta_i = t_i$ for each $i$ $(1 \leq i \leq 5)$. In particular $\Theta_i$ belongs to $M_1(\Gamma^2(3))$.*

*Proof* We investigate the action of $\Gamma = Sp(2, \mathbb{Z})$ to these functions. First we consider the element $\gamma(S) = \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}$, ${}^tS = S \in M_2(\mathbb{Z})$. By Proposition 2.1, for $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$ we have

$$\begin{aligned} \Theta_1|\gamma(S) &= \Theta_1, & t_1|\gamma(S) &= t_1, \\ \Theta_2|\gamma(S) &= \omega^{s_1}\Theta_2, & t_2|\gamma(S) &= \omega^{s_1} t_2, \\ \Theta_3|\gamma(S) &= \omega^{s_3}\Theta_3, & t_3|\gamma(S) &= \omega^{s_3} t_3, \\ \Theta_4|\gamma(S) &= \omega^{s_1 - s_2 + s_3}\Theta_4, & t_4|\gamma(S) &= \omega^{s_1 - s_2 + s_3} t_4, \\ \Theta_5|\gamma(S) &= \omega^{s_1 + s_2 + s_3}\Theta_5, & t_5|\gamma(S) &= \omega^{s_1 + s_2 + s_3} t_5. \end{aligned}$$

In particular $\Theta_1, \dots, \Theta_5$ are linearly independent, since any of $\Theta_i$ is non-zero because of Fourier expansion.

Next we consider the action of the element $J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$. By Proposition 2.1, we can show

$$\begin{pmatrix} t_1|J \\ t_2|J \\ t_3|J \\ t_4|J \\ t_5|J \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & -1 & 2 & -1 & -1 \\ 1 & 2 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 2 \\ 1 & -1 & -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}.$$

On the other hand, we have

$$\begin{aligned} \Theta_0|J &= \det \begin{pmatrix} 18 & 0 \\ 0 & 6 \end{pmatrix}^{-1} \det(i1_2) \sum_{a,b \in \mathbb{Z}/6\mathbb{Z}} \sum_{\substack{0 \leq s,u \leq 17 \\ 0 \leq t,v \leq 5}} \exp 2\pi i \operatorname{Tr}(\frac{1}{6^4} \begin{pmatrix} 6a & 18a \\ 6b & 18b \end{pmatrix} \begin{pmatrix} 18 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 2s & 2u \\ 6t & 6v \end{pmatrix}) \Theta \begin{bmatrix} 2s & 2u \\ 6t & 6v \end{bmatrix} \\ &= -\frac{1}{3 \cdot 6^2} \sum_{\substack{s,u \\ t,v}} \sum_{a,b} \exp 2\pi i (\frac{(s+3t)a + (u+3v)b}{6}) \Theta \begin{bmatrix} 2s & 2u \\ 6t & 6v \end{bmatrix} \\ &= -\frac{1}{3} \sum_{\substack{s+3t \equiv 0 \bmod 6 \\ u+3v \equiv 0 \bmod 6}} \Theta \begin{bmatrix} 2s & 2u \\ 6t & 6v \end{bmatrix}. \end{aligned}$$

Now all the vectors ${}^t(2s, 6t) \bmod 36$ for $0 \leq s \leq 17$, $0 \leq t \leq 5$ such that $s + 3t \equiv 0 \bmod 6$ are given by follows.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ 18 \end{pmatrix}, \quad \begin{pmatrix} 12 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 18 \\ 18 \end{pmatrix}, \quad \begin{pmatrix} 24 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 30 \\ 18 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 24 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 12 \\ 24 \end{pmatrix}, \quad \begin{pmatrix} 18 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 24 \\ 24 \end{pmatrix}, \quad \begin{pmatrix} 30 \\ 6 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 12 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ 30 \end{pmatrix}, \quad \begin{pmatrix} 12 \\ 12 \end{pmatrix}, \quad \begin{pmatrix} 18 \\ 30 \end{pmatrix}, \quad \begin{pmatrix} 24 \\ 12 \end{pmatrix}, \quad \begin{pmatrix} 30 \\ 30 \end{pmatrix}.$$

One can see that the first, second and the third row of this list is equal to the vectors ${}^t(6a, 18a)$, ${}^t(6a, 18a-12)$, and ${}^t(6a, 18a+12)$ modulo 36 ($0 \le a \le 5$) respectively. Moreover we have

$$\Theta \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \Theta \begin{bmatrix} -x & -y \\ z & w \end{bmatrix} = \Theta \begin{bmatrix} x & y \\ -z & -w \end{bmatrix}$$

by definition. In particular

$$\Theta \begin{bmatrix} 6a & 6b \\ 18a & 18b-12 \end{bmatrix} = \Theta \begin{bmatrix} 6a & 6b \\ -18a & -18b+12 \end{bmatrix} = \Theta \begin{bmatrix} 6a & 6b \\ 18a & 18b+12 \end{bmatrix}.$$

Using this, we have

$$\Theta_1|J = -\frac{1}{3}(\Theta_1 + 2\Theta_2 + 2\Theta_3 + 2\Theta_4 + 2\Theta_5).$$

Next we consider $\Theta_2|J$. A By the similar calculation, we have

$$\begin{aligned}\Theta_2|J &= -\frac{1}{3} \sum_{\substack{s+3t\equiv 0 \bmod 6 \\ u+3v\equiv 0 \bmod 6}} e^{-2\pi i t/3} \Theta \begin{bmatrix} 2s & 2u \\ 6t & 6v \end{bmatrix} \\ &= -\frac{1}{3}(\Theta_1 + \Theta_3 + \Theta_3 + \omega^2\Theta_2 + \omega^2\Theta_4 + \omega^2\Theta_5 + \omega\Theta_2 + \omega\Theta_5 + \omega\Theta_4) \\ &= -\frac{1}{3}(\Theta_1 - \Theta_2 + 2\Theta_3 - \Theta_4 - \Theta_5).\end{aligned}$$

Similarly we have

$$\begin{pmatrix} \Theta_1|J \\ \Theta_2|J \\ \Theta_3|J \\ \Theta_4|J \\ \Theta_5|J \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & -1 & 2 & -1 & -1 \\ 1 & 2 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 2 \\ 1 & -1 & -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \\ \Theta_5 \end{pmatrix}.$$

Now we consider the natural immersion $M_1(\Gamma^2(3)) \to M_1(\Gamma^2(36))$. We write the image of this map $V$. Since $\Gamma = Sp(2, \mathbb{Z})$ is generated by $\gamma(S)$ and $J$ (cf. [Kl, Proposition 6, §3]), the above calculation shows that the subspace $W \subset M_1(\Gamma^2(36))$ spanned by $\Theta_1, \dots, \Theta_5$ is closed under the action of $\Gamma$. We claim $V = W$. Indeed we see that $V$ and $W$ are isomorphic as $\Gamma$-modules, thus all elements of $W$, in particular $\Theta_1, \dots, \Theta_5$, are invariant under the action of $\Gamma^2(3)$; this means $V = W$. Moreover since the action of $\Gamma$ on $V$ is irreducible (Lemma 3.1), we have $t_i = a\Theta_i$, $a \neq 0 \in \mathbb{C}$ for each $i$, by Schur's lemma. We see $a = 1$ by comparing the Fourier coefficients. This completes the proof of the theorem. □

## 4. List of defining equations of degree 3

Let $A$ be an abelian surface and $L_0$ be a principal ample line bundle. For $L = L_0^3$, since $\dim \operatorname{Sym}^3 H^0(A, L) = 165$ and $\dim H^0(A, L^3) = 81$, the dimension of the kernel of the natural map $\operatorname{Sym}^3 H^0(A, L) \to H^0(A, L^3)$ is 84. By Theorem 1.1, the kernel of degree 3 is essentially gives all the defining equations. We use the same notation of §3.

Let $W_3 = \{0, 3, 6\}$, and $\widehat{Z}_6^+$ be the set of all the character $\rho$ of $Z_6$ such that $\rho^2 \equiv 1$, that is, all the character of $W_3^2 \bmod 9 \cong (\mathbb{Z}/3\mathbb{Z})^2$. We define the character $\rho_1, \dots, \rho_4 \in \widehat{Z}_6^+$ by

$$\begin{cases} \rho_1\left(\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right) = 1, \\ \rho_1\left(\begin{smallmatrix} 0 \\ 3 \end{smallmatrix}\right) = \omega. \end{cases} \quad \begin{cases} \rho_2\left(\begin{smallmatrix} 0 \\ 3 \end{smallmatrix}\right) = 1, \\ \rho_2\left(\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right) = \omega. \end{cases} \quad \begin{cases} \rho_3\left(\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}\right) = 1, \\ \rho_3\left(\begin{smallmatrix} 0 \\ 3 \end{smallmatrix}\right) = \omega. \end{cases} \quad \begin{cases} \rho_4\left(\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right) = 1, \\ \rho_4\left(\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right) = \omega. \end{cases}$$

For $\rho \in \widehat{Z}_6^+$, we define

$$\theta^\rho \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \sum_{a,b \in \mathbb{Z}/6\mathbb{Z}} \rho \begin{pmatrix} 3a \\ 3b \end{pmatrix} \Theta \begin{bmatrix} 6a-2x & 6b-2z \\ 6a-2y & 6b-2w \end{bmatrix},$$

with $\Theta$ given in (6).

**Theorem 4.1.** *The following list contains all of the* 84 *linearly independent relations of degree* 3.

$$\begin{aligned}\sum_{(a,b)\in K(L)_1} X_{a,b}^3 &= 3\ \frac{\theta^1\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^1\left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right)}(X_{0,0}X_{6,0}X_{12,0}+X_{0,6}X_{6,6}X_{12,6}+X_{0,12}X_{6,12}X_{12,12})\\ &= 3\ \frac{\theta^1\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^1\left(\begin{smallmatrix}0&0\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{0,6}X_{0,12}+X_{6,0}X_{6,6}X_{6,12}+X_{12,0}X_{12,6}X_{12,12})\\ &= 3\ \frac{\theta^1\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^1\left(\begin{smallmatrix}0&6\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{6,6}X_{12,12}+X_{0,6}X_{6,12}X_{12,0}+X_{0,12}X_{12,6}X_{6,0})\\ &= 3\ \frac{\theta^1\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^1\left(\begin{smallmatrix}0&6\\0&12\end{smallmatrix}\right)}(X_{0,0}X_{6,12}X_{12,6}+X_{6,6}X_{0,12}X_{12,0}+X_{12,12}X_{0,6}X_{6,0})\end{aligned}$$

$$X_{0,0}^3+X_{6,0}^3+X_{12,0}^3-X_{0,6}^3-X_{6,6}^3-X_{12,6}^3 = 3\ \frac{\theta^{\rho_1}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_1}\left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right)}(X_{0,0}X_{6,0}X_{12,0}-X_{0,6}X_{6,6}X_{12,6})$$

$$X_{0,0}^3+X_{6,0}^3+X_{12,0}^3-X_{0,12}^3-X_{6,12}^3-X_{12,12}^3 = 3\ \frac{\theta^{\rho_1}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_1}\left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right)}(X_{0,0}X_{6,0}X_{12,0}-X_{0,12}X_{6,12}X_{12,12})$$

$$X_{0,0}^3+X_{0,6}^3+X_{0,12}^3-X_{6,0}^3-X_{6,6}^3-X_{6,12}^3 = 3\ \frac{\theta^{\rho_2}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_2}\left(\begin{smallmatrix}0&0\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{0,6}X_{0,12}-X_{6,0}X_{6,6}X_{6,12})$$

$$X_{0,0}^3+X_{0,6}^3+X_{0,12}^3-X_{12,0}^3-X_{12,6}^3-X_{12,12}^3 = 3\ \frac{\theta^{\rho_2}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_2}\left(\begin{smallmatrix}0&0\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{0,6}X_{0,12}-X_{12,0}X_{12,6}X_{12,12})$$

$$X_{0,0}^3+X_{6,6}^3+X_{12,12}^3-X_{0,6}^3-X_{6,12}^3-X_{12,0}^3 = 3\ \frac{\theta^{\rho_3}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_3}\left(\begin{smallmatrix}0&6\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{6,6}X_{12,12}-X_{0,6}X_{6,12}X_{12,0})$$

$$X_{0,0}^3+X_{6,6}^3+X_{12,12}^3-X_{0,12}^3-X_{6,0}^3-X_{12,6}^3 = 3\ \frac{\theta^{\rho_3}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_3}\left(\begin{smallmatrix}0&6\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{6,6}X_{12,12}-X_{0,12}X_{6,0}X_{12,6})$$

$$X_{0,0}^3+X_{6,12}^3+X_{12,6}^3-X_{6,0}^3-X_{12,12}^3-X_{0,6}^3 = 3\ \frac{\theta^{\rho_4}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_4}\left(\begin{smallmatrix}0&6\\0&12\end{smallmatrix}\right)}(X_{0,0}X_{6,12}X_{12,6}-X_{6,0}X_{12,12}X_{0,6})$$

$$X_{0,0}^3+X_{6,12}^3+X_{12,6}^3-X_{12,0}^3-X_{0,12}^3-X_{6,6}^3 = 3\ \frac{\theta^{\rho_4}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_4}\left(\begin{smallmatrix}0&6\\0&12\end{smallmatrix}\right)}(X_{0,0}X_{6,12}X_{12,6}-X_{12,0}X_{0,12}X_{6,6})$$

$$\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{2a,6+2b}^2X_{2a,12+2b} = \frac{\theta^{\rho}\left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right)}{\theta^{\rho}\left(\begin{smallmatrix}0&6\\4&0\end{smallmatrix}\right)}(\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,6+2b}X_{12+2a,6+2b}X_{2a,12+2b})$$

$$\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,2b}^2X_{12+2a,2b} = \frac{\theta^{\rho}\left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho}\left(\begin{smallmatrix}4&0\\0&6\end{smallmatrix}\right)}(\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,6+2b}X_{6+2a,12+2b}X_{12+2a,2b})$$

$$\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{2a,6+2b}^2X_{2a,2b} = \frac{\theta^{\rho}\left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right)}{\theta^{\rho}\left(\begin{smallmatrix}0&6\\2&0\end{smallmatrix}\right)}(\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,6+2b}X_{12+2a,6+2b}X_{2a,2b})$$

$$\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,2b}^2X_{2a,2b}) = \frac{\theta^{\rho}\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho}\left(\begin{smallmatrix}2&0\\0&6\end{smallmatrix}\right)}(\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,6+2b}X_{6+2a,12+2b}X_{2a,2b})$$

$$\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,6+2b}^2X_{12+2a,12+2b} = \frac{\theta^{\rho}\left(\begin{smallmatrix}4&0\\4&0\end{smallmatrix}\right)}{\theta^{\rho}\left(\begin{smallmatrix}4&0\\4&6\end{smallmatrix}\right)}(\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,12+2b}X_{6+2a,2b}X_{12+2a,12+2b})$$

$$\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,6+2b}^2X_{12+2a,2b} = \frac{\theta^{\rho}\left(\begin{smallmatrix}4&0\\2&0\end{smallmatrix}\right)}{\theta^{\rho}\left(\begin{smallmatrix}4&0\\2&6\end{smallmatrix}\right)}(\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,12+2b}X_{6+2a,2b}X_{12+2a,2b})$$

$$\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{6+2a,6+2b}^2X_{2a,12+2b} = \frac{\theta^{\rho}\left(\begin{smallmatrix}2&0\\4&0\end{smallmatrix}\right)}{\theta^{\rho}\left(\begin{smallmatrix}2&6\\4&0\end{smallmatrix}\right)}(\sum_{a,b\in W_3}\rho\begin{pmatrix}a\\b\end{pmatrix}X_{12+2a,6+2b}X_{2a,6+2b}X_{2a,12+2b})$$

$$\sum_{a,b\in W_3} \rho\begin{pmatrix} a \\ b \end{pmatrix} X^2_{6+2a,6+2b}X_{2a,2b} = \frac{\theta^\rho\left(\begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix}\right)}{\theta^\rho\left(\begin{smallmatrix} 2 & 6 \\ 2 & 0 \end{smallmatrix}\right)}\Big(\sum_{a,b\in W_3} \rho\begin{pmatrix} a \\ b \end{pmatrix} X_{6+2a,12+2b}X_{6+2a,6+2b}X_{2a,2b}\Big)$$

*Here in the last 8 equations, $\rho$ runs all the characters of $\widehat{Z}_6^+$.*

*Proof* The generators of the kernel of the map $\mathrm{Sym}^3 H^0(X,L) \to H^0(X,L^3)$ have the form

$$\theta_{(y_1',y_2'),\rho}(0)P_{(y_1,y_2,y_3),\rho}(\mathbf{X}) - \theta_{(y_1,y_2),\rho}(0)P_{(y_1,y_2,y_3),\rho}(\mathbf{X}).$$

There are 3 types of generators as:

$$\begin{cases} \text{type I} & P_{(y_1,y_2,y_3),\rho}(\mathbf{X}) = 0 \ \text{ or } \ P_{(y_1',y_2',y_3'),\rho}(\mathbf{X}) = 0; \\ \text{type II} & P_{(y_1,y_2,y_3),\rho}(\mathbf{X}) = \alpha P_{(y_1',y_2',y_3'),\rho}(\mathbf{X}), \quad \alpha \in \mathbb{C}^\times; \\ \text{type III} & \text{otherwise.} \end{cases}$$

**Claim:** All the generators of type I and type II are identically zero, that is, $\theta_{(y_1,y_2),\rho} = 0$ (resp. $\theta_{(y_1,y_2),\rho} = \alpha\theta_{(y_1',y_2'),\rho}$) for the coefficient of generators of type I (resp. type II).

We shall prove the claim. There are 84 elements of type III as in the above list, and one can show that any of the coefficients $\theta_{(y_1,y_2),\rho}$ is not identically zero as functions in $\tau$. Let $U$ be the open set in $\mathbb{H}_2$ on which all the coefficients of these elements do not vanish. Then, for all $\tau \in U$, these elements of type III are linearly independent, and the number of the linearly independent generators of the kernel is exactly 84. Hence, all the generators of type I and II, which are scalar multiple of single $P_{(y_1,y_2,y_3),\rho}$, must be zero for $\tau \in U$. But $\theta$ are holomorphic functions, these equalities hold for all $\tau \in \mathbb{H}_2$, and we prove the claim.

This shows that all the generators are type III, and we complete the proof of the theorem. □

**Open problem** Among these cubic relations, one might be interested in finding the relations derived from quadratic relations in $\mathrm{Sym}^2 H^0(A,L)$. The author tried their problem but did not succeed.

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