UTMS 2003–46 December 15, 2003

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by

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December 15, 2003

Abstract

We study the global stability in determination of a coefficient in an acoustic equation from data of the solution in a subboundary over a time interval. Providing regular initial data, without any assumption on an observation subboundary, we prove the logarithmic stability estimate in the inverse problem with a single measurement. Moreover the exponent in the stability estimate depends on the regularity of initial data.

1 Introduction

In this paper, we discuss the uniqueness and stability in determining a coefficient in an acoustic equation from data of the solution on a subboundary over a time interval. We will formulate our problem as follows: In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \le 3$, with sufficiently smooth boundary $\Gamma = \partial\Omega$, we consider an acoustic equation

$$\begin{cases} \partial_t^2 u(x,t) - \mathrm{div}(a(x)\nabla u(x,t)) = 0 & \text{in } \ Q \equiv \Omega \times [-T,T] \\ u(x,0) = \Phi_0(x), \quad \partial_t u(x,0) = \Phi_1(x) & \text{in } \ \Omega \\ \partial_\nu u(x,t) = 0 & \text{on } \ \Sigma \equiv \Gamma \times [-T,T]. \end{cases} \tag{1.1}$$

Here $\nu = \nu(x)$ denotes the unit outward normal vector and we set $\partial_\nu u = \nabla u \cdot \nu$. We denote the solution to (1.1) by u_a. The unknown coefficient $a \in C^2(\overline{\Omega})$ is assumed to be a positive function $a(x) > 0$ for all $x \in \overline{\Omega}$.
Let $\Gamma_1 \subset \Gamma$ be given arbitrarily. A question of our inverse problem is whether or not we can conclude $a(x) = b(x)$, $x \in \Omega$, by

$$u_a(x,t) = u_b(x,t); \qquad (x,t) \in \Sigma_1 \equiv \Gamma_1 \times]-T, T[. \tag{1.2}$$

Throughout this paper, we assume that unknown a and b coincide near the boundary Γ.

From the physical viewpoint, our inverse problem is the determination of the bulk modulus $a(x)$ in acoustic equation (1.1) which is considered in a nonhomogeneous medium.

In [12], Imanuvilov and Yamamoto consider an inverse problem concerning the determination of the coefficient $a(x)$, $x \in \Omega$ from data $u_{|\omega_0 \times [0,T]}$, where $\omega_0 \subset \Omega$ is a subdomain. More precisely, in the case where ω_0 satisfies the geometric condition: $\partial\omega_0 \supset \{x \in \Gamma;\, (x - x_0)\cdot\nu \geq 0\}$ with some $x_0 \notin \overline{\Omega}$, an L^2-estimate of Hölder type was proved, provided that a, b satisfy a priori uniform boundedness condition, compatible conditions and some positivity condition. The key is a Carleman estimate for a hyperbolic operator in an H^{-1}-space.
As a result of this geometric condition, $\omega_0 \subset \Omega$ cannot be an arbitrary subdomain. For example, in the case of $\Omega = \{x; |x| < R\}$, the geometric condition requires that ω_0 should be a neighbourhood of a subboundary which is larger than the half of Γ. The geometric condition is also a sufficient condition for an observability inequality by observations in $\omega_0 \times]0, T[$ (see [2]).

Our inverse problem is formulated with a single measurement. The main methodology is based on an L^2-weighted inequality called a Carleman estimate, and was introduced by Bukhgeim and Klibanov [5]. Furthermore, as for applications of Carleman estimates to inverse problems, we can refer to Bellassoued [3], Bukhgeim [4], Bukhgeim, Cheng, Isakov and Yamamoto [6], Imanuvilov and Yamamoto [9] - [12], Isakov [13] - [15], Isakov and Yamamoto [16], Khaĭdarov [18], [19], Klibanov [20], [21], Klibanov and Timonov [22], Kubo [23], Puel and Yamamoto [29], Yamamoto [35]. Most of those papers treat the determination of the coefficient $p(x)$ in the zeroth order term of a hyperbolic equation $\partial_t^2 u(t,x) - \Delta u(t,x) + p(x)u(t,x) = 0$. As for observability inequalities, by means of a Carleman estimate and a similar type of estimates, see Kazemi and Klibanov [17], Lasiecka, Triggiani and Yao [24].

Except for the one-dimensional spatial case and [12], the argument in the above papers requires us to suitably change initial values $(n+1)$-times because an unknown coefficient a appears in the divergence form, and a, $\partial_i a$, $1 \leq i \leq n$, are regarded as independent unknown functions. For such an inverse hyperbolic problem of determining multiple functions by the corresponding number of measurements, we refer to [13], [19]. Note that the machinery used in [13] and [19], cannot take advantage of the dependence of $(n+1)$ unknown functions a, $\partial_1 a$,...,$\partial_n a$, so that they are treated as $(n+1)$ independent unknowns. As a consequence, such an approach requires several measurements. On the other hand, in the case of $n = 1$, a change $v = a\partial_x u$ of variables reduces (1.1) to a hyperbolic equation of the form $\partial_t^2 v - a\partial_x^2 v = 0$, so that the existing results imply stability in the inverse hyperbolic problem with a single measurement.

Our main result is the stability in the inverse problem, and the main achievements of this paper are

- a single measurement in determining a single coefficient of the principal term
- arbitrariness of the observation subboundary Γ_1
- the improvement of the exponent in the stability estimates for our inverse problem.

Our key idea is a combination of the Carleman estimate proved in [3] and the Fourier-Bros-Iagolnitzer (FBI) transformation introduced by Robbiano [30], [31]. We use the idea of [30], [31] to apply the Fourier-Bros-Iagolnitzer transformation and change the problem near the boundary where we can apply an elliptic Carleman estimate.

1.1 Notations and preliminary definition

To formulate our results, we need to introduce some notations. First of all, without loss of generality, we may assume that $0 \notin \overline{\Omega}$.
Let

$$\mathcal{D} = \sup_{x\in\Omega} |x| . \tag{1.3}$$

Let $\omega \subset \Omega$ be a given arbitrary neighbourhood of the boundary Γ and $\eta = \eta(x)$ a smooth function in ω.

Throughout this paper, let us consider the admissible set $\Lambda = \Lambda(M, k, \omega, \eta, \theta_0, \theta_1)$ of unknown coefficients a, b:

$$\Lambda = \left\{ a \in C^{k+2}(\overline{\Omega});\ \|a\|_{C^{k+2}(\overline{\Omega})} \le M;\ a = \eta \text{ in } \ \omega,\ \ a(x) > \theta_1,\ \ \frac{|\nabla a(x)|}{a(x)} < \frac{\theta_0}{2\mathcal{D}},\ x \in \overline{\Omega} \right\} \tag{1.4}$$

where $k \in \mathbb{N} \cup \{0\}$, the constants $M > 0$, $0 < \theta_0 < 1$ and $\theta_1 > 0$ are given.
Let us take the product space $\mathcal{H}^k(\Omega) = H^{k+2}(\Omega) \oplus H^{k+1}(\Omega)$ as the state space of our system. The norm in $\mathcal{H}^k(\Omega)$ is chosen as follows:

$$\|(\Phi_0, \Phi_1)\|^2_{\mathcal{H}^k(\Omega)} = \|\Phi_0\|^2_{H^{k+2}(\Omega)} + \|\Phi_1\|^2_{H^{k+1}(\Omega)} \quad \text{for any} \quad (\Phi_0, \Phi_1) \in \mathcal{H}^k(\Omega). \tag{1.5}$$

Furthermore we assume that the observation data are measured by the norm:

$$\epsilon(\Sigma_1) = \sum_{j=2}^{3} \left\| \partial_t^j (u_a - u_b) \right\|^2_{H^1(\Sigma_1)} . \tag{1.6}$$

Definition 1 *For* (Φ_0, Φ_1), *we define* Φ_p $(p = 2, 3, ...)$ *inductively by*

$$\Phi_p(x) = div(a(x)\nabla\Phi_{p-2}(x)). \tag{1.7}$$

Then we say that the data (Φ_0, Φ_1) *satisfy the k-th order compatibility conditions with respect to a if*

$$(\Phi_0, \Phi_1) \in \mathcal{H}^k(\Omega) \tag{1.8}$$

and

$$\partial_\nu \Phi_p = 0 \quad in\ \Gamma, \quad p = 0, ..., k. \tag{1.9}$$

We remark that if $a \in \Lambda$ and (Φ_0, Φ_1) satisfies the k-th order compatibilty conditions with respect to a, then (Φ_0, Φ_1) satisfies also the k-th order compatibilty conditions with respect to all $b \in \Lambda$, because we have $a(x) = b(x)$ near the boundary Γ by the definition of Λ.

Finally let the Sobolev spaces $W^{m,p}(\Omega)$ be defined for $p \geq 1$ and an integer $m \geq 0$ by

$$W^{m,p}(\Omega) = \{u;\ u \in L^p(\Omega),\ \partial^\alpha u \in L^p(\Omega), \text{for } \ |\alpha| \leq m\}. \tag{1.10}$$

1.2 Statement of Main Results

Before stating the main results, we recall the following lemma on the unique existence of a strong solution to problem (1.1), which we shall use repeatedly in the sequel. The proof is based on [28], for example. We can also refer to [8].

Lemma 1.1 *Let* (Φ_0, Φ_1) *satisfy the k-th order compatibility conditions and let* $a \in \Lambda$. *Then there exists a unique solution* $u = u_a$ *to (1.1) whithin the following class:*

$$u \in \bigcap_{j=0}^{k+2} C^{k+2-j}(-T, T; H^j(\Omega)). \tag{1.11}$$

Moreover there exists a positive constant C(M) such that

$$\sum_{j=0}^{k+2} \|u_a\|_{C^{k+2-j}(-T,T;H^j(\Omega))} \leq C(M)\, \|(\Phi_0, \Phi_1)\|_{\mathcal{H}^k(\Omega)} \tag{1.12}$$

The main results of this paper can be stated as follows:

Theorem 1 *(H^1-stability) Let* $T > 0$ *be sufficiently large for* Ω, ω, η, T *and the constants* M, θ_0, θ_1 *in definition (1.4) of* Λ *and let* $k \in \mathbb{N}$ *satisfy* $k \geq 5$. *Moreover let* (Φ_0, Φ_1) *satisfy the k-th order compatibility conditions and*

$$\nabla\Phi_0(x) \cdot x \neq 0, \qquad for\ all\ \ x \in \overline{\Omega}. \tag{1.13}$$

Then there exist constants $C_k > 0$ *and* $\mu \in]0, 1[$ *such that the following estimate holds:*

$$\|\nabla(a - b)\|^2_{L^2(\Omega)} \leq C_k \left[\log\left(2 + \frac{C_k}{\epsilon(\Sigma_1)}\right)\right]^{-\mu(k-2)}, \tag{1.14}$$

for all $a, b \in \Lambda$.
Here we note that $\epsilon(\Sigma_1)$ *is given by (1.6), and the constants* C_k *and* $\mu \in]0,1[$ *are dependent on* k, Ω, ω, T, M, *and independent of* $a, b \in \Lambda$.

Theorem 2 *(L^2-stability) Let* $T > 0$ *be sufficiently large for* Ω, ω, η, T, *and the constants in (1.4), and let* $k \geq 4$. *Moreover let* (Φ_0, Φ_1) *satisfy the* k*-th order compatibility conditions and (1.13). Then there exist constants* $C_k > 0$ *and* $\mu \in]0,1[$ *such that the following estimate holds:*

$$\|a - b\|^2_{L^2(\Omega)} \leq C_k \left[\log \left(2 + \frac{C_k}{\epsilon(\Sigma_1)} \right) \right]^{-\mu(k-2)}, \qquad (1.15)$$

for all $a, b \in \Lambda$.

By Theorem 2, we can readily derive the uniqueness in the inverse problem:

Corollary 1.1 *(Uniqueness) Under the assumptions in Theorem 2, for all* $a, b \in \Lambda$, *we have the uniqueness:*

$$u_a(x,t) = u_b(x,t), \quad (x,t) \in \Sigma_1 \quad \textit{imply} \quad a(x) = b(x) \quad \textit{for all} \quad x \in \Omega. \qquad (1.16)$$

1.3 Comments on the existing papers

1. Thanks to the extra information $a = b$ in a neighbourhood ω of $\partial\Omega$, the sharp unique continuation by Robbiano [30], Robbiano and Zuily [32], Tataru [33], implies $u_a = u_b$ and $\nabla u_a = \nabla u_b$ on $\partial(\Omega \setminus \overline{\omega}) \times (-T, T)$, provided that $T > 0$ is sufficiently large. Therefore the method in Imanuvilov and Yamamoto [12] directly yields the uniqueness in our inverse problem. However our main result is concerned with the stability in the inverse problem, and the direct combination of the existing results in [12] and [30], [33] does not work. For our purpose, we will use the Fourier-Bros-Iagolnitzer transformation according to Robbiano [31], rather than [33].

2. The techniques developed in this paper may be applied, with appropriate modifications, to more complex inverse hyperbolic problems (e.g., identification of multiple coefficients of terms of higher order in a hyperbolic equation).

3. In [23], Kubo gives some Carleman estimates including boundary values, so that he shows the unique continuation across a lateral boundary for hyperbolic equations and the uniqueness in hyperbolic inverse problems by the above unique continuation. It is remarked in [23] that the uniqueness can be proved for observation on a more general subboundary part $\Gamma_1 \subset \Gamma$ and the characterization for such a subboundary Γ_1 is related to the uniform Lapatinskii condition.

4. Here we do not need to discuss the uniform Lopatinskii condition (see [34]) and to study Carleman estimates with a reduced number of boundary traces, because in the formulation of our inverse problem, we have extra information near the whole boundary, that is, $a(x) = b(x)$ near Γ.

5. Since Bukhgeim and Klibanov [5], the uniqueness in the inverse problems has been studied by the Carleman estimate (e.g., [4], [6], [13] - [15], [18], [20], [23]). As the existing papers concerning the stability, see [9] - [12], [16], [19], [29], [35]. In particular, Imanuvilov and Yamamoto [11] proved a global and both-sided Lipschitz stability estimate in the determination of the coefficient $p(x)$ of the zeroth order term of a hyperbolic equation $(\partial_t^2 - \Delta + p)u = 0$ with the lateral Neumann data $\partial_\nu u = 0$ and initial data $u_0 > 0$ on $\overline{\Omega}$.

6. This paper employs a new Carleman estimate. A technical advantage of the new Carleman estimate is that it holds in the whole cylindrical domain Q (note that the classical one holds in level sets bounded by the weight function). As for general treatments of Carleman estimates, see Hörmander [7], Isakov [14], Tataru [34]. In Lavrent'ev, Romanov and Shishat·skiĭ[25], Carleman estimates were derived by a direct pointwise manner.

7. We further have to assume $|(\nabla\Phi_0(x)\cdot x)| > 0$ in a subset of Ω where one wants to determine $a(x)$. We do not know the uniqueness, in general, even in the case where $\{x \in \Omega\backslash\omega;\ (\nabla\Phi_0(x)\cdot x) = 0\}$ is a set of zero Lebesgue measure. This non-degeneracy condition is very restrictive in many cases, but the relaxation of the non-degeneracy condition of Φ_0 is an open problem.

The remainder of the paper is organized as follows. In Section 2, we give some estimates which are used for the proof of the main results. In Section 3, we prove Theorems 1 and 2 on the basis of the weak observation estimate. Section 4 is devoted to the proof of the weak observation estimate.

2 Preliminary Estimates

In this section we first derive several preliminary estimates. We shall use the following notations. We choose $\varrho, \varrho_1, \varrho_2 > 0$ such that

$$\omega(8\varrho) \equiv \{x \in \Omega,\ \mathrm{dist}(x, \Gamma) \le 8\varrho\} \subset \omega, \tag{2.1}$$

and

$$\omega(\varrho_1, \varrho_2) \equiv \{x \in \Omega,\ \varrho_1 \le \mathrm{dist}(x, \Gamma) \le \varrho_2\} \subset \omega; \quad \varrho_1 < \varrho_2 < 8\varrho. \tag{2.2}$$

We set

$$\omega_T(\varrho) = \omega(\varrho) \times [-T, T], \tag{2.3}$$

$$\omega_T(\varrho_1, \varrho_2) = \omega(\varrho_1, \varrho_2) \times [-T, T]. \tag{2.4}$$

For α such that $0 < \alpha < T$, we set

$$\begin{aligned} &Q_\alpha = \Omega \times [-T+\alpha, T-\alpha] \subset Q \\ &Q_\alpha(\varrho) = \Omega(\varrho) \times [-T+\alpha, T-\alpha]\,; \qquad \Omega(\varrho) = \Omega\backslash\overline{\omega(\varrho)} \end{aligned} \tag{2.5}$$

We shall begin with the first step in our analysis.

2.1 Carleman Estimate

Here we show the Carleman estimate which is the starting point of the proof of the Theorem 1. In order to prove a Carleman estimate, we have to assume a condition called the pseudoconvexity (e.g., [7], [14]) where the coefficient of the principal term is involved. Since such a coefficient is unknown in our inverse problem, we need to establish a Carleman estimate with one possible explicit characterization (1.4) of coefficients for the pseudoconvexity, and we will argue similarly to Bellassoued [3]. Moreover for our stability estimates, unlike [7], [14], we require a Carleman estimate for functions which have not compact supports.

For formulating our Carleman estimate, we introduce the function $\psi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ of class C^1 by setting

$$\psi(x,t) = |x|^2 - \gamma_0 |t|^2 \quad \text{for all } x \in \Omega, \quad -T \leq t \leq T, \tag{2.6}$$

where $T > 0$ and $0 < \gamma_0 < 1$ are selected as follows. We fix $\delta > 0$ and $\gamma_0 > 0$ such that

$$\gamma_0 T^2 > \max_{x \in \overline{\Omega}} |x|^2 + \delta, \quad 0 < \gamma_0 < (1-\theta_0)\theta_1. \tag{2.7}$$

Therefore, by (2.6) and (2.7), we have the following properties:

$$\psi(x,0) = |x|^2 > 0, \quad \psi(x,-T) = \psi(x,T) < -\delta \text{ for all } x \in \Omega. \tag{2.8}$$

We next introduce a function $\varphi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ by setting

$$\varphi(x,t) = e^{\beta \psi(x,t)}, \quad \beta > 0 \tag{2.9}$$

where $\beta \geq 1$ is a large parameter.
By (2.8), there exist $\alpha \in]0, \frac{T}{2}[$ and $d \in]0,1[$ such that

$$\varphi(x,t) \leq d, \quad \text{for all } (x,t) \in Q \backslash \overline{Q_{2\alpha}}. \tag{2.10}$$

Now we will consider the following second-order hyperbolic operator

$$P(x,D) = \partial_t^2 - \mathrm{div}(a(x)\nabla). \tag{2.11}$$

Finally we introduce the following notation $\nabla_{x,t} v(t,x) = \left(\frac{\partial v}{\partial x_1}, ..., \frac{\partial v}{\partial x_n}, \frac{\partial v}{\partial t} \right) = (\nabla v, \partial_t v)$.
The following Carleman estimate holds:

Proposition 2.1 *Let $T > \frac{\max_{x\in\overline{\Omega}} |x|}{\sqrt{(1-\theta_0)\theta_1}}$. Then we can choose $\beta_* > 0$ satisfying the following property: For any $\beta > \beta_*$, we can choose $\tau_* = \tau_*(\beta) > 0$ such that there exists a constant $C = C(\beta) > 0$, independent of τ, such that for all $\tau \geq \tau_*$, we have*

$$\tau \int_{Q_\alpha(3\varrho)} e^{2\tau\varphi} \left(|\nabla_{x,t} v|^2 + \tau^2 |v|^2 \right) dxdt \leq C \int_Q e^{2\tau\varphi} |P(x,D)v|^2 \, dxdt$$

$$+C\tau\int_{\omega_T(\varrho,3\varrho)} e^{2\tau\varphi}\left(|\nabla_{x,t}v|^2+\tau^2|v|^2\right)dxdt$$
$$+\tau\int_{Q\setminus Q_\alpha} e^{2\tau\varphi}\left(|\nabla_{x,t}v|^2+\tau^2|v|^2\right)dxdt. \tag{2.12}$$

whenever $v\in H^1(Q)$ and the right hand side is finite.

Proof . Inequality (2.12) can be deduced from a more general Theorem 2 in [3] as follows. The Hessian of $\vartheta(x)\equiv|x|^2$ with respect to the metric $g=a(x)^{-1}dx$ is given by

$$\mathbb{D}^2\vartheta(x)(X,X)=2(X\cdot x)\left(\frac{\nabla a(x)\cdot X}{a(x)}\right)+2|X|^2\left(1-\frac{\nabla a(x)\cdot x}{2a(x)}\right),\quad X=\sum_{i=1}^n\xi_i\frac{\partial}{\partial x_i}. \tag{2.13}$$

Since $a\in\Lambda$, we have

$$\begin{aligned}\mathbb{D}^2\vartheta(x)(X,X) &\geq 2|X|^2\left(1-\frac{\theta_0}{2}\right)-2|X|^2\left(\frac{|\nabla a(x)|\,|x|}{2a(x)}\right)\\ &\geq 2(1-\theta_0)|X|^2\geq 2(1-\theta_0)\theta_1|X|_g^2.\end{aligned} \tag{2.14}$$

Therefore we can apply the following Carleman estimate from [3]:

$$\tau\int_{Q_\alpha} e^{2\tau\varphi}\left(|\nabla_{x,t}v|^2+\tau^2|v|^2\right)dxdt\leq C\int_Q e^{2\tau\varphi}|P(x,D)v|^2dxdt$$
$$+C\tau\int_\Sigma e^{2\tau\varphi}\left(|\nabla_{x,t}v|^2+\tau^2|v|^2\right)dxdt$$
$$+\tau\int_{Q\setminus Q_\alpha} e^{2\tau\varphi}\left(|\nabla_{x,t}v|^2+\tau^2|v|^2\right)dxdt. \tag{2.15}$$

We introduce a cut-off function χ satisfying $0\leq\chi\leq1$, $\chi\in C_0^\infty(\mathbb{R}^n)$ and

$$\chi(x)=\begin{cases}0 & x\in\omega(\varrho)\\ 1 & x\in\Omega(3\varrho).\end{cases} \tag{2.16}$$

We apply (2.15) to $\tilde v=\chi v$ and obtain

$$\tau\int_{Q_\alpha} e^{2\tau\varphi}\left(|\nabla_{x,t}\tilde v|^2+\tau^2|\tilde v|^2\right)dxdt\leq C\int_Q e^{2\tau\varphi}|P(x,D)\tilde v|^2dxdt$$
$$+\tau\int_{Q\setminus Q_\alpha} e^{2\tau\varphi}\left(|\nabla_{x,t}\tilde v|^2+\tau^2|\tilde v|^2\right)dxdt. \tag{2.17}$$

Furthermore

$$P(x,D)\tilde v=\chi P(x,D)v+[P,\chi]v,$$

where $[A,B]$ stands for the commutator of operators A and B. Since $[P,\chi]$ is a first order differential operator and is supported in $\omega(\varrho,3\varrho)$, we see (2.12). •

Next we will show the following Carleman estimate for a first order partial differential operator. The function $\varphi(x,t)$ can be written as:

$$\varphi(x,t) = e^{\beta\psi(x,t)} =: \rho(x)\sigma(t),$$

where $\rho(x) \geq 1$ and $\sigma(t) \leq 1$ are defined by

$$\rho(x) = e^{\beta|x|^2} \geq 1, \ \forall x \in \Omega \quad \sigma(t) = e^{-\beta\gamma t^2} \leq 1, \ \forall t \in [-T,T].$$

We consider a first order partial differential equation

$$A(x,D)v = \sum_{i=1}^{n} \gamma_i(x)\partial_i v + \gamma_0(x)v, \quad x \in \Omega \tag{2.18}$$

where

$$\gamma_0 \in C(\overline{\Omega}), \quad \gamma = (\gamma_1, ..., \gamma_n) \in \left[C^1(\overline{\Omega})\right]^n \tag{2.19}$$

and

$$|\gamma(x)\cdot x| \geq c_0 > 0, \quad \text{on } \overline{\Omega}, \tag{2.20}$$

with a constant $c_0 > 0$. Then

Lemma 2.1 *In addition to (2.20), we assume that $\|\gamma_0\|_{C(\overline{\Omega})} \leq M$ and $\|\gamma_i\|_{C^1(\overline{\Omega})} \leq M$, $1 \leq i \leq n$. Then for sufficiently large $\beta > 0$, there exist constants $\tau_* > 0$ and $C > 0$ such that*

$$\tau \int_\Omega |v(x)|^2 e^{2\tau\rho(x)} dx \leq C \int_\Omega |A(x,D)v(x)|^2 e^{2\tau\rho(x)} dx$$

for all $v \in H_0^1(\Omega)$ and all $\tau > \tau_*$.

Proof . We multiply the both sides of (2.18) by $v(x)e^{2\tau\rho(x)}$ and using the divergence theorem, we obtain

$$\begin{aligned}
&\int_\Omega A(x,D)v(x)\cdot v(x)e^{2\tau\rho(x)}dx = \int_\Omega \nabla v(x)\cdot(e^{2\tau\rho(x)}v(x)\gamma(x))dx + \int_\Omega \gamma_0(x)|v(x)|^2 dx \\
= & -\int_\Omega v(x)\mathrm{div}(e^{2\tau\rho(x)}v(x)\gamma(x))dx + \int_\Omega \gamma_0(x)|v(x)|^2 e^{2\tau\rho(x)}dx \\
= & -\int_\Omega |v|^2 e^{2\tau\rho(x)}\mathrm{div}(\gamma(x))dx - 2\tau\int_\Omega |v(x)|^2 \nabla\rho\cdot\gamma(x)e^{2\tau\rho(x)}dx \\
& -\int_\Omega e^{2\tau\rho(x)}v(x)\nabla v(x)\cdot\gamma(x)dx + \int_\Omega \gamma_0(x)|v(x)|^2 e^{2\tau\rho(x)}dx.
\end{aligned} \tag{2.21}$$

By (2.20), we obtain

$$|\nabla\rho(x).\gamma(x)| \geq 2c_0, \quad \text{on } \overline{\Omega},$$
$$\nabla v(x)\cdot\gamma(x) = A(x,D)v - \gamma_0(x)v(x)$$

and so in terms of (2.21) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\tau\int_\Omega |v(x)|^2 e^{2\tau\rho(x)}dx &\leq C\int_\Omega |A(x,D)v(x)\cdot v(x)| e^{2\tau\rho(x)}dx + C\int_\Omega |v(x)|^2 e^{2\tau\rho(x)}dx \\
&\leq C\int_\Omega |A(x,D)v(x)|^2 e^{2\tau\rho(x)}dx + C\int_\Omega |v(x)|^2 e^{2\tau\rho(x)}dx.
\end{aligned}$$

Then for large τ, we can complete the proof of Lemma 2.1. •

2.2 Weak observation estimate

The following proposition shows the stability in the continuation of solutions in class (1.11) of a hyperbolic equation from lateral boundary data on an arbitrarily small part Γ_1 of $\partial\Omega$, and gives the corresponding stability in the continuation where the uniqueness was proved by Robbiano [30], Tataru [33].

Proposition 2.2 *Let (Φ_0, Φ_1) satisfy the k-th order compatibility conditions with $k \geq 5$. There exist sufficiently large $T > 0$ and $C(k) >$ such that the following estimate holds.*

$$\sum_{j=2}^{3} \left\| \partial_t^j (u_a - u_b) \right\|^2_{H^1(\omega_T(\varrho, 3\varrho))} \leq C(k) \left[\log \left(2 + \frac{C(k)}{\epsilon(\Sigma_1)} \right) \right]^{-(k-2)} \tag{2.22}$$

for all $a, b \in \Lambda$. Moreover for $k \geq 4$, we have

$$\sum_{j=2}^{3} \left\| \partial_t^j (u_a - u_b) \right\|^2_{L^2(\omega_T(\varrho, 3\varrho))} \leq C(k) \left[\log \left(2 + \frac{C(k)}{\epsilon(\Sigma_1)} \right) \right]^{-(k-2)}, \tag{2.23}$$

for all $a, b \in \Lambda$. Here the constant $C(k)$ is dependent on Ω, ω, T, M and independent of $a, b \in \Lambda$.

As a related result, see Robbiano [31]. To prove Proposition 2.2, we use the idea of Robbiano [30], [31] to apply the Fourier-Bros-Iagolnitzer transformation and the proof is given in Section 4.

3 Proof of the main result

This section is devoted to the proof of Theorems 1 and 2. The key is the combination of Proposition 2.2 and the existing method (e.g., [10] - [12]).

3.1 Linearized inverse problem

First of all, we consider the difference $w = u_a - u_b$. Then

$$\begin{cases} \partial_t^2 w(x,t) - \mathrm{div}(a(x)\nabla w(x,t)) = F(x,t) & \text{in } \Omega \times [-T, T] \\ w(x,0) = \partial_t w(x,0) = 0 & \text{in } \Omega \\ \partial_\nu w(x,t) = 0 & \text{on } \Gamma \times [-T, T] \end{cases} \tag{3.1}$$

where the function F is given by

$$F(x,t) = \mathrm{div}(f(x)\nabla u_b(x,t)), \quad f(x) = a(x) - b(x), \quad (x,t) \in Q. \tag{3.2}$$

Let $k \geq 4$ and let us recall regularity (1.12) for u_a and u_b. In this subsection, we discuss a linearized inverse problem of determining f from $w_{|\Gamma_1 \times [0,T]}$ in a series of Lemmata 3.1 -

3.4.

Let us set $v = \partial_t w$. Then we have

$$\begin{cases} \partial_t^2 v(x,t) - \mathrm{div}(a(x)\nabla v(x,t)) = F_1(x,t) & \text{in } Q = \Omega \times [-T,T] \\ v(x,0) = 0, \quad \partial_t v(x,0) = \mathrm{div}(f(x)\nabla \Phi_0) & \text{in } \Omega \\ \partial_\nu v(x,t) = 0 & \text{on } \Sigma = \Gamma \times [0,T] \end{cases} \tag{3.3}$$

where F_1 is given by

$$F_1(x,t) = \partial_t F(x,t) = \mathrm{div}(f(x)\partial_t \nabla u_b(x,t)). \tag{3.4}$$

Now we introduce the following notations:

$$z_j(x,t) = \partial_t^j v(x,t), \quad F_j(x,t) = \partial_t^j F(x,t), \quad (x,t) \in Q, \quad j = 1,2. \tag{3.5}$$

Then we will prove

Lemma 3.1 *Let $\|a\|_{C^2(\overline{\Omega})} \leq M$. Then there exists $\beta_* > 0$ such that for all $\beta > \beta_*$ there exist $\tau_* > 0$ and a constant $C > 0$ such that*

$$\begin{aligned} \tau \int_{Q_\alpha(3\varrho)} \left(|\nabla_{x,t} z_j(x,t)|^2 + \tau^2 |z_j(x,t)|^2 \right) e^{2\tau\varphi} dxdt \quad \leq \quad & C \Big[\int_Q |F_{1+j}(x,t)|^2 e^{2\tau\varphi} dxdt + C_k \tau^3 e^{2d\tau} \\ & + e^{C\tau} \|z_j\|^2_{H^1(\omega_T(\varrho,3\varrho))} \Big] \end{aligned} \tag{3.6}$$

for all $\tau > \tau_*$, $j = 1,2$.

Proof . The function z_j, $j = 1,2$, solves the following hyperbolic equation:

$$\partial_t^2 z_j(x,t) - \mathrm{div}(a(x)\nabla z_j(x,t)) = F_{1+j}(x,t), \quad (x,t) \in Q. \tag{3.7}$$

We apply Proposition 2.1 to obtain

$$\begin{aligned} \tau \int_{Q_\alpha(3\varrho)} e^{2\tau\varphi} \left(|\nabla_{x,t} z_j|^2 + \tau^2 |z_j|^2 \right) dxdt \quad \leq \quad & C \int_Q e^{2\tau\varphi} |F_{1+j}|^2 dxdt \\ & + C\tau \int_{\omega_T(\varrho,3\varrho)} e^{2\tau\varphi} \left(|\nabla_{x,t} z_j|^2 + \tau^2 |z_j|^2 \right) dxdt \\ & + \tau \int_{Q \setminus Q_\alpha} e^{2\tau\varphi} \left(|\nabla_{x,t} z_j|^2 + \tau^2 |z_j|^2 \right) dxdt \end{aligned} \tag{3.8}$$

provided that $\tau > 0$ is large enough.

We now estimate the last term in (3.8). It follows from (2.10) that we can choose $\alpha > 0$ sufficiently small such that

$$\tau \int_{Q \setminus Q_\alpha} e^{2\tau\varphi} \left(|\nabla_{x,t} z_j|^2 + \tau^2 |z_j|^2 \right) dxdt \leq C\tau^3 e^{2d\tau} \|z_j\|^2_{H^1(Q)} \tag{3.9}$$

where $d < 1$. Henceforth in Section 3, $C > 0$ denotes generic constants.
Next, since (Φ_0, Φ_1) satisfies the k-th compatibility conditions with respect to a, it satisfies also the k-th compatibility conditions with respect to b by $a = b$ near Γ. Hence by (1.12), there exists a constant $C_k > 0$ such that for $j = 1, 2$ we have

$$\sup_{t\in[-T,T]} \left[\|z_j(\cdot,t)\|^2_{H^1(\Omega)} + \|\partial_t z_j(\cdot,t)\|^2_{L^2(\Omega)}\right]$$
$$= \sup_{t\in[-T,T]} \left[\left\|\partial_t^{j+1} w(\cdot,t)\right\|^2_{H^1(\Omega)} + \left\|\partial_t^{j+2} w(\cdot,t)\right\|^2_{L^2(\Omega)}\right] \le C_k. \tag{3.10}$$

Substituting (3.10) into (3.9), we have

$$\tau\int_{Q\setminus Q_\alpha} e^{2\tau\varphi}\left(|\nabla_{x,t} z_j|^2 + \tau^2 |z_j|^2\right) dxdt \le C_k \tau^3 e^{2d\tau}. \tag{3.11}$$

Applying inequality (3.11) to inequality (3.8), we complete the proof of (3.6). •

Lemma 3.2 *Let $\phi(x) = div(f(x)\nabla\Phi_0(x))$. Then there exists a constant $C > 0$ such that the following estimate hold:*

$$\int_\Omega \left(|\nabla\phi(x)|^2 + |\phi(x)|^2\right) e^{2\tau\rho(x)} dx \le C\left[\tau\sum_{j=1}^{2}\int_{Q_\alpha(3\varrho)} \left(|\nabla z_j|^2 + |z_j|^2\right) e^{2\tau\varphi} dxdt + C_k\tau^3 e^{2d\tau}\right] \tag{3.12}$$

provided that τ is large.

Proof . We introduce a cut-off function $\chi_1 \in C_0^\infty(\mathbb{R})$ satisfying $0 \le \chi_1 \le 1$ and

$$\chi_1(t) = \begin{cases} 1, & \text{for} \quad |t| \le T - 2\alpha \\ 0, & \text{for} \quad |t| \ge T - \alpha. \end{cases} \tag{3.13}$$

By direct computations, we have

$$\int_{\Omega(3\varrho)} \chi_1^2(0)\,|\partial_i z_1(x,0)|^2 e^{2\tau\rho(x)} dx = \int_{-T}^{0} \frac{d}{dt}\left(\int_{\Omega(3\varrho)} \chi_1(t)^2 |\partial_i z_1|^2 e^{2\tau\varphi} dx\right) dt$$
$$= \int_{-T}^{0}\int_{\Omega(3\varrho)} 2\chi_1(t)^2(\partial_i z_2)(\partial_i z_1) e^{2\tau\varphi} dxdt + \int_{-T}^{0}\int_{\Omega(3\varrho)} 2\chi_1(t)\chi_1'(t)\,|\partial_i z_1|^2 e^{2\tau\varphi} dxdt$$
$$+ \int_{-T}^{0}\int_{\Omega(3\varrho)} 2\chi_1(t)^2 \tau\partial_t\varphi\,|\partial_i z_1|^2 e^{2\tau\varphi} dxdt, \quad 1 \le i \le n. \tag{3.14}$$

Therefore, because

$$\chi_1(0)\partial_i z_1(x,0) = \partial_i \mathrm{div}(f(x)\nabla\Phi_0(x)) = \partial_i\phi(x),$$

and $\chi_1'(t)$ is supported in $[-T, -T+2\alpha] \cup [T-2\alpha, T]$, by the Cauchy-Schwarz inequality, we obtain

$$\int_{\Omega(3\varrho)} |\partial_i\phi(x)|^2 e^{2\tau\rho(x)} dx \quad \le \quad C\tau\int_{Q_\alpha(3\varrho)} \left(|\nabla z_2(x,t)|^2 + |\nabla z_1(x,t)|^2\right) e^{2\tau\varphi} dxdt$$

$$+\int_{Q\setminus Q_{2\alpha}} |\nabla z_1(x,t)|^2 e^{2\tau\varphi} dxdt. \tag{3.15}$$

Similarly we have

$$\int_{\Omega(3\varrho)} |\phi(x)|^2 e^{2\tau\rho(x)} dx \leq C\tau \int_{Q_\alpha(3\varrho)} \left(|z_2(x,t)|^2 + |z_1(x,t)|^2\right) e^{2\tau\varphi} dxdt$$
$$+\int_{Q\setminus Q_{2\alpha}} |z_1(x,t)|^2 e^{2\tau\varphi} dxdt. \tag{3.16}$$

It follows from (2.10) and (3.10) that we can choose $\alpha > 0$ sufficiently small, so that

$$\int_{Q\setminus Q_{2\alpha}} e^{2\tau\varphi} \left(|\nabla z_1(x,t)|^2 + |z_1(x,t)|^2\right) dxdt \leq Ce^{2d\tau} \|z_1\|^2_{H^1(Q)} \leq C_k e^{2d\tau}. \tag{3.17}$$

Combining (3.15) - (3.17), we obtain

$$\int_{\Omega(3\varrho)} \left(|\nabla\phi(x)|^2 + |\phi(x)|^2\right) e^{2\tau\rho(x)} dx \leq C\left[\tau \sum_{j=1}^{2} \int_{Q_\alpha(3\varrho)} \left(|\nabla z_j|^2 + |z_j|^2\right) e^{2\tau\varphi} dxdt + C_k e^{2d\tau}\right].$$

Using that $\phi(x) = 0$ in $\Omega\setminus\Omega(3\varrho)$, by $f = 0$ in $\Omega\setminus\Omega(3\varrho)$, we obtain (3.12). •

Lemma 3.3 *There exists a constant $C > 0$ such that*

$$\tau \int_\Omega \left(|\nabla f(x)|^2 + |f(x)|^2\right) e^{2\tau\rho} dx \leq C \int_\Omega \left(|\nabla\phi(x)|^2 + |\phi(x)|^2\right) e^{2\tau\rho(x)} dx$$

for all large $\tau > 0$.

Proof . We have

$$\mathrm{div}(\partial_i f(x)\nabla\Phi_0(x)) = \partial_i\phi(x) - \mathrm{div}(f\partial_i\nabla\Phi_0(x)) \quad \text{for all } i = 1,...,n.$$

Therefore

$$\int_\Omega \left(|\mathrm{div}((\partial_i f)\nabla\Phi_0)|^2 + |\mathrm{div}(f\nabla\Phi_0)|^2\right) e^{2\tau\rho} dx$$
$$\leq \int_\Omega \left(|\nabla\phi|^2 + |\phi|^2\right) e^{2\tau\rho(x)} dx + C\int_\Omega \left(|f|^2 + |\nabla f|^2\right) e^{2\tau\rho} dx, \quad 1 \leq i \leq n. \tag{3.18}$$

Since $f = 0$ near the boundary Γ and $\nabla\Phi_0 \cdot x \neq 0$, we can apply Lemma 2.1 respectively with the choice $v = f$ and $v = \partial_i f$ and obtain

$$\tau \int_\Omega \left(|\partial_i f(x)|^2 + |f(x)|^2\right) e^{2\tau\rho} dx \leq C \int_\Omega \left(|\mathrm{div}((\partial_i f)\nabla\Phi_0)|^2 + |\mathrm{div}(f\nabla\Phi_0)|^2\right) e^{2\tau\rho} dx. \tag{3.19}$$

Inserting (3.19) into the left hand side of (3.18) and choosing $\tau > 0$ large, we obtain

$$\tau \int_\Omega \left(|\nabla f(x)|^2 + |f(x)|^2\right) e^{2\tau\rho} dx \leq C \int_\Omega \left(|\nabla\phi(x)|^2 + |\phi(x)|^2\right) e^{2\tau\rho(x)} dx.$$

The proof is complete. •

Lemma 3.4 *There exists a constant $C > 0$ such that*

$$\sum_{j=1}^{2} \int_Q e^{2\tau\varphi} |F_{j+1}(x,t)|^2 \, dxdt \le C \int_Q e^{2\tau\rho(x)} \left(|\nabla f(x)|^2 + |f(x)|^2 \right) dx$$

for all $\tau > 0$.

Proof . Since

$$F_{j+1}(x,t) = \partial_t^{j+1} F(x,t) = \operatorname{div}(f(x)\partial_t^{j+1} \nabla u_b(x,t)), \quad j = 1, 2, \ (x,t) \in Q,$$

we have

$$\sum_{j=1}^{2} \int_Q e^{2\tau\varphi} |F_{j+1}(x,t)|^2 \, dxdt \le C \sum_{j=2}^{3} \int_Q e^{2\tau\varphi} \left(|\nabla f|^2 + |f|^2 \right) \left\| \partial_t^j u_b(\cdot,t) \right\|_{W^{2,\infty}(\Omega)}^2 dxdt. \tag{3.20}$$

By the Sobolev embedding theorem (e.g., Adams [1]), we have

$$H^{k-1}(\Omega) \hookrightarrow W^{2,\infty}(\Omega), \quad n \le 3, \ k \ge 5.$$

Using (1.12), we obtain

$$\sum_{j=2}^{3} \left[\sup_{t \in [-T,T]} \left\| \partial_t^j u_b(\cdot,t) \right\|_{W^{2,\infty}(\Omega)}^2 \right] \le C \sum_{j=2}^{3} \left[\sup_{t \in [-T,T]} \left\| \partial_t^j u_b(\cdot,t) \right\|_{H^{k-1}(\Omega)}^2 \right] \le C. \tag{3.21}$$

Substituting (3.21) in (3.20), we see that

$$\sum_{j=1}^{2} \int_Q e^{2\tau\varphi} |F_{j+1}(x,t)|^2 \, dxdt \le C \int_\Omega e^{2\tau\rho(x)} \left(|\nabla f(x)|^2 + |f(x)|^2 \right) dx.$$

The proof of Lemma 3.4 is complete. •

3.2 Proof of main results

3.2.1 Proof of Theorem 1

In terms of Lemmata 3.1 - 3.4, we will now complete the proof of Theorem 1.
By Lemmata 3.2 and 3.3, we obtain

$$\tau \int_\Omega e^{2\tau\rho(x)} \left(|\nabla f(x)|^2 + |f(x)|^2 \right) dx \le C \int_\Omega \left(|\nabla \phi(x)|^2 + |\phi(x)|^2 \right) e^{2\tau\rho(x)} dx$$
$$\le C\tau \sum_{j=1}^{2} \int_{Q_\alpha(3\varrho)} \left(|\nabla z_j|^2 + |z_j|^2 \right) e^{2\tau\varphi} dxdt + C_k \tau^3 e^{2d\tau}. \tag{3.22}$$

On the other hand, combining Lemma 3.1 and (3.22), we obtain

$$\begin{aligned}
&\tau\int_\Omega e^{2\tau\rho(x)}\left(|\nabla f(x)|^2+|f(x)|^2\right)dx\\
\le\quad & C\sum_{j=1}^{2}\int_Q |F_{j+1}(x,t)|^2 e^{2\tau\varphi}dxdt + C_k\tau^3 e^{2d\tau} + Ce^{C\tau}\|z_j\|^2_{H^1(\omega_T(\varrho,3\varrho))}. \qquad (3.23)
\end{aligned}$$

Combining (3.23) and Lemma 3.4, we obtain

$$\begin{aligned}
\tau\int_\Omega e^{2\tau\rho(x)}\left(|\nabla f(x)|^2+|f(x)|^2\right) \le\quad & C_k\int_\Omega e^{2\tau\rho(x)}\left(|\nabla f(x)|^2+|f(x)|^2\right)dx + C_k\tau^3 e^{2d\tau}\\
+\quad & e^{C\tau}\sum_{j=2}^{3}\left\|\partial_t^j(u_a-u_b)\right\|^2_{H^1(\omega_T(\varrho,3\varrho))}. \qquad (3.24)
\end{aligned}$$

Then the first term of the right hand side of (3.24) can be absorbed into the left hand side if we take large $\tau>0$.
Since $\rho(x)\ge 1$, we obtain

$$\begin{aligned}
&\int_\Omega\left(|\nabla f(x)|^2+|f(x)|^2\right)dx \le C_k\tau^2 e^{(2d-2)\tau} + e^{(C-2)\tau}\sum_{j=2}^{3}\left\|\partial_t^j(u_a-u_b)\right\|^2_{H^1(\omega_T(\varrho,3\varrho))}\\
\le\quad & C_k e^{-D_1\tau} + e^{D_2\tau}\sum_{j=2}^{3}\left\|\partial_t^j(u_a-u_b)\right\|^2_{H^1(\omega_T(\varrho,3\varrho))}. \qquad (3.25)
\end{aligned}$$

At the last inequality, we used: By $0<d<1$, we can choose $D_1>0$ such that $\tau^2 e^{(2d-2)\tau}\le e^{-D_1\tau}$ for sufficiently large $\tau>0$. Then we apply Proposition 2.2, so that

$$\|\nabla(a-b)\|^2_{L^2(\Omega)} \le C_k e^{-D_1\tau} + e^{D_2\tau}\left(\log\left(2+\frac{C(k)}{\epsilon(\Sigma_1)}\right)\right)^{-(k-2)}.$$

Setting

$$\tau = -\frac{1}{D_1+D_2}\log\left(\left(\log\left(2+\frac{C(k)}{\epsilon(\Sigma_1)}\right)\right)^{-(k-2)}\right),$$

we obtain desired estimate (1.14) with $\mu=\frac{D_1}{D_1+D_2}$.

3.3 Proof of Theorem 2

By the Sobolev imbedding theorem, noting that the spatial dimension ≤ 3, we have

$$H^{k+2}(\Omega)\hookrightarrow W^{4,\infty}(\Omega),\quad k\ge 4.$$

Let (Φ_0,Φ_1) satisfy the k-th order compatibility conditions such that $\nabla\Phi_0\cdot x\ne 0$ for all $x\in\overline{\Omega}$. Then there exists a constant $C_k>0$ such that

$$\sup_{t\in[-T,T]}\left[\|u_a(\cdot,t)\|_{W^{4,\infty}(\Omega)}\right] \le C\sup_{t\in[-T,T]}\left[\|u_a(\cdot,t)\|_{H^{k+2}(\Omega)}\right]\le C_k.$$

By Theorem 1 in Imanuvilov and Yamamoto [12], we obtain

$$\|a-b\|^2_{L^2(\Omega)} \le C\left(\sum_{j=2}^{3}\left\|\partial_t^j(u_a-u_b)\right\|^2_{L^2(\omega_T(\varrho,3\varrho))}\right)^{\mu}. \tag{3.26}$$

Applying (2.23) in (3.26), we obtain (1.15) as is desired.

4 Proof of the weak observation

We will now prove Proposition 2.2. This will be done in terms of the Fourier-Bros-Iagolnitzer (FBI) transformation. Let v be a given solution to

$$\partial_t^2 v - \operatorname{div}(a(x)\nabla v) = R(x,t) \quad \text{in}\, Q = \Omega\times[-T,T] \tag{4.1}$$

with the Neumann boundary condition

$$\partial_\nu v(x,t) = 0 \quad \text{on}\, \Sigma = \Gamma\times[-T,T]. \tag{4.2}$$

Here and henceforth we assume that

$$R(x,t) = 0, \quad (x,t)\in\omega\times[-T,T]. \tag{4.3}$$

4.1 Preliminaries and elliptic estimation

Denote for $r>0$

$$\begin{aligned}&\Omega_r = \Omega\times]-r,r[; \quad \omega_r(\varrho,3\varrho) = \omega(\varrho,3\varrho)\times]-r,r[,\\ &\Gamma_r = \Gamma\times]-r,r[, \quad \Gamma_{1,r} = \Gamma_1\times]-r,r[.\end{aligned} \tag{4.4}$$

We fix $m\in\mathbb{N}$ such that

$$\gamma \equiv 1-\frac{1}{2m} > \frac{1}{2}, \quad \text{and}\;\; 2m > k-2 \tag{4.5}$$

and for $z\in\mathbb{C}$ we define

$$K(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\xi}e^{-\xi^{2m}}d\xi. \tag{4.6}$$

Then the even function $K(z)$ is holomorphic and there exist positive constants A, c_0, c_1, c_2 such that for $\alpha \equiv \frac{2m}{2m-1} = \frac{1}{\gamma}$, we have ([27])

$$\begin{cases} |K(z)| + |K'(z)| \le Ae^{c_0|Im z|^\alpha}, & \forall\, z\in\mathbb{C},\\[2ex] |K(z)| \le Ae^{-c_1|z|^\alpha} & \text{if } |Im\, z| \le c_2\,|Re\, z|\,. \end{cases} \tag{4.7}$$

For $\lambda\ge 1$ and $z\in\mathbb{C}$, we set

$$K_\lambda(z) = \lambda^\gamma K(\lambda^\gamma z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\xi}e^{-(\xi/\lambda^\gamma)^{2m}}d\xi. \tag{4.8}$$

Then by (4.7) we have

$$|K_\lambda(z)| + |K'_\lambda(z)| \le A\lambda^{2\gamma}e^{c_0\lambda} \quad \text{for} \quad |Im\, z| \le 1, \tag{4.9}$$

and, by the second inequality of (4.7), we see that there exists a constant $C_3 > 0$ such that for sufficiently large $T > 0$, we have

$$|K_\lambda(z)| \le Ae^{-C_3\lambda T} \quad \text{for all } z \in \mathbb{C} \quad \text{such that } |Imz| \le 1,\, |Rez| \ge \frac{T}{3}. \tag{4.10}$$

We define a cut-off function $\theta \in C_0^\infty(\mathbb{R})$ defined by

$$\theta(t) = \begin{cases} 1 & |t| \le (T-2), \\ 0 & |t| \ge (T-1). \end{cases} \tag{4.11}$$

Henceforth C_j, C denote generic constants which are independent of λ, T, γ, r, τ. We introduce the Fourier-Bros-Iagolnitzer (FBI) transformation T_λ. It is defined for $u \in \mathcal{S}(\mathbb{R}^{n+1})$, the space of rapidly decreasing functions, by

$$u_{\lambda,t}(x,s) = T_\lambda u(z,x) = \int_{\mathbb{R}} K_\lambda(z-y)\theta(y)u(x,y)dy, \quad z = t + is. \tag{4.12}$$

In the sequel we assume that T is sufficiently large, $s \in [-3r, 3r]$ and $t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$. We introduce a cut-off function χ_2 satisfying $0 \le \chi_2 \le 1$, $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ and

$$\chi_2(x) = \begin{cases} 1 & if\ x \in \omega(6\varrho), \\ 0 & if\ x \in \Omega(7\varrho). \end{cases} \tag{4.13}$$

Let $v(x,t)$ be a solution to (4.1). We set $u(x,t) = \chi_2(x)v(x,t)$, and we have

$$\partial_t^2 u - \mathrm{div}(a(x)\nabla u) = -\mathrm{div}(a(x)v(x,t)\nabla\chi_2(x)) - a(x)\nabla v(x,t)\cdot\nabla\chi_2 \quad \text{in } Q = \Omega\times[-T,T] \tag{4.14}$$

and

$$\partial_\nu u(x,t) = 0 \quad \text{on } \Sigma = \Gamma \times [-T,T], \tag{4.15}$$

where we have used $\chi_2(x)R(x,t) = 0$ by (4.3).
In connection with the operator $\partial_t^2 - \mathrm{div}(a(x)\nabla)$, we define an elliptic operator by

$$Q = \partial_s^2 + \mathrm{div}(a(x)\nabla). \tag{4.16}$$

Since

$$\partial_s \int_{\mathbb{R}} K_\lambda(is+t-y)\theta(y)u(x,y)dy = i\int_{\mathbb{R}} K_\lambda(is+t-y)\partial_y\left[\theta(y)u(x,y)\right]dy, \tag{4.17}$$

by (4.11) and integration by parts, we have

$$\begin{aligned} Qu_{\lambda,t}(x,s) &= F_{\lambda,t}(x,s) + G_{\lambda,t}(x,s), \quad &(x,s) \in \Omega_{3r} \\ \partial_\nu u_{\lambda,t}(x,s) &= 0, &(x,s) \in \Sigma_{3r} \end{aligned} \tag{4.18}$$

where

$$F_{\lambda,t}(x,s) = -\int_{\mathbb{R}} K_\lambda(z-y)\left(2\theta'(y)\partial_t u(x,y) + \theta''(y)u(x,y)\right)dy \tag{4.19}$$

and

$$G_{\lambda,t}(x,s) = \int_{\mathbb{R}} K_\lambda(z-y)\theta(y)\{\mathrm{div}(a(x)v(x,y)\nabla\chi_2(x)) + a(x)\nabla v(x,y)\cdot\nabla\chi_2(x)\}dy. \tag{4.20}$$

Here we have used also (4.14). Since θ' and θ'' are supported in $|y| \geq (T-2)$, by (4.10) we obtain

$$\|F_{\lambda,t}\|_{L^2(\Omega_{3r})} \leq C_4 e^{-C_3\lambda T}\|u\|_{H^1(Q)}, \quad \forall t \in \left[-\frac{T}{2}, \frac{T}{2}\right]. \tag{4.21}$$

Moreover, in terms of (4.9), there exists $C_5 > 0$, independent of T, such that

$$\|u_{\lambda,t}\|_{H^1(\Omega_{3r})} \leq C_4 e^{C_5\lambda}\|u\|_{H^1(Q)}, \quad \forall t \in \left[-\frac{T}{2}, \frac{T}{2}\right]. \tag{4.22}$$

By (4.20) and (4.13), we easily obtain

$$G_{\lambda,t}(x,s) = 0, \quad \forall x \in \omega(6\varrho). \tag{4.23}$$

Let K be a compact in $\overline{\Omega_{3r}}$ and $\widetilde{\psi}(x,s)$ be a C^1-function satisfying $|\nabla_{x,s}\widetilde{\psi}| \neq 0$ on K. Let

$$\widetilde{\varphi}(x,s) = e^{-\beta\widetilde{\psi}(x,s)}, \tag{4.24}$$

where $\beta > 0$ is sufficiently large. Then the following Carleman estimate holds true (see for example, [7], [27]): There exists $\tau_0 > 0$ such that

$$C\tau\left\|e^{\tau\widetilde{\varphi}}u\right\|^2_{H^1_\tau(\Omega_{3r})} \leq \left\|e^{\tau\widetilde{\varphi}}Qu\right\|^2_{L^2(\Omega_{3r})} + \tau\left\|e^{\tau\widetilde{\varphi}}u\right\|^2_{H^1_\tau(\Gamma_{3r})} + \tau\left\|e^{\tau\widetilde{\varphi}}\partial_\nu u\right\|^2_{L^2(\Gamma_{3r})} \tag{4.25}$$

whenever $u \in C_0^\infty(K)$ and $\tau > \tau_0$.
Here and henceforth we set

$$\|u\|^2_{H^1_\tau(\Omega_{3r})} = \|\nabla_{s,x}u\|^2_{L^2(\Omega_{3r})} + \tau^2\|u\|^2_{L^2(\Omega_{3r})} \tag{4.26}$$

and

$$\|u\|^2_{H^1_\tau(\Gamma_{3r})} = \|u\|^2_{H^1(\Gamma_{3r})} + \tau^2\|u\|^2_{L^2(\Gamma_{3r})}. \tag{4.27}$$

We further introduce a cut-off function χ_3 satisfying $0 \leq \chi_3 \leq 1$, $\chi_3 \in C_0^\infty(\mathbb{R})$, and

$$\chi_3(\eta) = \begin{cases} 0 & if\ \eta \leq \frac{1}{2},\ \eta \geq 8 \\ 1 & if\ \eta \in \left[\frac{3}{4}, 7\right]. \end{cases} \tag{4.28}$$

Now we proceed to the estimation near Γ_1.

4.2 Estimation near the boundary part Γ_1

We shall begin to estimate $u_{\lambda,t}$ in a ball $B_1 = B(x^{(1)}, r) = \{x \in \mathbb{R}^n; |x - x^{(1)}| < r\}$ over a small interval $]-r, r[$ by the velocity trace (in the normal direction) in the given part $\Gamma_{1,3r} = \Gamma_1 \times [-3r, 3r] \subset \Gamma_{3r}$.

Lemma 4.1 *Let $u_{\lambda,t}$ be a solution to (4.18). Then there exists $B_1^* \equiv B_1(x^{(1)}, r) \times [-r, r] \subset \Omega_r$ and $\nu_0 \in]0,1[$ such that the following estimate holds:*

$$\|u_{\lambda,t}\|_{H^1(B_1^*)} \le C \left(\|F_{\lambda,t}\|_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|_{H^1(\Gamma_{1,3r})} \right)^{\nu_0} \|u_{\lambda,t}\|^{1-\nu_0}_{H^1(\Omega_{3r})} \tag{4.29}$$

for some positive constant C.

Proof . Let us choose $\delta > 0$ and $x^{(0)} \in \mathbb{R}^n \backslash \overline{\Omega}$ such that

$$\delta < \frac{\varrho}{4}, \quad \overline{B(x^{(0)}, \delta)} \cap \overline{\Omega} = \emptyset, \quad B(x^{(0)}, 2\delta) \cap \Omega \neq \emptyset, \quad B(x^{(0)}, 4\delta) \cap \Gamma \subset \Gamma_1. \tag{4.30}$$

That is, $x^{(0)}$ is an outer point of $\overline{\Omega}$ and is near Γ_1. We define the functions $\psi_0(x, s)$ and $\varphi_0(x, s)$ by

$$\psi_0(x, s) = \left|x - x^{(0)}\right|^2 + s^2, \quad \varphi_0(x, s) = e^{-\frac{\beta}{\delta^2}\psi_0(x,s)}. \tag{4.31}$$

Denote

$$w_{\lambda,t}(x, s) = \chi_3 \left(\frac{\psi_0}{\delta^2} \right) u_{\lambda,t}(x, s). \tag{4.32}$$

Taking into account $\partial_\nu u_{\lambda,t} = 0$ on Γ and applying Carleman estimate (4.25), we obtain

$$C\tau \|e^{\tau\varphi_0} w_{\lambda,t}\|^2_{H^1_\tau(\Omega_r)} \le \|e^{\tau\varphi_0} Q w_{\lambda,t}\|^2_{L^2(\Omega_{3r})} + \tau \left\| e^{\tau\varphi_0} \left(\chi_3 \left(\frac{\psi_0}{\delta^2} \right) u_{\lambda,t} \right) \right\|^2_{H^1(\Gamma_{3r})} \tag{4.33}$$

for $\tau > \tau_0$. Therefore by (4.18), (4.32), (4.23) and (4.28), we have

$$\begin{aligned} Q w_{\lambda,t}(x, s) &= \chi_3 \left(\frac{\psi_0}{\delta^2} \right) Q u_{\lambda,t}(x, s) + \left[Q, \chi_3 \left(\frac{\psi_0}{\delta^2} \right) \right] u_{\lambda,t}(x, s) \\ &= \chi_3 \left(\frac{\psi_0}{\delta^2} \right) (F_{\lambda,t}(x, s) + G_{\lambda,t}(x, s)) + \left[Q, \chi_3 \left(\frac{\psi_0}{\delta^2} \right) \right] u_{\lambda,t}(x, s) \\ &= \chi_3 \left(\frac{\psi_0}{\delta^2} \right) F_{\lambda,t}(x, s) + \left[Q, \chi_3 \left(\frac{\psi_0}{\delta^2} \right) \right] u_{\lambda,t}(x, s). \end{aligned} \tag{4.34}$$

Since $\left[Q, \chi_3 \left(\frac{\psi_0}{\delta^2}\right)\right]$ is supported in

$$\left|x - x^{(0)}\right|^2 + s^2 \le \frac{3}{4}\delta^2, \quad 7\delta^2 \le \left|x - x^{(0)}\right|^2 + s^2 \le 8\delta^2, \tag{4.35}$$

taking (4.30) into account, we see that $|x - x^{(0)}| \ge \delta$ for all $x \in \overline{\Omega}$ and $\Omega \cap \{x; |x - x^{(0)}| \le \frac{3}{4}\delta^2\} \neq \emptyset$, so that we obtain

$$C\tau e^{2\tau e^{-4\beta}} \|u_{\lambda,t}\|^2_{H^1_\tau((\delta^2 \le \psi_0 \le 4\delta^2)\cap\Omega)} \le e^{2\tau e^{-7\beta}} \|u_{\lambda,t}\|^2_{H^1(\psi_0 \le 8\delta^2)} + e^{2\tau e^{-\beta}} \|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})}$$

$$+\tau e^{2\tau e^{-\beta}} \|u_{\lambda,t}\|^2_{H^1(\Gamma_{1,3r})}. \tag{4.36}$$

We can select $r > 0$ and $x^{(1)} \in \Omega$ such that

$$dist(x^{(1)}, \Gamma) \geq 4r, \quad B_1^* = B(x^{(1)}, r) \times [-r, r] \subset \left\{\delta^2 \leq \psi_0(x, s) \leq 4\delta^2\right\}. \tag{4.37}$$

This is possible because the second condition in (4.30) implies the existence of $x^{(1)} \in \Omega$ such that $|x^{(1)} - x^{(0)}| < 2\delta$. Therefore, for sufficiently small $r > 0$, condition (4.37) is satisfied. Then for $\tau > \tau_0$ we have

$$\|u_{\lambda,t}\|^2_{H^1(B_1^*)} \leq C_6 e^{C_6\tau} \left[\|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|^2_{H^1(\Gamma_{1,3r})}\right] + e^{-C_7\tau} \|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})}. \tag{4.38}$$

Now minimize the right hand side with respect to τ, with

$$\tau = \frac{1}{C_6 + C_7} \log \frac{\|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})}}{\|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|^2_{H^1(\Gamma_{1,3r})}} \tag{4.39}$$

and we obtain

$$\|u_{\lambda,t}\|^2_{H^1(B_1^*)} \leq \left(\|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|^2_{H^1(\Gamma_{1,3r})}\right)^{\nu_0} \left(\|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})}\right)^{1-\nu_0}, \tag{4.40}$$

where $\nu_0 = \frac{C_7}{C_6+C_7}$, provided that the right hand side of (4.39) $\geq \tau_0$. If the right hand side $\leq \tau_0$, then

$$\|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})} \leq C_8 \left[\|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|^2_{H^1(\Gamma_{1,3r})}\right].$$

Therefore

$$\begin{aligned} & \|u_{\lambda,t}\|^2_{H^1(B_1^*)} \leq \|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})} \\ = \quad & \left(\|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})}\right)^{\nu_0} \left(\|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})}\right)^{1-\nu_0} \\ \leq \quad & C_8^{\nu_0} \left(\|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|^2_{H^1(\Gamma_{1,3r})}\right)^{\nu_0} \left(\|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})}\right)^{1-\nu_0}. \end{aligned}$$

This completes the proof of the lemma. •

4.3 Estimation in $\omega_r(\varrho, 3\varrho)$

In this subsection we extend the estimation from B_1^* to $\omega_r(\varrho, 4\varrho)$. To accomplish this, we use the techniques developed in [31]. This will be done by continuing estimates (4.29). Let $B(x^{(j)}, r)$, $2 \leq j \leq N$, be a finite covering of $\omega(\varrho, 4\varrho)$. We can assume that $x^{(j)}$ satisfies dist $(x^{(j)}, \Gamma) \geq 4r$. In the sequel, we assume without loss of generality that

$$B(x^{(j+1)}, r) \subset B(x^{(j)}, 2r), \tag{4.41}$$

and we set

$$B_j^* = B(x^{(j)}, r) \times]-r, r[, \quad 2 \leq j \leq N.$$

Lemma 4.2 *Let $u_{\lambda,t}$ be a solution to (4.18). Then there exist a constant $\nu \in]0,1[$ and $C > 0$ such that the following estimate holds:*

$$\|u_{\lambda,t}\|_{H^1(B^*_{k+1})} \leq C \left(\|F_{\lambda,t}\|_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|_{H^1(B^*_k)} \right)^{\nu} \|u_{\lambda,t}\|^{1-\nu}_{H^1(\Omega_{3r})}, \quad k \geq 1. \tag{4.42}$$

Proof . In order to prove (4.42), we define the functions $\psi_k(x,s)$ and $\varphi_k(x,s)$ by

$$\psi_k(x,s) = \left|x - x^{(k)}\right|^2 + s^2, \quad \varphi_k(x,s) = e^{-\frac{\beta}{r^2}\psi_k(x,s)}. \tag{4.43}$$

Moreover we set

$$w_{\lambda,t}(x,s) = \chi_3\left(\frac{\psi_k}{r^2}\right) u_{\lambda,t}(x,s).$$

By applying Carleman estimate (4.25) in the interior domain, we obtain

$$C_9 \tau \left\|e^{\tau\varphi_k} w_{\lambda,t}\right\|^2_{H^1_\tau} \leq \left\|e^{\tau\varphi_k} Q w_{\lambda,t}\right\|^2_{L^2}. \tag{4.44}$$

In the same way as (4.34), we have

$$Q w_{\lambda,t}(x,s) = \chi_3\left(\frac{\psi_k}{r^2}\right) F_{\lambda,t}(x,s) + \left[Q, \chi_3\left(\frac{\psi_k}{r^2}\right)\right] u_{\lambda,t}(x,s). \tag{4.45}$$

Since $\left[Q, \chi_3\left(\frac{\psi_k}{r^2}\right)\right]$ is supported in

$$\frac{r^2}{2} \leq |x - x_0|^2 + s^2 \leq r^2, \quad 7r^2 \leq |x - x_0|^2 + s^2 \leq 8r^2, \tag{4.46}$$

we combine (4.44) and (4.45), so that

$$\begin{aligned} C_{10} \tau e^{2\tau e^{-5\beta}} \|u_{\lambda,t}\|^2_{H^1_\tau(r^2 \leq \psi_k \leq 5r^2)} \leq\ & e^{2\tau e^{-\beta/2}} \|u_{\lambda,t}\|^2_{H^1(\psi_k \leq r^2)} + e^{2\tau e^{-7\beta}} \|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})} \\ & + e^{2\tau e^{-\beta/2}} \|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})}, \end{aligned} \tag{4.47}$$

and hence

$$\begin{aligned} C_{11} e^{2\tau e^{-5\beta}} \|u_{\lambda,t}\|^2_{H^1_\tau(\psi_k \leq 5r^2)} \leq\ & e^{2\tau e^{-\beta/2}} \|u_{\lambda,t}\|^2_{H^1(\psi_k \leq r^2)} + e^{2\tau e^{-7\beta}} \|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})} \\ & + e^{2\tau e^{-\beta/2}} \|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})}. \end{aligned} \tag{4.48}$$

Thus we obtain

$$\|u_{\lambda,t}\|^2_{H^1(\psi_k \leq 5r^2)} \leq e^{C_{12}\tau} \left[\|u_{\lambda,t}\|^2_{H^1(\psi_k \leq r^2)} + \|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})} \right] + e^{-C_{13}\tau} \|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})}. \tag{4.49}$$

Now minimizing the right hand side with respect to τ, for some $\mu \in]0,1[$, we obtain

$$\|u_{\lambda,t}\|^2_{H^1(\psi_k \leq 5r^2)} \leq C \left(\|F_{\lambda,t}\|^2_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|^2_{H^1(\psi_k \leq r^2)} \right)^{\nu} \left(\|u_{\lambda,t}\|^2_{H^1(\Omega_{3r})} \right)^{1-\nu}. \tag{4.50}$$

Since

$$B^*_{k+1} \subset \left\{\psi_k(s,x) \leq 5r^2\right\}, \quad \left\{\psi_k(x,s) \leq r^2\right\} \subset B^*_k, \tag{4.51}$$

we obtain (4.42). This completes the proof of the lemma. •

Lemma 4.3 *Let $u_{\lambda,t}$ be a solution to (4.18). Then there exists a constant $C > 0$ such that*

$$\|u_{\lambda,t}\|_{H^1(B_n^*)} \leq C \left(\|F_{\lambda,t}\|_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|_{H^1(B_1^*)} \right)^{\nu^n} \left(\|u_{\lambda,t}\|_{H^1(\Omega_{3r})} \right)^{1-\nu^n}, \quad n \geq 1. \quad (4.52)$$

Here $\nu \in]0,1[$ is the constant given in Lemma 4.2.

Proof . Put

$$\alpha_k = \|u_{\lambda,t}\|_{H^1(B_k^*)}, \quad A = \|F_{\lambda,t}\|_{L^2(\Omega_{3r})}, \quad B = C^{\frac{1}{1-\nu}} \|u_{\lambda,t}\|_{H^1(\Omega_{3r})}. \quad (4.53)$$

By (4.42) we have

$$\alpha_{k+1} \leq B^{1-\nu}(\alpha_k + A)^{\nu}. \quad (4.54)$$

Applying Lemma 4 in [26], we obtain for all $\widetilde{\mu} \in]0, \nu^n]$

$$\alpha_n \leq 2^{\frac{1}{1-\nu}} B^{1-\widetilde{\mu}}(\alpha_1 + A)^{\widetilde{\mu}}. \quad (4.55)$$

This completes the proof of the lemma. •

Lemma 4.4 *Let $u_{\lambda,t}$ be a solution to (4.18). Then there exist $C > 0$ and $\alpha_1 > 0$ such that for all $n \in \mathbb{N}$, there exist $C(n) > 0$ and $T(n) > 0$ such that*

$$C \|u_{\lambda,t}\|^2_{H^1(B_n^*)} \leq e^{-\alpha_1 \lambda} \|u\|^2_{H^1(Q)} + e^{C(n)\lambda} \|u\|^2_{H^1(\Sigma_1)} \quad (4.56)$$

for all $t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$ where $T > T(n)$.

Proof . By Lemma 4.3 and the Young inequality, we easily obtain

$$\|u_{\lambda,t}\|_{H^1(B_n^*)} \leq C\epsilon^p \|u_{\lambda,t}\|_{H^1(\Omega_{3r})} + C\epsilon^{-p'} \left[\|F_{\lambda,t}\|_{L^2(\Omega_{3r})} + \|u_{\lambda,t}\|_{H^1(B_1^*)} \right], \quad (4.57)$$

for all $\epsilon > 0$. Here

$$p = \frac{1}{1-\mu}, \quad p' = \frac{1}{\mu}, \quad \text{and} \quad \mu = \nu^n. \quad (4.58)$$

Using estimates (4.21) and (4.22), we have for all $t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$

$$\|u_{\lambda,t}\|_{H^1(B_n^*)} \leq C\epsilon^p e^{C_5\lambda} \|u\|_{H^1(Q)} + C\epsilon^{-p'} \left[e^{-C_3 T\lambda} \|u\|_{H^1(Q)} + \|u_{\lambda,t}\|_{H^1(B_1^*)} \right]. \quad (4.59)$$

Selecting in (4.59)

$$\epsilon = e^{-\frac{2C_5}{p}\lambda},$$

we obtain

$$\|u_{\lambda,t}\|_{H^1(B_n^*)} \leq Ce^{-C_5\lambda} \|u\|_{H^1(Q)} + Ce^{-(C_3T - \frac{2C_5 p'}{p})\lambda} \|u\|_{H^1(Q)} + Ce^{\frac{2C_5 p'}{p}\lambda} \|u_{\lambda,t}\|_{H^1(B_1^*)} \quad (4.60)$$

for all $t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$ and $\lambda > 0$. Take T sufficiently large such that

$$C_3 T - \frac{2C_5 p'}{p} > C_5 \tag{4.61}$$

and we obtain from (4.60)

$$\|u_{\lambda,t}\|_{H^1(B_n^*)} \le Ce^{-C_5\lambda} \|u\|_{H^1(Q)} + Ce^{C_{14}\lambda} \|u_{\lambda,t}\|_{H^1(B_1^*)} \tag{4.62}$$

where we set

$$C_{14} = \frac{2C_5 p'}{p}. \tag{4.63}$$

Similarly to (4.59), we obtain from Lemma 4.1 and the Young inequality

$$\|u_{\lambda,t}\|_{H^1(B_1^*)} \le C\epsilon^{p_0} e^{C_5\lambda} \|u\|_{H^1(Q)} + C\epsilon^{-p_0'} \left[e^{-C_3T\lambda} \|u\|_{H^1(Q)} + \|u_{\lambda,t}\|_{H^1(\Gamma_{1,3r})}\right] \tag{4.64}$$

where

$$p_0 = \frac{1}{1-\nu_0}, \quad p_0' = \frac{1}{\nu_0}. \tag{4.65}$$

Selecting $\epsilon = e^{-\left(\frac{2C_5+C_{14}}{p_0}\right)\lambda}$, we obtain for some positive constant C_{15}:

$$\|u_{\lambda,t}\|_{H^1(B_1^*)} \le Ce^{-(C_5+C_{14})\lambda} \|u\|_{H^1(Q)} + Ce^{-(C_3T - \frac{(2C_5+C_{14})p_0'}{p_0})\lambda} \|u\|_{H^1(Q)} + e^{C_{15}\lambda} \|u_{\lambda,t}\|_{H^1(\Gamma_{1,3r})}. \tag{4.66}$$

Take T large such that

$$C_3T - \frac{(2C_5 + C_{14})p_0'}{p_0} > C_5 + C_{14}. \tag{4.67}$$

Then, by (4.66), we obtain

$$\|u_{\lambda,t}\|_{H^1(B_1^*)} \le Ce^{-(C_5+C_{14})\lambda} \|u\|_{H^1(Q)} + Ce^{C_{15}\lambda} \|u_{\lambda,t}\|_{H^1(\Gamma_{1,3r})} \tag{4.68}$$

and, applying (4.68) in (4.62), we have

$$\|u_{\lambda,t}\|_{H^1(B_n^*)} \le Ce^{-C_5\lambda} \|u\|_{H^1(Q)} + e^{C(n)\lambda} \|u_{\lambda,t}\|_{H^1(\Gamma_{1,3r})} \tag{4.69}$$

for some positive constant $C(n)$. This completes the proof of (4.56). •

We fix $T > \max_{1\le n\le N} T(n)$. Addition of inequalities (4.69) for $n \in \{1, ..., N\}$ yields for all $t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$

$$\|u_{\lambda,t}\|_{H^1(\omega_r(\varrho,3\varrho))} \le Ce^{-\sigma\lambda} \|u\|_{H^1(Q)} + Ce^{C\lambda} \|u\|_{H^1(\Gamma_{1,r})} \tag{4.70}$$

for some positive constants σ and C.

4.4 End of the proof of Proposition 2.2

We shall complete the proof of Proposition 2.2 in this subsection.

Lemma 4.5 *Let u be a solution to (4.14). Let $T_1 = T/2 - r$. Then there exist $C > 0$ and $C_{16} > 0$ such that*

$$\|u\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))} \le C\left(\lambda^{-(k-2)}\|u\|^2_{H^{k-1}(Q)} + e^{C_{16}\lambda}\|u\|^2_{H^1(\Sigma_1)}\right) \tag{4.71}$$

and

$$\|u\|^2_{L^2(\omega_{T_1}(\varrho,3\varrho))} \le C\left(\lambda^{-(k-2)}\|u\|^2_{H^{k-2}(Q)} + e^{C_{16}\lambda}\|u\|^2_{H^1(\Sigma_1)}\right). \tag{4.72}$$

Proof . We set $u_\lambda(x,t) = u_{\lambda,t}(x,0)$ for $s = 0$. Then we have

$$\begin{aligned} u_\lambda(x,t) &= \int_{\mathbb{R}} K_\lambda(t-y)\theta(y)u(x,y)dy \\ &= (K_\lambda * \theta u)(x,t), \end{aligned} \tag{4.73}$$

where

$$K_\lambda(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\xi}e^{-(\xi/\lambda^\gamma)^{2m}}d\xi. \tag{4.74}$$

Then we have

$$\widehat{\theta u}(x,\tau) - \widehat{u_\lambda}(x,\tau) = (1-\widehat{K_\lambda})\widehat{\theta u}(x,\tau). \tag{4.75}$$

Furthermore we can immediately verify that

$$\left|1-\widehat{K_\lambda}\right| \le \left(\frac{|\tau|}{\lambda^\gamma}\right)^{k-2} \le \left(\frac{|\tau|}{\sqrt{\lambda}}\right)^{k-2}, \tag{4.76}$$

so that we obtain for $T_1 = T/2 - r$

$$\|u-u_\lambda\|^2_{L^2(\omega_{T_1}(\varrho,3\varrho))} \le C\lambda^{-(k-2)}\|u\|^2_{H^{k-2}(Q)}. \tag{4.77}$$

Similarly we have

$$\|u-u_\lambda\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))} \le C\lambda^{-(k-2)}\|u\|^2_{H^{k-1}(Q)}. \tag{4.78}$$

Hence

$$\begin{aligned} \|u\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))} &\le C\left[\|u-u_\lambda\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))} + \|u_\lambda\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))}\right] \\ &\le C\left[\lambda^{-(k-2)}\|u\|^2_{H^{k-1}(Q)} + \|u_\lambda\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))}\right]. \end{aligned} \tag{4.79}$$

On the other hand, by the Cauchy formula (Lemma 4 in [31]) and (4.70), by an argument similar to [31], we obtain

$$\|u_\lambda\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))} \le Ce^{-C_{16}\lambda}\|u\|^2_{H^1(Q)} + Ce^{C_{17}\lambda}\|u\|^2_{H^1(\Sigma_1)}. \tag{4.80}$$

This completes the proof of (4.71). The proof of (4.72) is similar. •

We now turn to the proof of Proposition 2.2. By (4.71) and $u = \chi_2 v$, we obtain

$$\|v\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))} \le C\lambda^{-(k-2)} \|v\|^2_{H^{k-1}(Q)} + Ce^{C\lambda} \|v\|^2_{H^1(\Sigma_1)}. \tag{4.81}$$

If we take $v = \partial_t^3(u_a - u_b)$, then we obtain

$$\left\|\partial_t^3(u_a - u_b)\right\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))} \le C(k)\lambda^{-(k-2)} + Ce^{C\lambda} \left\|\partial_t^3(u_a - u_b)\right\|^2_{H^1(\Sigma_1)} \tag{4.82}$$

where we have used

$$\sup_{t\in[-T,T]} \left[\sum_{j=0}^{k+2} \left\|\partial_t^{k+2-j}(u_a - u_b)(\cdot,t)\right\|^2_{H^j(\Omega)}\right] \le C(k). \tag{4.83}$$

Similarly, if we take $v = \partial_t^2(u_a - u_b)$, then we obtain

$$\left\|\partial_t^2(u_a - u_b)\right\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))} \le C(k)\lambda^{-(k-2)} + Ce^{C\lambda} \left\|\partial_t^2(u_a - u_b)\right\|^2_{H^1(\Sigma_1)}. \tag{4.84}$$

In terms of (4.84) and (4.82), we obtain

$$\sum_{j=2}^{3} \left\|\partial_t^j(u_a - u_b)\right\|^2_{H^1(\omega_{T_1}(\varrho,3\varrho))} \le C(k)\lambda^{-(k-2)} + Ce^{C\lambda}\epsilon(\Sigma_1). \tag{4.85}$$

Selecting

$$\lambda = \frac{1}{2C}\log\left(2 + \frac{C(k)}{\epsilon(\Sigma_1)}\right), \tag{4.86}$$

we obtain (2.22), our conclusion. The proof of (2.23) is similar.
Acknowledgements. This work has been done during the stay of the first named author at Graduate School of Mathematical Sciences of the University of Tokyo in October in 2003. The second named author was partly supported by Grants 15340027 and 15654015 from the Japan Society for the Promotion of Science and Sanwa Systems Development Co., Ltd. (Tokyo).

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