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Abstract: In this paper, we establish the Littlewood-Paley-Stein inequality on an infinite dimensional setting. We show this inequality under a weaker condition than the lower boundedness of Bakry-Emery's Γ_2. We also discuss a relationship of Sobolev norms. As an example, we handle certain infinite dimensional diffusion processes associated with stochastic partial differential equations(=SPDEs, in abbreviation).

1 Framework and the Results

In this paper, we give a remark on the Littlewood-Paley-Stein inequality. After the Meyer's celebrated work [6], many authors studied this inequality by a probabilistic approach. Especially, Shigekawa-Yoshida [8] studied to symmetric diffusion processes on a general state space. In [8], they assumed that Bakry-Emery's Γ_2 is bounded from below. To define Γ_2, they also assumed the existence of a suitable core $\mathcal{A}$ which is stable under the operation of the semigroup and the infinitesimal generator. However their assumption is serious when we face to several infinite dimensional diffusion processes. Because it is difficult to check the existence of such a core $\mathcal{A}$.

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In this paper, we show that the Littlewood-Paley-Stein inequality also holds under the *gradient estimate condition* (G) even if we do not assume that $\mathcal{A}$ is stable under the operations of the semigroup and the infinitesimal generator. Our condition seems somewhat weaker than the lower boundedness of Γ_2. So we can handle certain infinite dimensional diffusion processes represented by the solution of SPDEs. We describe the details in Section 4.

We introduce the framework that we work in this paper. Let X be a Souslin space, that is, the continuous image of a separable complete metric space. Suppose we are given a Borel probability measure μ on X and a local μ-symmetric quasi-regular Dirichlet form $\mathcal{E}$ in $L^2(X;\mu)$ with the domain $\mathcal{D}(\mathcal{E})$. Then there exists a μ-symmetric Hunt diffusion process $\mathbb{M} := (X_t, \{P_x\}_{x\in X})$ associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. We denote the infinitesimal generator and the transition semigroup by L and $\{P_t\}_{t\geq 0}$, respectively. Since $\{P_t\}$ is μ-symmetric, it can be extended to the semi-group on $L^p(X;\mu)$. We shall also denote it by $\{P_t\}$. We also denote this generator in $L^p(X;\mu)$ by L_p and the domain by $\mathrm{Dom}(L_p)$, respectively if we have to specify the acting space. We assume that $\mathbf{1} \in \mathrm{Dom}(L_p)$ and $L\mathbf{1} = 0$ where $\mathbf{1}$ denotes the function that is identically equal to 1. Hence the diffusion process $\mathbb{M}$ is conservative.

Throughout this paper, we impose the following condition.

(A): There exists a subspace $\mathcal{A}$ of $\mathrm{Dom}(L_2)$, dense in $\mathcal{D}(\mathcal{E})$, such that $f^2 \in \mathrm{Dom}(L_1)$ holds for any $f \in \mathcal{A}$.

Under this condition, the form $\mathcal{E}$ admits a carré du champ, namely there exists a unique positive symmetric and continuous bilinear form Γ from $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ into $L^1(X;\mu)$ such that

$$\mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(h, fg) = 2\int_X h\Gamma(f,g)\,d\mu$$

holds for any $f, g, h \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X;\mu)$. In particular,

$$\Gamma(f,g) = \frac{1}{2}\big\{L_1(fg) - (L_2 f)g - f(L_2 g)\big\}$$

holds for $f, g \in \mathrm{Dom}(L_2)$. For further information, see section 1.4 of Bouleau-Hirsch [3]. In the sequel, we also use the notation $\Gamma(f) := \Gamma(f,f)$ for the simplicity.

The following condition is crucial in this paper.

(G): For $f \in \mathcal{D}(\mathcal{E})$, there exists a constant $R \in \mathbb{R}$ such that the following inequality holds for any $t \in [0,\infty)$:

$$\Gamma(P_t f)^{1/2} \leq e^{Rt} P_t\big\{\Gamma(f)^{1/2}\big\}. \tag{1.1}$$

Here we give a remark on this condition. If we can see $\mathcal{A}$ is stable under the operations of $\{P_t\}$ and L,

$$\Gamma_2(f) \geq -R\Gamma(f), \quad f \in \mathcal{A} \tag{1.2}$$

implies (1.1). Especially, (1.2) means that the Ricci curvature is bounded by $-R$ from below in the case where X is a finite dimensional Riemannian manifold. See Proposition

2.3 in Bakry [2] for the detail. Hence it seems that our condition (G) is slight weaker than (1.1).

Let us introduce the Littlewood-Paley G-functions. To do this, we recall the subordination of a semigroup. For $t \geq 0$, we define a probability measure λ_t on $[0, \infty)$ by

$$\lambda_t(ds) := \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds.$$

In terms of the Laplace transform, this measure is characterized as

$$\int_0^\infty e^{-\gamma s} \lambda_t(ds) = e^{-\sqrt{\gamma} t}, \quad \gamma > 0.$$

Then for $\alpha > 0$, the subordination $\{Q_t^{(\alpha)}\}$ of $\{P_t\}$ is defined by

$$Q_t^{(\alpha)} f := \int_0^\infty e^{-\alpha s} P_s f \, \lambda_t(ds), \quad f \in L^p(X; \mu).$$

Then $\{Q_t^{(\alpha)}\}$ is a strongly continuous contraction semigroup on $L^p(X; \mu)$. The infinitesimal generator of $\{Q_t^{(\alpha)}\}$ in $L^2(X; \mu)$ is $-\sqrt{\alpha - L}$. In the case of $L^p(X; \mu)$, this operator will be clearly denoted by $-\sqrt{\alpha - L_p}$ when the dependence of p is significant.

For $f \in L^2(X; \mu) \cap L^p(X; \mu)$, we define Littlewood-Paley's G-functions as follows:

$$g_f^{\rightarrow}(x,t) := \left| \frac{\partial}{\partial t} (Q_t^{(\alpha)} f)(x) \right|, \qquad G_f^{\rightarrow}(x) := \left(\int_0^\infty t g_f^{\rightarrow}(x,t)^2 \, dt \right)^{1/2},$$

$$g_f^{\uparrow}(x,t) := \left\{ \Gamma(Q_t^{(\alpha)} f) \right\}^{1/2}(x), \qquad G_f^{\uparrow}(x) := \left(\int_0^\infty t g_f^{\uparrow}(x,t)^2 \, dt \right)^{1/2},$$

$$g_f(x,t) := \sqrt{(g_f^{\rightarrow}(x,t))^2 + (g_f^{\uparrow}(x,t))^2}, \qquad G_f(x) := \left(\int_0^\infty t g_f(x,t)^2 \, dt \right)^{1/2}.$$

Then we can obtain the following theorem. In below, the notation $\|u\|_{L^p(X;\mu)} \lesssim \|v\|_{L^p(X;\mu)}$ stands for $\|u\|_{L^p(X;\mu)} \leq C_p \|v\|_{L^p(X;\mu)}$, where C_p is a positive constant depending only on p.

Theorem 1.1 (Littlewood-Paley-Stein Inequality) *For any $1 < p < \infty$ and $\alpha > R$, the following inequalities holds for $f \in L^2(X; \mu) \cap L^p(X; \mu)$:*

$$\|G_f\|_{L^p(X;\mu)} \lesssim \|f\|_{L^p(X;\mu)}, \tag{1.3}$$

$$\|f\|_{L^p(X;\mu)} \lesssim \|G_f^{\rightarrow}\|_{L^p(X;\mu)}. \tag{1.4}$$

Although it is weaker than the celebrated Meyer equivalence of norms, we can establish the following relationship as a by-product of Theorem 1.1.

Theorem 1.2 *For any $p \geq 2$, $q > 1$ and $\alpha > R$, the following inequality holds for $f \in L^p(X; \mu)$:*

$$\left\| \Gamma\{(\sqrt{\alpha - L_p})^{-q} f\}^{1/2} \right\|_{L^p(X;\mu)} \lesssim \|f\|_{L^p(X;\mu)}.$$

This means the following inclusion holds:

$$\mathrm{Dom}\big((\sqrt{1 - L_p})^q\big) \subset W^{1,p}(X; \mu) := \big\{ f \in L^p(X; \mu) \cap \mathcal{D}(\mathcal{E}) \mid \Gamma(f)^{1/2} \in L^p(X; \mu) \big\}.$$

2 Proof of Theorem 1.1

We first prove Theorem 1.1 by a probabilistic method. The original idea is due to Meyer [6]. The reader is referred to see also Bakry [1], Shigekawa-Yoshida [8], and Yoshida [9]. Although many parts are merely repetition of them with slight modification, we give the proof for the completeness.

In the case $p = 2$, (1.3) is proved as equality by using spectral resolution of L. See Proposition 3.1 in Shigekawa-Yoshida [8] for the proof.

Using the standard duality argument, (1.4) is derived from (1.3). See Theorem 4.4 in Shigekawa-Yoshida [8] for the detail. Therefore we only need to show (1.3).

For the case $p < 2$ and $p > 2$, we need some preparation. We recall the diffusion process $\mathbb{M} = (X_t, \{P_x\}_{x\in X})$ associated to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. From now on, we write $P_x^{\uparrow}$ in place of P_x. Let $(B_t, P_a^{\rightarrow})$ be one-dimensional Brownian motion starting at $a \in \mathbb{R}$ with the generator $\frac{\partial^2}{\partial a^2}$. We shall denote the hitting time at 0 of $\{B_t\}$ by τ, namely

$$\tau := \inf\{t > 0 \mid B_t = 0\}.$$

Set $Y_t := (X_t, B_t)$, $t \geq 0$ and $\mathbb{P}_{(x,a)} := P_x^{\uparrow} \otimes P_a^{\rightarrow}$. Then $\tilde{\mathbb{M}} := (Y_t, \{\mathbb{P}_{(x,a)}\})$ is a diffusion process on $X \times \mathbb{R}$ with the (formal) generator $L + \frac{\partial^2}{\partial a^2}$. Put $\mathbb{P}_{\mu\otimes\delta_a} := \int_X \mathbb{P}_{(x,a)}\mu(dx)$.

We shall denote the integration with respect to $P_x^{\uparrow}, P_a^{\rightarrow}, \mathbb{P}_{(x,a)}$ and $\mathbb{P}_{\mu\otimes\delta_a}$ by $\mathbb{E}_x^{\uparrow}, \mathbb{E}_a^{\rightarrow}, \mathbb{E}_{(x,a)}$ and $\mathbb{E}_{\mu\otimes\delta_a}$, respectively.

The following relation is fundamental. See [6] for the proof.

Lemma 2.1 *Let $\eta : E \times [0,\infty) \to [0,\infty)$ be a measurable function. Then*

$$\mathbb{E}_{\mu\otimes\delta_a}\Big[\int_0^\tau \eta(X_t, B_t)dt\Big] = \int_X \mu(dx)\int_0^\infty (a\wedge t)\eta(x,t)\,dt \tag{2.1}$$

and

$$\mathbb{E}_{\mu\otimes\delta_a}\Big[\int_0^\tau \eta(X_t, B_t)dt\Big| X_\tau = x\Big] = \int_X \mu(dx)\int_0^\infty (a\wedge t)Q_t\eta(\cdot,t)(x)\,dt. \tag{2.2}$$

For $f \in \mathcal{A}$ and $\alpha > 0$, we consider $u(x,a) := Q_t^{(\alpha)}f(x)$. Then the following identity holds:

$$\left(\frac{\partial^2}{\partial a^2} + L - \alpha\right)u(x,a) = 0.$$

Next we set

$$M_t^{[u]} := u(X_{t\wedge\tau}, B_{t\wedge\tau}) - u(X_0, B_0) - \alpha\int_0^{t\wedge\tau} u(X_s, B_s)\,ds.$$

Then we have the following proposition.

Proposition 2.2 *$\{M_t^{[u]}\}_{t\geq 0}$ is a martingale under $\mathbb{P}_{\mu\otimes\delta_a}$ whose quadratic variation is given by*

$$\langle M^{[u]}\rangle_t = 2\int_0^{t\wedge\tau} g_f(X_s, B_s)^2\,ds.$$

Proof. In this proof, we denote one-dimensional Lebesgue measure by m. We consider the semigroup on $L^p(X \times \mathbb{R}; \mu \otimes m)$ associated to the diffusion process $\{Y_t\}$ by $\{\hat{P}_t\}$ and its generator by $\hat{L}_p$.

For $n \in \mathbb{N}$ we choose a cut-off function $\chi_n \in C^\infty(\mathbb{R})$ such that is 0 on $(-\infty, 1/(2n))$ and 1 on $(1/n, +\infty)$. We set $v_n(x, a) := u(x, a)\chi_n(a)$. Then we have the following lemma.

Lemma 2.3 (1) $v_n \in \mathrm{Dom}(\hat{L}_2)$ *and it satisfies*

$$\hat{L}_2 v_n = \alpha u \qquad \text{on} \ \ \{(x, a) \in X \times \mathbb{R} \mid a > 1/n\}.$$

(2) $v_n^2 \in \mathrm{Dom}(\hat{L}_1)$ *and it satisfies*

$$\hat{L}_1(v_n^2) = 2\Big\{ \Big(\frac{\partial u}{\partial a}\Big)^2 + \Gamma(u) + \alpha u^2 \Big\} \qquad \text{on} \ \ \{(x, a) \in X \times \mathbb{R} \mid a > 1/n\}.$$

Proof. We only give a proof of the assertion (1). Due to the condition (A), the assertion (2) can be proved in the same way.

By Theorem 2.1.3 of Pazy [7], it is sufficient to prove

$$\lim_{t \to 0} \frac{1}{t}(\hat{P}_t v_n - v_n) = \alpha v_n + 2\chi_n' \frac{\partial u}{\partial a} + \chi_n'' u \qquad \text{weakly in } L^2(X \times \mathbb{R}; \mu \otimes m), \tag{2.3}$$

since the right hand side of (2.3) is equal to αu on $\{(x, a) \in X \times \mathbb{R} \mid a > 1/n\}$.

Let $\varphi \in L^2(X \times \mathbb{R}; \mu \otimes m)$ and set $\psi(s, x, a) := \mathbb{E}_x^{\uparrow}[\varphi(X_s, a)]$. For fixed $\varepsilon > 0$, we consider the function $\Phi : \mathbb{R} \times (\varepsilon, +\infty) \to \mathbb{R}$ defined by

$$\Phi(r_1, r_2) := \int_X \{v_n(x, r_1) - v_n(x, a)\}\psi(r_2, x, a)\, \mu(dx).$$

Since we easily see $\Phi \in C_b^2(\mathbb{R} \times (\varepsilon, +\infty))$, we can apply Itô's formula to $\Phi(B_t, t)$. Then we obtain

$$\begin{aligned}
&\int_X \{v_n(x, B_t) - v_n(x, a)\}\psi(t, x, a)\mu(dx) - \int_X \{v_n(x, B_\varepsilon) - v_n(x, a)\}\psi(\varepsilon, x, a)\mu(dx) \\
&\quad = \text{martingale} + \int_\varepsilon^t \int_X \{v_n(x, B_s) - v_n(x, a)\} L\psi(s, x, a)\mu(dx)ds \\
&\qquad + \int_\varepsilon^t \int_X \Big\{\alpha v_n(x, B_s) - \chi_n(B_s)Lu(x, B_s) + 2\chi_n'(B_s)\frac{\partial u}{\partial a}(x, B_s) + \chi_n''(B_s)u(x, B_s)\Big\} \\
&\qquad\qquad \times \psi(s, x, a)\mu(dx)ds.
\end{aligned}$$

By taking the expectation with respect to $(B_t, P_a^{\rightarrow})$, we obtain

$$\begin{aligned}
&\int_X \mathbb{E}_a^{\rightarrow}[v_n(x, B_t) - v_n(x, a)]\psi(t, x, a)\mu(dx) - \int_X \mathbb{E}_a^{\rightarrow}[v_n(x, B_\varepsilon) - v_n(x, a)]\psi(\varepsilon, x, a)\mu(dx) \\
&= \int_\varepsilon^t \int_X \mathbb{E}_a^{\rightarrow}[v_n(x, B_s) - v_n(x, a)] L\psi(s, x, a)\mu(dx)ds \\
&\quad + \int_\varepsilon^t \int_X \mathbb{E}_a^{\rightarrow}\Big[\alpha v_n(x, B_s) - \chi_n(B_s)Lu(x, B_s) + 2\chi_n'(B_s)\frac{\partial u}{\partial a}(x, B_s) + \chi_n''(B_s)u(x, B_s)\Big] \\
&\qquad\qquad \times \psi(s, x, a)\mu(dx)ds.
\end{aligned} \tag{2.4}$$

Since $\{X_t\}$ is μ-symmetric and $\{X_t\}$ and $\{B_t\}$ are independent, the left hand side of (2.4) is equal to

$$\int_X \{\hat{P}_t v_n(x,a) - \mathbb{E}_x^{\uparrow}[v_n(X_t,a)]\}\varphi(x,a)\mu(dx) \\ - \int_X \{\hat{P}_\varepsilon v_n(x,a) - \mathbb{E}_x^{\uparrow}[v_n(X_\varepsilon,a)]\}\varphi(x,a)\mu(dx),$$

and the first term of the right hand side of (2.4) is

$$\int_\varepsilon^t \int_X \{L\hat{P}_s v_n(x,a) - L\mathbb{E}_x^{\uparrow}[v_n(X_s,a)]\}\varphi(x,a)\mu(dx)ds.$$

Therefore, by integrating with respect to $m(da)$ and letting $\varepsilon \to 0$ in (2.4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_X \{\hat{P}_t v_n(x,a) - \mathbb{E}_x^{\uparrow}[v_n(X_t,a)]\}\varphi(x,a)\mu(dx)m(da) \\
&\quad = \int_0^t \int_{\mathbb{R}}\int_X \{L\hat{P}_s v_n(x,a) - L\mathbb{E}_x^{\uparrow}[v_n(X_s,a)]\}\varphi(x,a)\,\mu(dx)\,m(da)\,ds \\
&\qquad + \int_0^t \int_{\mathbb{R}}\int_X \mathbb{E}_a^{\rightarrow}\Big[\alpha v_n(x,B_s) - \chi_n(B_s)Lu(x,B_s) + 2\chi_n'(B_s)\frac{\partial u}{\partial a}u(x,B_s) \\
&\qquad\qquad + \chi_n''(B_s)u(x,B_s)\Big] \times \psi(s,x,a)\,\mu(dx)\,m(da)\,ds.
\end{aligned} \tag{2.5}$$

On the other hand, since $v_n(\cdot,a) \in \mathrm{Dom}(L_2)$ for any fixed $a \in \mathbb{R}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_X \{\mathbb{E}_x^{\uparrow}[v_n(X_t,a)] - v_n(x,a)\}\varphi(x,a)\mu(dx)\,m(da) \\
&\quad = \int_0^t \int_{\mathbb{R}}\int_X \chi_n(a)\mathbb{E}_x^{\uparrow}[Lu(X_s,a)]\varphi(x,a)\,\mu(dx)\,m(da)\,ds.
\end{aligned} \tag{2.6}$$

Therefore by adding (2.5) and (2.6), dividing by t and letting $t \to 0$, we have shown (2.3). ∎

Continuation of the proof of Proposition 2.2. Let τ_n be the hitting time of $\{B_t\}$ at $1/n$, namely $\tau_n := \inf\{t > 0 \mid B_t = 1/n\}$. Then by Proposition 2.3, we can show in the same way as the proof of the first half of Fukushima-Ohshima-Takeda [4] that

$$M_t^{(n)} := u(X_{t\wedge\tau_n}, B_{t\wedge\tau_n}) - u(X_0,B_0) - \alpha\int_0^{t\wedge\tau_n} u(X_s,B_s)\,ds \tag{2.7}$$

and

$$\begin{aligned}
N_t^{(n)} :=&\, u(X_{t\wedge\tau_n}, B_{t\wedge\tau_n})^2 - u(X_0,B_0)^2 \\
&- 2\int_0^{t\wedge\tau_n}\Big\{\Big(\frac{\partial u}{\partial a}(X_s,B_s)\Big)^2 + \Gamma(u)(X_s,B_s) + \alpha u(X_s,B_s)^2\Big\}\,ds
\end{aligned} \tag{2.8}$$

are martingales. Note that (2.8) gives a semi-martingale decomposition of $u(X_{t\wedge\tau_n}, B_{t\wedge\tau_n})^2$.

On the other hand, (2.7) and Itô's formula lead us that

$$u(X_{t\wedge\tau_n}, B_{t\wedge\tau_n})^2 = u(X_0, B_0)^2 + 2\int_0^{t\wedge\tau_n} u(X_s, B_s)\, dM_s^{(n)} + 2\alpha \int_0^{t\wedge\tau_n} u(X_s, B_s)^2 ds + \langle M^{(n)}\rangle_t.$$

Therefore, by the uniqueness of semi-martingale decomposition of $u(X_{t\wedge\tau_n}, B_{t\wedge\tau_n})^2$, we have

$$\begin{aligned}\langle M^{(n)}\rangle_t &= 2\int_0^{t\wedge\tau_n} \left\{ \left(\frac{\partial u}{\partial a}(X_s, B_s)\right)^2 + \Gamma(u)(X_s, B_s)\right\} ds \\ &= 2\int_0^{t\wedge\tau_n} g_f(X_s, B_s)^2\, ds.\end{aligned}$$

Here we put $\eta = (g_f)^2$ in (2.1). Then we have

$$\begin{aligned}\mathbb{E}_{\mu\otimes\delta_a}\left[2\int_0^{t\wedge\tau} g_f(X_s, B_s)^2\, ds\right] &= \int_X \mu(dx) \int_0^\infty (s\wedge a) g_f(x, s)^2\, ds \\ &\le \int_X \mu(dx) \int_0^\infty s g_f(x, s)^2\, ds \\ &= \|G_f\|^2_{L^2(X;\mu)} = \|f\|^2_{L^2(X;\mu)} < +\infty,\end{aligned}$$

where we use (1.3) in the case of $p = 2$ for the last line. Therefore $\{M_t^{(n)}\}$ converges to a martingale as $n \to \infty$.

Finally, we note that τ_n converges to τ as $n \to \infty$ almost surely. Then $\{M_t^{(n)}\}$ converges to $\{M_t\}$ as $n \to \infty$. Thus we obtain the conclusion. ∎

We return to the proof of Theorem 1.1. Firstly, we work in the case $p < 2$. We put $U_t := u(X_{t\wedge\tau}, B_{t\wedge\tau})$. Then by Proposition 2.2, we have

$$dU_t = dM_t^{[u]} + \alpha U_t dt.$$

We apply Itô's formula to U_t^2. Then we have

$$\begin{aligned}d(U_t^2) &= 2U_t dM_t^{[u]} + 2\alpha U_t^2 dt + d\langle M^{[u]}\rangle_t \\ &= 2U_t dM_t^{[u]} + 2\big(g_f(X_t, B_t)^2 + \alpha U_t^2\big) dt.\end{aligned}$$

Let $\varepsilon > 0$. By applying Itô's formula to $(U_t^2 + \varepsilon)^{p/2}$ again, we also have

$$\begin{aligned}d\big(U_t^2 + \varepsilon\big)^{p/2} &= p\big(U_t^2 + \varepsilon\big)^{p/2-1} U_t dM_t^{[u]} \\ &\quad + p\big(U_t^2 + \varepsilon\big)^{p/2-1}\big(g_f(X_t, B_t)^2 + \alpha U_t^2\big) dt \\ &\quad + \frac{p(p-2)}{2}\big(U_t^2 + \varepsilon\big)^{p/2-2} U_t^2 d\langle M^{[u]}\rangle_t \\ &\ge p\big(U_t^2 + \varepsilon\big)^{p/2-1} U_t dM_t^{[u]} + p(p-1)\big(U_t^2 + \varepsilon\big)^{p/2-1} g_f(X_t, B_t)^2 dt,\end{aligned}$$

where we use $p < 2$ for the last line.

Hence by taking the expectation, we have

$$\begin{aligned}
&\mathbb{E}_{\mu\otimes\delta_a}\left[p(p-1)\int_0^\tau \left(U_t^2+\varepsilon\right)^{p/2-1} g_f(X_t,B_t)^2\,dt\right] \\
&\le \mathbb{E}_{\mu\otimes\delta_a}\left[(U_\tau^2+\varepsilon)^{p/2}-(U_0^2+\varepsilon)^{p/2}\right] \\
&\le \mathbb{E}_{\mu\otimes\delta_a}\left[\left(U_\tau^2+\varepsilon\right)^{p/2}\right] \\
&= \mathbb{E}_{\mu\otimes\delta_a}\left[\left(u(X_\tau,B_\tau)^2+\varepsilon\right)^{p/2}\right] \\
&= \mathbb{E}_{\mu\otimes\delta_a}\left[\left(f(X_\tau)^2+\varepsilon\right)^{p/2}\right] = \int_X (|f(x)|^2+\varepsilon)^{p/2}\,\mu(dx). \qquad (2.9)
\end{aligned}$$

Here, by recalling (2.1), the left hand side of (2.9) is equal to

$$p(p-1)\int_X \mu(dx)\int_0^\infty (t\wedge a)(u(x,t)^2+\varepsilon)^{p/2-1}g_f(x,t)^2\,dt.$$

Therefore, by letting $\varepsilon\to 0$ and $a\to\infty$, we have

$$p(p-1)\int_X \mu(dx)\int_0^\infty t|u(x,t)|^{p-2}g_f(x,t)^2\,dt \le \int_X |f(x)|^p\,\mu(dx).$$

Now we recall the maximal inequality $\|u^*\|_{L^p(X;\mu)} \lesssim \|f\|_{L^p(X;\mu)}$, where $u^*(x) := \sup_{s\ge 0}|P_s f(x)|$. Therefore

$$\begin{aligned}
\|G_f\|^p_{L^p(X;\mu)} &= \int_X \mu(dx)\left\{\int_0^\infty t|u(x,t)|^{2-p}|u(x,t)|^{p-2}g_f(x,t)^2\,dt\right\}^{p/2} \\
&\le \int_X \mu(dx)\left\{\int_0^\infty t|u^*(x)|^{2-p}|u(x,t)|^{p-2}g_f(x,t)^2\,dt\right\}^{p/2} \\
&\le \left\{\int_X |u^*(x)|^p\,\mu(dx)\right\}^{\frac{2-p}{2}}\left\{\int_X\int_0^\infty t|u(x,t)|^{p-2}g_f(x,t)^2\,dt\,\mu(dx)\right\}^{p/2} \\
&\lesssim \left\{\int_X |f(x)|^p\,\mu(dx)\right\}^{\frac{2-p}{2}}\left\{\int_X |f(x)|^p\,\mu(dx)\right\}^{p/2} = \|f\|^p_{L^p(X;\mu)}.
\end{aligned}$$

This completes the proof of (1.3) in the case $p < 2$.

In the case $p > 2$, we need additional functions, namely H-functions defined by

$$\begin{aligned}
H_f^{\rightarrow}(x) &:= \left\{\int_0^\infty tQ_t^{(0)}(g_f^{\rightarrow}(\cdot,t)^2)(x)\,dt\right\}^{1/2}, \\
H_f^{\uparrow}(x) &:= \left\{\int_0^\infty tQ_t^{(0)}(g_f^{\uparrow}(\cdot,t)^2)(x)\,dt\right\}^{1/2}, \\
H_f(x) &:= \left\{\int_0^\infty tQ_t^{(0)}(g_f(\cdot,t)^2)(x)\,dt\right\}^{1/2}.
\end{aligned}$$

Then we have the following proposition.

Proposition 2.4 *For $p > 2$ and $\alpha > R$, there exists a constant c_p, depending only on p, such that the following inequality holds for any $f \in L^p(X;\mu)$;*

$$\|H_f\|_{L^p(X;\mu)} \lesssim \|f\|_{L^p(X;\mu)}.$$

Due to (2.2) and Proposition 2.2, this proposition can be proved in the same way as the proof of Proposition 4.2 in Shigekawa-Yoshida [8], replacing A_t by $\alpha \int_0^{t\wedge\tau} u(X_s, B_s)^2\, ds$. So we shall omit the proof.

Next we study the relationship between G-functions and H-functions. In the proof of this proposition, the condition (G) plays a key role.

Proposition 2.5 (1) *For any $f \in \mathcal{A}$, the following inequality holds;*

$$G_f^{\uparrow} \leq 2H_f^{\uparrow}.$$

(2) *For any $f \in \mathcal{A}$ and $\alpha > R$, the following inequality holds;*

$$G_f^{\rightarrow} \leq 2H_f^{\rightarrow}.$$

Proof. We shall only give a proof of the assertion (1). The assertion (2) can be proved in the same way.

Taking the condition (G) into account, we have the following estimate for any $\alpha > R$ and $f \in \mathcal{A}$:

$$\begin{aligned}
\Gamma(Q_t^{(\alpha)} f)^{1/2} &\leq \int_0^\infty e^{-\alpha s} \Gamma(P_s f)^{1/2}\, \lambda_t(ds) \\
&\leq \int_0^\infty e^{-\alpha s} \big\{ e^{Rt} P_s\big(\Gamma(f)^{1/2}\big) \big\}\, \lambda_t(ds) \\
&= Q_t^{(\alpha - R)}\big(\Gamma(f)^{1/2}\big).
\end{aligned} \tag{2.10}$$

Combining (2.10) and Schwarz's inequality, we have

$$\begin{aligned}
g_f^{\uparrow}(x, 2t)^2 &= \Gamma(Q_{2t}^{(\alpha)} f)(x) \\
&= \Gamma\big\{Q_t^{(\alpha)}(Q_t^{(\alpha)} f)\big\}(x) \\
&\leq Q_t^{(\alpha - R)}\big\{\Gamma(Q_t^{(\alpha)} f)\big\}(x) \\
&= Q_t^{(0)}\big(g_f^{\uparrow}(\cdot, t)^2\big)(x).
\end{aligned} \tag{2.11}$$

From this estimate, we have

$$\begin{aligned}
(G_f^{\uparrow}(x))^2 &= 4 \int_0^\infty t g_f^{\uparrow}(x, 2t)^2\, dt \\
&\leq 4 \int_0^\infty t Q_t^{(0)}\big(g_f^{\uparrow}(\cdot, t)^2\big)(x)\, dt \\
&= 4(H_f^{\uparrow}(x))^2,
\end{aligned}$$

where we change the variable t to $2t$ in the first line, and use (2.11) in the second line. This completes the proof. ∎

It is clear that Proposition 2.4 and Proposition 2.5 concludes (1.3) in the case $p > 2$. This completes of the proof of Theorem 1.1.

3 Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we make a preparation parallel to Yoshida [9]. Let ν be a finite signed measure on $[0,\infty)$. We denote by $\hat{\nu}$ and $\|\nu\| := \int_0^\infty |\nu|(ds)$ the Laplace transform and the total variation of ν, respectively. For $\alpha > 0$, we define a bounded operator $\hat{\nu}(\alpha - L)$ on $L^p(X;\mu)$, $1 \le p < \infty$ by

$$\hat{\nu}(\alpha - L)f := \int_{[0,\infty)} e^{-\alpha s} P_s f \; \nu(ds).$$

Thus we easily have

$$\|\hat{\nu}(\alpha - L)f\|_{L^p(X;\mu)} \le \|\nu\| \cdot \|f\|_{L^p(X;\mu)}, \;\; f \in L^p(X;\mu). \tag{3.1}$$

Here we give a remark in the case of $p = 2$. In this case, this operator is represented by

$$\hat{\nu}(\alpha - L) := \int_{[0,\infty)} \hat{\nu}(\alpha + \lambda) dE_\lambda,$$

where $\{E_\lambda\}_{\lambda \ge 0}$ is the spectral decomposition of $-L$ in $L^2(X;\mu)$.

By Lemma 2.3 in [1], there exist finite signed measures ν_1 and ν_2 such that the Laplace transform are given by $\hat{\nu}_1(\lambda) = \frac{\sqrt{1+\lambda}}{1+\sqrt{\lambda}}$ and $\hat{\nu}_2(\lambda) = \frac{1+\sqrt{\lambda}}{\sqrt{1+\lambda}}$, respectively. For $\varepsilon > 0$, we denote by $\nu_i^{(\varepsilon)}$, $i = 1, 2$ the image measure of ν_i under the mapping $\lambda \mapsto \lambda/\varepsilon$. Then we have

$$\hat{\nu}_1^{(\varepsilon)}(\lambda) = \frac{\sqrt{\varepsilon + \lambda}}{\sqrt{\varepsilon} + \sqrt{\lambda}}, \quad \|\nu_1^{(\varepsilon)}\| \le \|\nu_1\|, \tag{3.2}$$

$$\hat{\nu}_2^{(\varepsilon)}(\lambda) = \frac{\sqrt{\varepsilon} + \sqrt{\lambda}}{\sqrt{\varepsilon + \lambda}}, \quad \|\nu_2^{(\varepsilon)}\| \le \|\nu_2\|. \tag{3.3}$$

(3.1), (3.2) and (3.3) imply the resulting operators $\frac{\sqrt{\varepsilon+(\alpha-L)}}{\sqrt{\varepsilon}+\sqrt{\alpha-L}}$ and $\frac{\sqrt{\varepsilon}+\sqrt{\alpha-L}}{\sqrt{\varepsilon+(\alpha-L)}}$ on $L^p(X;\mu)$ have the operator norms not more than $\|\nu_1\|$ and $\|\nu_2\|$, respectively. We also have

$$\Big(\frac{\sqrt{\varepsilon + (\alpha - L)}}{\sqrt{\varepsilon} + \sqrt{\alpha - L}}\Big)\Big(\frac{\sqrt{\varepsilon} + \sqrt{\alpha - L}}{\sqrt{\varepsilon + (\alpha - L)}}\Big) = \Big(\frac{\sqrt{\varepsilon} + \sqrt{\alpha - L}}{\sqrt{\varepsilon + (\alpha - L)}}\Big)\Big(\frac{\sqrt{\varepsilon + (\alpha - L)}}{\sqrt{\varepsilon} + \sqrt{\alpha - L}}\Big) = I.$$

Then we obtain the following relation for $q > 1$:

$$\begin{aligned}\big(\sqrt{\varepsilon + (\alpha - L)}\big)^{-q} &= (\sqrt{\varepsilon} + \sqrt{\alpha - L})^{-q}\Big(\frac{\sqrt{\varepsilon + (\alpha - L)}}{\sqrt{\varepsilon} + \sqrt{\alpha - L}}\Big)^{-q} \\ &= (\sqrt{\varepsilon} + \sqrt{\alpha - L})^{-q}\Big(\frac{\sqrt{\varepsilon} + \sqrt{\alpha - L}}{\sqrt{\varepsilon + (\alpha - L)}}\Big)^{q}. \end{aligned} \tag{3.4}$$

Now we are in a position to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Firstly, we fix $\beta \in \mathbb{R}$ and $\varepsilon > 0$ such that $\alpha = \beta + \varepsilon$ and $\beta > R$. Let $f \in L^2(X;\mu) \cap L^p(X;\mu)$ and put

$$g := \Big(\frac{\sqrt{\varepsilon} + \sqrt{\beta - L}}{\sqrt{\varepsilon + (\beta - L)}}\Big)^q f.$$

By (3.4), we have

$$\begin{aligned}\Gamma\big\{(\sqrt{\alpha-L})^{-q}f\big\} &= \Gamma\big\{(\sqrt{\varepsilon}+\sqrt{\beta-L})^{-q}g\big\}\\ &\le \left\{\frac{1}{\Gamma(q)}\int_0^\infty t^{q-1}e^{-\sqrt{\varepsilon}t}\Gamma(Q_t^{(\beta)}g)^{1/2}\,dt\right\}^2.\end{aligned}$$

Using Theorem 1.1, we have the following estimate by recalling $q>1$.

$$\begin{aligned}\big\|\Gamma\big\{\sqrt{\alpha-L})^{-q}f\big\}^{1/2}\big\|_{L^p(d\mu)} &\le \frac{1}{\Gamma(q)}\Big\|\int_0^\infty t^{q-1}e^{-\sqrt{\varepsilon}t}\Gamma(Q_t^{(\beta)}g)^{1/2}dt\Big\|_{L^p(X;\mu)}\\ &\le \frac{1}{\Gamma(q)}\Big\|\Big(\int_0^\infty t^{2q-3}e^{-\varepsilon t}dt\Big)^{1/2}\Big(\int_0^\infty t\Gamma(Q_t^{(\beta)}g)dt\Big)^{1/2}\Big\|_{L^p(X;\mu)}\\ &= \frac{1}{\Gamma(q)}\cdot\Big(\frac{\Gamma(2q-2)}{\varepsilon^{2q-2}}\Big)^{1/2}\|G_g^{\uparrow}\|_{L^p(X;\mu)}\\ &\lesssim \|g\|_{L^p(X;\mu)}\\ &\le \|\nu_1\|^q\cdot\|f\|_{L^p(X;\mu)}.\end{aligned}$$

This completes proof. ∎

4 Example: Diffusion Process associated with SPDE

In this section, we give an example. This is studied in Kawabi [5]. We consider the solution of an infinite dimensional stochastic differential equation which is called the time dependent Ginzburg-Landau type SPDE

$$dX_t(x)=\big\{\Delta_x X_t(x)-\nabla U(X_t(x))\big\}dt+\sqrt{2}dW_t(x),\quad x\in\mathbb{R},\ t>0, \tag{4.1}$$

where $U(z):\mathbb{R}^d\to\mathbb{R}$, $\Delta_x=d^2/dx^2$, $\nabla=(\partial/\partial z_i)_{i=1}^d$ and $(W_t)_{t\ge0}$ is a white noise process. This dynamics is called the $P(\phi)_1$-time evolution which has its origin in Parisi and Wu's stochastic quantization model.

At the beginning, we introduce a triplet (E,H,μ). (In this section, we denote the state space by E instead of X.) For fixed $\bar\lambda>0$, we introduce a Hilbert space $E:=L^2(\mathbb{R},e^{-2\bar\lambda\chi(x)}dx),\bar\lambda>0$, where $\chi\in C^\infty(\mathbb{R},\mathbb{R})$ is a positive symmetric convex function satisfying $\chi(x)=|x|$ for $|x|\ge1$. E has an inner product defined by

$$(X,Y)_{\bar\lambda}:=\int_{\mathbb{R}}\big(X(x),Y(x)\big)_{\mathbb{R}^d}e^{-2\bar\lambda\chi(x)}dx,\quad X,Y\in E.$$

We also define a suitable subspace of $C(\mathbb{R},\mathbb{R}^d)$. For functions of $C(\mathbb{R},\mathbb{R}^d)$, we define

$$|||X|||_\lambda:=\sup_{x\in\mathbb{R}}|X(x)|e^{-\lambda\chi(x)}\quad\text{for }\lambda>0.$$

Let

$$\mathcal{C}:=\bigcap_{\lambda>0}\big\{X(x)\in C(\mathbb{R},\mathbb{R}^d)\mid |||X|||_\lambda<\infty\big\}.$$

With the system of norms $|\!|\!| \cdot |\!|\!|_\lambda$, $\mathcal{C}$ becomes a Fréchet space. We easily see that the densely inclusion $\mathcal{C} \subset E \cap C(\mathbb{R}, \mathbb{R}^d)$ holds with respect to the topology of E. We regard these spaces as the state spaces of our dynamics.

Let μ be a (U-)Gibbs measure. This means that the regular conditional probability satisfies the following DLR-equation for every $r \in \mathbb{N}$ and μ-a.e. $\xi \in \mathcal{C}$:

$$\mu(dw|\mathcal{B}_r^*)(\xi) = Z_{r,\xi}^{-1} \exp\big(- \int_{-r}^{r} U(w(x))dx \big) \mathcal{W}_{r,\xi}(dw),$$

where $\mathcal{B}_r^*$ is the σ-field generated by $\mathcal{C}|_{[-r,r]^c}$, $\mathcal{W}_{r,\xi}$ is the path measure of the Brownian bridge on $[-r, r]$ with a boundary condition $\mathcal{W}_{r,\xi}\big(w(r) = \xi(r), w(-r) = \xi(-r)\big) = 1$ and $Z_{r,\xi}$ is the normalization constant.

Next we impose the conditions on the potential function U as follows:

(U1) $U \in C^2(\mathbb{R}^d, \mathbb{R})$ and there exists a constant $K_1 \in \mathbb{R}$ such that $\nabla^2 U(z) \geq -K_1$ holds for any $z \in \mathbb{R}^d$.

(U2) There exist $K_2 > 0$ and $p > 0$ such that $|\nabla U(z)| \leq K_2(1 + |z|^p)$ holds for any $z \in \mathbb{R}^d$.

(U3) $\lim_{|z|\to\infty} U(z) = \infty$.

Typical example of U satisfying above conditions is a double-well potential. That is, $U(z) = a(|z|^4 - |z|^2)$, $a > 0$.

The conditions (U1) and (U2) imply that the SPDE (4.1) has the unique solution $X \in C([0, \infty), \mathcal{C})$ for an initial data $w \in \mathcal{C}$. Moreover if we add the condition (U3), a Gibbs measure μ is a reversible measure of this dynamics.

Now we introduce the relationship between this dynamics and a certain Dirichlet form. We denote $H := L^2(\mathbb{R}, \mathbb{R}^d)$ and

$$\begin{aligned} \mathcal{FC}_b^\infty \;\; := \;\; \Big\{ f(w) = \tilde{f}(\langle w, \phi_1 \rangle, \cdots, \langle w, \phi_n \rangle) \;\Big|\; & n \in \mathbb{N},\ \{\phi_k\}_{k=1}^\infty \subset C_0^\infty(\mathbb{R}), \\ f = \tilde{f}(\alpha_1,\ \cdots, \alpha_n) \in C_b^\infty(\mathbb{R}^n),\ & \langle w, \phi_k \rangle := \int_{\mathbb{R}} (w(x), \phi_k(x))_{\mathbb{R}^d} dx \Big\}. \end{aligned}$$

For $f \in \mathcal{FC}_b^\infty$, we define the Fréchet derivative $Df : E \longrightarrow H$ by

$$Df(w)(x) := \sum_{k=1}^{n} \frac{\partial \tilde{f}}{\partial \alpha_k}(\langle w, \phi_1 \rangle, \cdots, \langle w, \phi_n \rangle)\phi_k(x), \quad x \in \mathbb{R}. \tag{4.2}$$

We consider a symmetric bilinear form $\mathcal{E}$ which is given by

$$\mathcal{E}(f) = \int_E |Df(w)|_H^2 \mu(dw), \quad f \in \mathcal{FC}_b^\infty.$$

We also define $\mathcal{E}_1(f) := \mathcal{E}(f) + \|f\|_{L^2(X;\mu)}^2$ and $\mathcal{D}(\mathcal{E})$ by the completion of $\mathcal{FC}_b^\infty$ with respect to $\mathcal{E}_1^{1/2}$-norm. For $f \in \mathcal{D}(\mathcal{E})$, we also denote Df by the closed extension of (4.2).

By virtue of the $C_0^\infty(\mathbb{R}, \mathbb{R}^d)$-quasi-invariance and the strictly positive property of the Gibbs measure μ, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(E; \mu)$, i.e., $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a closed

Markovian symmetric bilinear form. Hence the condition (A) holds. Moreover our dynamics is associated with this Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. See Proposition 2.3 in [5] for the detail. We note that $\Gamma(f) = |Df|_H^2$ in this case.

Then the following gradient estimate of the transition semigroup $\{P_t\}$ holds for any $f \in \mathcal{D}(\mathcal{E})$:

$$|D(P_t f)(w)|_H \leq e^{K_1 t} P_t\big(|Df|_H\big)(w) \quad \text{for } \mu\text{-a.e. } w \in E.$$

See Proposition 2.4 in [5] for the detail. Therefore Theorem 1.1 and Theorem 1.2 hold for $\alpha > K_1$.

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