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Compact Lie group actions on compact manifolds of nonpositive curvature

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Compact Lie Group Actions on Compact Manifolds of Nonpositive Curvature

Dedicated to Professor T. Ochiai on his sixtieth birthday

By XU Bin

Abstract

Let G be a homotopically trivial and effective compact Lie group action on a compact manifold N of nonpositive curvature. Under certain assumptions on N we prove that if G has dimension equal to rank of Center $\pi_1(N)$, then G must be connected. Furthermore, if on N there exists a point having negative definite Ricci tensor, then we show that G is the trivial group.

1 Introduction

A closed manifold N is *aspherical* if its universal covering $\tilde{N}$ is contractible. A lot of information has been obtained by Conner-Raymond [2]-[4] regarding group actions on such manifolds. One of their main results is that a connected compact Lie group acting effectively on a compact aspherical manifold is a torus of dimension not greater than rank of Center $\pi_1(N)$. On the other hand a classical theorem of Bieberbach states that any finite group can act freely on some torus, which is a simple example of aspherical manifold. A group action on N is *homotopically trivial* if any element of this group, which defines a homeomorphism of N, is homotopic to the identity. Gottlieb-Lee-Özaydin [9] showed that a compact Lie group acting effectively and homotopically trivially on a compact aspherical manifold is Abelian.

A celebrated theorem due to Hadamard and E. Cartan says that any simply connected complete Riemannian manifold of nonpositive sectional curvature is

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diffeomorphic to the Euclidean space. Compact manifolds of nonpositive sectional curvature become a special class of aspherical manifolds, on which we shall refine the result of Gottlieb-Lee-Özaydin in two ways as follows. The first way by which we shall do the refinement is to consider the compact Lie group actions on the nonpositive curved manifolds which have "negative" curvatures in the sense of the following

Definition 1.1 We call a Riemannian manifold *quasi-negatively curved* if and only if it is of nonpositive sectional curvature which is strictly negative at some point. We call a Riemannian manifold *Ricci-negatively curved* if and only if it is of nonpositive sectional curvature and has negative definite Ricci tensor. We call a Riemannian manifold *quasi-Ricci-negatively curved* if and only if it is of nonpositive sectional curvature and has a point at which the Ricci tensor is negative definite.

If a compact Riemannian manifold has negative semi-definite Ricci tensor and has a point at which the Ricci tensor is negative definite, then Bochner [1] showed its isometry group is finite. In [14] Sampson used his rigidity theorem (cf Theorem 4 in section [14]) of harmonic mappings to a compact quasi-negatively curved manifold to show that its isometry group is finite, and no two of its elements are homotopic. Due to Frankel [8], the same result as Sampson's also holds for a compact Ricci-negatively curved manifold. These two results of Samspon and Frankel have intersection with Gottlieb-Lee-Özaydin's that any homotopically trivial element in the isometry groups of the above two classes of Riemannian manifolds must be the identity. In section 2 we shall establish a rigidity theorem (cf Theorem 2.1) of harmonic mappings to a quasi-Ricci-negatively curved manifold, from which we generalize the results by Sampson and Frankel, and refine the one by Gottlieb-Lee-Özaydin in the following

Theorem 1.1 *Let (N, g_0) be a compact, connected and quasi-Ricci-negatively curved Riemannian manifold and G a compact Lie group acting effectively and smoothly on it. Then G is finite and no two elements of it are homotopic.*

In [12] Lawson and Yau showed that the isometry group $I(N)$ of a compact manifold N of nonpositive sectional curvature has dimension equal to rank of Center $\pi_1(N)$, and that the identity component $I^0(N)$ is a torus which is generated by the parallel vector fields on N. By Conner-Raymond's result, we know that the isometry group $I(N)$ attains the maximal dimension among the compact group actions on N. A simple but nontrivial example of Gottlieb-Lee-Özaydin's result, is the group $\langle \exp(2\pi\sqrt{-1}/m)\rangle \cong \mathbf{Z}/m\mathbf{Z}$ acting on the circle $S^1 = \{\exp(\sqrt{-1}\theta) : \theta \in [0,\ 2\pi)\}$ by multiplication. We are suggested by Lawson-Yau's result to let G attain the maxiaml dimension equal to rank of Center $\pi_1(N)$, by which we refine the result of Gottlieb-Lee-Özaydin as the following

Theorem 1.2 *Let (N, g_1) be a real analytic Riemannian compact connected manifold of nonpositive sectional curvature and G a compact Lie group acting effectively, real analytically and homotopically trivially on N. Suppose that either N is orientable or N has dimension 2 and that the dimension of G is equal to rank of* Center $\pi_1(N)$. *Then G must be connected.*

Remark 1.1 It might be unnecessary in Theorem 1.2 that we assume the restrictions on N as following:
• the Riemannian metric g_1 is real analytic,
• N is orientable when dim $N > 2$.
They are originated from the tools (Theorem SY in section 2 and Lemma 3.1) of proving this theorem.

This paper is organized as follows. In Section 2 we cite the rigidity theorem of Schoen-Yau [15] and prove a rigidity theorem (cf Theorem 2.1) of the harmonic mappings to compact quasi-Ricci-negatively curved manifolds. The first one of them will be used to prove Theorem 1.2, and the other to Theorem 1.1. In Section 3, we prepare two lemmata for Theorem 1.1. The last section of this paper consists of the proof to Theorem 1.1 and Theorem 1.2 and a generalization of Theorem 1.1 (cf Theorem 4.1).

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2 Rigidity theorems of harmonic mappings

Theorem SY (cf Theorem 4 in Schoen-Yau [15]) *Suppose M, N are compact connected real analytic Riemannian manifolds and N has nonpositive sectional curvatures. Suppose $h : M \to N$ is a surjective harmonic map and its induced map $h_* : \pi_1(M) \to \pi_1(N)$ is also surjective. Then the space of surjective harmonic maps homotopic to h is represented by $\{\beta \circ h | \beta \in I^0(N)\}$.*

Theorem 2.1 *Let $h_0 : M \to N$ be a harmonic mappings, where M is compact and N of of nonpositive sectional curvature . Suppose that there exists a point p in M such that the followings hold:*
(a) *at $h_0(p)$ the Ricci tensor of N is negative definite,*
(b) *the differential map $dh_0(p)$ of h_0 at p is surjective.*
Then h_0 is the only harmonic mapping in its homotopy class.

Before the proof of the above theorem, we make a quick review on the formula for second variation of the energy (cf J. Jost [10]) for a family of harmonic mappings.

Let M be a compact, and N a complete Riemannian manfiolds of dimension m and n, respectively. In local coordinates, the metric tensor of M is written as

$$(\gamma_{\alpha\beta})_{\alpha,\beta=1,\cdots,m} \ ,$$

and the one of N as

$$(g_{ij})_{i,j=1,\cdots,n} \ .$$

We shall use the notation

$$(\gamma^{\alpha\beta})_{\alpha,\beta=1,\cdots,m} = (\gamma_{\alpha\beta})^{-1}_{\alpha,\beta=1,\cdots,m} \text{ (inverse metric tensor)}.$$

Let $f : M \to N$ be a smooth map and $f^{-1}TN$ the pullback bundle on M by f of the tangent bundle TN of N. $f^{-1}TN$ has the metric $(g_{ij}(f(x))$, and the cotangent bundle T^*M of course has the metric $(\gamma_{\alpha\beta})$. Then the energy density $e(f)$ is defined as

$$\frac{1}{2}||df||^2 \ ,$$

which is the square of the norm of the differential of f as a section of the Riemannian bundle $T^*M \otimes f^{-1}TN$. The *energy* of $f : M \to N$ is

$$E(f) := \int_M e(f) dM$$

with dM the volume form of M. The smooth map $f : M \to N$ is *harmonic* if and only if it is a critical point of the energy functional E.

Let

$$\begin{array}{rcl} F_t(x) & = & F(x,t) \\ F & : & M \times (-\epsilon, \epsilon) \to N \end{array}$$

be a family of smooth maps between Riemannian manfiolds, in which

$$F_0(x) = F(x,0) = f \ .$$

Then $W := \frac{\partial F}{\partial t}|_{t=0}$ is a section of $f^{-1}TN$. Let ∇ denote the Levi-Civita connection in $f^{-1}TN$ and R^N the curvature tensor of N. Then we have the following

Fact E For the second variation of energy the equality

$$\frac{\partial^2}{\partial t^2} E(F_t)|_{t=0} = \int_M ||\nabla W||^2_{f^{-1}TN} - \int_M \text{trace}_M \langle R^N(df, W)W, df\rangle_{f^{-1}TN}$$

holds provided that $F(x, \cdot)$ is geodesic for every x.

Then we recall a result on homotopic harmonic mappings by Hartman [7].

Fact H (cf [10]) Assume that N is a complete manifold of nonpositive sectional curvature. Let $f_0, f_1 : M \to N$ be homotopic harmonic mappings. Then there exists a family $f_t : M \to N,\ t \in [0,\ 1]$, of harmonic mappings connecting them, for which the energy $E(f_t)$ is independent of t, and for which every curve $\gamma_x(t) := f_t(x)$ is geodesic, and $||\frac{\partial}{\partial t}\gamma_x(t)||$ is independent of x and t.

Proof of Theorem 2.1 Let $h_1 : M \to N$ be a harmonic map homotopic to h_0. We can find a family $h_t : M \to N,\ t \in [0,\ 1]$, of harmonic mappings connecting them with the property as Fact H. By Fact E, since N has nonpositive sectional curvature,

$$\begin{aligned} 0 &= \frac{d^2}{dt^2}E(h_t) \\ &= \int_M (||\nabla \frac{\partial}{\partial t}\gamma_x(t)||^2 - \mathrm{trace}_M \langle R^N(dh_t, \frac{\partial}{\partial t}\gamma_x(t))\frac{\partial}{\partial t}\gamma_x(t), dh_t\rangle_{f^{-1}TN}) \geq 0\ . \end{aligned}$$

Hence we obtain that for any $x \in M$ and any $t \in [0,\ 1]$,

$$\mathrm{trace}_M \langle R^N(dh_t, \frac{\partial}{\partial t}\gamma_x(t))\frac{\partial}{\partial t}\gamma_x(t), dh_t\rangle_{f^{-1}TN}) = 0,$$

in particular,

$$\mathrm{trace}_M \langle R^N(dh_0(p), \frac{\partial}{\partial t}\gamma_p(t))\frac{\partial}{\partial t}\gamma_p(t), dh_0(p)\rangle_{f^{-1}TN} = 0\ .$$

By assumptions (a), (b) and the following Fact T, we can see that the tangent vector $\frac{\partial}{\partial t}\gamma_p(t)$ of the geodesic $\gamma_p(t)$ vanishes. By Fact H every geodesic $\gamma_x(t),\ x \in M,\ t \in [0,\ 1]$ degenerates into a point. That is, $h_0 = h_1$.

Fact T Let R be a $n \times n$ real symmetric matrix having nonpostive eigenvalues and negative trace. Suppose B is a $m \times n$ real matrix with full rank. Then the matrix BRB^t also has negative trace, where B^t is the transposed matrix of B.

Proof Without loss of generality, we can assume R to be the diagonal matrix

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

in which $\lambda_1 < 0$ and $\lambda_2, \cdots, \lambda_n \leq 0$. We denote $B = (b_{ij})_{1 \leq i \leq m,\ 1 \leq j \leq n}$. Since B has rank n, there exists a nonzero element b_{i1} in the first row of B. By computation, we know

$$\mathrm{trace}\ BRB^t \leq \lambda_1 b_{i1}^2 < 0\ .$$

3 Two lemmata

Lemma 3.1 *Let M and N be compact connected smooth Riemannian manifolds of the same dimension and $f : M \to N$ a smooth mapping. Assume deg $f = m \neq 0$ if M is orientable, and deg $f \equiv 1 \pmod 2$ if M is non-orientable. Define a subgroup A of $I(M)$ by*

$$A = \{\alpha \in I(M) | f \circ \alpha = f\}.$$

Then if M is orientable, the order of A divides m. In particular, in case of deg $f = \pm 1$, the group A is trivial. If M is non-orientable, the order of A is an odd integer.

Proof. We firstly prove the part in which M is orientable. Taking a regualr value $y_0 \in N$ whose preimages under f are $x_1, \cdots, x_{2k+m}$, then

$$m = \text{deg } f = \Sigma_{i=1}^{2k+m} \text{sgn det } Jf(x_i),$$

where $Jf(x_i)$ is the Jacobian matrix of f at point x_i. Without loss of generality, we set $m \geq 1$ and

$$\text{sgn det } Jf(x_i) = 1, \text{ for } 1 \leq i \leq k + m;$$

$$\text{sgn det } Jf(x_j) = -1, \text{ for } k + m + 1 \leq j \leq 2k + m.$$

The group A acts on the set $f^{-1}(y_0)$ by the definition of A. We claim that A acts freely on this set. In fact if an element $\beta \in A$ has x_1 as its fixed point, then its differential at point x_1 is the identity map since $df(x_1) \circ d\beta(x_1) = df(x_1)$. Since M is connected, β must be the identity of M.

We also claim that A should preserve the orientation of M. Otherwise let $\bar{A}$ be the subgroup of A whose elements preserve the orientation of M, then we have $[A : \bar{A}] = 2$. Hence we can set

$$\bar{A} = \{g_1, \cdots, g_n\}, \quad A = \{g_1, \cdots, g_n, gg_1, \cdots, gg_n\}$$

and then the equalities

$$\text{sgn det } Jf(x) = \text{sgn det } Jf(g_i(x)), \text{ sgn det } Jf(x) = -\text{sgn det } Jf(gg_i(x))$$

hold for any $x \in f^{-1}(y_0)$ and $1 \leq i \leq n$. As A acts freely on $f^{-1}(y_0)$, we obtain deg f=0 and a contradiction. Since A preserves the orientation of M and acts freely on the set $\{x_1, \cdots, x_{2k+m}\}$, A should act freely on the two set $\{x_1, \cdots, x_{k+m}\}$ and $\{x_{k+m+1}, \cdots, x_{2k+m}\}$ respectively so that the order of A divides m.

When M is non-orientable, we know the number of the set $f^{-1}(y_0)$ is odd because of deg $f \equiv 1 \pmod 2$. By the same way, we can show that A acts freely on $f^{-1}(y_0)$ so that the order of A is also odd.

Remark 3.1 Schoen and Yau proved a result (cf. Theroem 4 (i) of [15]) more general than Lemma 3.1. However we prefer a simple proof for our case.

Lemma 3.2 *Let M be a non-orientable smooth manifold and M' its orientable double covering. Then a diffeomorphism of M can be lifted to that of M'.*

Proof. Let Γ and Γ' be the fundamental groups of M and M' respectively, which are two deck transformation groups acting freely and discontinuously on the universal covering space $\tilde{M}$ of M and M'. And Γ' is the normal subgroup of Γ of index 2 which consists of all the orientation-preserving elements in Γ . Let $f : M \to M$ be a diffeomorphism and $\tilde{f} : \tilde{M} \to \tilde{M}$ a lifting on $\tilde{M}$ of f. If an element γ of Γ preserves the orientation of $\tilde{M}$, then so does the one $\tilde{f} \circ \gamma \circ \tilde{f}^{-1}$ of Γ. That is, $\tilde{f} \circ \Gamma' \circ \tilde{f}^{-1} = \Gamma'$. Therefore there exists a diffeomorphism f' of M' such that $\tilde{f}$ is one of its liftings on $\tilde{M}$, which tells us that f' is the lifting on M' of f.

4 Compact group actions

Proof of Theorem 1.1 At first we prove the finiteness of G. We only need to show that the identity component G^0 is trivial. Since G is a compact Lie group acting on N, we can choose a Riemannian metric g such that G action on (N, g) is isometric. By Eells-Sampson [6], there exists a harmonic mapping $h : (N, g) \to (N, g_0)$ homotopic to the identity so that h is surjective. We shall prove

Claim 1 h is the only harmonic mapping in its homotopy class.

Since (N, g_0) is quasi-Ricci-negatively curved , there exists an open subset U of N on which (N, g_0) has negative definite Ricci tensor. On the other hand, by virtue of Sard's theorem and the surjectivity of h, we know the regular value set of h is a dense subset of N so that there exists a regular point of h whose image belongs to U. Then we apply Theorem 2.1 complete the proof of Claim 1.

By Claim 1 G^0 is contained in the set

$$\{\alpha \in G : h \circ \alpha = h\},$$

which is a finite set by Lemma 3.1. That is, G^0 is trivial.

Then we prove that no two element of G are homotopic. We only need to prove that an element α of G homotopic to the identity is the identity. If N is orientable, then by Claim 1 and Lemma 3.1 it follows from that h has degree 1. In the following we assume that N is non-orientable. Let N' be its orientable double covering space and $\gamma : N' \to N'$ the deck transformation. By Lemma 3.2 α has two liftings $\tilde{\alpha}, \gamma \circ \tilde{\alpha}$ on N', in which $\tilde{\alpha}$ is homotopic to $\mathrm{id}_{N'}$. Since the order of $\tilde{\alpha}$ is equal to that of $\alpha \le |G|$, by the previous result in the orientable case, we can see that $\tilde{\alpha}$ is $\mathrm{id}_{N'}$ and then α is the identity of N. QED

By Lemma 3.1 and the above argument, in fact we have proved

Theorem 4.1 *Let M, N be compact connected Riemannian manifolds. Suppose that $h : M \to N$ be the only harmonic mapping in its homotopy class. Suppose that*
(i) *h have degree 1 in case that N is orientable,*
(ii) *M is diffeomorphic to N and h is homotopic the identity in case that N is non-orientable.*
Then the isometry group of M is finite, and no two elements of it are homotopic.

Let (N, g_1) be a real analytic Riemannian manifold of of nonpositive sectional curvature and G a Lie compact group acting effectively, real analytically and homotopically trivally on N. As in the proof of Theorem 1.1 we also have a harmonic mapping $h : (N, g) \to (N, g_1)$ homotopic to the identity and G acts isometrically on (N, g). For an element α of G, since h and $h \circ \alpha$ are two homotopic harmonic mappings satisfying the condition of Theorem SY, there exists one and only one element β of $I^0(N, g_1)$ such that

$$h \circ \alpha = \beta \circ h \ ,$$

which leads to a Lie group homomorphism

$$\rho : G \to I^0(N, g_1), \ \rho(\alpha) := \beta \ .$$

We have the following

Claim 2 If N is orientable, then ρ is a monomorphism.

Claim 2 follows from that the kernel of ρ is trivial by Lemma 3.1.

Proof of Theorem 1.2 If N is orientable, then by Claim 2 we have a monomorphism $\rho : G \to I^0(N, g_1)$, the image $I^0(N)$ of which is a torus of dimension equal to rank of Center $\pi_1(N)$ (cf Lawson-Yau [15]). Sine $\dim G = \dim\ I^0(N, g_1)$, ρ must be an isomorhism so that G is a torus. Here we explain the zero dimensional torus group acting on N to be the trivial group.

If N has dimension 2, by a result of Jost-Schoen [11], we can take the harmonic mapping h as above to be a diffeomorphism of N. This tells us the homomorphism ρ is injective so that we complete the proof.

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