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Sharp estimates of the modified Hardy Littlewood maximal operator on the non-homogeneous space via covering lemmas

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Sharp estimates of the modified Hardy Littlewood maximal operator on the non-homogeneous space via covering lemmas

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Abstract

In this paper we consider the modified maximal operator on the separable metric space. Define $M_k f(x) = \sup_{r>0} \frac{1}{\mu(B(x,kr))} \int_{B(x,r)} |f(y)| d\mu(y)$ and $M_{k,uc} f(x) = \sup_{x \in B(y,r)} \frac{1}{\mu(B(y,kr))} \int_{B(y,r)} |f(z)| d\mu(z)$ respectively. We investigate in what parameter k the weak $(1,1)$-inequality holds for M_k and $M_{k,uc}$ in general metric space and Euclidean space. The proofs are sharper than the method of Vitali's covering lemma. When we investigate $\mathbf{R}^d$, we prove a new covering lemma of $\mathbf{R}^d$. In any case we will prove our results are best possible. In connection with this we consider the dual inequality of Stein type and its applications.

Key words: maximal operator, covering lemma, non-homogeneous

1 Introduction

Let (X,d) be a separable metric space endowed with a Radon measure μ such that all the balls are non-degenerate. We say that a ball B with positive radius is non-degenerate if $\mu(B) > 0$. In [6] the modified maximal operator is introduced as $\tilde{M} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,3r))} \int_{B(x,r)} |f(y)| d\mu(y)$, where $B(x,r)$ is an open ball with radius $r > 0$ and center $x \in X$. They showed that $\mu(\{x \in X \mid \tilde{M} f(x) > \lambda\}) \le \frac{1}{\lambda} \int_X |f(x)| d\mu(x)$.

Motivated to this, we define $M_k f(x) = \sup_{r>0} \frac{1}{\mu(B(x,kr))} \int_{B(x,r)} |f(y)| d\mu(y)$. Section 2 is devoted to the study of the weak-$(1,1)$ property of M_k. The strong-(p,p) property and weak-$(1,1)$ property of M_k still hold if $k \ge 2$.

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Yutaka Terasawa showed the following theorem.

Theorem 1.1 *Suppose that $k > 2$. Then we have M_k is weak-$(1,1)$ bounded. And the weak-$(1,1)$ constant is less than 1 :*

$$\mu(\{x \in X \mid M_k f(x) > \lambda\}) \leq \frac{1}{\lambda} \int_X |f(x)| d\mu(x).$$

He proved the theorem using an outer measure, which is different from the method we are going to use.

We shall prove the theorem with $k = 2$ using a new covering lemma.

Theorem 1.2 *M_2 is weak-$(1,1)$ bounded. And the weak-$(1,1)$ constant is less than 1 :*

$$\mu(\{x \in X \mid M_2 f(x) > \lambda\}) \leq \frac{1}{\lambda} \int_X |f(x)| d\mu(x).$$

Of course the strong-(∞,∞) property is clear, by interpolation we only need to prove the weak-$(1,1)$ property to obtain the strong-(p,p) property.

Yutaka Terasawa proposed the following question. In what parameter k does M_k satisfy the weak-$(1,1)$ estimate? Motivated to this we will consider the following problem concerning to the weak-$(1,1)$ property.

[1] In $\mathbf{R}^n$, when we consider $M_{k,uc}$ or M_k, in what parameter k do M_k and $M_{k,uc}$ satisfy the weak-$(1,1)$ estimate respectively?

[2] Does there exist a separable metric space such that M_k is weak-$(1,1)$ bounded only if $k \geq 2$?

We will give answers to these questions.

Theorem 1.2 is sharp. We will show this sharpness in Section 2.2 by making an example whose property is summarized below.

Proposition 1.1 *There exist a separable space (X,d) and a measure μ such that M_k is bounded if and only if $k \geq 2$. And the weak-$(1,1)$ norm of M_2 is 1 on this space.*

Next we develop applications of this weak-type inequality and the covering lemma used to prove Theorem 1.2. First we derive the dual inequality by the method used in Theorem 1.2. Next using the duality inequality carefully, we derive the Fefferman-Stein type vector-valued inequality for the nonhomogenuous space. The result we will get is the following.

Theorem 1.3 *If $p, q > 1$, then we have*

$$\left\| \left(\sum_{j \in \mathbf{Z}} (M_l f_j)^p \right)^{\frac{1}{p}} \right\|_q \leq C_{p,q} \left\| \left(\sum_{j \in \mathbf{Z}} |f_j|^p \right)^{\frac{1}{p}} \right\|_q,$$

if l is large enough. $l \geq 22$ will do.

This vector-valued inequality automatically yields the one of the maximal operator of the singular integral appearing in [3]. We will quote the definition of singular integral from [6].

Definition 1.1 We say that μ satisfies the growth condition if

$$\mu(B(x,r)) \le Cr^n \text{ for all } r > 0.$$

Here n is a positive constant that can be different from the (geometric or Euclidean) dimension of X.

Definition 1.2 Let μ and n be as above, the singular integral operator is a bounded linear operator $T : L^2(X) \to L^2(X)$ that satisfies the following:

There exists a function K that satisfies three properties listed below.

(1) There exists $C > 0$ such that $|K(x,y)| \le \dfrac{C}{d(x,y)^n}$.

(2) There exist $\epsilon > 0$ and $C > 0$ such that $|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le C\dfrac{d(x,z)^\epsilon}{d(x,y)^{\epsilon+n}}$, if $d(x,y) > 2d(x,z)$.

(3) If f is a bounded measurable function with bounded support, then we have

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y) \text{ for all } x \notin \operatorname{supp}(f).$$

Definition 1.3 We also define the maximal operator of the truncated integral by the formula

$$T^*f(x) = \sup_{r>0}\left|\int_{\{y \in X \ | \ d(x,y)>r\}} K(x,y)f(y)d\mu(y)\right|.$$

Theorem 1.4 *Let K, T and μ be ones appearing in the above definitions. If $1 < p, q < \infty$, then we have*

$$\left\|\left(\sum_{j\in\mathbf{Z}}(T^*f_j)^p\right)^{1/p}\right\|_q \le C_{p,q}\left\|\left(\sum_{j\in\mathbf{Z}}|f_j|^p\right)^{1/p}\right\|_q.$$

When we consider the Euclidean space endowed with a standard distance, we have M_1 is weak-$(1,1)$ bounded. This is due to the Besikovitch's covering lemma.

For the proof of that lemma, see [2]. We just cite it below for completeness and comparison with our Theorem 1.5. We state it in the form different from the one stated in [2].

Definition 1.4 Let $\{B_\lambda\}_{\lambda \in L}$ be a family of balls in the metric space. We say $\{B_\lambda\}_{\lambda \in L}$ is disjoint if $B_{\lambda_1} \cap B_{\lambda_2} = \emptyset$ for $\lambda_1 \neq \lambda_2$. Let $\mathcal{B}$ be a family of balls and $\{B_\lambda\}_{\lambda \in L_1}, \ldots, \{B_\lambda\}_{\lambda \in L_N}$ be subfamilies of $\mathcal{B}$. We say $\{B_\lambda\}_{\lambda \in L_1}, \ldots, \{B_\lambda\}_{\lambda \in L_N}$ are disjoint subfamilies if $\{B_\lambda\}_{\lambda \in L_j}$ is disjoint for all j.

Lemma 1.1 *Let $\{B_\lambda\}_{\lambda \in L}$ be a family of balls. Suppose the diameters of balls $\{B_\lambda\}_{\lambda \in L}$ are bounded. Then there exists an integer N depending only on the dimension that has the following property:*

There are N disjoint subfamilies $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$ such that all the centers of balls $\{B_\lambda\}_{\lambda \in L}$ belong to a ball in some $\mathcal{G}_j$.

The proof that M_1 is weak-$(1,1)$ bounded is omitted, since the proof is analogous to the proof of Theorem 1.6.

Parallel to this we also define the uncentered maximal operator

$$M_{k,uc}f(x) = \sup_{x \in B(y,r)} \frac{1}{\mu(B(y,kr))} \int_{B(y,r)} |f(z)| d\mu(z).$$

We devote Section 3 to the study of $M_{k,uc}$. Similar example appearing in the proposition 3 shows that there exist (X,d) and μ such that $M_{k,uc}$ is bounded if and only if $k \geq 3$. In Euclidean setting, endowed with a standard distance, M_1 is weak-$(1,1)$ is bounded by Besikovitch covering lemma. But as for the uncentered version, $M_{k,uc}$ is bounded only if $k > 1$. We show this by proving a new covering lemma (Theorem 1.5).

Theorem 1.5 *For all $k > 1$ there exists an integer $N = N_k$, depending only on the dimension and k, that satisfies the following:*

Let $\{B(x_\lambda, r_\lambda)\}_{\lambda \in L}$ be a family of balls in Euclidean space endowed with a standard distance. Suppose that $\sup_{\lambda \in L} r_\lambda < \infty$.

Then we can take disjoint subfamilies

$$\{B(x_\rho, r_\rho)\}_{\rho \in L_1}, \{B(x_\rho, r_\rho)\}_{\rho \in L_2}, \ldots, \{B(x_\rho, r_\rho)\}_{\rho \in L_N}$$

such that $\bigcup_{\lambda \in L} B(x_\lambda, r_\lambda) \subset \bigcup_{j=1,\ldots,N} \bigcup_{\rho \in L_j} B(x_\rho, kr_\rho)$.

Theorem 1.6 *$M_{k,uc}$ is bounded, if $k > 1$.*

For the uncentered version with $k = 1$, see [5]: There exists a measure such that $M_{1,uc}$ is not weak-$(1,1)$ bounded.

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2 The centered maximal operator

2.1 A covering lemma

The first covering lemma is the refinement of the Vitali's covering lemma, which leads us to obtain the weak-$(1,1)$ boundedness of M_2. And it is used in application again.

Lemma 2.1 *Let $\delta > 0$. Suppose we have a family of n balls $\{B(x_j, r_j)\}_{j=1,\ldots,n}$. Then we can take a subfamily $\{B(x_j, r_j)\}_{j \in A}$ such that*

(1) $\{B(x_j, r_j)\}_{j \in A}$ *is disjoint.*

(2) $\displaystyle\bigcup_{j=1,\ldots,n} B(x_j, \delta r_j) \subset \bigcup_{j \in A} B(x_j, (2+\delta) r_j)$.

Remark 2.1 This is an extension of the Vitali's covering lemma: The lemma is precisely Vitali's covering lemma if $\delta = 1$.

Proof. We select j_1 so that $r_{j_1} = \max\{r_1, \ldots, r_n\}$. If

$$\bigcup_{j=1,\ldots,n} \{B(x_j, \delta r_j)\} \subset B(x_{j_1}, (2+\delta) r_{j_1}),$$

we have nothing else to do. Let us assume otherwise in the sequel. We define

$$\Lambda_1 = \{j \in \{1, \ldots, n\} \mid B(x_j, \delta r_j) \text{ is not contained in } B(x_1, (2+\delta) r_1)\}.$$

We inductively define the subsets of $\{1, \ldots, n\}$ and $j_1, \ldots, j_p \in \{1, \ldots, n\}$ as follows:
Suppose that $j_1, \ldots, j_{q-1}$ and the subsets $\Lambda_1, \ldots, \Lambda_{q-1} \subset \{1, \ldots, n\}$ are defined. Then we take j_q so that

$$r_{j_q} = \max_{j \in \Lambda_1 \cap \ldots \cap \Lambda_{q-1}} r_j \text{ with } j_q \in \Lambda_1 \cap \ldots \cap \Lambda_{q-1}$$

and we define

$$\Lambda_q = \left\{ j \in \{1, \ldots, n\} \mid B(x_j, \delta r_j) \text{ is not contained in } \bigcup_{j=1,\ldots,q-1} B(x_j, (2+\delta) r_j) \right\}.$$

This procedure will be stopped because we are dealing with the finite number of the balls. Suppose we have stopped after we selected j_p and Λ_p. We will verify that $A = \{j_1, \ldots, j_p\}$ satisfies all the requirement of the lemma.

To verify this we fix $j \in \{1, \ldots, n\}$. We have three possibilities.

(a) $j \in \{j_1, \ldots, j_p\}$.

(b) $r_{j_1} = r_j$ and $j \notin \{j_1, \ldots, j_p\}$.

(c) $r_{j_k} > r_j \geq r_{j_{k+1}}$ for some $k \in \{1, \dots, p-1\}$ and $j \notin \{j_1, \dots, j_p\}$.

We want to show that $B(x_j, \delta r_j) \subset \cup_{j\in A} B(x_j, (2+\delta) r_j)$. If (a) happens, this inclusion is clear. We assume (c) in the sequel. The rest of the possibility can be dealt similarly. Assuming (c) we have $B(x_j, \delta r_j) \subset \bigcup_{j=j_1, j_2, \dots, j_k} B(x_j, (2+\delta) r_j)$ by the definition of $\Lambda_1, \dots, \Lambda_k$ and $r_{j_{k+1}}$. Thus our claim is justified.

Moreover the balls $\{B(x_j, r_j)\}_{j \in A}$ are disjoint. Indeed suppose $j < j'$, so that we have $r_j \geq r_{j'}$. By the definition of Λ_j, we have $B(x_{j'}, \delta r_{j'})$ is not contained in $B(x_j, (2+\delta) r_j)$, since $j < j'$. Thus the center $x_{j'}$ is not an element of $B(x_j, 2r_j)$. This implies $d(x_j, x_{j'}) \geq 2r_j$. Furthermore as noted, we have $r_j \geq r_{j'}$. Combining them, we obtain $\{B(x_j, r_j)\}_{j\in A}$ is disjoint.

2.2 Proof of Theorem 1.2

First of all let us remark the following fact, which is often used in the sequel.

Remark 2.2 $B(x, r)$ is an open ball with radius $r > 0$ and center $x \in X$. We use $\overline{B}(x, r)$ to denote a closed ball with radius $r > 0$ and center $x \in X$. By Radon property, we can replace $B(x, r)$ by $\overline{B}(x, r)$ in the definition of M_k and $M_{k,uc}$. Thus for all measurable $f : X \to \mathbf{C}$ we have $M_k f(x) \to M_{k_0} f(x)$ as $k \to k_0$.

Noting this remark, we shall prove Theorem 1.2. Fix $\lambda > 0$. By Remark 2.2, it follows that

$$\bigcup_{k>2} \{x \in X \mid M_k f(x) > \lambda\} = \{x \in X \mid M_2 f(x) > \lambda\}.$$

Let $\delta > 0$ and $k = 2 + \delta$. We define $E_k = \{x \in X \mid M_k f(x) > \lambda\}$. For all $x \in E_k$, by its definition there exists $r_x > 0$ such that $\dfrac{1}{\mu(B(x, kr_x))} \displaystyle\int_{B(x,r_x)} |f(y)| d\mu(y) > \lambda$. Since μ is a Radon measure, E_k is an open set. Since X is separable, so with the aid of the Linderöf covering theorem we can take $x_j \in E_k, j = 1, 2, \dots$ such that $E_k \subset \bigcup_{j\in\mathbf{N}} B(x_j, \delta r_{x_j})$.

By Lemma 2.1 there exists $A \subset \{1, \dots, n\}$ such that

$$B(x_j, \delta r_{x_j})_{j=1,\dots,n} \subset \bigcup_{l\in A} B(x_l, (2+\delta) r_{x_l}) \text{ and } \{B(x_l, r_{x_l})\}_{l\in A} \text{ is disjoint.}$$

By the definition of E_k, we also have

$$\mu(B(x_l, (2+\delta) r_{x_l})) \leq \frac{1}{\lambda} \int_{B(x_l, r_{x_l})} |f(x)| d\mu(x).$$

Putting them together, we obtain

$$\mu(\bigcup_{j=1,2,\dots,n} B(x_j, \delta r_{x_j})) \leq \mu(\bigcup_{l\in A} B(x_l, r_{x_l})) \leq \sum_{l\in A} \mu(B(x_l, r_{x_l}))$$

$$\leq \sum_{l \in A} \frac{1}{\lambda} \int_{B(x_l, r_{x_l})} |f(x)| d\mu(x) \leq \frac{1}{\lambda} \int_X |f(x)| d\mu(x).$$

Letting n tend to infinity in the above inequality, $\mu(E_k) \leq \frac{1}{\lambda} \int_X |f(x)| d\mu(x)$ is derived. As is noted in Remark 2.2, we have $\bigcup_{k>2} \{x \in X \mid M_k f(x) > \lambda\} = \{x \in X \mid M_2 f(x) > \lambda\}$. Tending $k \downarrow 2$, we get $\mu(\{x \in X \mid M_2 f(x) > \lambda\}) \leq \frac{1}{\lambda} \int_X |f(x)| d\mu(x)$.

Remark 2.3 This theorem is an extension of the result [1]. In [1], Carleson proved on the Euclidean space with normal distance. He used the method appeared in [4]. But we cannot apply it to the metric space with non-doubling measure, because we cannot use the Lebesgue differential theorem in general separable metric space.

Corollary 2.1 *We have for* $p > 1$

$$\|M_2 f\|_p \leq C_p \|f\|_p.$$

Proof. Since $\|M_2 f\|_\infty \leq \|f\|_\infty$ is trivial, by interpolation we obtain the desired inequality.

2.3 An example showing sharpness of Theorem 1.2

Next we want to construct a space where M_k is not bounded if $k < 2$. First we define a set on which the distance and the measure will be defined. The distance is quite different from the one of the usual Euclidean space.

Definition 2.1 Let D be a closed unit disk on the complex plane. Define X as a direct product of a countable copies of D.

In what follows $[\cdot]$ is used to denote the Gauss sign.

Definition 2.2 We define a function d as follows: Take $\mathbf{x} = \{x_n\}_{n \in \mathbf{N}}$ and $\mathbf{y} = \{y_n\}_{n \in \mathbf{N}}$. Then there exists $r \leq 2$ such that $|x_n - y_n| \leq r$ for all $n \leq 2 + [\log_{10} \frac{1}{r}]$ We define $d(\mathbf{x}, \mathbf{y})$ as an infimum of such $r > 0$.

Remark 2.4 r that appears in the definition does exist: $r = 2$ will be enough.

Lemma 2.2 *This function defines a distance.*

Proof. We will omit the proof, since it is easy to show the lemma, noting next remark.

Remark 2.5 If $r > 0$, the open ball $B(x,r)$ is precisely the set

$$\left\{ \mathbf{y} = \{y_n\}_{n \in \mathbf{N}} \in X \mid |x_n - y_n| \le r \text{ for all } n \le 2 + \left[\log_{10} \frac{1}{r} \right] \right\}.$$

Lemma 2.3 *The space X, endowed with a distance function d, is a separable subspace.*

Proof. Using standard argument, this lemma is easy to show. So we will omit the proof.

This is the distance space we work on. Next let us define the measure.

Definition 2.3 Let $\mathcal{B}$ be a σ-algebra generated by $B(\mathbf{x}, r)$ with $\mathbf{x} \in X$ and $r > 0$.

The following proposition ensures the measurability of a maximal function.

Proposition 2.1 *The σ-algebra $\mathcal{B}$ is nothing but the one generated by the cylinder sets of the form $A_1 \times \ldots \times A_l \times D \times D \times \ldots$, where $A_1, \ldots, A_l$ are Bore-measurable sets of D.*

Proof. Put $\Delta(a,r) = \{z \in D \mid |z - a| < r\}$, where $a \in D$ and $r > 0$. Let us show that $\Delta(a_1, r_1) \times \Delta(a_2, r_2) \times \ldots \times \Delta(a_l, r_l) \times D \times D \times \ldots$ is $\mathcal{B}$-measurable, if $a_1, a_2, \ldots, a_l \in D$ and $r_1, r_2, \ldots, r_l > 0$. In fact $B(\mathbf{a}, 10^{-l+2})$ is contained in $\mathcal{B}$ for all $\mathbf{a} \in X$. Since $\Delta(a_1, r_1) \times \Delta(a_2, r_2) \times \ldots \times \Delta(a_l, r_l) \times D \times D \times \ldots$ is expressible as a countable union and intersection of balls of the form $B(\mathbf{a}, 10^{-l+2})$, where $\mathbf{a} \in X$, $\Delta(a_1, r_1) \times \Delta(a_2, r_2) \times \ldots \times \Delta(a_l, r_l) \times D \times D \times \ldots$ is $\mathcal{B}$-measurable. Thus σ-algebra $\mathcal{B}$ contains σ-algebra generated by cylinder sets. The reverse inclusion is clear so our claim is justified.

Remark 2.6 The topology induced by this distance is the same as product topology induced by Euclidean topology of D. This can be shown using the same idea as that of Proposition 2.1.

Definition 2.4 We define $f : D \to \mathbf{R}$ as follows:
First we define $a_n = \prod_{l=1}^{n+2} l!$. We define annulus A_n and B_n with $n = 0, 1, \ldots$ as

$$A_n = \left\{ z \in D \mid 3^{-n} \left(1 - \frac{1}{a_n^3} \right) < |z| < 3^{-n} \right\}$$

$$B_n = \left\{ z \in D \mid 3^{-n-1} < |x| < 3^{-n} \left(1 - \frac{1}{a_n^3} \right) \right\}.$$

And we define the density function f as

$$f(z) = \begin{cases} l/a_n & \text{on } A_n \\ l/a_{n+3}^5 & \text{on } B_n \\ 0 & \text{otherwise.} \end{cases}$$

We define $\nu = f(z)dz$, where dz is the Lebesgue measure on D. Constant l is taken so that $\nu(D) = 1$. Let μ_n be a measure on $D \times D \times \ldots \times D$ (n-tuple)defined as $\mu_n = \nu \times \nu \times \ldots \times \nu$. Since $\nu(D) = 1$, we can use Kolmogorov's extension theorem to define μ as a countable product of ν.

Firstly, let us examine the property of the (complicated) function f. It is summarized as a lemma below.

Lemma 2.4 (a) *The ball is non-degenerate if its radius is positive. And there are infinitely many integers n such that $B\left(\mathbf{x}, \frac{1}{3^n}\right)$ and $B\left(\mathbf{x}, \frac{1}{2\cdot 3^n}\right)$ are made up of the product of the same number of nontrivial balls and the closed unit disks, where a ball is nontrivial means that it is a proper subset of D.*

(b) *Define $\Omega_1 = \Omega_{1,n}$ as*

$$\Omega_1 = \{x + \sqrt{-1}y \in D \mid (x - 3^{-n})^2 + y^2 \leq 4 \cdot 9^{-n}\}.$$

Then we have

$$\lim_{n\to\infty} \frac{\nu(\Omega_1)}{2\pi l/9^n a_n^4} = 1.$$

(c) *Let $\frac{3}{2} < k < 2$. Define $\Omega_2 = \Omega_{2,n}$ as*

$$\Omega_2 = \{x + \sqrt{-1}y \in D \mid (x - 3^{-n})^2 + y^2 \leq k^2 \cdot 9^{-n}\}.$$

Then we have

$$\lim_{n\to\infty} \frac{\nu(\Omega_2)}{2\pi l/9^n a_n^4} = C_k,$$

where C_k is a geometric constant strictly less than 1 that depends on k.

(d) *Define $\Omega_3 = \Omega_{3,n}$ as*

$$\Omega_3 = \{x + \sqrt{-1}y \in D \mid x^2 + y^2 \leq 9^{-n}\}.$$

Then we have

$$\lim_{n\to\infty} \frac{\nu(\Omega_3)}{2\pi l/9^n a_n^4} = 1.$$

Proof. (a) is clear because $\log_{10} 3 = 0.4771\ldots$ and $f(x)$ is dx-almost everywhere positive, where dx is a Lebesgue measure on D. Let us prove (b). Note that $\{x + \sqrt{(-1)}y \in D \mid x^2 + y^2 \leq 3^{-n}\} \subset \Omega_1 \subset \{x + \sqrt{(-1)}y \in D \mid x^2 + y^2 \leq 3^{1-n}\}$.

Hence we have $\Omega_1 \cap A_k = A_k, \Omega_1 \cap B_k = B_k$, if $n \leq k$. $\Omega_1 \cap A_k = \emptyset, \Omega_1 \cap B_k = \emptyset$, if $n > k+1$. Taking this into account, we will estimate $\Omega_1 \cap A_k$ with $n < k$, $\Omega_1 \cap B_k$ with $n < k$, $\Omega_1 \cap A_n$, $\Omega_1 \cap B_n$, $\Omega_1 \cap A_{n-1}$, and $\Omega_1 \cap B_{n-1}$ respectively. Firstly a little long computation leads us to a rough estimate that $\nu(\Omega_1 \cap A_{n-1})$ is bounded by $C/9^n a_{n-1}^6$, which is less than $C/9^n a_n^4$ if n is large. As for $\nu(\Omega_1 \cap B_{n-1})$, it is bounded by $C/9^n a_{n+2}^5$, which is also less than $C/9^n a_n^4$ if n is large. Furthermore we have $\lim_{n\to\infty} \dfrac{\nu(\Omega_1 \cap A_n)}{2\pi l/9^n a_n^4} = 1$. This is due to the fact that the Lebesgue measure of A_n is equal to

$$\pi \left(3^{-n}\right)^2 - \pi \left(3^{-n}\left(1 - \frac{1}{a_n^3}\right)\right)^2 \approx \pi \times 2 \times 9^{-n} \frac{1}{a_n^3}$$

and that on A_n we have $f(x) = \dfrac{l}{a_n}$.

We also have

$$\nu\left(\Omega_1 \cap \bigcup_{k:n<k} A_k\right) \le C/9^n a_{n+1}^3 a_n$$

and

$$\nu\left(\Omega_1 \cap \bigcup_{k:n\le k} B_k\right) \le C/9^n a_{n+3}^5 a_n.$$

With these estimates we obtain (b).

(c) and (d) follow similarly. But the proof of (c) is the crucial point of the proof of unboundedness of M_k with $k < 2$. So we will point out what counts.

The essential difference lies in the estimate of $\nu(\Omega_2 \cap A_n)$. With k fixed geometric observation shows that there exists C_k strictly less than 1 such that $\dfrac{\nu(\Omega_2 \cap A_n)}{\nu(A_n)} \to C_k$ as $n \to \infty$. More precisely, C_k is given by the following formula:

$$C_k = \frac{|\{(x,y) \mid x^2 + y^2 = 1, (x-1)^2 + y^2 \le k^2\}|}{|\{(x,y) \mid x^2 + y^2 = 1\}|} = \frac{1}{\pi} \cos^{-1}\left(\frac{2-k^2}{2}\right),$$

here $|E|$ means arc length of an arc E. This is the critical point of (c) and the rest is quite similar to that of (b), so the detail is omitted.

Remark 2.7 Our calculation shows

$$\nu(\Omega_1) = \frac{2\pi l}{9^n a_n^4}\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

This will be used in Remark 2.9.

Under this measure, we will show that M_k is bounded only if $k \ge 2$. Before proving this, we introduce one more notation.

Notation 2.1 For a positive measure α on X we denote $\sup_{r>0} \dfrac{\alpha(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, kr))}$ by $M_k\alpha(\mathbf{x})$. Let $\delta_{\mathbf{0}}$ the Dirac measure at $\mathbf{0} = (0, 0, \ldots)$.

Proposition 2.2 *M_k is not bounded, if $k < 2$.*

Proof. We may assume $\frac{3}{2} < k < 2$, since M_k is decreasing as k increases. Suppose we have M_k is bounded. We want to derive a contradiction. We begin with constructing an approximation of Dirac delta. Let $g_r = \dfrac{1_{B(\mathbf{0},r)}}{\mu(B(\mathbf{0},r))}$. $M_k g_r$ tends pointwise to $M_k \delta_{\mathbf{0}}$ as $r \to 0$, where $\delta_{\mathbf{0}}$ is a point mass at $\mathbf{0}$. Let $\mathbf{a}$ be a point defined as follows:
Put $K_n = 2 + [\log_{10} 3^n]$. Define $\mathbf{a} = (a_j)_{j \in \mathbf{N}}$ as

$$a_j = \begin{cases} 3^{-n} & (j \leq K_n) \\ 0 & (\text{otherwise}). \end{cases}$$

Take $\lambda = \dfrac{1}{\mu(B(\mathbf{a}, k3^{-n}))}$. We have $\{x \in X \mid M_k\delta_{\mathbf{0}}(x) > \lambda\} \supset B(\mathbf{0}, 3^{-n})$. Thus we have $\mu(B(\mathbf{0}, 3^{-n})) \leq \dfrac{C}{\lambda}$. So we have $\mu(B(\mathbf{0}, 3^{-n}))/\mu(B(\mathbf{a}, k3^{-n})) \leq C$. By the definition of ν and Lemma 2.4 (b), there are infinitely many integers n such that

$$\left\{ \frac{\nu(\{x + \sqrt{-1}y \in D \mid x^2 + y^2 = 9^{-n}\})}{\nu(\{x + \sqrt{-1}y \in D \mid (x - 3^{-n})^2 + y^2 = k^2 \cdot 9^{-n}\})} \right\}^{(2+[\log_{10} 3^n])} \leq C.$$

Take limit of this quantity as n, running through such a integer, to infinity. With the aid of (c) and (d) of Lemma 2.4 contradiction is obtained, since C_k is strictly less than 1.

By construction we can check the following:

Proposition 2.3 *In the above space the measure of $B(x,r)$ grows in the polynomial order of any degree, that is, $\mu(B(x,r))/r^n$ is bounded for all positive integer n.*

Remark 2.8 This proposition can be interpreted that the dimension of the space is "infinity" according to the terminology of [6]. But this proposition cannot be improved in the sense that $\mu(B(x,r))/r^n$ is bounded uniformly on n.

Remark 2.9 Using Lemma 2.4 (b), (d) and Remark 2.7, the proof of the Theorem 1.2 and the reproduction of the proof with the parameter of k changed into 2 shows that the Theorem 1.2 is sharp in the following sense: We cannot take the weak-$(1,1)$ constant strictly less than 1 in general.

2.4 Proof of Theorem 1.3 and Theorem 1.4

As an application of Lemma 2.1 and Theorem 1.2 we will prove Theorem 1.3 and Theorem 1.4. First of all we get a weighted inequality of the Stein type, using again Lemma 2.1.

Proposition 2.4 *We have* $\displaystyle\int_{\{M_7 f \geq \lambda\}} |g(x)| d\mu(x) \leq \frac{1}{\lambda} \int_X |f(x)| M_2 g(x) d\mu(x),.$

Proof. Take $k > 7$. Let $E = \{x \in X \mid M_k f \geq \lambda\}$. By the definition of E, for all $x \in E$ there exists r_x such that

$$\frac{1}{\mu(B(x, kr_x))} \int_{B(x,r_x)} |f(x)| d\mu(x) > \lambda.$$

Since E is an openset, again by the Linderöf theorem, there exist $x_j, j = 1, 2, \ldots$ such that $E \subset \cup_{j \in \mathbf{N}} B(x_j, ar_{x_j})$. We will take $a = (k-7)/20$.

We claim that

$$\int_{\bigcup_{j=1,\ldots,n} B(x_j, ar_{x_j})} |g(x)| d\mu(x) \leq \frac{1}{\lambda} \int_X |f(x)| M_2 g(x) d\mu(x).$$

By Lemma 2.1 there exists a subfamily of balls $\{B(x_j, ar_{x_j})\}_{j \in \Lambda}$ ($\Lambda \subset \{1, 2, \ldots, n\}$) that satisfies the following properties.

(a) $\{B(x_j, r_{x_j})\}_{j \in \Lambda}$ is disjoint. (b) $\displaystyle\bigcup_{j=1,\ldots,n} B(x_j, ar_{x_j}) \subset \bigcup_{j \in \Lambda} B(x_j, (2+a) r_{x_j})$.

Note that we have, for all $x \in B(x_j, r_{x_j})$,

$$\begin{aligned} &\frac{1}{\mu(B(x_j, kr_{x_j}))} \int_{B(x_j, (2+a) r_{x_j})} |g(y)| d\mu(y) \\ \leq &\frac{1}{\mu(B(x, (6+2a) r_{x_j}))} \int_{B(x, (3+a) r_{x_j})} |g(y)| d\mu(y) \ \leq M_2 g(x). \end{aligned}$$

Using this, we obtain $\displaystyle\int_{\bigcup_{j=1,\ldots,n} B(x_j, ar_{x_j})} |g(x)| d\mu(x)$

$$\begin{aligned} &\leq \int_{\bigcup_{j \in \Lambda} B(x_j, (2+a) r_{x_j})} |g(x)| d\mu(x) \leq \sum_{j \in \Lambda} \int_{B(x_j, (2+a) r_{x_j})} |g(x)| d\mu(x) \\ &\leq \sum_{j \in \Lambda} \frac{1}{\lambda} \int_{B(x_j, r_{x_j})} |f(x)| d\mu(x) \frac{1}{\mu(B(x_j, kr_{x_j}))} \int_{B(x_j, (2+a) r_{x_j})} |g(x)| d\mu(x) \\ &\leq \sum_{j \in \Lambda} \frac{1}{\lambda} \int_{B(x_j, r_{x_j})} |f(x)| M_2 g(x) d\mu(x) \leq \int_X |f(x)| M_2 g(x) d\mu(x). \end{aligned}$$

By the definition of E we have

$$\int_{\{M_k f > \lambda\}} |g(x)| d\mu(x) \leq \frac{1}{\lambda} \int_X |f(x)| M_2 g(x) d\mu(x).$$

Letting $k \downarrow 7$, we finally obtain $\displaystyle\int_{\{M_7 f > \lambda\}} |g(x)| d\mu(x) \leq \frac{1}{\lambda} \int_X |f(x)| M_2 g(x) d\mu(x)$. Thus we have finished.

Corollary 2.2 *If $p > 1$, then we have*

$$\int_X (M_7 f(x))^p |g(x)| d\mu(x) \leq C_p \int_X |f(x)|^p M_2 g(x) d\mu(x).$$

Proof. For the positive function w we denote $\| \cdot \|_{\infty,w}$ is a L^∞-norm of the function with respect to the weighted measure $w d\mu$. Since

$$\|M_7 f\|_{\infty,|g|} \leq \|f\|_{\infty,M_2 g}$$

is clear, this is again just a matter of the interpolation of this inequalities and the last results.

Remark 2.10 We use the following analogous which is used below to obtain Theorem 1.3. The proof is only the change of the parameters k of Theorem 2.4 and Corollary 2.2 respectively.

Proposition 2.5 *Let X, μ be as above.*

(a) *The estimate*

$$\int_{\{M_{22} f \geq \lambda\}} |g(x)| d\mu(x) \leq \frac{1}{\lambda} \int_X |f(x)| M_7 g(x) d\mu(x)$$

holds for all $\lambda > 0$.

(b) *If $p > 1$, then we have*

$$\int_X (M_{22} f(x))^p |g(x)| d\mu(x) \leq C_p \int_X |f(x)|^p M_7 g(x) d\mu(x).$$

At last we are in the position of proving Theorem 1.3. If $q \geq p > 1$, a little more can be said. We have the following.

Theorem 2.1 *If $q \geq p > 1$, then*

$$\left\| \left(\sum_{j \in \mathbf{Z}} (M_7 f_j)^p \right)^{1/p} \right\|_q \leq C_{p,q} \left\| \left(\sum_{j \in \mathbf{Z}} |f_j|^p \right)^{1/p} \right\|_q .$$

Proof. The case when $p = q$ is trivial. Assume that $p < q$. Put $r = \frac{q}{p}$ and let r' be a conjugate exponent of r.

By using Theorem 1.2 and Corollary 2.2 we get to

$$\left\| \left(\sum_{j \in \mathbf{Z}} (M_7 f_j)^p \right)^{1/p} \right\|_q^p = \sup_{\|g\|_{r'}=1, g \geq 0} \int \left(\sum_{j \in \mathbf{Z}} (M_7 f_j)^p \right) g d\mu$$

$$= \sup_{\|g\|_{r'}=1, g\geq 0} \sum_{j\in\mathbf{Z}} \int (M_7 f_j)^p g d\mu \leq C_{p,q} \sup_{\|g\|_{r'}=1, g\geq 0} \sum_{j\in\mathbf{Z}} \int |f_j|^p M_2 g d\mu$$
$$\leq C_{p,q} \sup_{\|g\|_r'=1, g\geq 0} \left\{ \int \left(\sum_{j\in\mathbf{Z}} |f_j|^p \right)^r d\mu \right\}^{1/r} \left\{ \int (M_2 g)^{r'} d\mu \right\}^{1/r'}$$
$$\leq C_{p,q} \left\{ \int \left(\sum_{j\in\mathbf{Z}} |f_j|^p \right)^r d\mu \right\}^{1/r} \sup_{\|g\|_r'=1, g\geq 0} \left(\int g^{r'} d\mu \right)^{1/r'}$$
$$= C_{p,q} \left\{ \int \left(\sum_{j\in\mathbf{Z}} |f_j|^p \right)^r d\mu \right\}^{1/r}.$$

Taking $\frac{1}{p}$-th power of both sides, we obtain

$$\left\| \left(\sum_{j\in\mathbf{Z}} (M_7 f_j)^p \right)^{1/p} \right\|_q \leq C_{p,q} \left\| \left(\sum_{j\in\mathbf{Z}} |f_j|^p \right)^{1/p} \right\|_q.$$

For the case that $p > q$, we use the Proposition 2.5 and Theorem 2.1

Proof. (of Theorem 1.3) By Theorem 2.1, it remains to show when $p > q$. Let $p > q$ in what follows and take another $r < p$ so close to p that $\dfrac{qr}{p} > 1$. According to $(L^{\frac{qr}{p}}\text{-}L^{(\frac{qr}{p})'})$ -duality we have

$$\left\| \left(\sum_{j\in\mathbf{Z}} (M_{22} f_j)^p \right)^{1/p} \right\|_q^{\frac{p}{r}} = \sup_{\{\|g\|_{(\frac{qr}{p})'}=1, g\geq 0\}} \int \left(\sum_{j\in\mathbf{Z}} (M_{22} f_j)^p \right)^{1/r} g d\mu$$

Keeping this in mind, let us fix positive g with $\|g\|_{(\frac{qr}{p})'} = 1$.

Note that $p > q$ implies $\left(\dfrac{qr}{p}\right)' > r'$, so that we are in the position of using Theorem 2.1 with parameter $\left(\dfrac{qr}{p}\right)' > r'$. We also use Proposition 2.5 to obtain

$$\int \left(\sum_{j\in\mathbf{Z}} (M_{22} f_j)^p \right)^{1/r} g d\mu = \sup_{\sum_k h_k^{r'}=1 \ \ h_k\geq 0} \int \left(\sum_{j\in\mathbf{Z}} ((M_{22} f_j)^{\frac{p}{r}} h_j) g \right) d\mu$$

(Below we will write sup instead of $\displaystyle\sup_{\sum_k h_k^{r'}=1 \ \ h_k\geq 0}$.)

$$\leq C_{p,q} \sup \int \left(\sum_{j\in\mathbf{Z}} |f_j|^{\frac{p}{r}} \right) M_7(h_j g) d\mu$$
$$\leq C_{p,q} \sup \int \left(\sum_{j\in\mathbf{Z}} |f_j|^p \right)^{1/r} \left(\sum_{j\in\mathbf{Z}} (M_7 h_j g)^{r'} \right)^{1/r'} d\mu$$

$$
\begin{aligned}
&\leq C_{p,q} \sup \left\{ \int \left(\sum_{j \in \mathbf{Z}} |f_j|^p \right)^{q/p} d\mu \right\}^{p/qr} \times \left\{ \int \left(\sum_{j \in \mathbf{Z}} (M_7 h_j g)^{r'} \right)^{(qr/p)'/r'} d\mu \right\}^{1/(qr/p)'} \\
&\leq C_{p,q} \left\{ \int \left(\sum_{j \in \mathbf{Z}} |f_j|^p \right)^{q/p} d\mu \right\}^{p/qr} \times \sup \left\{ \int \left(\sum_{j \in \mathbf{Z}} (|h_j g|^{r'}) \right)^{(qr/p)'/r'} d\mu \right\}^{1/(qr/p)'} \\
&= C_{p,q} \left\{ \int \left(\sum_{j \in \mathbf{Z}} |f_j|^p \right)^{q/p} d\mu \right\}^{p/qr} .
\end{aligned}
$$

Putting together this and first observation we finish the proof.

This vector-valued inequality is different from the one that appeared in the [6] only in that we enlarged k by 22 or 7 times not by three times.

We will assume the assumption posed on Definition 1.1 and Definition 1.2 until the end of this section. With minor modification of the results of [6] we obtain

Theorem 2.2 [6] *We have for $\beta > 1$ and large l*

$$
T^* f(x) \leq C_{\beta,l}(M_l(Tf)(x)) + C_{\beta,l}\{(M_l|f|^\beta)(x)^{\frac{1}{\beta}}\}.
$$

The next result is due to García-Cuerva [3].

Theorem 2.3 [3] *If $1 < p, q < \infty$, we have*

$$
\left\| \left(\sum_{j \in \mathbf{Z}} |Tf_j|^p \right)^{1/p} \right\|_q \leq C_{p,q} \left\| \left(\sum_{j \in \mathbf{Z}} |f_j|^p \right)^{1/p} \right\|_q .
$$

Combining these results and Theorem 1.3, we obtain Theorem 1.4.

3 The uncentered maximal operator on the Euclidean space

In this section we examine the uncentered maximal operator. The result by [6] which appeared in the introduction is sharp: We can construct a similar example of the space on which $M_{k,uc}$ is not bounded if $k < 3$, using an idea of Section 2.3. Hence in this section we limit ourselves to the space $\mathbf{R}^d$ with Euclidean distance. If the space is Euclidean and the ball is defined by a standard distance, we shall show that $\tilde{M}_{k,uc}$ is bounded if $k > 1$. This is best possible as [5] shows: As in [5] for $\mu = \exp(x^2 + y^2)dxdy$ in $\mathbf{R}^2$, $M_{1,uc}$ is not weak-$(1,1)$ bounded.

3.1 Another covering lemma (Proof of Theorem 1.5)

We want a substitute of Besikovitch's covering lemma. This Theorem 1.5 is a covering lemma for our purpose. This may be viewed also as a substitute of Vitali's covering lemma. To prove Theorem 1.5, firstly we prove it by posing another assumption.

Lemma 3.1 *Let $\{B(x_\lambda, r_\lambda)\}_{\lambda\in L}$ be a family of balls and assume that*

$$\frac{\sup_{\lambda\in L} r_\lambda}{\inf_{\lambda\in L} r_\lambda} < \sqrt{k}.$$

Then we can take disjoint subfamilies as in Theorem 1.5.

Proof. First of all by scaling, we may normalize to have $\sup\limits_{\lambda\in L} r_\lambda = 1$. (We are working on the Euclidean space. So we are able to multiply the scalar.)

In this part we divide the family of balls. More precisely we proceed as follows: Let $\mathcal{Q}_0$ be a family of dyadic cubes of side length 1. Here we are now considering cubes of the form $\mathrm{Q} = \Pi_{j=1}^d [m_j, m_j+1)$, where $m_j, j = 1, \ldots, d$ are integers. We abbreviate the dyadic cubes in $\mathcal{Q}_0$ to "cubes" for short. Let Q^0 be $[0,1)^d$. We divide the cubes into subfamily:

If $\overrightarrow{m} = (m_1, m_2, \ldots, m_d)$ is an element of $\{0,1,2,3\}^d$, we put

$$\mathcal{Q}_{\overrightarrow{m}} = \{\mathrm{Q} \in \mathcal{Q}_0 \mid \mathrm{Q} - (\overrightarrow{p} + \overrightarrow{m}) = \mathrm{Q}^0 \text{ for some } \overrightarrow{p} \in (4\mathbf{Z})^d\}.$$

Next we define $L_{\overrightarrow{m}}$ as

$$L_{\overrightarrow{m}} = \{\lambda \in L \mid x_\lambda \text{ is contained in some cube in } \mathcal{Q}_{\overrightarrow{m}}\}.$$

Note that the cubes in $\mathcal{Q}_{\overrightarrow{m}}$ satisfy the following property: Suppose that Q and Q' are both in $\mathcal{Q}_{\overrightarrow{m}}$ and that Q and Q' are different, then the distance between the two cubes is larger than 3. Hence if the center of B is in Q and the center of B' is in Q', then B and B' are disjoint.

Taking into account of the preceding paragraphs we may assume that all the centers of the balls are in Q^0. In fact once this is proved, by the last paragraph we can take the balls satisfying the property of this lemma from $\mathcal{Q}_{\overrightarrow{m}}$ for any $\overrightarrow{m} \in \{0,1,2,3\}^d$. For any $\overrightarrow{m} \in \{0,1,2,3\}^d$, we obtain families $\mathcal{B}^{(1)}_{\overrightarrow{m}}, \ldots, \mathcal{B}^{(N_k)}_{\overrightarrow{m}}$. Translation shows the number N_k is not dependent on $\overrightarrow{m}$. So our desired family is

$$\bigcup_{\overrightarrow{m}\in\{0,1,2,3\}^d,\ j\le N_k} \mathcal{B}^{(j)}_{\overrightarrow{m}}.$$

So in what follows let us assume that all the centers of the balls are in Q^0 and that $\sup\limits_{\lambda\in L} r_\lambda = 1$ by normalization.

First take a ball $B(x_{\lambda_1}, r_{\lambda_1})$ arbitrarily from the family $\{B(x_\lambda, r_\lambda)\}_{\lambda\in L}$.

The assumption $\dfrac{1}{\inf_{\lambda\in L} r_\lambda} < \sqrt{k}$ ensures that the radius of the ball is between $\dfrac{1}{\sqrt{k}}$ and 1. Thus the ball $B(x_{\lambda_1}, kr_{\lambda_1})$ contains all the ball $B(x_\lambda, r_\lambda)$ such that $d(x_\lambda, x_{\lambda_1})$ is less than $\sqrt{k}-1$.

Next take a ball $B(x_{\lambda_2}, r_{\lambda_2})$ such that $d(x_{\lambda_2}, x_{\lambda_1}) \geq \sqrt{k}-1$. We may choose it arbitrarily as long as this condition is satisfied. As in the proceeding paragraph, the ball $B(x_{\lambda_2}, kr_{\lambda_2})$ contains all the ball $B(x_\lambda, r_\lambda)$ such that $d(x_\lambda, x_{\lambda_2})$ is less than $\sqrt{k}-1$.

In this way we repeatedly take a ball $B(x_{\lambda_p}, r_{\lambda_p})$ such that $d(x_{\lambda_p}, x_{\lambda_j}) \geq \sqrt{k}-1$ for all $j = 1, 2, \ldots, p-1$. This procedure will be stopped at qth step when we obtain $\bigcup_{\lambda\in L} B(x_\lambda, r_\lambda) \subset \bigcup_{p=1}^{q} B(x_{\lambda_p}, kr_{\lambda_p})$.

In fact this procedure stops in finite times: Precisely speaking, q appearing in the last part is bounded by the constant that depends only on $k>1$ and d. Let us show this. Since all the radius of the ball is at most 1, all the ball is contained in $[-1,2]^d$. By the construction of $\{x_{\lambda_a}\}$, we have $d(x_{\lambda_a}, x_{\lambda_b}) \geq \sqrt{k}-1$ for all $a < b \leq q$. Thus we have q disjoint balls whose radii are more than $\dfrac{\sqrt{k}-1}{2}$. Precisely speaking, $\left\{B\left(x_j, \dfrac{\sqrt{k}-1}{2}\right)\right\}_{j=1,\ldots,q}$ is disjoint. And we have $B\left(x_j, \dfrac{\sqrt{k}-1}{2}\right)$ is contained in $[-1,2]^d$ for all $j = 1, \ldots, q$. Hence we have $q\left(\dfrac{\sqrt{k}-1}{2}\right)^d V \leq 3^d$, where V is volume of a unit ball. Thus q is bounded by the quantity which depends only on $k>1$ and d. We put this bound N. If q is less than N, we formally define $L_j = \emptyset$ for $j > q$. Peacing together these observations we are done.

Next we prove Theorem 1.5, that is, we want to eliminate the assumption

$$\frac{\sup_{\lambda\in L} r_\lambda}{\inf_{\lambda\in L} r_\lambda} < \sqrt{k}.$$

Proof. (of Theorem 1.5) Again we may assume that $\sup_{\lambda\in L} r_\lambda = 1$. First we take the subfamilies $\mathcal{B}_{j,p}$ inductively as follows (j runs through all the positive integers and p through $[1, N]$, where N is a number obtained in the Lemma 3.1): First we define X_1 as

$$X_1 = \left\{B(x_\lambda, r_\lambda) \mid r_\lambda > \frac{1}{\sqrt{k}}\right\}$$

Let $\mathcal{B}_{1,p}$ be families obtained from X_1, using the Lemma 3.1. Suppose we have obtained the families of the balls $\mathcal{B}_{l,p}$ with $l = 1, \ldots, j, p = 1, \ldots, N$ and that X_l

with $l = 1, \dots, j$ are defined as the subsets of $\{B(x_\lambda, r_\lambda)\}_{\lambda \in L}$. Then we define

$$X_{j+1} = \{B(x_\lambda, r_\lambda) \mid \frac{1}{\sqrt{k}^j} \geq r_\lambda > \frac{1}{\sqrt{k}^{j+1}},$$

$$B(x_\lambda, r_\lambda) \text{ is not contained in } \bigcup_{p=1,\dots,N} \bigcup_{l=1,\dots,j} \bigcup_{B \in \mathcal{B}_{l,p}} kB.\},$$

where kB is an abbreviation of $B(x, kr)$ when $B = B(x, r)$. And we apply the Lemma 3.1 to this subset to obtain $\mathcal{B}_{j+1,p}$ with $p = 1, \dots, N$, which enjoy the following properties:

$$\{\mathcal{B}_{j+1,p}\}_{1 \leq p \leq N} \text{ are disjoint subfamilies and } \bigcup_{B \in X_{j+1}} B \subset \bigcup_{p=1,\dots,N} \bigcup_{B \in \mathcal{B}_{j+1,p}} kB.$$

By the definition of X_j we have,

$$B(x_\lambda, r_\lambda) \in X_j \text{ implies } r_\lambda \leq \frac{1}{\sqrt{k}^{j-1}}$$

and

$$B(x_\lambda, r_\lambda) \text{ is not contained in } \bigcup_{l=1}^{j-1} \bigcup_{p=1}^{N} \bigcup_{B \in \mathcal{B}_{l,p}} kB.$$

Next we claim that there is an integer N', which depends only on k, that satisfies the following:

$$\text{If } |j - l| > N', B' \in \cup_p \mathcal{B}_{j,p}, \text{ and } B \in \cup_p \mathcal{B}_{l,p}, \text{then we have } B \cap B' = \emptyset.$$

In fact suppose that $B \cap B' \ni x$ and $l > j$. Then by property noted above, there exists $y \in B \backslash kB'$. Let c be the center of B'. If E is a subset of $\mathbf{R}^d$, $\mathrm{diam}(E)$ denotes the diameter of E. Under this notation and setting we have

$$d(x, y) \leq \mathrm{diam}(B), \; d(c, y) \geq \frac{k}{2}\mathrm{diam}(B'), \text{ and } d(c, x) \leq \frac{1}{2}\mathrm{diam}(B').$$

Thus $\frac{(k-1)}{2}\mathrm{diam}(B') < \mathrm{diam}(B)$. By the construction of X_j, $\mathrm{diam}(B) \leq \frac{2}{\sqrt{k}^{l-1}}$ and $\mathrm{diam}(B') \geq \frac{2}{\sqrt{k}^j}$, hence we have $\frac{2}{\sqrt{k}^{l-1}} \geq \frac{(k-1)}{\sqrt{k}^j}$. Since $k > 1$, this is possible only if the difference of j and l is small, that is, $\frac{\log \frac{2\sqrt{k}}{k-1}}{\log \sqrt{k}} > l - j$.

Put $\mathcal{G}_{j,p} = \bigcup_{i \,:\, i \equiv j \mod N'} \mathcal{B}_{i,p}$. Then $\{\mathcal{G}_{j,p}\}_{j \leq N', p \leq N}$ does satisfy all the demands of the theorem.

Using this covering Lemma we can prove Theorem 1.6.
Proof. (of Theorem 1.6)

Put

$$E = \{x \in \mathbf{R}^d \mid M_k f(x) > \lambda\}.$$

By definition of E, for all $x \in E$, there exists r_x such that

$$\mu(B(x, kr_x)) \leq \frac{1}{\lambda} \int_{B(x,r_x)} |f(x)| d\mu(x).$$

By Theorem 1.5, there exists N_k and $x_1, x_2, \ldots$ such that

$$1_E(x) \leq \sum_j 1_{B(x_j, kr_{x_j})}(x) \text{ and } \sum_j 1_{B(x_j, r_{x_j})}(x) \leq N_k$$

for all $x \in \mathbf{R}^d$.

Using this we obtain

$$\mu(E) \leq \mu(\bigcup_j B(x_j, kr_{x_j})) \leq \sum_j \mu(B(x_j, kr_{x_j})) \leq \sum_j \frac{1}{\lambda} \int_{B(x_j, r_{x_j})} |f(x)| d\mu(x)$$

$$= \sum_j \frac{1}{\lambda} \int_{\mathbf{R}^d} 1_{B(x_j, r_{x_j})} |f(x)| d\mu(x) \leq \frac{N_k}{\lambda} \int_{\mathbf{R}^d} |f(x)| d\mu(x).$$

Thus we have finished.

We consider another application of this covering lemma. This covering lemma allows us to obtain various estimates.

Theorem 3.1 *We have the dual inequality*

$$\int_{\{M_{b,uc} f \geq \lambda\}} |g(x)| d\mu(x) \leq \frac{C_{a,b}}{\lambda} \int_{\mathbf{R}^d} M_{a,uc} g(x) |f(x)| d\mu(x).$$

if $b > a > 1$, where $C_{a,b}$ is a positive constant depending on a, b and d.

Proof. Fix $R > 0$. We "cut off" the maximal function. Put

$$M^R_{b,uc} f(x) = \sup_{x \in B(z,r), R > r > 0} \frac{1}{\mu(B(z, br))} \int_{B(z,r)} |f(y)| d\mu(y).$$

(This notation is rather complicated but we want to emphasize that we are considering maximal operator with radii less than R.)

Fix $\lambda > 0$. We set

$$E_b = \{x \in \mathbf{R}^{\mathrm{d}} \mid M^R_{b,uc} f(x) > \lambda\}.$$

For all $x \in E_b$ by its definition there exists $r_x < R$ and y_x such that

$$\frac{1}{\mu(B(y_x, br_x))} \int_{B(y_x, r_x)} |f(z)| d\mu(z) > \lambda \text{ and } x \in B(y_x, r_x).$$

Note that $\sup_{x \in E_b} r_x$ is at most R. So we can apply Theorem 1.5. Applying the theorem with $k = \dfrac{b}{a} > 1$, we obtain a countable subset $A \subset E_b$ such that $\{B(y_x, r_x)\}_{x \in A}$ satisfies

$$\bigcup_{x \in E_b} B(y_x, r_x) \subset \bigcup_j B\left(x_j, \frac{b}{a} r_j\right) \text{ and } \sum_j 1_{B\left(x_j, \frac{b}{a} r_j\right)} \leq C_{a,b}.$$

Using these properties, we have $\displaystyle\int_{E_b} g(x) d\mu(x)$

$$\leq \int_{\bigcup_j B(x_j, \frac{b}{a} r_j)} g(x) d\mu(x) \leq \sum_j \frac{1}{\mu(B(x_j, br_j))} \int_{B(x_j, \frac{b}{a} r_j)} g(x) d\mu(x) \times \mu(B(x_j, br_j))$$

$$\leq \sum_j \inf_{x \in B(x_j, r_j)} M^R_{a,uc} g(x) \frac{1}{\lambda} \int_{B(x_j, r_j)} |f(y)| d\mu(y) \leq \frac{C_{a,b}}{\lambda} \int_{\mathbf{R}^d} |f(y)| M^R_{a,uc,} g(y) d\mu(y).$$

To obtain the last inequality we used $\displaystyle\sum_j 1_{B(x_j, \frac{b}{a} r_j)} \leq C_{a,b}$ Tending R to ∞, we are done.

As a corollary we have another estimate.

Theorem 3.2 *If $p, q, k > 1$, then* $\displaystyle\left\| \left(\sum_{j \in \mathbf{Z}} (M_{k,uc} f_j)^p \right)^{1/p} \right\|_q \leq C_{p,q,k} \left\| \left(\sum_{j \in \mathbf{Z}} |f_j|^p \right)^{1/p} \right\|_q.$

The proof is obtained by changing the parameters in Theorem 1.3 suitably.

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