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1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbf{T}}, \mathbf{P})$ be a filtered probability space, $\{(W_t^1, W_t^2)\}_{t\in\mathbf{T}}$ be a $l+l'$-dimensional $\mathcal{F}_t-$Brownian Motion, $\mathbf{T} = [0,T], T > 0$, and $b : \mathbf{T}\times\mathbf{R}^d \to \mathbf{R}^d$ and $\sigma_1 : \mathbf{T}\times\mathbf{R}^d \to \mathbf{R}^d\otimes\mathbf{R}^l$ be continuous functions. For each $\varepsilon \in [0,\infty)$, we consider the following stochastic differential equation

$$X_t(\varepsilon) = x_0 + \int_0^t b(s, X_s(\varepsilon))\ ds + \varepsilon\int_0^t \sigma_1(s, X_s(\varepsilon))\ dW_s^1, \qquad t \in \mathbf{T}. \tag{1}$$

Let $F : [0,\infty)\times\mathbf{T}\times\mathbf{R}^d\times\mathbf{R}^{l'} \to \mathbf{R}^{l'}$ and $\sigma_2 : \mathbf{T}\times\mathbf{R}^{l'} \to \mathbf{R}^{l'}\otimes\mathbf{R}^{l'}$ be bounded Lipshitz continuous functions, and consider the following stochastic differential equation

$$Y_t(\varepsilon) = \int_0^t F(\varepsilon, s, X_s(\varepsilon), Y_s(\varepsilon))ds + \int_0^t \sigma_2(s, Y_s(\varepsilon))dW_s^2. \tag{2}$$

We think that $X_t(\varepsilon)$ is a system process and $Y_t(\varepsilon)$ is a observation process. Let $\mathcal{G}_t(\varepsilon) = \sigma(Y_s(\varepsilon); 0 \le s \le t), t \ge 0, \varepsilon \ge 0$. Our aim is to obtain the approximate expression of $\mathbf{E}[g(X_t(\varepsilon))|\mathcal{G}_t(\varepsilon)]$ as $\varepsilon \downarrow 0$ for an arbitrary bounded smooth function $g(x)$.

We assume the following

$A.1$, The SDE (1) has a unique strong solution for all $\varepsilon \ge 0$.

$A.2$, There is a $\eta > 0$ such that $b(t,x)$ and $\sigma_1(t,x)$ are smooth in the region

$D_\eta = \{(t,x) \in \mathbf{T}\times\mathbf{R}^d; |x - X_t(0)| \le \eta\}$ and F is smooth in $[0,1]\times D_\eta\times\mathbf{R}^{l'}$.

$A.3$, $\sigma_2(t,x)^{-1}$ exists and is bounded in (t,x).

Our main theorem is the following.

Theorem 1

For any bounded smooth function $g(x)$ *and* $t > 0$*, there exist measurable functionals* $h^{(k)} : C(\mathbf{T}; \mathbf{R}^{d'}) \to \mathbf{R}$, $k = 0, 1, 2, \cdots$, *satisfying*

$$\lim_{\varepsilon \to 0} \mathbf{E}\left[\left|\frac{1}{\varepsilon^n}\{\mathbf{E}[g(X_t(\varepsilon))|\mathcal{G}_t(\varepsilon)(\varepsilon)] - \sum_{k=0}^{n} \varepsilon^k h^{(k)}(Y_\cdot(\varepsilon))\}\right|^p\right] = 0,$$

for any $p > 1$ *and* $n \in \mathbf{N}$.

In Section 3 we give an example of functionals $h^{(k)}$ related to a certain problem in finance.

2 Proof of Theorem

Let $G(\varepsilon, t, x, y) = \sigma_2(t, y)^{-1} F(\varepsilon, t, x, y)$, and let

$$\begin{aligned}
&\alpha(\varepsilon, t, X_\cdot(\varepsilon), y_\cdot) \\
&= \exp\left\{\int_0^t G(\varepsilon, s, X_s(\varepsilon), y_s)\sigma_2(t, y)^{-1}\, dy_s - \frac{1}{2}\int_0^t |G(\varepsilon, s, X_s(\varepsilon), y_s)|^2 ds\right\} \\
&= \exp\left\{\int_0^t G(\varepsilon, s, X_s(\varepsilon), y_s)\, dW_s^2 + \frac{1}{2}\int_0^t |G(\varepsilon, s, X_s(\varepsilon), y_s)|^2 ds\right\}.
\end{aligned}$$

Let $\mathbf{Q}(\varepsilon)$ be a probability measure defined by $d\mathbf{Q}(\varepsilon) = \alpha(\varepsilon, t, X_\cdot(\varepsilon), Y(\varepsilon)_\cdot)^{-1} d\mathbf{P}$. Then, $\mathbf{Q}(\varepsilon)|_{\sigma(X(\varepsilon))} = \mathbf{P}|_{\sigma(X(\varepsilon))}$ and $\widetilde{W}_t(\varepsilon) = W_t^2 + \int_0^t G(\varepsilon, s, X_s(\varepsilon), Y_s(\varepsilon))ds$ is a $\mathcal{F}_t-$Wiener process under $\mathbf{Q}(\varepsilon)$.

Under this probability measure $\mathbf{Q}(\varepsilon)$,

$$Y_t(\varepsilon) = \int_0^t \sigma_2(s, Y_s(\varepsilon)) d\widetilde{W}_s(\varepsilon), \quad t \in [0, T].$$

is independent of $X_t(\varepsilon), t \in [0, T]$. Let $\{\widetilde{Y}_t\}_{t \in \mathbf{T}}$ be the solution of the following S.D.E.

$$\widetilde{Y}_t = \int_0^t \sigma_2(s, \widetilde{Y}_s) dW_s^2.$$

Then, the distribution of $\{(X_t(\varepsilon), Y_t(\varepsilon))\}_{t\in\mathbf{T}}$ under $\mathbf{Q}(\varepsilon)$ is equal to the distribution of $\{(X_t(\varepsilon), \widetilde{Y}_t)\}_{t\in\mathbf{T}}$ under $\mathbf{P}$. So we have

$$\begin{aligned}\mathbf{E}[g(X_t(\varepsilon))|\mathcal{G}_t(\varepsilon)] &= \frac{\mathbf{E}^{\mathbf{Q}(\varepsilon)}[g(X_t(\varepsilon))\alpha(\varepsilon,t,X.(\varepsilon),Y.(\varepsilon))|\mathcal{G}_t(\varepsilon)]}{\mathbf{E}^{\mathbf{Q}(\varepsilon)}[\alpha(\varepsilon,t,X.(\varepsilon),Y.(\varepsilon))|\mathcal{G}_t(\varepsilon)]} \\ &= \frac{\mathbf{E}[g(X_t(\varepsilon))\alpha(\varepsilon,t,X.(\varepsilon),y.)]|_{y.=Y.(\varepsilon)}}{\mathbf{E}[\alpha(\varepsilon,t,X.(\varepsilon),y.)]|_{y.=Y.(\varepsilon)}}\end{aligned}$$

Note that, there exists a constant C such that

$$\mathbf{E}[|f(\widetilde{X}_t(\varepsilon), Y.(\varepsilon))|] = \mathbf{E}[|f(\widetilde{X}_t(\varepsilon), \widetilde{Y}.)\alpha(\varepsilon,t,X.(\varepsilon),\widetilde{Y}.)|] \le C\mathbf{E}[|f(\widetilde{X}_t(\varepsilon), \widetilde{Y}.)|^2]$$

for an arbitary functional f(x,y.). So Theorem 1 follows from the following Lemma .

Lemma 2

For any arbitrary bounded smooth function $\tilde{g}(x)$ for which its derivatives are bounded for any $t > 0$, and there exist functionals
$\widetilde{h}^{(k)} : C(\mathbf{T};\mathbf{R}^{d'}) \to \mathbf{R}$, $k = 0,1,2,\cdots$, such that

$$\lim_{\varepsilon\to 0}\mathbf{E}\left[\left|\frac{1}{\varepsilon^n}\{\mathbf{E}[\tilde{g}(X_t(\varepsilon))\alpha(\varepsilon,t,X.(\varepsilon),y.)]|_{y.=\widetilde{Y}.} - \sum_{k=0}^{n}\varepsilon^k\widetilde{h}^{(k)}(\widetilde{Y}.)\}\right|^p\right] = 0,$$

for any $p \in (1,\infty)$ and $n \in \mathbf{N}$.

Before proving this Lemma, we make some preparations.

Let $\widetilde{b}(t,x) : \mathbf{T}\times\mathbf{R}^d \to \mathbf{R}^d$ and $\widetilde{\sigma}_1(t,x) : \mathbf{T}\times\mathbf{R}^d \to \mathbf{R}^d\otimes\mathbf{R}^l$, be bounded smooth functions such that their all derivatives are bounded and

$$\widetilde{\sigma}_1(t,x) = \sigma_1(t,x), \quad \widetilde{b}(t,x) = b(t,x) \qquad \text{for } (t,x) \in D_\eta.$$

We define $\{\widetilde{X}_t(\varepsilon)\}_{t\in\mathbf{T}}$ to be a solution of the stochastic differential equation.

$$\widetilde{X}_t(\varepsilon) = x_0 + \int_0^t \widetilde{b}(s,\widetilde{X}_s(\varepsilon))\,ds + \varepsilon\int_0^t \widetilde{\sigma}_1(s,\widetilde{X}_s(\varepsilon))\,dW_s^1. \qquad (3)$$

Proposition 3

There exists constants C and γ such that

$$\mathbf{P}((t,\widetilde{X}_t(\varepsilon)) \notin D_\eta \text{ for some } t \in \mathbf{T}) \le \mathbf{P}(\sup_{t\in\mathbf{T}} \|X_t(\varepsilon) - \widetilde{X}_t(\varepsilon)\| \neq 0) \le Ce^{-\gamma/\varepsilon^2},$$

for any $0 < \varepsilon \le 1$.

Proof.

Let $Z_t(\varepsilon) = \widetilde{X}_t(\varepsilon) - X_t(0)$. Then $Z_t(\varepsilon)$ satisfies

$$Z_t(\varepsilon) = \int_0^t \{\widetilde{b}(s, Z_s(\varepsilon) + X_s(0)) - \widetilde{b}(s, X_s(0))\}ds + \varepsilon \int_0^t \widetilde{\sigma}_1(s, Z_s(\varepsilon) + X_s(0))dW_s^1.$$

We have

$$\|Z_t(\varepsilon)\| \le K \int_0^t \|Z_s(\varepsilon)\|ds + \varepsilon \| \int_0^t \widetilde{\sigma}_1(s, Z_s(\varepsilon) + X_t(0))dW_s^1\|, \quad \varepsilon \in (0,1], t \in \mathbf{T}.$$

Let $M_t = \int_0^t \widetilde{\sigma}_1(s, Z_s(\varepsilon) + X_t(0))dW_s^1$. Then for each i,there is a 1-dimensioned Brownian Motion $B(t)$ such that $M_t^i = B(\langle M^i \rangle_t)$. Note that

$$\langle M^i \rangle_t = \int_0^t \{\widetilde{\sigma}_1^i(s, Z_s(\varepsilon) + X_s(0))\}^2 ds \le k^2 T, \quad \text{where } k = \sup_{t\in\mathbf{T},\ x\in\mathbf{R}} \|\widetilde{\sigma}_1(t,x)\|.$$

So there are absolute constants A and A', such that

$$\mathbf{P}(\sup_{t\in\mathbf{T}} |M_t^i| > \eta'/\varepsilon) \le \mathbf{P}(\sup_{0\le t\le k^2T} |B_t| > \eta'/\varepsilon) \le Ae^{-A'\eta'^2 k^2 T/\varepsilon^2}$$

for any $\varepsilon \in (0,1], \eta' > 0$.

If $\sup_{t\in\mathbf{T}} \|M_t\| \le l\eta'/\varepsilon$, then $\|Z_t(\varepsilon)\| \le K \int_0^t \|Z_s(\varepsilon)\|ds + l\eta'$. So we have

$$\mathbf{P}(\sup_{t\in\mathbf{T}} \|Z_t(\varepsilon)\| > l\eta' e^{KT}) \le Ale^{-A'\eta'^2 k^2 T/\varepsilon^2}$$

from Gronwall's inequality. Letting $\eta' = \eta(le^{KT})^{-1}$, we have by the pathwise uniqueness of S.D.E.,

$$\mathbf{P}(\sup_{t\in\mathbf{T}} \|X_t(\varepsilon) - \widetilde{X}_t(\varepsilon)\| \neq 0) \le \mathbf{P}(\sup_{t\in\mathbf{T}} \|\widetilde{X}_t(\varepsilon) - X_t(0)\| > \eta) \le Ce^{-C'\eta^2/\varepsilon^2}.$$

This completes the proof. □

The following is due to Kunita [5], Theorem 4.6.4, p.172.

Proposition 4

$\{\widetilde{X}_t(\varepsilon)\}_{t\in\mathbf{T}}$ is smooth in ε in $\mathbf{L}^p$-sence , for any $p \in (1,\infty)$.

Therefore there exists L^p bounded continuous process $\{\widetilde{X}_t^{(k)}\}_{t\in\mathbf{T}}, k \in \mathbf{N}$, such that

$$\lim_{\varepsilon\to 0}\mathbf{E}\left[\sup_{t\in\mathbf{T}}\left\|\frac{1}{\varepsilon^n}\left\{\widetilde{X}_t(\varepsilon)-\widetilde{X}_t(0)-\sum_{k=1}^{n}\varepsilon^k\widetilde{X}_t^{(k)}\right\}\right\|^p\right]=0,\quad p\in(1,\infty), n\in\mathbf{N}.$$

$G(\varepsilon,t,x,y)$ has a Taylor expantion in x at $\widetilde{X}_t(0)$, so there exist smooth functions $G^{(k)}(t,x,y):\mathbf{T}\times\mathbf{R}^{d(k+1)}\times\mathbf{R}^{(l')}\to\mathbf{R}$ which are polynomials in x and satisfy

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{np}}\mathbf{E}\left[\sup_{t\in\mathbf{T}}\left\|G(\varepsilon,t,\widetilde{X}_t(\varepsilon),\widetilde{Y}_t)-\sum_{k=0}^{n}\varepsilon^k G^{(k)}(t,\widetilde{X}_t(0),\widetilde{X}_t^{(1)},\widetilde{X}_t^{(2)},\cdots,\widetilde{X}_t^{(k)},\widetilde{Y}_t)\right\|^p\right]=0$$

for any $n\in\mathbf{N}$ and $p\in(1,\infty)$.

Let us denote $G^{(k)}(t,y)=G^{(k)}(t,\widetilde{X}_t(0),\widetilde{X}_t^{(1)},\widetilde{X}_t^{(2)},\cdots,\widetilde{X}_t^{(k)},y)$
and $\alpha(\varepsilon,t,y.)=\alpha(\varepsilon,t,\widetilde{X}.(\varepsilon),y.)$
and $\alpha^{(0)}(t,y.)=\exp\left\{\int_0^t G^{(0)}(s,y_s)\sigma(s,y_s)^{-1}\,dy_s+\frac{1}{2}\int_0^t|G^{(0)}(s,y_s)|^2ds\right\}$.

Proposition 5

For any $t\in\mathbf{T}$ and $p>1$ and $n\in\mathbf{N}$,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{np}}\mathbf{E}\left[\sup_{t\in\mathbf{T}}\left|\alpha(\varepsilon,t,\widetilde{Y}.)-\alpha^{(0)}(\varepsilon,t,\widetilde{Y}.)\sum_{k'=0}^{n}\frac{1}{k'}\left\{\int_0^t\sum_{k=1}^{n}\varepsilon^k G^{(k)}(s,\widetilde{Y}_s)dW_s^2\right.\right.\right.$$
$$\left.\left.\left.+\frac{1}{2}\int_0^t\{|\sum_{k=0}^{n}\varepsilon^k G^{(k)}(s,\widetilde{Y}_s)|^2-|G^{(0)}(s,\widetilde{Y}_s)|^2\}ds\right\}^{k'}\right|^p\right]=0.$$

Proof.

Let

$$\widetilde{G}(\varepsilon,y.)=\int_0^t G(\varepsilon,t,\widetilde{X}_s(\varepsilon),y_s)dW_s^2+\frac{1}{2}\int_0^t|G(\varepsilon,t,\widetilde{X}_s(\varepsilon),y_s)|^2ds$$

$$-\int_0^t G^{(0)}(s,y_s)dW_s^2 - \frac{1}{2}\int_0^t |G^{(0)}(s,y_s)|^2 ds.$$

Then

$$\alpha(\varepsilon,t,Y.) = \alpha^{(0)}(t,Y.)\exp\widetilde{G}(\varepsilon,Y.),$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{np}}\mathbf{E}\Bigg[\sup_{t\in\mathbf{T}}\Bigg|\{\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\}^{k'} - \Bigg\{\int_0^t\sum_{k=1}^n \varepsilon^k G^{(k)}(s,\widetilde{Y}.)dW_s^2 - \frac{1}{2}\int_0^t\left(|G^{(0)}(s,\widetilde{Y}.)+\sum_{k=1}^n\varepsilon^k G^{(k)}(s,\widetilde{Y}.)|^2 - |G^{(0)}(s,\widetilde{Y}.)|^2\right)ds\Bigg\}^{k'}\Bigg|^p\Bigg] = 0$$

for any $n, k' \in \mathbf{N}$ and $p\in(1,\infty)$.

On the other hand,

$$e^x = \sum_{k=0}^n \frac{x^k}{k!} + \int_0^x\left\{\int_0^{y_1}\left\{\int_0^{y_2}\cdots\left\{\int_0^{y_n} e^{y_{n+1}}dy_{n+1}\right\}\cdots dy_3\right\}dy_2\right\}dy_1.$$

So we have

$$\left|e^x - \sum_{k=0}^n\frac{x^k}{k!}\right| \le \frac{|x|^{n+1}}{(n+1)!}e^{|x|}.$$

Therefore,

$$\mathbf{E}\left[\left|\exp\{\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\} - \sum_{k'=0}^n\frac{\{\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\}^{k'}}{k'!}\right|^p\right] \le \mathbf{E}\left[\left\{\frac{\left|\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\right|^{n+1}}{(n+1)!}\exp\left|\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\right|\right\}^p\right].$$

We see that,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{nq}}\mathbf{E}\left[\sup_{t\in\mathbf{T}}\left||\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)|^{n+1}\right|^q\right] = 0, \qquad \text{for any} \quad \in(1,\infty),$$

and

$$\begin{aligned}\mathbf{E}\left[\left\{\exp\left|\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\right|\right\}^q\right] &\le \mathbf{E}\left[\exp\{q\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\}\right] + \mathbf{E}\left[\exp\{-q\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\}\right]\\ &\le 2e^{\frac{1}{2}q^2K^2T} \quad \text{for any } q\in(1,\infty).\end{aligned}$$

by the following Proposition.

Proposition 6

Let $G:\mathbf{T}\times\Omega\to\mathbf{R}^{l'}$ is adapted and satisfy $|G_t|\le K$ for some constant K. Then,

$$\mathbf{E}\left[\exp\{q\int_0^T G_s dW_s^2\}\right] \le e^{\frac{1}{2}q^2K^2T}, \quad \textit{for any } q\in\mathbf{R}.$$

Proof.

$$\mathbf{E}\left[\exp\left\{\int_0^T qG_s dW_s^2 - \frac{1}{2}\int_0^T |qG_s|^2 ds\right\}\right] \leq 1.$$

So we have,

$$\begin{aligned}
&\mathbf{E}\left[\exp\left\{\int_0^T qG_s dW_s^2\right\}\right] \\
&= \mathbf{E}\Bigg[\exp\left\{\int_0^T qG_s dW_s^2 - \frac{1}{2}\int_0^T |qG_s|^2 ds\right\}\exp\left\{\frac{1}{2}\int_0^T q^2|G_s|^2 ds\right\}\Bigg] \\
&\leq \mathbf{E}\left[\exp\left\{\int_0^T qG_s dW_s^2 - \frac{1}{2}\int_0^T |qG_s|^2 ds\right\}\right]\exp\left\{\frac{1}{2}\int_0^T q^2 K^2 ds\right\} \\
&\leq e^{\frac{1}{2}q^2K^2T}.
\end{aligned}$$

□

Therefore,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{np}}\mathbf{E}\left[\left|\exp\{\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\} - \sum_{k'=0}^{n}\frac{\{\widetilde{G}(\varepsilon,\widetilde{Y}_\cdot)\}^{k'}}{k'!}\right|^p\right] = 0.$$

So,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{np}}\mathbf{E}\Bigg[\sup_{t\in\mathbf{T}}\Bigg|\alpha(\varepsilon,t,\widetilde{Y}.) - \alpha^{(0)}(t,\widetilde{Y}.)\sum_{k'=0}^{n}\frac{1}{k'}\Bigg\{\int_0^t\sum_{k=1}^{n}\varepsilon^k G^{(k)}(s,\widetilde{Y}_s)dW_s^2 \\
&\quad +\frac{1}{2}\int_0^t\left(|\sum_{k=0}^{n}\varepsilon^k G^{(k)}(s,\widetilde{Y}_s)|^2 - |G^{(0)}(s,\widetilde{Y}_s)|^2\right)ds\Bigg\}^{k'}\Bigg|^p\Bigg] \\
&\leq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{np}}\mathbf{E}\left[\sup_{t\in\mathbf{T}}\left|\alpha^{(0)}(\widetilde{Y}.)\left\{\exp\{\widetilde{G}(\varepsilon,\widetilde{Y}.)\} - \sum_{k'=0}^{n}\frac{\{\widetilde{G}(\varepsilon,\widetilde{Y}.)\}^{k'}}{k'!}\right\}\right|^p\right] \\
&\quad +\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{np}}\mathbf{E}\Bigg[\sup_{t\in\mathbf{T}}|\alpha^{(0)}(\widetilde{Y}.)|^p\Bigg|\sum_{k'=0}^{n}\frac{1}{k'}\Bigg\{\{\widetilde{G}(\varepsilon,y_t)\}^{k'} \\
&\qquad -\left\{\int_0^t\sum_{k=1}^{n}\varepsilon^k G^{(k)}(\widetilde{Y}_s)dW_s^2 + \frac{1}{2}\int_0^t\left(|\sum_{k=0}^{n}\varepsilon^k G^{(k)}(s,\widetilde{Y}_s)|^2 - |G^{(0)}(s,\widetilde{Y}_s)|^2\right)ds\right\}^{k'}\Bigg\}\Bigg|^p\Bigg] \\
&= 0.
\end{aligned}$$

Because $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^q}\mathbf{E}[\sup_{t\in\mathbf{T}}|\alpha^{(0)}(\widetilde{Y}.)|^q] = 0$ for any $q \in (1,\infty)$, we have our assertion.

This complete the proof of Proposition 5 . □

Now let us prove Lemma 2.

$\widetilde{g}(\widetilde{X}_t(\varepsilon))$ has an asymptotic expantion in ε, because $\widetilde{X}_t(\varepsilon)$ has an asymptotic expantion by proposition 4 and because $\widetilde{g}$ is smooth and bounded. Also $\alpha(t, \widetilde{X}.(\varepsilon), y.)$ also has an asymptotic expantion by Proposition 5. So, there exists $\widetilde{h}^{(k)}(y.)$ such that

$$\lim_{\varepsilon\to 0}\mathbf{E}\left[\left|\frac{1}{\varepsilon^n}\left\{\mathbf{E}[\widetilde{g}(\widetilde{X}_t(\varepsilon))\alpha(\varepsilon,t,\widetilde{X}.(\varepsilon),\widetilde{Y}.)]-\sum_{k=0}^n \varepsilon^k\widetilde{h}^{(k)}(\widetilde{Y}.)\right\}\right|^p\right]=0$$

for any $t \geq 0$, $p \in (1,\infty)$ and $n \in \mathbf{N}$. $\widetilde{h}^{(k)}(y.)$ satisfy

$$\begin{aligned}
&\left|\frac{1}{\varepsilon^n}\{\mathbf{E}[\widetilde{g}(X_t(\varepsilon))\alpha(\varepsilon,t,X.(\varepsilon),\widetilde{Y}.)]-\sum_{k=0}^n \varepsilon^k\widetilde{h}^{(k)}(\widetilde{Y}.)\}\right| \\
&\quad\leq \left|\frac{1}{\varepsilon^n}\{\mathbf{E}[\widetilde{g}(X_t(\varepsilon))\alpha(\varepsilon,t,X.,\widetilde{Y}.)|\mathcal{G}_t(\varepsilon)]-\mathbf{E}[\widetilde{g}(\widetilde{X}_t(\varepsilon))\alpha(\varepsilon,t,\widetilde{X}.,\widetilde{Y}.)|\mathcal{G}_t(\varepsilon)]\}\right|^p \\
&\qquad+\left|\frac{1}{\varepsilon^n}\{\mathbf{E}[\widetilde{g}(\widetilde{X}_t(\varepsilon))\alpha(\varepsilon,t,\widetilde{X}.,\widetilde{Y}.)]-\sum_{k=0}^n \varepsilon^k\widetilde{h}^{(k)}(\widetilde{Y}.)\}\right|^p.
\end{aligned}$$

We have,

$$\begin{aligned}
&\left|\frac{1}{\varepsilon^n}\{\mathbf{E}[\widetilde{g}(X_t(\varepsilon))\alpha(\varepsilon,t,X.,\widetilde{Y}.)|\mathcal{G}_t(\varepsilon)]-\mathbf{E}[\widetilde{g}(\widetilde{X}_t(\varepsilon))\alpha(\varepsilon,t,\widetilde{X}.,\widetilde{Y}.)|\mathcal{G}_t(\varepsilon)]\}\right|^p \\
&\quad\leq \left|\frac{1}{\varepsilon^n}\mathbf{E}[1_{\{X_t(\varepsilon)\neq\widetilde{X}_t(\varepsilon)\}}\{|\widetilde{g}(X_t(\varepsilon))\alpha(\varepsilon,t,X.,\widetilde{Y}.)|+|\widetilde{g}(\widetilde{X}_t(\varepsilon))\alpha(\varepsilon,t,\widetilde{X}.,\widetilde{Y}.)|\}|\mathcal{G}_t(\varepsilon)]\}\right|^p \\
&\quad\leq \frac{1}{\varepsilon^n}\{\mathbf{P}(X_t(\varepsilon)\neq\widetilde{X}_t(\varepsilon))\}^{\frac{p}{2}}\mathbf{E}\left[\{|\widetilde{g}(X_t(\varepsilon))\alpha(\varepsilon,t,X.,\widetilde{Y}.)|+|\widetilde{g}(\widetilde{X}_t(\varepsilon))\alpha(\varepsilon,t,\widetilde{X}.,\widetilde{Y}.)|\}^{\frac{p}{2}}\right].
\end{aligned}$$

We have, $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\{\mathbf{P}(X_t(\varepsilon)\neq\widetilde{X}_t(\varepsilon))\}^{\frac{p}{2}}=0$ by Proposition 3.

And $\mathbf{E}\left[\{|\widetilde{g}(X_t(\varepsilon))\alpha(\varepsilon,t,X.,\widetilde{Y}.)|+|\widetilde{g}(\widetilde{X}_t(\varepsilon))\alpha(\varepsilon,t,\widetilde{X}.,\widetilde{Y}.)|\}^{\frac{p}{2}}\right]$ is bounded. Because $\widetilde{g}$ is bounded and

$$\mathbf{E}\left[|\alpha(\varepsilon,t,X.,\widetilde{Y}.)|^q\right]\leq \exp\left\{\frac{(q^2-q)}{2}K^2T\right\},\quad \text{for any } q\in(1,\infty).$$

by same argument in the proof of the Proposition 6.

Therefore,

$$\lim_{\varepsilon\to 0}\mathbf{E}\left[\left|\frac{1}{\varepsilon^n}\{\mathbf{E}[\widetilde{g}(X_t(\varepsilon))\alpha(\varepsilon,t,X.(\varepsilon),\widetilde{Y}.)]-\sum_{k=0}^n \varepsilon^k\widetilde{h}^{(k)}(\widetilde{Y}.)\}\right|^p\right]=0.$$

This complete the proof. □

3 Example

At the frictionless market, if we suppose no arbitrage, then one bond will have a unique price. But we can only know the price that distored by many reasons. In this section , we use above theorem for C.I.R and Vasicek model.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbf{T}}, \mathbf{P})$ be a probability space, and $\mathbf{T} = [0, T]$ be a parameter of time, and $a, b, \alpha, \beta, \varepsilon$ are constants satisfying $a \in (0, \infty), b, \alpha, \beta \in [0, \infty), \varepsilon \in [0, 1]$, $a\beta + b\alpha \geq 0$ and x_0 satisfies $\alpha x_0 + \beta > 0$. And let $\{(W_t^1, W_t^2)_{t\in\mathbf{T}}\}$ be a 2-dim $\mathcal{F}_t$-Brownian Motion. We assume that spot rate process $\{X_t(\varepsilon)\}_{t\in\mathbf{T}}$ satisfies following stochastic differential equation.

$$X_t(\varepsilon) = x_0 + \int_0^t (-aX_s(\varepsilon) + b)\ ds + \varepsilon \int_0^t \sqrt{\alpha X_t(\varepsilon) + \beta}\ dW_s^1.$$

We regard $X_t(\varepsilon)$ as the spot rate process and $\mathbf{P}$ is a risk neutoral measure. The 0-cupon bond price $F(\varepsilon, t, X_t(\varepsilon))$ with maturity T, is given by

$$F(\varepsilon, t, X_t(\varepsilon)) = \mathbf{E}\left[\exp\left(-\int_t^T X_s(\varepsilon) ds\right) |\mathcal{F}_t\right], \qquad t \in T.$$

Here

$$F(\varepsilon, t, x) = \exp\{A(\varepsilon, T - t) + B(\varepsilon, T - t)x\}$$

where A and B satisfies following differential equation.

$$B'(\varepsilon, t) = -aB(\varepsilon, t) + \frac{1}{2}\varepsilon^2\alpha\{B(\varepsilon, t)\}^2 - 1, \qquad B(\varepsilon, 0) = 0$$

$$A'(\varepsilon, t) = bB(\varepsilon, t) + \frac{1}{2}\varepsilon^2\beta\{B(\varepsilon, t)\}^2, \qquad A(\varepsilon, 0) = 0$$

We assume that we can only observe the process $\{Y_t(\varepsilon)\}_{t\in\mathbf{T}}$ given by

$$Y_t(\varepsilon) = \int_0^t F(\varepsilon, s, X_s(\varepsilon)) ds + \sigma W_t^2.$$

Let $\alpha(\varepsilon, t, X.(\varepsilon), y.) = \exp\left\{\int_0^t \frac{1}{\sigma} F(\varepsilon, s, X_s(\varepsilon)) dy_s - \frac{1}{2}\int_0^t |\frac{1}{\sigma} F(\varepsilon, s, X_s(\varepsilon))|^2 ds\right\}$ and $\mathcal{G}_t(\varepsilon) = \sigma(Y_s(\varepsilon); 0 \leq s \leq t)$. Then,

$$\mathbf{E}[F(\varepsilon, t, X_t(\varepsilon))|\mathcal{G}_t(\varepsilon)] \quad = \frac{\mathbf{E}^{\mathbf{Q}(\varepsilon)}[F(\varepsilon, t, X_t(\varepsilon))\alpha(\varepsilon, t, X.(\varepsilon), y.)]|_{y.=Y.(\varepsilon)}}{\mathbf{E}^{\mathbf{Q}(\varepsilon)}[\alpha(\varepsilon, t, X.(\varepsilon), y.)]|_{y.=Y.(\varepsilon)}}$$

Remark 1

$a\beta + b\alpha \geq 0$ is the condition for $X_t(\varepsilon)$ to be well defined.

Remark 2

This model is called the C.I.R model if $\beta = 0$, and is called the Vasicek model if $\alpha = 0$.

Remark 3

$F(\varepsilon, t, x)$ is bounded on $\left\{ x \in \mathbf{R}; x \geq -\dfrac{\beta}{\alpha} \right\}$, because $B'(\varepsilon, t) < 0$.

Now we show an asymptotic expantion of $\mathbf{E}[F(\varepsilon, t, X_t(\varepsilon))\alpha(\varepsilon, t, X., y.)]$.

Let η be a constant satisfies $\eta < \dfrac{1}{2}\left\{ x_0 + \dfrac{\beta}{\alpha} \right\}$, and let

$$D_\eta = \left\{ (t, x) \in \mathbf{T} \times \mathbf{R}; x_0 - \eta \leq x \leq x_0 \exp^{-aT} + \frac{b}{a} + \eta \right\}.$$

Then $-ax + b$ and $\sqrt{\alpha x + \beta}$ are smooth and bounded in the region D_η, and $F(\varepsilon, t, x)$ is, too. So there exists $\{\widetilde{X}_t(\varepsilon)\}_{t \in \mathbf{T}}$ that has an asymptotic expantion

$$\lim_{\varepsilon \to 0} \mathbf{E}\left[\left\| \frac{1}{\varepsilon^n} \left\{ \widetilde{X}_t(\varepsilon) - \widetilde{X}_t(0) - \sum_{k=1}^{n} \varepsilon^k \widetilde{X}_t^{(k)} \right\} \right\|^p \right] = 0.$$

For example ,

$$\widetilde{X}_t(0) = \begin{cases} \dfrac{b}{a} + (x_0 - \dfrac{1}{2})e^{-at} & a \neq 0 \\ \\ x_0 + bt & a = 0 \end{cases}$$

$$\widetilde{X}_t^{(1)} = e^{-at} \int_0^t e^{as} \sqrt{\alpha \widetilde{X}_s(0) + \beta} dW_s^1$$

$$\widetilde{X}_t^{(2)} = e^{-at} \int_0^t e^{as} \frac{\alpha}{\sqrt{\alpha \widetilde{X}_s(0) + \beta}} \widetilde{X}_s^{(1)} dW_s^1.$$

Let $\widetilde{F}(\varepsilon, t, x) : \mathbf{T} \times \mathbf{R} \to \mathbf{R}$ be the bounded smooth function that is equal to $F(\varepsilon, t, x)$ in the region D_η. Then we have

$$\lim_{\varepsilon \to 0} \mathbf{E}\left[\left| \mathbf{E}[F(\varepsilon, t, X_t(\varepsilon))\alpha(\varepsilon, t, X., y.)]|_{y.=\sigma W^2.} - \mathbf{E}[\widetilde{F}(t, \widetilde{X}_t(\varepsilon))\alpha(\varepsilon, t, \widetilde{X}., y.)]|_{y.=\sigma W^2.} \right|^p \right] = 0.$$

We show an asymptotic expantion of $\mathbf{E}[\widetilde{F}(\varepsilon, t, \widetilde{X}_t(\varepsilon))\alpha(\varepsilon, t, \widetilde{X}., y.)]$.

There exist $F_k(t)$ such that

$$\lim_{\varepsilon \to 0} \mathbf{E}\left[\sup_{t \in \mathbf{T}} \left| \frac{1}{\varepsilon^n} \left\{ \widetilde{F}(\varepsilon, t, \widetilde{X}_t(\varepsilon)) - \sum_{k=0}^{n} \varepsilon^k F_k(t) \right\} \right|^p \right] = 0$$

for any $p > 1$ and $n \in \mathbf{N}$.

For example, there exist $A_0(t), A_2(t), B_0(t)$ and $B_2(t)$ such that

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbf{T}}\left|\frac{1}{\varepsilon^2}\left\{A(\varepsilon,t)-A_0(t)-\varepsilon^2 A_2(t)\right\}\right|=0$$

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbf{T}}\left|\frac{1}{\varepsilon^2}\left\{B(\varepsilon,t)-B_0(t)-\varepsilon^2 B_2(t)\right\}\right|=0$$

for any $p > 1$ and $n \in \mathbf{N}$.

$$A_0(t)=\begin{cases} -\dfrac{b}{a^2}e^{-at}-\dfrac{b}{a}t+\dfrac{b}{a^2} & a\neq 0\\[2ex] -\dfrac{1}{2}bt^2 & a=0\end{cases}$$

$$A_2(t)=\begin{cases} \dfrac{b\alpha+a\beta}{4a^2}\left\{-e^{-2at}+\dfrac{4}{a}e^{-at}+\dfrac{2}{a}t+\dfrac{4}{a}-1\right\}+\dfrac{bt}{a^2}e^{-at} & a\neq 0\\[2ex] \dfrac{1}{24}\alpha bt^4+\dfrac{1}{6}\beta t^3 & a=0\end{cases}$$

$$B_0(t)=\begin{cases} \dfrac{1}{a}(e^{-at}-1) & a\neq 0\\[2ex] -t & a=0\end{cases}$$

$$B_2(t)=\begin{cases} \dfrac{\alpha}{2a^3}(e^{-2at}-2ate^{-at}+1) & a\neq 0\\[2ex] \dfrac{1}{6}\alpha t^3 & a=0.\end{cases}$$

Using these, we have

$$F_0(t)=\exp\left\{A_0(T-t)+B_0(T-t)\widetilde{X}_t(0)\right\}$$

$$F_1(t)=F_0(t)B_0(T-t)\widetilde{X}_t^{(1)}$$

$$F_2(t)=F_0(t)\{A_2(T-t)+B_2(T-t)\widetilde{X}_t(0)+B_0(T-t)\widetilde{X}_t^{(2)}+\frac{1}{2}B_0(T-t)^2(\widetilde{X}_t^{(1)})^2\}.$$

So, there exist founctionals $G_k(y.)$ such that

$$\lim_{\varepsilon\to 0}\mathbf{E}\left[\sup_{t\in\mathbf{T}}\left|\frac{1}{\varepsilon^n}\left\{\left(\int_0^t F(\varepsilon,s,\widetilde{X}_s(\varepsilon))\sigma dW_s^2-\frac{1}{2}\int_0^t|F(\varepsilon,s,\widetilde{X}_s(\varepsilon))|^2ds\right)-\sum_{k=0}^n\varepsilon^k G_k(t,\sigma W^2.)\right\}\right|^p\right]=0$$

for any $t \geq 0$, $p \in (1,\infty)$ and $n \in \mathbf{N}$.

For example,

$$G_0(y.)=\int_0^t F_0(s)dy_s-\frac{1}{2}\int_0^t|F_0(s)|^2ds$$

$$G_1(y.)=\int_0^t F_1(s)dy_s-\int_0^t F_0(s)F_1(s)ds$$

$$G_2(y.) = \int_0^t F_2(s)dy_s - \int_0^t \left\{ F_0(s)F_2(s) + \frac{1}{2}|F_1(s)|^2 \right\} ds.$$

Furthermore, there exist founctionals $\alpha_k(y.)$ such that

$$\lim_{\varepsilon\to 0} \mathbf{E}\left[\sup_{t\in\mathbf{T}} \left\| \frac{1}{\varepsilon^n} \left\{ \alpha(\varepsilon, t, \widetilde{X}.(\varepsilon), \sigma W^2.) - \sum_{k=0}^n \varepsilon^k \alpha_k(\sigma W^2.) \right\} \right\|^p \right] = 0.$$

for any $t \geq 0$, $p \in (1,\infty)$ and $n \in \mathbf{N}$.

For example,

$$\alpha_0(y.) = e^{G_0(y.)}$$

$$\alpha_1(y.) = e^{G_0(y.)} G_1(y.)$$

$$\alpha_2(y.) = e^{G_0(y.)} \left\{ \frac{1}{2} G_1(y.)^2 + G_2(y.) \right\}.$$

Therefore,

$$\lim_{\varepsilon\to 0} \mathbf{E}\left[\sup_{t\in\mathbf{T}} \left| \frac{1}{\varepsilon^n} \left\{ \mathbf{E}[F(\varepsilon, t, \widetilde{X}_t(\varepsilon))\alpha(\varepsilon, t, \widetilde{X}., y.)]|_{y.=\sigma W^2.} - \sum_{k=0}^n \varepsilon^k h^{(k)}(\sigma W^2.) \right\} \right|^p \right] = 0.$$

for any $t \geq 0$, $p \in (1,\infty)$ and $n \in \mathbf{N}$.

For example,

$$\begin{aligned} h^{(0)}(y.) &= F_0(t)e^{G_0(y.)} \\ h^{(1)}(y.) &= \mathbf{E}[\{F_0(t)G_1(y.) + F_1(y.)\}e^{G_0(y.)}] = 0 \end{aligned}.$$

Let $v(t) = \mathbf{E}[\{X_t^{(1)}\}^2] = e^{-2at} \int_0^t e^{2as}(\alpha X_s(0) + \beta)ds$. Then, $h^{(2)}(y.)$ is as follows.

$$\begin{aligned} h^{(2)}(y.) = {} & e^{G_0(y.)} F_t(0)\{A_2(T-t) + B_2(T-t)\widetilde{X}_t(0)\} + \tfrac{1}{2} e^{G_0(y.)} F_t(0)(B_0(T-t))^2 v(t) \\ & + e^{G_0(y.)} y_t (F_0(t))^2 \{A_2(T-t) + B_2(T-t)\widetilde{X}_t(0)\} \\ & - e^{G_0(y.)} F_0(t) \int_0^t y_s \{A_2(T-s) + B_2(T-s)\widetilde{X}_s(0)\}' ds \\ & - e^{G_0(y.)} F_0(t) \int_0^t F_0(s)\{A_2(T-s) + B_2(T-s)\widetilde{X}_s(0)\} ds \\ & - e^{G_0(y.)} F_0(t) \int_0^t \{F_0(s)B_0(T-s)\}^2 v(s) ds \\ & + \tfrac{1}{2} e^{G_0(y.)} y_t \{F_0(t)B_0(T-t)\}^2 v(t) \\ & - \tfrac{1}{2} e^{G_0(y.)} F_t(0) \int_0^t y_s \{F_s(0)(B_0(T-s))^2 v(s)\}' ds \\ & + e^{G_0(y.)} y_t \{F_0(t)B_0(T-t)\}^2 v(t) \end{aligned}$$

$$-e^{G_0(y.)}F_0(t)B_0(T-t)e^{-at}\int_0^t (F_0(s))^2B_0(T-s)e^{as}v(s)ds$$
$$+e^{G_0(y.)}F_0(t)B_0(T-t)e^{-at}\int_0^t y_s\{F_0(s)B_0(T-s)e^{as}v(s)\}'ds$$
$$+e^{G_0(y.)}(y_t)^2(F_0(t))^3(B_0(T-t))^2v(t)$$
$$-e^{G_0(y.)}F_0(t)\int_0^t (y_s)^2\{(F_0(s)B_0(T-s)e^{-as})'\}^2e^{2as}v(s)ds$$
$$-e^{G_0(y.)}F_0(t)\int_0^t (y_s)^2(F_0(s))^2(B_0(T-s))^2(\alpha\widetilde{X}_s(0)+\beta)ds$$
$$-e^{G_0(y.)}F_0(t)\int_0^t (F_0(s))^4(B_0(T-s))^2v(s)ds.$$

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