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LONG RANGE SCATTERING FOR NONLINEAR SCHRÖDINGER EQUATIONS IN ONE AND TWO SPACE DIMENSIONS

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Abstract. We study the scattering theory for the nonlinear Schrödinger equations with cubic and quadratic nonlinearities in one and two space dimensions, respectively. For example, the nonlinearities are sum of gauge invariant term and non-gauge invariant terms such as $\lambda_0|u|^2u+\lambda_1u^3+\lambda_2u\bar{u}^2+\lambda_3\bar{u}^3$ in one dimensional case, where $\lambda_0\in\mathbb{R}$ and $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$. The scattering theory for these equations belongs to the long range case. We show the existence and uniqueness of global solutions for those equations which approach a given modified free profile. The same problem for the nonlinear Schrödinger equation with the Stark potentials is also considered.

1. Introduction

We study the global existence and the asymptotic behavior of solutions for the nonlinear Schrödinger equation in one or two space dimensions:

$$i\partial_t u=-\frac{1}{2}\Delta u+f_n(u),\quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{NLS}$$

where $n=1,2$ and u is a complex valued unknown function of (t,x). Here $f_n(u)$ is a nonlinear term of the form

$$f_1(u)=\lambda_0|u|^2u+\lambda_1u^3+\lambda_2u\bar{u}^2+\lambda_3\bar{u}^3,\quad \text{when } n=1, \tag{1.1}$$

$$f_2(u)=\lambda_0|u|u+\lambda_1u^2+\lambda_2\bar{u}^2,\quad \text{when } n=2, \tag{1.2}$$

where $\lambda_0\in\mathbb{R}$ and $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$. We also consider the following Schrödinger equation with the Stark effect in the same space dimensions:

$$i\partial_t u=-\frac{1}{2}\Delta u+V(x)u+F_n(u),\quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{NLS-S}$$

where $n=1,2$ and u is a complex valued unknown function of (t,x). Here $F_n(u)$ and V are a nonlinearity and a linear potential, respectively, of the form

$$F_1(u) = \lambda_0|u|^2u + \lambda_1u^3 + \lambda_2\bar{u}^3, \quad \text{when } n = 1, \tag{1.3}$$

$$F_2(u) = \lambda_0|u|u + \lambda_1u^2 + \lambda_2\bar{u}^2 + \lambda_3u\bar{u}, \quad \text{when } n = 2, \tag{1.4}$$

$$V(x) = E \cdot x, \tag{1.5}$$

where $\lambda_0 \in \mathbb{R}$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and $E \in \mathbb{R}^n \setminus \{0\}$. f_n and F_n are critical power nonlinearities between the short range case and the long range one in n space dimensions ($n = 1, 2$). The above potential V is a Stark potential with a constant electric field E. In this paper, we prove the existence of modified wave operators to the equations (NLS) and (NLS-S) for small final states.

A large amount of works have been devoted to the global existence and the asymptotic behavior of solutions for the nonlinear Schrödinger equation (see, e.g, [1, 3, 5, 6, 8, 11, 12, 16, 17, 18, 19, 20, 23, 28, 29, 30, 31]). In the scattering theory for the linear Schrödinger equation, (ordinary) wave operators are defined as follows. Assume that for a solution of the free Schrödinger equation with given initial data ϕ, there exists a unique time global solution u for the perturbed Schrödinger equation such that u behaves like the given free solution as $t \to \infty$ (this case is called the short range case, and otherwise we call the long range case). Then we define a wave operator W_+ by the mapping from ϕ to $u|_{t=0}$. In the long range case, ordinary wave operators do not exist and we have to construct modified wave operators including a suitable phase correction in their definition. For the nonlinear Schrödinger equation, we can define the wave operators and introduce the modified wave operators in the same way.

We first recall several known results concerning the scattering problem for the equation (NLS) in the case of power nonlinearity $f(u) = |u|^{p-1}u$. We consider the existence of wave operators $W_\pm$ for the equation (NLS). For the equation (NLS), the wave operator W_+ is defined as follows. Let Σ be L^2 or a dense subset of it. Let $\phi \in \Sigma$, and let

$$u_0(t, x) \equiv (U(t)\phi)(x),$$

where

$$U(t) \equiv e^{\frac{it}{2}\Delta}.$$

Note that u_0 is the solution to the Cauchy problem of the free Schrödinger equation

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = 0, \\ u(0, x) = \phi(x). \end{cases}$$

If there exists a unique global solution u of the equation (NLS) such that

$$\|u(t) - u_0(t)\|_{L^2} \to 0,$$

as $t \to +\infty$, then a mapping

$$W_+ \colon \phi \mapsto u(0)$$

is well-defined on Σ. We call the mapping W_+ a wave operator. ϕ is called a final state, final data, a scattered state or scattered data. It is known that when $p > 1 + 2/n$, the solution for the equation (NLS) has free profile as $t \to \pm\infty$, that is, there exist the wave operators $W_\pm$. On the other hand, when $1 \le p \le 1 + 2/n$, non-trivial solutions for that equation have no free profile, that is, we cannot define the wave operators (see, e.g., [1, 5, 6, 16, 20, 29, 30, 31]). Intuitive meaning of these facts is as follows. Recalling the well-known time decay estimates $\|u_0(t)\|_{L^2(\mathbb{R}^n)} = \|\phi\|_{L^2(\mathbb{R}^n)} = O(1)$, and $\|u_0(t)\|_{L^\infty(\mathbb{R}^n)} = O(t^{-n/2})$, we see that $\||u_0(t)|^p\|_{L^2(\mathbb{R}^n)} = O(t^{-n(p-1)/2})$. Roughly speaking, according to the linear scattering theory (the Cook-Kuroda method), wave operators exist if and only if $\||u_0(t)|^p\|_{L^2(\mathbb{R}^n)}$ is integrable with respect to t over the interval $[1, \infty)$, that is, $p > 1 + 2/n$.

There are several results concerning the long range scattering for the equation (NLS) in the case of $p = 1 + 2/n$. In the long range case, we consider the existence of the modified wave operators. The modified wave operator $\widetilde{W}_+$ is defined as follows. We construct a suitable modified free profile A_+, and a unique solution u for the equation (NLS) which approaches A_+ in L^2 as $t \to \infty$. The mapping

$$\widetilde{W}_+ \colon A_+(0) \mapsto u(0)$$

is called the modified wave operator. Ozawa [20] and Ginibre and Ozawa [5] proved the existence of modified wave operators for small final data in one space dimension and in two and three space dimensions, respectively. The proof of above results is mainly based on the method of phase correction. More precisely, they put a modified free profile of the form $A(t) = U(t)e^{-iS(t,-i\nabla)}\phi$, where ϕ is a given final state and $U(t) = e^{it\Delta/2}$, and chose the phase function S such that $\|\mathcal{L}A(t) - |A(t)|^{2/n}A(t)\|_{L^2}$ decays faster than $\||U(t)\phi|^{2/n}U(t)\phi\|_{L^2} = O(t^{-1})$, where $\mathcal{L} = i\partial_t + (1/2)\Delta$. Recently, Ginibre and Velo [9] have partially extended above results in the case of the nonlinearity $a(t)|u|^2u$ without assuming the restriction on the size of final data, where $a(t)$ has a suitable growth rate with respect to t. The phase correction method is applicable only for gauge invariant nonlinearities such as $\lambda|u|^{p-1}u$, where $\lambda \in \mathbb{R}$. We cannot apply this method to non-gauge invariant nonlinearities such as u^p or $|u|^{p-1}u + u^p$.

Recently, for the nonlinear Schrödinger equations with non-gauge invariant nonlinearities, for example u^3 in one space dimension, Moriyama, the second author and Tsutsumi [17] have proved the existence of the wave operators for small final data. The main idea of their proof is as follows. Using the oscillation of the asymptotics for u_0^3, they could construct a suitable approximate function $A = u_0 + u_2$ such that $(i\partial_t + \frac{1}{2}\Delta)u_2 - u_0^3$ and u_2 decays faster than u_0^3 and u_0 as $t \to \infty$, respectively. On the other hand, by using similar idea, the first author showed the existence of the wave operators for the Klein-Gordon-Schrödinger and the Zakharov equations and the existence of the modified wave operators for the wave-Schrödinger and the Maxwell-Schrödinger equations in [24, 25, 26, 27].

In this paper, we consider the nonlinearities which are sum of gauge invariant terms and non-gauge invariant terms, that is, (1.1) and (1.2), and prove that the equation (NLS) has a unique solution which approaches a given modified free profile. This implies the existence of the modified wave operators. We also consider the same problem for the equation (NLS-S), that is, the nonlinear Schrödinger equation with the Stark potential. (The space-time behavior of propagators and the scattering problem for the Schrödinger equation with Stark effect were studied in [4, 13, 14, 15, 21, 22]).

Before stating our main results, we introduce several notations.

Notations. We use the following symbols:

$$\partial_0 = \partial_t = \frac{\partial}{\partial t}.$$

When $n = 1$,

$$\partial = \partial_x = \frac{\partial}{\partial x}, \quad \Delta = \partial_x^2.$$

When $n = 2$,

$$x = (x_1, x_2) \in \mathbb{R}^2,$$

$$\partial_j = \frac{\partial}{\partial x_j} \quad \text{for } j = 1, 2,$$

$$\partial^\alpha = \partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2} \quad \text{for a multi-index } \alpha = (\alpha_1, \alpha_2),$$

$$\nabla = (\partial_1, \partial_2), \quad \Delta = \partial_1^2 + \partial_2^2.$$

Let

$$L^q \equiv L^q(\mathbb{R}^n) = \left\{ \psi \colon \|\psi\|_{L^q} = \left(\int_{\mathbb{R}^n} |\psi(x)|^q \, dx \right)^{1/q} < \infty \right\} \text{ for } 1 \le q < \infty,$$

$$L^\infty \equiv L^\infty(\mathbb{R}^n) = \left\{ \psi \colon \|\psi\|_{L^\infty} = \operatorname{ess.sup}_{x \in \mathbb{R}^n} |\psi(x)| < \infty \right\}.$$

We denote the set of rapidly decreasing functions on $\mathbb{R}^n$ by $\mathcal{S}$. Let $\mathcal{S}'$ be the set of tempered distributions on $\mathbb{R}^n$. For $w \in \mathcal{S}'$, we denote the Fourier transform of w by $\hat{w}$. For $w \in L^1(\mathbb{R}^n)$, $\hat{w}$ is represented as

$$\hat{w}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x) e^{-ix\cdot\xi} \, dx.$$

For $s, m \in \mathbb{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space L^2 as follows:

$$H^{s,m} \equiv \{\psi \in \mathcal{S}' \colon \|\psi\|_{H^{s,m}} \equiv \|(1+|x|^2)^{m/2}(1-\Delta)^{s/2}\psi\|_{L^2} < \infty\}.$$

We also denote $H^{s,0}$ by H^s. For $1 \le p \le \infty$ and a positive integer k, we define the Sobolev space W_p^k corresponding to the Lebesgue space L^p by

$$W_p^k \equiv \left\{ \psi \in L^p \colon \|\psi\|_{W_p^k} \equiv \sum_{|\alpha| \le k} \|\partial^\alpha \psi\|_{L^p} < \infty \right\}.$$

Note that for a positive integer k, $H^k = W_2^k$ and the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W_2^k}$ are equivalent.

For $s > 0$, we define the homogeneous Sobolev spaces $\dot{H}^s$ by the completion of $\mathcal{S}$ with respect to the norm

$$\|w\|_{\dot{H}^s} \equiv \|(-\Delta)^{s/2} w\|_{L^2}. \tag{1.6}$$

If $s < 0$, we set

$$\dot{H}^s \equiv \{w \in \mathcal{S}' \colon (-\Delta)^{s/2} w \in L^2\}.$$

Then $\dot{H}^s$ is a Banach space with the norm (1.6) for $s > 0$. On the other hand, $\dot{H}^s$ is a semi-normed space with the semi-norm (1.6) for $s < 0$.

Let $U(t)$ be the free Schrödinger group, that is,

$$U(t) \equiv e^{it\Delta/2}.$$

The Schrödinger operator $-(1/2)\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, where the potential V is defined by (1.5). H denotes the self-adjoint realization of that operator defined on $C_0^\infty(\mathbb{R}^n)$ and we define the unitary group U_H generated by H:

$$U_H(t) \equiv e^{-itH}.$$

C denotes a constant and so forth. They may differ from line to line, when it does not cause any confusion.

Our results are as follows:

Theorem 1.1. *Let $n = 1$. Assume that $\phi \in H^{0,3} \cap \dot{H}^{-4}$ and that*

$$\delta_1 \equiv \|\phi\|_{H^{0,3}} + \|\phi\|_{\dot{H}^{-4}} \tag{1.7}$$

is sufficiently small. Let d be a constant such that $1/2 < d < 1$. Then there exists a unique solution u for the equation (NLS) *with* (1.1) *satisfying*

$$u \in C([0,\infty); L^2),$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\|u(t)-U(t)e^{-i|\cdot|^2/2t}e^{-iS_1(t,-i\nabla)}\phi\|_{L^2}\right] < \infty,$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|u(s)-U(s)e^{-i|\cdot|^2/2s}e^{-iS_1(s,-i\nabla)}\phi\|_{L^\infty}^4\, ds\right)^{1/4}\right] < \infty,$$

where

$$S_1(t,\xi) \equiv \lambda_0|\hat{\phi}(\xi)|^2 \log t. \tag{1.8}$$

Furthermore the modified wave operator $W_+\colon \phi \mapsto u(0)$ *is well-defined.*

A similar result holds for negative time.

Theorem 1.2. *Let* $n = 2$. *Assume that* $\phi \in H^{0,4} \cap \dot{H}^{-4}$, $x\phi \in \dot{H}^{-2}$ *and that*

$$\delta_2 \equiv \|\phi\|_{H^{0,4}} + \|\phi\|_{\dot{H}^{-4}} + \|x\phi\|_{\dot{H}^{-2}} \tag{1.9}$$

is sufficiently small. Let d *be a constant such that* $1/2 < d < 1$. *Then there exists a unique solution* u *for the equation* (NLS) *with* (1.2) *satisfying*

$$u \in C([0,\infty); L^2),$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\|u(t)-U(t)e^{-i|\cdot|^2/2t}e^{-iS_2(t,-i\nabla)}\phi\|_{L^2}\right] < \infty,$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|u(s)-U(s)e^{-i|\cdot|^2/2s}e^{-iS_2(s,-i\nabla)}\phi\|_{L^4}^4\, ds\right)^{1/4}\right] < \infty,$$

where

$$S_2(t,\xi) \equiv \lambda_0|\hat{\phi}(\xi)| \log t. \tag{1.10}$$

Furthermore the modified wave operator $W_+\colon \phi \mapsto u(0)$ *is well-defined.*

A similar result holds for negative time.

Remark 1.1. The phase functions S_1 and S_2 defined by (1.8) and (1.10), respectively, were first introduced by Ozawa [20].

Remark 1.2. In Theorems 1.1 and 1.2, the singular conditions on the final state ϕ such as $\phi \in \dot{H}^{-4}$ and $x\phi \in \dot{H}^{-2}$ are assumed. They yield that the Fourier transform $\hat{\phi}$ of the final state ϕ vanishes at the origin.

Theorem 1.3. *Let* $n = 1$. *Assume that* $\phi \in H^2 \cap H^{0,2}$ *and that*

$$\eta_1 \equiv \|\phi\|_{H^2} + \|\phi\|_{H^{0,2}} \tag{1.11}$$

is sufficiently small. Let d be a constant such that $1/2 < d < 1$. Then there exists a unique solution u for the equation (NLS-S) *with* (1.3) *and* (1.5) *satisfying*

$$u \in C([0,\infty); L^2),$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\|u(t) - U_H(t)e^{-i|\cdot|^2/2t}e^{-iS_1(t,-i\nabla)}\phi\|_{L^2}\right] < \infty,$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|U(s)(U_H(-s)u(s) - e^{-i|\cdot|^2/2s}e^{-iS_1(s,-i\nabla)}\phi)\|_{L^\infty}^4\, ds\right)^{1/4}\right] < \infty,$$

where S_1 is defined by (1.8).

Furthermore the modified wave operator $W_+ \colon \phi \mapsto u(0)$ is well-defined.

A similar result holds for negative time.

Theorem 1.4. *Let $n = 2$. Assume that $\phi \in H^2 \cap H^{0,3}$ and that*

$$\eta_2 \equiv \|\phi\|_{H^2} + \|\phi\|_{H^{0,3}} \tag{1.12}$$

is sufficiently small. Let d be a constant such that $1/2 < d < 1$. Then there exists a unique solution u for the equation (NLS-S) *with* (1.4) *and* (1.5) *satisfying*

$$u \in C([0,\infty); L^2),$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\|u(t) - U_H(t)e^{-i|\cdot|^2/2t}e^{-iS_2(t,-i\nabla)}\phi\|_{L^2}\right] < \infty,$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|U(s)(U_H(-s)u(s) - e^{-i|\cdot|^2/2s}e^{-iS_2(s,-i\nabla)}\phi)\|_{L^4}^4\, ds\right)^{1/4}\right] < \infty,$$

where S_2 is defined by (1.8).

Furthermore the modified wave operator $W_+ \colon \phi \mapsto u(0)$ is well-defined.

A similar result holds for negative time.

Our main idea of proof is as follows. We begin with Theorem 1.1. First, we determine the principal term u_p of the asymptotic profile. As is mentioned above, $\lambda_0|u|^2u$ is a long range effect in one space dimension. To overcome this difficulty, we introduce the modified free dynamics u_p of the Dollard type for the Schrödinger equation such that $(i\partial_t + \frac{1}{2}\Delta)u_p - \lambda_0|u_p|^2u_p$ decays faster than $\lambda_0|u_p|^2u_p$ by using the method of phase correction as in Ozawa [20]. Next we note that as is mentioned above, we can not apply the method of phase correction to non-gauge invariant nonlinearities such as u^3. To overcome this difficulty, we construct a suitable second term u_r of the Schrödinger part such that $(i\partial_t + \frac{1}{2}\Delta)u_r - (\lambda_1 u_p^3 + \lambda_2 u_p\bar{u}_p^2 + \lambda_3\bar{u}_p^3)$ decays faster than $\lambda_1 u_p^3 + \lambda_2 u_p\bar{u}_p^2 + \lambda_3\bar{u}_p^3$, and we put an asymptotic profile $A = u_p + u_r$ so that the Cook-Kuroda method is applicable as in [24, 26]. This idea is also applicable to the coupled systems of the Schrödinger equations and the second order hyperbolic equations (see [10, 24, 25, 26, 27]). Theorem 1.2 can be proved in the same way as above. For the equation (NLS-S), by using change of variables (Proposition 3.1) which will be mentioned later, we may consider the equation (NLS-S′) and prove Propositions 3.2 and 3.3 in the same way as in the proof of Theorems 1.1 and 1.2.

2. Proof of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. Let $n = 1, 2$, and let A be a given asymptotic profile of the equation (NLS), namely an approximate solution for that equation as $t \to \infty$. We introduce the following function:

$$r \equiv \mathcal{L}A - f_n(A), \tag{2.1}$$

where

$$\mathcal{L} \equiv i\partial_t + \frac{1}{2}\Delta. \tag{2.2}$$

The function r is difference between the left hand sides and the right hand ones in the equation (NLS) substituted $u = A$.

When $n = 1$, using the Strichartz estimates (see, e.g., Yajima [32]), we can prove the following proposition by the contraction argument exactly in the same way as in the proof of Theorem 1 in Ozawa [20].

Proposition 2.1. *Let $n = 1$. Assume that there exists a constant $\delta_1' > 0$ such that*

$$\|A(t)\|_{L^2} \leq \delta_1',$$
$$\|A(t)\|_{L^\infty} \leq \delta_1'(1+t)^{-1/2},$$
$$\|r(t)\|_{L^2} \leq \delta_1' \frac{(\log(2+t))^2}{(1+t)^2}$$

for $t \geq 0$ and that $\delta_1' > 0$ is sufficiently small. Let d be a constant such that $1/2 < d < 1$. Then there exists a unique solution u for the equation (NLS) *with* (1.1) *satisfying*

$$u \in C([0,\infty); L^2),$$
$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\|u(t) - A(t)\|_{L^2}\right] < \infty,$$
$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|u(s) - A(s)\|_{L^\infty}^4\, ds\right)^{1/4}\right] < \infty.$$

A similar result holds for negative time.

When $n=2$, using the Strichartz estimates (see, e.g., Yajima [32]), we obtain the following proposition by the contraction argument exactly in the same way as in the proof of Proposition 3.1 in Ginibre and Ozawa [5].

Proposition 2.2. *Let $n=2$. Assume that there exists a constant $\delta_2'>0$ such that*

$$\begin{aligned}\|A(t)\|_{L^2} &\le \delta_2',\\ \|A(t)\|_{L^\infty} &\le \delta_2'(1+t)^{-1},\\ \|r(t)\|_{L^2} &\le \delta_2'\frac{(\log(2+t))^2}{(1+t)^2}\end{aligned}$$

for $t\ge 0$ and that $\delta_2'>0$ is sufficiently small. Let d be a constant such that $1/2<d<1$. Then there exists a unique solution u for the equation (NLS) *with* (1.2) *satisfying*

$$\begin{gathered}u\in C([0,\infty);L^2),\\ \sup_{t\ge 2}\left[\frac{t^d}{(\log t)^2}\|u(t)-A(t)\|_{L^2}\right]<\infty,\\ \sup_{t\ge 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|u(s)-A(s)\|_{L^4}^4\,ds\right)^{1/4}\right]<\infty.\end{gathered}$$

A similar result holds for negative time.

We construct an asymptotic profile A for the equation (NLS) which satisfies the assumptions of Propositions 2.1 or 2.2 when $n=1$ or $n=2$, respectively. We find an asymptotic profile of the form $A=u_p+u_r$, where u_p and u_r are a principal term and a remainder one, respectively. Namely u_p is a modified free profile of the Schrödinger equation.

First, we construct a principal term u_p of the asymptotic profile. It is well-known that if $n=1$ and $\lambda_1=\lambda_2=\lambda_3=0$, the equation (NLS) does not have a solution with a free profile (see, e.g., Barab [1] and Tsutsumi [29]), that is, there do not exist ordinary wave operators. Ozawa [20] proved the existence of modified wave operators in the above case. A similar result holds in the case of $n=2$ and $\lambda_1=\lambda_2=0$. These mean that the nonlinear term $\lambda_0|u|^{2/n}u$ yields a long range effect. As in Ozawa [20] or Ginibre and Ozawa [5], we construct a modified free profile u_p of the Dollard type by the method of phase correction such that $\mathcal{L}u_p-\lambda_0|u_p|^{2/n}u_p$ decays faster than $\lambda_0|u_p|^{2/n}u_p$.

Let ϕ be a given final data, S_n the real valued function defined by (1.8) and let

$$\begin{aligned}u_p(t,x)\equiv&(U(t)e^{-i|\cdot|^2/2t}e^{-iS_n(t,-i\nabla)}\phi)(x)z(t)\\ =&\frac{1}{(it)^{n/2}}\hat{\phi}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS_n(t,x/t)}z(t),\end{aligned}\tag{2.3}$$

where z is a non-negative function in $C^\infty(\mathbb{R}_t;\mathbb{R})$ such that $z(t)=0$ for $|t|\le 1/2$, $z(t)=1$ for $|t|\ge 1$. Then for $t\ge 1$,

$$\begin{aligned}&\mathcal{L}u_p(t,x)-\lambda_0|u_p(t,x)|^{2/n}u_p(t,x)\\&=-\frac{1}{2t^2}(U(t)e^{-i|\cdot|^2/2t}|\cdot|^2e^{-iS_n(t,-i\nabla)}\phi)(x).\end{aligned}\tag{2.4}$$

Lemma 2.1. *Let $n=1,2$. There exists a constant $C>0$ such that for $t\ge 0$,*

$$\|u_p(t)\|_{L^2}=z(t)\|\phi\|_{L^2}\le\delta_n,$$
$$\|u_p(t)\|_{L^\infty}\le C\|\phi\|_{L^1}(1+t)^{-n/2}\le C\delta_n(1+t)^{-n/2},$$
$$\begin{aligned}\|\mathcal{L}u_p(t)-\lambda_0|u_p|^{2/n}u_p\|_{L^2}\le&\frac{z(t)}{2t^2}\|\Delta(e^{-iS(t,\cdot)}\hat{\phi})\|_{L^2}+\|\phi\|_{L^2}|\partial_t z(t)|\\\le&C(\|\phi\|_{H^{0,2}}+\|\phi\|_{H^{0,2}}^3)\frac{(\log(2+t))^2}{(1+t)^2}\\\le&C\delta_n\frac{(\log(2+t))^2}{(1+t)^2},\end{aligned}$$

where δ_1 and δ_2 are defined by (1.7) *and* (1.9)*, respectively.*

Proof. The first and the second inequalities follow from the definition (2.3) of u_p and the Young inequality. In the case of $n=1$ and $n=2$, the last estimate is proved in Lemma 2 in Ozawa [20] and Lemma 3.1 in Ginibre and Ozawa [5], respectively. ∎

We next consider effects caused by the nonlinear terms $f_n(u)-\lambda_0|u|^{2/n}u$ which are not gauge invariant. When $\lambda_0=0$, the existence of ordinary wave operators is proved under the suitable assumptions on the final data in [17]. This means that the contribution of the nonlinearity $f_n(u)-\lambda_0|u|^{2/n}u$ is negligible as $t\to\infty$. Since the time decay estimate $\|f_n(u_p(t))-\lambda_0|u_p(t)|^{2/n}u_p(t)\|_{L^2(\mathbb{R}^n)}=O(t^{-1})$ is not sufficient to prove Theorems 1.1 and 1.2 directly by the Cook-Kuroda method, we construct a second correction term u_r of the asymptotic profile A such that $\mathcal{L}u_r-f_n(u_p)+\lambda_0|u_p|^{2/n}u_p$ decays faster than $f_n(u_p)-\lambda_0|u_p|^{2/n}u_p$ as $t\to\infty$. In [24, 25, 26, 27], this idea is also applied to the scattering problem for the coupled systems of the Schrödinger equations and the second order hyperbolic equations.

It is sufficient to construct a second correction term u_r mentioned above for the nonlinear term which has a single non-gauge invariant nonlinearity, that is,

$$f_n(u)=\lambda_0|u|^{2/n}u+g_n(u),$$

where

$$g_n(u)\equiv\lambda u^l\bar{u}^m$$

and

$$(l,m)=(3,0),(1,2),(0,3), \quad \text{when } n=1,$$
$$(l,m)=(2,0),(0,2), \quad \text{when } n=2.$$

As is mentioned above, the estimate $\|g_n(u_p(t))\|_{L^2}=O(t^{-1})$ cannot imply the existence of a unique solution for the equation (NLS) which approaches u_p. Our aim is to find a second correction term u_r of the asymptotic profile A such that $\mathcal{L}u_r-g_n(u_p)$ decays faster than $g_n(u_p)$ as $t\to\infty$.

Note that

$$g_n(u_p)=\lambda\frac{1}{i^{na/2}t^{np/2}}\psi\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}z(t)^p,$$

where

$$\psi(x)=\hat{\phi}(x)^l\overline{\hat{\phi}(x)}^m, \tag{2.5}$$
$$p=l+m=1+\frac{2}{n}, \quad a=l-m. \tag{2.6}$$

We construct a second correction term u_r of the form

$$u_r(t,x)=\frac{1}{t^b}P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}z(t)^p, \tag{2.7}$$

where b is a positive constant and P is a complex valued function of $x\in\mathbb{R}^n$. We determine a constant b and a function P.

By a direct calculation, we have

$$\begin{aligned}
&\mathcal{L}u_r(t,x)\\
&=\mathcal{L}\left(\frac{1}{t^b}e^{ia|x|^2/2t}\right)P\left(\frac{x}{t}\right)e^{-iaS_n(t,x/t)}z(t)^p\\
&\quad+\frac{1}{t^b}e^{ia|x|^2/2t}\mathcal{L}\left[P\left(\frac{x}{t}\right)e^{-iaS_n(t,x/t)}\right]z(t)^p\\
&\quad+\nabla_x\left(\frac{1}{t^b}e^{ia|x|^2/2t}\right)\cdot\nabla_x\left[P\left(\frac{x}{t}\right)e^{-iaS_n(t,x/t))}\right]z(t)^p\\
&\quad+\frac{i}{t^b}P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}\partial_t(z(t)^p)\\
&=\frac{1}{t^b}\frac{a(1-a)}{2}\frac{|x|^2}{t^2}P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}z(t)^p\\
&\quad+\frac{1}{t^{b+1}}\left[i\left(\frac{an}{2}-b\right)P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}z(t)^p\right.\\
&\quad+a\lambda_0\left|\hat{\phi}\left(\frac{x}{t}\right)\right|^{2/n}P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}z(t)^p
\end{aligned}$$

$$
\begin{aligned}
&+ i(a-1)e^{ia|x|^2/2t}\frac{x}{t}\cdot\left(\nabla_y(P(y)e^{-iaS_n(t,y)})|_{y=x/t}\right)z(t)^p\Big] \\
&+ \frac{1}{t^{b+2}}\frac{1}{2}e^{ia|x|^2/2t}\left(\Delta_y(P(y)e^{-iaS_n(t,y)})|_{y=x/t}\right)z(t)^p \\
&+ \frac{i}{t^b}P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}\partial_t(z(t)^p).
\end{aligned}
\tag{2.8}
$$

Here we have used the fact $\partial_0 S(t,x) = t^{-1}\lambda_0|\hat{\phi}(x)|^{2/n}$. Of all the terms on the right hand side of (2.8), the first one is expected to decay most slowly in L^2 as $t\to\infty$. We put

$$
b = \frac{np}{2}, \tag{2.9}
$$

$$
P(x) = \frac{2\lambda}{i^{na/2}a(1-a)}\frac{1}{|x|^2}\psi(x) \tag{2.10}
$$

so that

$$
g_n(u_p(t,x)) = \frac{1}{t^b}\frac{a(1-a)}{2}\frac{|x|^2}{t^2}P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}z(t)^p. \tag{2.11}
$$

Here ψ is defined by (2.5). Then we see

$$
\begin{aligned}
&\mathcal{L}u_r(t,x) - g_n(u_p(t,x)) \\
&=\frac{1}{t^{np/2+1}}\Big[\frac{i(a-p)n}{2}P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}z(t)^p \\
&\quad + a\lambda_0\left|\hat{\phi}\left(\frac{x}{t}\right)\right|^{2/n}P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}z(t)^p \\
&\quad + i(a-1)e^{ia|x|^2/2t}\frac{x}{t}\cdot\left(\nabla_y(P(y)e^{-iaS_n(t,y)})|_{y=x/t}\right)z(t)^p\Big] \\
&\quad + \frac{1}{t^{np/2+2}}\frac{1}{2}e^{ia|x|^2/2t}\left(\Delta_y(P(y)e^{-iaS_n(t,y)})|_{y=x/t}\right)z(t)^p \\
&\quad + \frac{i}{t^{np/2}}P\left(\frac{x}{t}\right)e^{ia|x|^2/2t-iaS_n(t,x/t)}\partial_t(z(t)^p).
\end{aligned}
\tag{2.12}
$$

From the equality (2.12), we can expect that $\mathcal{L}u_r - g_n(u_p)$ decays faster than $g_n(u_p)$, and it is justified in Lemma 2.3 below.

Lemma 2.2. *Let S_1, S_2, P, δ_1 and δ_2 be defined by* (1.8), (1.10), (2.10), (1.7) *and* (1.9), *respectively, and let $n = 1, 2$. Assume $\delta_n \le 1$. Then there exists a constant $C > 0$ such that for $t \ge 2$,*

$$
\begin{aligned}
&\|Pe^{-iaS_n(t,\cdot)}\|_{H^2} \\
&\le\begin{cases} C(\log t)^2(\|\phi\|_{\dot{H}^{-4}} + \|\phi\|_{H^{0,3}}), & \text{when } n=1, \\ C(\log t)^2(\|\phi\|_{\dot{H}^{-4}} + \|x\phi\|_{\dot{H}^{-2}} + \|\phi\|_{H^{0,4}}), & \text{when } n=2, \end{cases} \\
&\le C\delta_n(\log t)^2, \\
&\quad\|x\cdot\nabla(Pe^{-iaS_n(t,\cdot)})\|_{L^2} \le C(\log t)(\|\phi\|_{\dot{H}^{-2}} + \|\phi\|_{H^{0,[n/2]+2}}) \\
&\qquad\qquad\qquad\qquad\qquad\quad \le C\delta_n(\log t).
\end{aligned}
$$

Here $[q]$ *denotes the maximal integer which is less than or equal to* $q \in \mathbb{R}$.

Proof. We note that

$$\begin{aligned} Pe^{-iaS_n} + \Delta(Pe^{-iaS_n}) =& e^{-iaS_n}(P + \Delta P - 2ia\nabla P \cdot \nabla S_n \\ & - iaP\Delta S_n - a^2 P|\nabla S_n|^2), \\ x \cdot \nabla(Pe^{-iaS_n}) =& (x \cdot \nabla P - iaPx \cdot \nabla S_n)e^{-iaS_n}. \end{aligned}$$

These imply

$$\begin{aligned} \|Pe^{-iaS_n(t)}\|_{H^2} \le& C(\|P\|_{L^2} + \|\Delta P\|_{L^2} + \|\nabla P \cdot \nabla S_n(t)\|_{L^2} \\ & + \|P\Delta S_n(t)|_{L^2} + \|P|\nabla S_n(t)|^2\|_{L^2}), \end{aligned} \tag{2.13}$$

$$\|x \cdot \nabla(Pe^{-iaS_n(t)})\|_{L^2} \le C(\|x \cdot \nabla P\|_{L^2} + \|x \cdot \nabla S_n(t)P\|). \tag{2.14}$$

By the definition of S_n, we have

$$\begin{aligned} \nabla S_n =& (p-1)\lambda_0(\log t)|\hat{\phi}|^{p-3}\mathrm{Re}(\overline{\hat{\phi}}\nabla\hat{\phi}) \\ \Delta S_n =& (p-1)\lambda_0(\log t)[|\hat{\phi}|^{p-3}(\mathrm{Re}(\overline{\hat{\phi}}\Delta\hat{\phi}) + |\nabla\hat{\phi}|^2) \\ & + (p-3)|\hat{\phi}|^{p-5}|\mathrm{Re}(\overline{\hat{\phi}}\nabla\hat{\phi})|^2], \end{aligned}$$

where $p = 1 + 2/n$. Substituting these into the inequalities (2.13) and (2.14) and using Hölder's inequality and the Sobolev embedding theorem $H^{[n/2]+1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, we have this lemma. Here we have noted $\delta_n^k \le \delta_n$ for $k \ge 1$. □

Lemma 2.3. *Let* $n = 1, 2$. *There exists a constant* $C > 0$ *such that for* $t \ge 0$ *such that*

$$\begin{aligned} \|u_r(t)\|_{L^2} \le& C\delta_n(1+t)^{-1}, \\ \|u_r(t)\|_{L^\infty} \le& C\delta_n(1+t)^{-np/2}, \\ \|\mathcal{L}u_r(t) - g_n(u_p(t))\|_{L^2} \le& C\delta_n \frac{(\log(2+t))^2}{(1+t)^2}, \end{aligned}$$

where δ_1 *and* δ_2 *are defined by* (1.7) *and* (1.9), *respectively.*

Proof. The first and the second estimates immediately follow from the definition of u_r. From the equality (2.12) and Lemma 2.2, we have the third inequality. □

Thus we have determined the asymptotic profile $A = u_p + u_r$. Let r be defined by (2.1). Then

$$\begin{aligned} r =& \mathcal{L}u_p - \lambda_0|u_p|^{p-1}u_p + \mathcal{L}u_r - g_n(u_p) \\ & - \lambda_0(|u_p + u_r|^{p-1}(u_p + u_r) - |u_p|^{p-1}u_p) \\ & - (g_n(u_p + u_r) - g_n(u_p)), \end{aligned} \tag{2.15}$$

where $p = 1 + 2/n$.

Lemma 2.4. *Let $n = 1, 2$. There exists a constant $C > 0$ such that for $t \geq 0$,*

$$\|A(t)\|_{L^2} \leq C\delta_n,$$
$$\|A(t)\|_{L^\infty} \leq C\delta_n(1+t)^{-n/2},$$
$$\|r(t)\|_{L^2} \leq C\delta_n \frac{(\log(2+t))^2}{(1+t)^2},$$

where δ_1 and δ_2 are defined by (1.7) *and* (1.9)*, respectively.*

Proof. The first and the second inequalities directly follow from Lemmas 2.1 and 2.3. We prove the last estimate. By the equality (2.15), Lemmas 2.1 and 2.3, we see that

$$\begin{aligned}\|r(t)\|_{L^2} \leq &\|\mathcal{L}u_p(t) - \lambda_0|u_p(t)|^{p-1}u_p(t)\|_{L^2} \\ &+ \|\mathcal{L}u_r(t) - g_n(u_p(t))\|_{L^2} \\ &+ C\||u_r(t)|(|u_p(t)|^{p-1} + |u_r(t)|^{p-1})\|_{L^2} \\ \leq &C\delta_n \frac{(\log(2+t))^2}{(1+t)^2}\end{aligned}$$

for $t \geq 0$. This completes the proof of this lemma. □

Proof of Theorems 1.1 and 1.2. We assume that $\delta_n > 0$ is sufficiently small. In view of Lemma 2.4, we see that the asymptotic profile A satisfies the assumptions of Propositions 2.1 and 2.2 for $\delta_n' = C\delta_n$, where $C > 0$ is a constant independent of δ_n and t. Recalling the first estimate in Lemma 2.3, we see that Theorems 1.1 and 1.2 follow immediately from Propositions 2.1 and 2.2, respectively. The proof of Theorems 1.1 and 1.2 is completed. □

3. Proof of Theorems 1.3 and 1.4

In this section, we prove Theoremx 1.3 and 1.4.

We consider the following nonlinear Schrödinger equation.

$$i\partial_t v = -\frac{1}{2}\Delta v + G_n(v), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{NLS-S$'$}$$

where $n = 1, 2$ and $G_n(v) = G_n(v, t, x)$ are given by

$$\begin{aligned}G_1(v) =&\lambda_0|v|^2 v + \lambda_1 v^3 e^{-2i(tE\cdot x - t^3|E|^2/3)} + \lambda_2 v\bar{v}^2 e^{2i(tE\cdot x - t^3|E|^2/3)} \\ &+ \lambda_3 \bar{v}^3 e^{4i(tE\cdot x - t^3|E|^2/3)}, \quad \text{when } n = 1,\end{aligned} \tag{3.1}$$

$$\begin{aligned}G_2(v) =&\lambda_0|v|v + \lambda_1 v^2 e^{-i(tE\cdot x - t^3|E|^2/3)} + \lambda_2 \bar{v}^2 e^{3i(tE\cdot x - t^3|E|^2/3)} \\ &+ \lambda_3 v\bar{v} e^{i(tE\cdot x - t^3|E|^2/3)}, \quad \text{when } n = 2.\end{aligned} \tag{3.2}$$

By a direct calculation, we obtain the following relation between a solution to the equation (NLS-S) and that to the equation (NLS-S$'$).

The following proposition is not essentially new but almost well-known. (See Cycon, Froese, Kirsch and Simon [4] and Kitada and Yajima [15]).

Proposition 3.1. *If v solves the equation* (NLS-S$'$), *then*

$$u(t,x) = v\left(t, x + \frac{t^2}{2}E\right) e^{-i(tE\cdot x + t^3|E|^2/6)}$$

solves the equation (NLS-S) *with* (1.5).

Conversely, if u solves the equation (NLS-S) *with* (1.5), *then*

$$v(t,x) = u\left(t, x - \frac{t^2}{2}E\right) e^{i(tE\cdot x - t^3|E|^2/3)}$$

solves the equation (NLS-S$'$).

Remark 3.1. Recently, the above change of variables has been applied to the nonlinear Schrödinger equation with the Stark effects and the gauge invariant nonlinearity by Carles and Nakamura [2].

According to Proposition 3.1, Theorems 1.3 and 1.4 are immediate consequences of Propositions 3.2 and 3.3 below, respectively.

Proposition 3.2. *Let $n = 1$. Assume that ϕ satisfies all the assumptions of Theorem 1.3. Let d be a constant such that $1/2 < d < 1$. Then there exists a unique solution v for the equation* (NLS-S$'$) *with* (3.1) *and* (1.5) *satisfying*

$$v \in C([0,\infty); L^2),$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\|v(t) - U(t)e^{-i|\cdot|^2/2t}e^{-iS_1(t,-i\nabla)}\phi\|_{L^2}\right] < \infty,$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|v(s) - U(s)e^{-i|\cdot|^2/2s}e^{-iS_1(s,-i\nabla)}\phi\|_{L^\infty}^4\, ds\right)^{1/4}\right] < \infty,$$

where S_1 is defined by (1.8).

A similar result holds for negative time.

Proposition 3.3. *Let $n = 2$. Assume that ϕ satisfies all the assumptions of Theorem 1.4. Let d be a constant such that $1/2 < d < 1$. Then there exists a unique solution v for the equation* (NLS-S$'$) *with* (3.2) *and* (1.5) *satisfying*

$$v \in C([0,\infty); L^2),$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\|v(t) - U(t)e^{-i|\cdot|^2/2t}e^{-iS_2(t,-i\nabla)}\phi\|_{L^2}\right] < \infty,$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|v(s) - U(s)e^{-i|\cdot|^2/2s}e^{-iS_2(s,-i\nabla)}\phi\|_{L^4}^4\, ds\right)^{1/4}\right] < \infty,$$

where S_2 is defined by (1.8).

A similar result holds for negative time.

In what follows, we shall prove Propositions 3.2 and 3.3.

Let $n = 1, 2$, and let B be a given asymptotic profile of the equation (NLS-S$'$), namely an approximate solution for that equation as $t \to \infty$. We introduce the following function:

$$R \equiv \mathcal{L}B - G_n(B), \tag{3.3}$$

where the linear operator $\mathcal{L}$ is defined by (2.2). The function R is difference between the left hand sides and the right hand ones in the equation (NLS-S$'$) substituted $v = B$.

When $n = 1$, we can prove the following proposition exactly in the same way as Proposition 2.1 (see the proof of Theorem 1 in Ozawa [20]).

Proposition 3.4. *Let $n = 1$. Assume that there exists a constant $\eta_1' > 0$ such that*

$$\|B(t)\|_{L^2} \le \eta_1',$$
$$\|B(t)\|_{L^\infty} \le \eta_1'(1+t)^{-1/2},$$
$$\|R(t)\|_{L^2} \le \eta_1' \frac{(\log(2+t))^2}{(1+t)^2}$$

for $t \ge 0$ and that $\eta_1' > 0$ is sufficiently small. Let d be a constant such that $1/2 < d < 1$. Then there exists a unique solution v for the equation (NLS-S$'$) *with* (3.1) *and* (1.5) *satisfying*

$$v \in C([0,\infty); L^2),$$
$$\sup_{t\ge 2}\left[\frac{t^d}{(\log t)^2}\|v(t) - B(t)\|_{L^2}\right] < \infty,$$
$$\sup_{t\ge 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|v(s) - B(s)\|_{L^\infty}^4\, ds\right)^{1/4}\right] < \infty.$$

A similar result holds for negative time.

When $n = 2$, we obtain the following proposition exactly in the same way as Proposition 2.2 (see the proof of Proposition 3.1 in Ginibre and Ozawa [5]).

Proposition 3.5. *Let $n = 2$. Assume that there exists a constant $\eta_2' > 0$ such that*

$$\|B(t)\|_{L^2} \le \eta_2',$$
$$\|B(t)\|_{L^\infty} \le \eta_2'(1+t)^{-1},$$
$$\|R(t)\|_{L^2} \le \eta_2' \frac{(\log(2+t))^2}{(1+t)^2}$$

for $t \ge 0$ and that $\eta_2' > 0$ is sufficiently small. Let d be a constant such that $1/2 < d < 1$. Then there exists a unique solution v for the *equation* (NLS-S′) *with* (3.2) *and* (1.5) *satisfying*

$$v \in C([0,\infty); L^2),$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\|v(t)-B(t)\|_{L^2}\right] < \infty,$$

$$\sup_{t\geq 2}\left[\frac{t^d}{(\log t)^2}\left(\int_t^\infty \|v(s)-B(s)\|_{L^4}^4\,ds\right)^{1/4}\right] < \infty.$$

A similar result holds for negative time.

We construct an asymptotic profile B for the equation (NLS-S′) which satisfies the assumptions of Propositions 3.4 or 3.5 when $n=1$ or $n=2$, respectively. We find an asymptotic profile of the form $B = v_p + v_r$, where v_p and v_r are a principal term and a remainder one, respectively. Namely v_p is a modified free profile of the Schrödinger equation.

First, we construct a principal term v_p. Since the first terms on the right hand sides of (3.1) and (3.2) are gauge invariant and they are expected to be dominant, we take the following function for a principal term v_p

$$\begin{aligned} v_p(t,x) \equiv& (U(t)e^{-i|\cdot|^2/2t}e^{-iS_n(t,-i\nabla)}\phi)(x)z(t)\\ =& \frac{1}{(it)^{n/2}}\hat{\phi}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS_n(t,x/t)}z(t), \end{aligned}$$

which is the exactly same one as that in the previous section (see (2.3)) and therefore the estimates in Lemma 2.1 with δ_n replaced by η_n hold for v_p.

We next construct a second correction term v_r such that $\mathcal{L}v_r - G_n(v_p) + \lambda_0|v_p|^{2/n}v_p$ decays faster than $G_n(v_p) - \lambda_0|v_p|^{2/n}v_p$ as $t\to\infty$.

It is sufficient to construct a second correction term v_r for the nonlinear term which has a single non-gauge invariant nonlinearity, that is,

$$G_n(v) = \lambda_0|v|^{2/n}v + g_n(v),$$

where

$$g_n(v) = g_n(v,t,x) \equiv \lambda v^l \bar{v}^m e^{-i(a-1)(tE\cdot x - t^3|E|^2/3)},$$

$$(l,m) = (3,0),(0,3), \quad \text{when } n=1,$$
$$(l,m) = (2,0),(1,1),(0,2), \quad \text{when } n=2$$

and $a = l-m$.

Note that

$$g_n(v_p) = \lambda\frac{1}{i^{na/2}t^{np/2}}\psi\left(\frac{x}{t}\right)e^{i\theta}z(t)^p,$$

where

$$\psi(x) = \hat{\phi}(x)^l \overline{\hat{\phi}(x)}^m, \tag{3.4}$$

$$p = l + m = 1 + \frac{2}{n} \tag{3.5}$$

and

$$\theta = a\left(\frac{|x|^2}{2t} - S_n\left(t, \frac{x}{t}\right)\right) - (a-1)\left(tE \cdot x - \frac{|E|^2}{3}t^3\right).$$

We construct a second correction term v_r of the form

$$v_r(t,x) = \frac{1}{t^b} P\left(\frac{x}{t}\right) e^{i\theta} z(t)^p, \tag{3.6}$$

where b is a positive constant and P is a complex valued function of $x \in \mathbb{R}^n$. We determine a constant b and a function P.

By a direct calculation, we have

$$\begin{aligned}
&\mathcal{L}v_r(t,x) \\
&= -\frac{(a+1)(a-1)|E|^2}{2t^{b-2}} P(y) e^{i\theta} z(t)^p \\
&\quad + \frac{1}{t^{b-1}} \Bigg[(a+1)(a-1) E \cdot y P(y) \\
&\quad - a(a-1)\lambda_0 \frac{\log t}{t} E \cdot (\nabla |\hat{\phi}|^{2/n})(y) P(y) \\
&\quad - \frac{1}{t}\left\{ \frac{1}{2} a(a-1)|y|^2 P(y) + i(a-1) E \cdot (\nabla P)(y) \right\} \\
&\quad + a(a-1)\lambda_0 \frac{\log t}{t^2} y \cdot (\nabla |\hat{\phi}|^{2/n})(y) P(y) \\
&\quad + \frac{1}{t^2}\Bigg\{ \left(\frac{i(2an-b)}{2} + a\lambda_0 |\hat{\phi}(y)|^{2/n} \right) P(y) \\
&\quad + (a-1) y \cdot \nabla P(y) \Bigg\} - \frac{1}{2} a^2 \lambda_0^2 \frac{(\log t)^2}{t^3} \left| (\nabla |\hat{\phi}|^{2/n})(y) \right|^2 P(y) \\
&\quad - \frac{a\lambda_0}{2} \frac{\log t}{t^3} \left\{ i(\Delta |\hat{\phi}|^{2/n})(y) P(y) + 2(\nabla |\hat{\phi}|^{2/n})(y) \cdot (\nabla P)(y) \right\} \\
&\quad + \frac{1}{2t^3} (\Delta P)(y) \Bigg] e^{i\theta} z(t)^p + \frac{i}{t^b} P(y) e^{i\theta} \partial_t (z(t)^p),
\end{aligned} \tag{3.7}$$

where $y = x/t$. Of all the terms on the right hand side of (3.7), the first one is expected to decay most slowly in L^2 as $t \to \infty$. We put

$$b = \frac{np}{2} + 2, \tag{3.8}$$

$$P(x) = -\frac{2\lambda}{i^{na/2}(a+1)(a-1)|E|^2} \psi(x) \tag{3.9}$$

so that

$$g_n(v_p(t,x)) = -\frac{(a+1)(a-1)|E|^2}{2t^{b-2}} P\left(\frac{x}{t}\right) e^{i\theta} z(t)^p. \tag{3.10}$$

Here ψ is defined by (3.4). Then we see that $\mathcal{L}v_r - g_n(v_p)$ decays faster than $g_n(v_p)$ from Lemma 3.1 below.

Remark 3.2. Suppose that $n = 1$ and that $(l, m) = (1, 2)$, that is, $g_n(v) = \lambda v \bar{v}^2 e^{2i(tE\cdot x - t^3|E|^2/3)}$. Then we have $a = -1$. Therefore the first term and the second one on the right hand side of (3.7) vanish and it is the third one that decays most slowly in L^2 as $t \to \infty$:

$$\mathcal{L}v_r(t,x) = -\frac{\log t}{t^b} 2\lambda_0 E(|\hat{\phi}|^2)'(y)P(y) + (\text{faster decay terms}).$$

Since $\log t$ appears in the numerator of the main term of $\mathcal{L}v_r$, $g_n(v_p)$ and the main term of $\mathcal{L}v_r$ cannot be equal to each other for any b and P. For v_r of another form

$$v_r(t,x) = \frac{1}{t^b(\log t)^c} P\left(\frac{x}{t}\right) e^{i\theta} z(t)^p,$$

we have

$$\begin{aligned}\mathcal{L}v_r(t,x) = &-\frac{1}{t^b(\log t)^{c-1}} 2\lambda_0 E(|\hat{\phi}|^2)'(y)P(y)e^{i\theta}z(t)^p \\ &+ \frac{1}{t^b(\log t)^c}\Big(-y^2P(y) + 2iEP'(y)\Big)e^{i\theta}z(t)^p \\ &+ (\text{faster decay terms})\end{aligned}$$

Therefore $\mathcal{L}v_r - g_n(v_p)$ decays faster than $g_n(v_p)$ for v_r with $b = 3/2, c = 1$ and $P = -\dfrac{\lambda i^{1/2}}{2\lambda_0 E}\dfrac{\hat{\phi}\bar{\hat{\phi}}^2}{(|\hat{\phi}|^2)'}$ under a suitable condition on ϕ. For this choice of v_r, however, a rather complicated or strong condition seems to be needed on ϕ because of the form of P and we can expect only $\|\mathcal{L}v_r - g_n(v_p)\|_{L^2} = O\big(\frac{1}{t\log t}\big)$ as $t \to \infty$, where we note that $\frac{1}{t\log t}$ is not integrable on $[c, \infty)$ for $c > 1$. For this reason, $F_1(u)$ does not contain a $u\bar{u}^2$-term in our result (see (1.3)).

Lemma 3.1. *Let $n = 1, 2$. There exists a constant $C > 0$ such that for $t \geq 0$ such that*

$$\|v_r(t)\|_{L^2} \leq C\eta_n(1+t)^{-3},$$
$$\|v_r(t)\|_{L^\infty} \leq C\eta_n(1+t)^{-np/2-2},$$
$$\|\mathcal{L}v_r(t) - g_n(v_p(t))\|_{L^2} \leq C\eta_n(1+t)^{-2},$$

where η_1 and η_2 are defined by (1.11) *and* (1.12)*, respectively.*

Proof. The first and the second estimates immediately follow from the definition of v_r. From the equalities (3.7) and (3.10), we have the third inequality. $\square$

Thus we have determined an asymptotic profile $B = v_p + v_r$. Let R be defined by (3.3). Then

$$\begin{aligned} R =& \mathcal{L}v_p - \lambda_0 |v_p|^{p-1} v_p + \mathcal{L}v_r - g_n(v_p) \\ & - \lambda_0(|v_p + v_r|^{p-1}(v_p + v_r) - |v_p|^{p-1} v_p) \\ & - (g_n(v_p + v_r) - g_n(v_p)), \end{aligned} \tag{3.11}$$

where $p = 1 + 2/n$. We can prove the following lemma in the same way as the proof of Lemma 2.4.

Lemma 3.2. *Let $n = 1, 2$. There exists a constant $C > 0$ such that for $t \geq 0$,*

$$\|B(t)\|_{L^2} \leq C\eta_n,$$

$$\|B(t)\|_{L^\infty} \leq C\eta_n (1 + t)^{-n/2},$$

$$\|R(t)\|_{L^2} \leq C\eta_n \frac{(\log(2 + t))^2}{(1 + t)^2},$$

where η_1 and η_2 are defined by (1.11) *and* (1.12)*, respectively.*

Proof of Theorems 1.3 and 1.4. We assume that $\eta_n > 0$ is sufficiently small. In view of Lemma 3.2, we see that the asymptotic profile B satisfies the assumptions of Propositions 3.4 and 3.5 for $\eta_n' = C\eta_n$, where $C > 0$ is a constant independent of η_n and t. Recalling the first estimate in Lemma 3.1, we see that Theorems 1.3 and 1.4 follow immediately from Propositions 3.4 and 3.5, respectively. The proof of Theorems 1.3 and 1.4 is completed. □

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