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Integral switching engine for special Clebsch-Gordan coefficients for the representations of $\mathfrak{gl}_3$ with respect to Gelfand-Zelevinsky basis

by

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Introduction

Before discussing the contents of this paper let us explain its motivation. We have been working on explicit formulas on generalized spherical functions on real semisimple Lie groups of low rank in these several years. Our interest is to have effectively computable results in real harmonic analysis for much deeper study of automorphic forms of many variables. To investigate geometric automorphic forms, one sometimes needs results on irreducible representations of a real semisimple group with non-trivial minimal K-types. In this direction we already have some results for real semisimple groups of split rank 2 ([6], [7], [5]).

To handle non-trivial K-types, it is necessary to describe the representation of the maximal compact subgroup K of a real semisimple Lie group G. When the complexified Lie algebra $\mathfrak{k}_{\mathbf{C}} = \mathfrak{k} \otimes_{\mathbf{R}} \mathbf{C}$ with $\mathfrak{k} = \mathrm{Lie}(K)$ is a direct sum of copies of $\mathfrak{sl}_2(\mathbf{C})$ and $\mathbf{C}$, this is quite easy. And it was one of the main reasons to have explicit results mentioned above successively.

But if $\mathfrak{k}_{\mathbf{C}}$ has larger simple factors, the problem becomes quite difficult. When K has only simple factors of type A and BD, and abelian factor, some authors could go through the hard computation using the Gelfand-Tsetlin basis (*cf.* Taniguchi [12], Tsuzuki [13]). In these computations the essential

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point is to describe explicitly the decomposition $V \otimes \mathrm{Ad}_{\mathfrak{p}}$ into irreducible components for a given finite dimensional representation V of K and the adjoint representation $\mathrm{Ad}_{\mathfrak{p}}$ of K on $\mathfrak{p}$, where $\mathfrak{p}$ is the complement of $\mathfrak{k}$ in the Cartan decomposition: $\mathrm{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}$. Even if the representation $\mathrm{Ad}_{\mathfrak{p}}$ is relatively small, this becomes a formidable problem for general K. However when $\mathfrak{k}_{\mathbf{C}}$ has only simple factors $\mathbf{C}, \mathfrak{sl}_2(\mathbf{C}), \mathfrak{sl}_3(\mathbf{C})$, we seem to have more tractable situation. To show this is the purpose of this paper.

Around mid 80's, Gelfand and Zelevinsky [4] defined the *canonical basis* in the representation spaces of $\mathfrak{gl}_3$ in their sense, and found an explicit relation between the Gelfand-Tsetlin basis and their basis. We call it *Gelfand-Zelevinsky basis* in this paper. According to the introduction of the paper [1] by Fomin and Zelevinsky, this basis is dual to the limit $q \to 1$ of the canonical basis in quantum groups, investigated by Kashiwara [8], [9] and Lusztig [10], [11].

We utilize this result to formulate Theorem 1 in §2, which gives the explicit formulas for the projectors from the tensor product $V \otimes V_{(1,0,0)}$ of an irreducible representation V of $\mathfrak{gl}_3$ and the standard representation $V_{(1,0,0)}$, to its irreducible components. Though we suppressed it, to formulate Theorem 1 we used the relation between Gelfand-Tsetlin basis and Gelfand-Zelevinsky basis. Once one can find the 'right formulas', the proof is given by direct computation.

In §3, we also give the explicit projectors from $V \otimes V_{(2,0,0)}$ to its (generically 6) irreducible components (Theorem 2 in §3), by applying Theorem 1 twice.

Our result in this paper is just simple computation, but it might have application for investigation of spherical functions with non-trivial K-types, and also it might contain some suggestion for general investigation of 'canonical' Clebsch-Gordan coefficients.

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Notations. For a Gelfand-Tsetlin pattern (which simply we may call G-pattern)

$$M = \begin{pmatrix} \mathbf{m}_3 \\ \mathbf{m}_2 \\ \mathbf{m}_1 \end{pmatrix} = \begin{pmatrix} m_{13}\ m_{23}\ m_{33} \\ m_{12}\ m_{22} \\ m_{11} \end{pmatrix}$$

of degree 3, we define

$$M \begin{pmatrix} i_{13}\ i_{23}\ i_{33} \\ i_{12}\ i_{22} \\ i_{11} \end{pmatrix} = \begin{pmatrix} m_{13}+i_{13} \quad m_{23}+i_{23} \quad m_{33}+i_{33} \\ m_{12}+i_{12} \quad m_{22}+i_{22} \\ m_{11}+i_{11} \end{pmatrix}.$$

If the vector $(i_{13}\ i_{23}\ i_{33})$ is zero, we omit the top row in the left hand side of the above defining equality. So the left hand side is written as

$$M\left(\begin{array}{c} i_{12}\ i_{22} \\ i_{11} \end{array}\right).$$

A convenient symbol is $M[k]$, which is defined by

$$M\left(\begin{array}{c} k\ \ -k \\ 0 \end{array}\right).$$

This means that it causes a 'twist' of weight k at the second row $\mathbf{m}_2$ in M.

Recall first that the *Gelfand-Zelevinsky basis* $\{f(M)\}$ (i.e. *the canonical basis* in this paper) *is parameterized by the same label set* $\{M\}$ *as the Gelfand-Tsetlin basis*. If any of the above shifts M' of M violates the conditions of Gelfand-Tsetlin pattern, i.e. if either

$$m'_{13} \geq m'_{12} \geq m'_{23} \geq m'_{22} \geq m'_{33}$$

or

$$m'_{12} \geq m'_{11} \geq m'_{22}$$

is not satisfied, then the corresponding vectors $f(M')$ in the canonical basis should be zero.

Functions in M. We set

$$\delta(M) = m_{12} + m_{22} - m_{11} - m_{23}.$$

Let $\chi_+(M)$ and $\chi_-(M)$ be the characteristic functions of the sets $\{M|\delta(M) > 0\}$ and $\{M|\delta(M) < 0\}$, respectively. More generally we introduce functions $\chi_\pm^{(i)}(M)$ by

$$\chi_+^{(i)}(M) = \left\{\begin{array}{ll} 1, & \delta(M) > i \\ 0, & \delta(M) \leq i \end{array}\right., \quad \chi_-^{(i)}(M) = \left\{\begin{array}{ll} 1, & \delta(M) < -i \\ 0, & \delta(M) \geq -i \end{array}\right..$$

Then we have $\chi_+(M) = \chi_+^{(0)}(M)$ and $\chi_-(M) = \chi_-^{(0)}(M)$.

We introduce 'piecewise-linear' functions $C_1(M)$, $\bar{C}_1(M)$ and $C_2(M)$ by

$$C_1(M) = \left\{\begin{array}{ll} m_{11} - m_{22}, & \text{if } \delta(M) \geq 0 \\ m_{12} - m_{23}, & \text{if } \delta(M) \leq 0 \end{array}\right., \quad \bar{C}_1(M) = \left\{\begin{array}{ll} m_{23} - m_{22}, & \text{if } \delta(M) \geq 0 \\ m_{12} - m_{11}, & \text{if } \delta(M) \leq 0 \end{array}\right.,$$

and

$$C_2(M) = C_1(M)\bar{C}_1(M).$$

Another expression of $C_1(M)$ and $\bar{C}_1(M)$ is

$$C_1(M) = \mathrm{Min}\{m_{11}-m_{22}, m_{12}-m_{23}\}, \quad \bar{C}_1(M) = \mathrm{Min}\{m_{23}-m_{22}, m_{12}-m_{11}\}.$$

1 The result of Gelfand-Zelevinsky

Firstly we recall the definition of the canonical basis in the sense of Gelfand and Zelevinsky. In the beginning let us consider the case of the Lie algebra $\mathfrak{gl}_n$.

Definition. A *weight* is an integral vector $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbf{Z}^n$ of length n. A weight γ is *dominant* if $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$.

Let E_{ij} $(1 \leq i, j \leq n)$ be the matrix unit of size n with 1 at the (i, j)-entry and 0 at other entries. As is well-known, any irreducible representation V of finite dimension of $\mathfrak{gl}_n$ splits into weight subspaces:

$$V = \oplus_\gamma V(\gamma).$$

Here

$$V(\gamma) = \{v \in V | E_{ii}v = \gamma_i v \text{ for all } i\} \neq \{0\}.$$

And there is the (unique) dominant weight λ s.t. $\lambda \geq \gamma$ in the lexicographical order. Therefore the representation V is labelled by such dominant weight λ, i.e. $V = V_\lambda$.

Now for another dominant weight $\nu = (\nu_1, \cdots, \nu_n)$, we set

$$V_\lambda(\gamma, \nu) = \{v \in V_\lambda(\gamma) | E_{i,i+1}^{\nu_i - \nu_{i+1}+1} v = 0, \text{ for } 1 \leq i \leq n-1\}.$$

Definition. A basis B in V_λ is called *proper* if each of subspaces $V_\lambda(\gamma, \nu)$ (for all possible γ, ν) is spanned by its subset, i.e.

$$V_\lambda(\gamma, \nu) = < B \cap V_\lambda(\gamma, \nu) > .$$

Theorem. (Gelfand-Zelevinsky)

1. Each irreducible finite dimensional representation of $\mathfrak{gl}_n$ has a proper basis.

2. In each irreducible finite dimensional representation of $\mathfrak{gl}_3$, there is only one proper basis up to scalar multiple. And this basis is called *canonical*.

Up to this point, the notion of the canonical basis has the ambiguity of scalar multiple. Gelfand and Zelevinsky normalized this scalar factor somehow to get the following formulas.

If $i \neq j$, the matrix E_{ij} is a generator of the root space of some root in $\mathfrak{gl}_3$ with respect to the Cartan subalgebra consisting of diagonal matrices. If $|i-j| = 1$, E_{ij} is a root vector of a simple root. There are 4 such simple root vectors E_{12}, E_{21}, E_{23} and E_{32}.

Proposition. (Gelfand-Zelevinsky) The action of simple root vectors on the canonical basis $\{f(M)\}$ of $V_{\mathbf{m_3}}$ is given as follows.

$$
\begin{aligned}
E_{12}f(M) &= (m_{12}-m_{11})f\Big(M\left(\begin{smallmatrix} 0 & & 0 \\ & 1 & \end{smallmatrix}\right)\Big) \\
&\quad +(m_{23}-m_{22})\chi_+(M)f\Big(M\left(\begin{smallmatrix} 0 & & 0 \\ & 1 & \end{smallmatrix}\right)[-1]\Big), \\
E_{21}f(M) &= (m_{11}-m_{22})f\Big(M\left(\begin{smallmatrix} 0 & & 0 \\ & -1 & \end{smallmatrix}\right)\Big) \\
&\quad +(m_{12}-m_{23})\chi_-(M)f\Big(M\left(\begin{smallmatrix} 0 & & 0 \\ & -1 & \end{smallmatrix}\right)[-1]\Big), \\
E_{23}f(M) &= (m_{13}-m_{12})f\Big(M\left(\begin{smallmatrix} 1 & & 0 \\ & 0 & \end{smallmatrix}\right)\Big) \\
&\quad +\big\{m_{13}-m_{12}-\delta(M)\big\}\chi_-(M)f\Big(M\left(\begin{smallmatrix} 1 & & 0 \\ & 0 & \end{smallmatrix}\right)[-1]\Big), \\
E_{32}f(M) &= (m_{22}-m_{33})f\Big(M\left(\begin{smallmatrix} 0 & & -1 \\ & 0 & \end{smallmatrix}\right)\Big) \\
&\quad +\big\{m_{22}-m_{33}+\delta(M)\big\}\chi_+(M)f\Big(M\left(\begin{smallmatrix} 0 & & -1 \\ & 0 & \end{smallmatrix}\right)[-1]\Big).
\end{aligned}
$$

Remark 1. In the formulas of E_{12} and E_{21}, we have

$$m_{23}-m_{22}=m_{12}-m_{11}-\delta(M), \quad m_{12}-m_{23}=m_{11}-m_{22}+\delta(M).$$

2 Tensor products with the standard representation

Generically the tensor product $V_{\mathbf{m_3}}\otimes V_{(1,0,0)}$ has three irreducible components: $V_{\mathbf{m_3}+(1,0,0)}, V_{\mathbf{m_3}+(0,1,0)}$ and $V_{\mathbf{m_3}+(0,0,1)}$. If either $\mathbf{m_3}+(0,1,0)$ or $\mathbf{m_3}+(0,0,1)$ is not dominant, the corresponding irreducible component does not occur. Thus for the dimension of the intertwining spaces, we have

$$
\begin{aligned}
\dim_{\mathbf{C}}\mathrm{Hom}(V_{\mathbf{m_3}}\otimes V_{(1,0,0)}, V_{\mathbf{m_3}+(1,0,0)}) &= 1, \\
\dim_{\mathbf{C}}\mathrm{Hom}(V_{\mathbf{m_3}}\otimes V_{(1,0,0)}, V_{\mathbf{m_3}+(0,1,0)}) &\le 1,
\end{aligned}
$$

and

$$\dim_{\mathbf{C}}\mathrm{Hom}(V_{\mathbf{m_3}}\otimes V_{(1,0,0)}, V_{\mathbf{m_3}+(0,0,1)}) \le 1.$$

Let $P_{(1,0,0)}$ be a non-zero generator of the first space, which is unique up to scalar multiple. And let $P_{(0,1,0)}$ or $P_{(0,0,1)}$ also be the generator of the second or the third space respectively, if either space is non-zero. Our purpose in this section is to give explicit expression of these projectors $P_{(1,0,0)}$, $P_{(0,1,0)}$ and $P_{(0,0,1)}$ in terms of canonical basis.

2.1 The projectors for $V_{\mathbf{m_3}} \otimes V_{(1,0,0)}$

Let $V_{\mathbf{m_3}}$ be the representation of $\mathfrak{gl}_3$ with canonical basis $\{f(M)\}$, and let $V_{(1,0,0)}$ be the standard representation. To denote the canonical basis of the standard representation $V_{(1,0,0)}$, we suppress the letter 'f' before G-patterns.

Theorem 1 Let $\{f'(M)\}$ be the canonical basis of the target representation.

Formula 1: The projector $P_{(1,0,0)} : V_{\mathbf{m_3}} \otimes V_{(1,0,0)} \to V_{\mathbf{m_3}+(1,0,0)}$.

1. $P_{(1,0,0)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\ 0\ 0 \\ 1\ 0 \\ 1 \end{smallmatrix}\right)\right) = f'\left(M\left(\begin{smallmatrix} 1\ 0\ 0 \\ 1\ 0 \\ 1 \end{smallmatrix}\right)\right).$

2. $P_{(1,0,0)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\ 0\ 0 \\ 1\ 0 \\ 0 \end{smallmatrix}\right)\right) = f'\left(M\left(\begin{smallmatrix} 1\ 0\ 0 \\ 1\ 0 \\ 0 \end{smallmatrix}\right)\right) + \chi_-(M) f'\left(M\left(\begin{smallmatrix} 1\ 0\ 0 \\ 0\ 1 \\ 0 \end{smallmatrix}\right)\right).$

3. $P_{(1,0,0)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\ 0\ 0 \\ 0\ 0 \\ 0 \end{smallmatrix}\right)\right) = f'\left(M\left(\begin{smallmatrix} 1\ 0\ 0 \\ 0\ 0 \\ 0 \end{smallmatrix}\right)\right).$

Formula 2: The projector $P_{(0,1,0)} : V_{\mathbf{m_3}} \otimes V_{(1,0,0)} \to V_{\mathbf{m_3}+(0,1,0)}$.

1. $P_{(0,1,0)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\ 0\ 0 \\ 1\ 0 \\ 1 \end{smallmatrix}\right)\right)$
$= -(m_{13} - m_{12}) f'\left(M\left(\begin{smallmatrix} 0\ 1\ 0 \\ 1\ 0 \\ 1 \end{smallmatrix}\right)\right) + \chi_+(M) D(M) f'\left(M\left(\begin{smallmatrix} 0\ 1\ 0 \\ 0\ 1 \\ 1 \end{smallmatrix}\right)\right).$

2. $P_{(0,1,0)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\ 0\ 0 \\ 1\ 0 \\ 0 \end{smallmatrix}\right)\right)$
$= -(m_{13} - m_{12}) f'\left(M\left(\begin{smallmatrix} 0\ 1\ 0 \\ 1\ 0 \\ 0 \end{smallmatrix}\right)\right) + C_1(M) f'\left(M\left(\begin{smallmatrix} 0\ 1\ 0 \\ 0\ 1 \\ 0 \end{smallmatrix}\right)\right).$

3. $P_{(0,1,0)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\ 0\ 0 \\ 0\ 0 \\ 0 \end{smallmatrix}\right)\right)$
$= (m_{12} - m_{23}) f'\left(M\left(\begin{smallmatrix} 0\ 1\ 0 \\ 0\ 0 \\ 0 \end{smallmatrix}\right)\right) + \chi_+(M) C_1(M) f'\left(M\left(\begin{smallmatrix} 0\ 1\ 0 \\ -1\ 1 \\ 0 \end{smallmatrix}\right)\right).$

Here $D(M) = -m_{13} + m_{12} - \delta(M)$.

Formula 3: The projector $P_{(0,0,1)} : V_{\mathbf{m_3}} \otimes V_{(1,0,0)} \to V_{\mathbf{m_3}+(0,0,1)}$.

1. $P_{(0,0,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\ 0\ 0 \\ 1\ 0 \\ 1 \end{smallmatrix}\right)\right)$
$= -(m_{13} - m_{12})(m_{22} - m_{33}) f'\left(M\left(\begin{smallmatrix} 0\ 0\ 1 \\ 1\ 0 \\ 1 \end{smallmatrix}\right)\right) + E(M) f'\left(M\left(\begin{smallmatrix} 0\ 0\ 1 \\ 0\ 1 \\ 1 \end{smallmatrix}\right)\right).$

2. $P_{(0,0,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\ 0\ 0 \\ 1\ 0 \\ 0 \end{smallmatrix}\right)\right)$
$= -(m_{13} - m_{12})(m_{22} - m_{33}) f'\left(M\left(\begin{smallmatrix} 0\ 0\ 1 \\ 1\ 0 \\ 0 \end{smallmatrix}\right)\right) + F(M) f'\left(M\left(\begin{smallmatrix} 0\ 0\ 1 \\ 0\ 1 \\ 0 \end{smallmatrix}\right)\right)$
$- \chi_-(M) C_2(M) f'\left(M\left(\begin{smallmatrix} 0\ 0\ 1 \\ -1\ 2 \\ 0 \end{smallmatrix}\right)\right).$

3. $P_{(0,0,1)}\left(f(M) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\right)$

$$= (m_{12}-m_{33}+1)(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\right) - C_2(M)f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ -1\,1 \\ 0 \end{smallmatrix}\right)\right).$$

Here

$$\begin{aligned} E(M) &= \bar{C}_1(M)\{m_{13} - m_{33} + 1 - C_1(M)\}, \\ F(M) &= -C_2(M) \\ &\quad -\chi_-(M)\{(m_{13} - m_{12})(m_{22} - m_{33}) - (m_{13} - m_{33} + 1)\delta(M)\}. \end{aligned}$$

Remark 3. As we can see in the next subsection, in order to prove Theorem 1, it suffices to show that any of three projector given above is a $\mathfrak{gl}_3$-homomorphism. But the actual method to find these formula is to use the relation between the canonical basis with the Gelfand-Tsetlin basis found by Gelfand-Zelevinsky [4]. To write this computation seems to take more space than the proof below.

2.2 Proof of Theorem 1

The proof is direct computation to check that either of three projectors is a $\mathfrak{gl}_3$-modules. The action of the Cartan subgroup is diagonal. Therefore the essential computation is those of simple root vectors $E_{i,i+1}$, $E_{i+1,i}$. The most complicated case is the formula 3, and other two cases are similar and much simpler. So we discuss only the formula 3 here. We have to confirm that $P_{(0,0,1)} \cdot E_{ij} = E_{ij} \cdot P_{(0,0,1)}$ for 4 simple root vectors E_{ij} $(i - j = \pm 1)$.

Let us check the action of E_{12}, say.

Claim 1. Apply E_{12} to the inside of $P_{(0,0,1)}$ in the left hand side of the formula 3-1. Then by using Proposition in §1 and the formulas 3-2 and 3-3, we have $\sum_{i=0}^{2} l_i f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 1\,0 \\ 2 \end{smallmatrix}\right)[-i]\right)$ with

$$\begin{aligned} l_0 &= -(m_{12} - m_{11})(m_{13} - m_{12})(m_{22} - m_{33}), \\ l_1 &= (m_{12} - m_{11})E\Big(M\left(\begin{smallmatrix} 0\,0 \\ 1 \end{smallmatrix}\right)\Big) \\ &\quad -(m_{23} - m_{22})\chi_+(M)(m_{13} - m_{12} + 1)(m_{22} + 1 - m_{33}), \\ l_2 &= (m_{23} - m_{22})\chi_+(M)E\Big(M\left(\begin{smallmatrix} 0\,0 \\ 1 \end{smallmatrix}\right)[-1]\Big). \end{aligned}$$

Claim 2. Apply E_{12} to the right hand side of the formula 3-1. Then we have $\sum_{i=0}^{2} r_i f'\Big(M\left(\begin{smallmatrix}0\,0\,1\\1\,0\\2\end{smallmatrix}\right)[-i]\Big)$ with

$$\begin{aligned}
r_0 &= -(m_{12}-m_{11})(m_{13}-m_{12})(m_{22}-m_{33}),\\
r_1 &= -(m_{13}-m_{12})(m_{22}-m_{33})(m_{23}-m_{22})\chi_+\Big(M\left(\begin{smallmatrix}0\,0\,1\\1\,0\\1\end{smallmatrix}\right)\Big)\\
&\quad +E(M)(m_{12}-m_{11}-1),\\
r_2 &= E(M)(m_{23}-m_{22}-1)\chi_+\Big(M\left(\begin{smallmatrix}0\,0\,1\\0\,1\\1\end{smallmatrix}\right)\Big).
\end{aligned}$$

We have to confirm the equalities $l_i = r_i$ $(i=0,1,2)$. The only nontrivial case is when $i=1$. The main ingredient to show this equality is the difference relation:

$$E\Big(M\left(\begin{smallmatrix}0\,0\\1\end{smallmatrix}\right)\Big) = \begin{cases} E(M) - \bar{C}_1(M), & \text{if } \delta(M) > 0,\\ E(M) - \big\{(m_{13}-m_{33}+1)-(m_{12}-m_{23})\big\}, & \text{if } \delta(M) \le 0.\end{cases}$$

The remaining formulas for the operator E_{12} are the following.

Claim 3. We have

$$P_{(0,0,1)}\cdot E_{12}(\text{inside of } P_{(0,0,1)} \text{ in LHS of (3-2)}) = \sum_{i=0}^{2} l'_i f'\Big(M\left(\begin{smallmatrix}0\,0\,1\\1\,0\\1\end{smallmatrix}\right)[-i]\Big),$$

and

$$P_{(0,0,1)}\cdot E_{12}(\text{inside of } P_{(0,0,1)} \text{ in LHS of (3-3)}) = \sum_{i=0}^{2} l''_i f'\Big(M\left(\begin{smallmatrix}0\,0\,1\\0\,0\\1\end{smallmatrix}\right)[-i]\Big),$$

with

$$\begin{aligned}
l'_0 &= -(m_{12}-m_{11}+1)(m_{13}-m_{12})(m_{22}-m_{33}),\\
l'_1 &= -\chi_+(M)(m_{23}-m_{22})(m_{13}-m_{12}+1)(m_{22}+1-m_{33})\\
&\quad +(m_{12}-m_{11})F\Big(M\left(\begin{smallmatrix}0\,0\\1\end{smallmatrix}\right)\Big)+E(M),\\
l'_2 &= -\chi_-\Big(M\left(\begin{smallmatrix}0\,0\\1\end{smallmatrix}\right)\Big)(m_{12}-m_{11})C_2\Big(M\left(\begin{smallmatrix}0\,0\\1\end{smallmatrix}\right)\Big)\\
&\quad +\chi_+(M)(m_{23}-m_{22})F\Big(M\left(\begin{smallmatrix}0\,0\\1\end{smallmatrix}\right)[-1]\Big),
\end{aligned}$$

and

$$\begin{aligned}
l''_0 &= (m_{12}-m_{11})(m_{12}-m_{33}+1)(m_{22}-m_{33}),\\
l''_1 &= \chi_+(M)(m_{23}-m_{22})(m_{12}-m_{33})(m_{22}-m_{33}+1)\\
&\quad -(m_{12}-m_{11})C_2\Big(M\left(\begin{smallmatrix}0\,0\\1\end{smallmatrix}\right)\Big),\\
l''_2 &= -\chi_+(M)(m_{23}-m_{22})C_2\Big(M\left(\begin{smallmatrix}0\,0\\1\end{smallmatrix}\right)[-1]\Big).
\end{aligned}$$

Claim 4. We have

$$E_{12}(\text{RHS of (3-2)}) = \sum_{i=0}^{2} r'_i f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 1\,0 \\ 1 \end{smallmatrix}\right)[-i]\right),$$

and

$$E_{12}(\text{RHS of (3-3)}) = \sum_{i=0}^{2} r''_i f'\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 0\,0 \\ 1 \end{smallmatrix}\right)[-i]\right),$$

with

$$\begin{aligned}
r'_0 &= -(m_{12}-m_{11}+1)(m_{13}-m_{12})(m_{22}-m_{33}),\\
r'_1 &= -(m_{13}-m_{12})(m_{22}-m_{33})(m_{23}-m_{22})\chi_+\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right)\\
&\quad +F(M)(m_{12}-m_{11}),\\
r'_2 &= -\chi_-(M)C_2(M)(m_{12}-m_{11}-1)\\
&\quad +\chi_+\left(M\left(\begin{smallmatrix} 0\,0\,1 \\ 0\,1 \\ 0 \end{smallmatrix}\right)\right)(m_{23}-m_{22}-1)F(M),
\end{aligned}$$

and

$$\begin{aligned}
r''_0 &= (m_{12}-m_{33}+1)(m_{12}-m_{11})(m_{22}-m_{33}),\\
r''_1 &= \chi_+(M)(m_{12}-m_{33}+1)(m_{22}-m_{33})(m_{23}-m_{22})\\
&\quad -(m_{12}-m_{11}-1)C_2(M),\\
r''_2 &= -C_2(M)(m_{23}-m_{22}-1)\chi_+(M).
\end{aligned}$$

To show $l''_i = r''_i$ for $i = 1, 2$, it is convenient to compute both side case by case, either when $\delta(M) > 0$ or when $\delta(M) \leq 0$. To show $l'_i = r'_i$ $(i = 2, 3)$, it would be better to divide the computation into 3 cases: $\delta(M) > 0, = 0$, and < 0.

We can discuss similarly for E_{21}, E_{23} and E_{32}.

□

2.3 The projectors for $V_{\mathbf{m}_3} \otimes V_{(0,0,-1)}$

Let $V_{\mathbf{m}_3}$ be the representation of $\mathfrak{gl}_3$ with canonical basis $\{f(M)\}$, and let $V_{(0,0,-1)}$ be the dual standard representation.

Theorem 1' Let $\{f'(M)\}$ be the canonical basis of the target representation.

Formula 1: The projector $P_{(0,0,-1)} : V_{\mathbf{m}_3} \otimes V_{(0,0,-1)} \to V_{\mathbf{m}_3+(0,0,-1)}$.

1. $P_{(0,0,-1)}\left(f(M) \otimes \left(\begin{smallmatrix} 0\,0\,-1 \\ 0\,-1 \\ -1 \end{smallmatrix}\right)\right) = f'\left(M\left(\begin{smallmatrix} 0\,0\,-1 \\ 0\,-1 \\ -1 \end{smallmatrix}\right)\right).$

2. $P_{(0,0,-1)}\left(f(M)\otimes\left(\begin{smallmatrix}0\;0\;-1\\0\;-1\\0\end{smallmatrix}\right)\right)=f'\left(M\left(\begin{smallmatrix}0\;0\;-1\\0\;-1\\0\end{smallmatrix}\right)\right)+\chi_+(M)f'\left(M\left(\begin{smallmatrix}0\;0\;-1\\-1\;0\\0\end{smallmatrix}\right)\right).$

3. $P_{(0,0,-1)}\left(f(M)\otimes\left(\begin{smallmatrix}0\;0\;-1\\0\;0\\0\end{smallmatrix}\right)\right)=f'\left(M\left(\begin{smallmatrix}0\;0\;-1\\0\;0\\0\end{smallmatrix}\right)\right).$

Formula 2: The projector $P_{(0,-1,0)}:V_{\mathbf{m}_3}\otimes V_{(0,0,-1)}\to V_{\mathbf{m}_3+(0,-1,0)}$.

1. $P_{(0,-1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}0\;0\;-1\\0\;-1\\-1\end{smallmatrix}\right)\right)$
 $=-(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}0\;-1\;0\\0\;-1\\-1\end{smallmatrix}\right)\right)+\chi_-(M)\bar{D}(M)f'\left(M\left(\begin{smallmatrix}0\;-1\;0\\-1\;0\\-1\end{smallmatrix}\right)\right).$

2. $P_{(0,-1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}0\;0\;-1\\0\;-1\\0\end{smallmatrix}\right)\right)$
 $=-(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}0\;-1\;0\\0\;-1\\0\end{smallmatrix}\right)\right)+\bar{C}_1(M)f'\left(M\left(\begin{smallmatrix}0\;-1\;0\\-1\;0\\0\end{smallmatrix}\right)\right).$

3. $P_{(0,-1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}0\;0\;-1\\0\;0\\0\end{smallmatrix}\right)\right)$
 $=(m_{23}-m_{22})f'\left(M\left(\begin{smallmatrix}0\;-1\;0\\0\;0\\0\end{smallmatrix}\right)\right)+\chi_-(M)\bar{C}_1(M)f'\left(M\left(\begin{smallmatrix}0\;-1\;0\\-1\;1\\0\end{smallmatrix}\right)\right).$

Here $\bar{D}(M)=-m_{22}+m_{33}+\delta(M)$.

Formula 3: The projector $P_{(-1,0,0)}:V_{\mathbf{m}_3}\otimes V_{(0,0,-1)}\to V_{\mathbf{m}_3+(-1,0,0)}$.

1. $P_{(-1,0,0)}\left(f(M)\otimes\left(\begin{smallmatrix}0\;0\;-1\\0\;-1\\-1\end{smallmatrix}\right)\right)$
 $=-(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}-1\;0\;0\\0\;-1\\-1\end{smallmatrix}\right)\right)+\bar{E}(M)f'\left(M\left(\begin{smallmatrix}-1\;0\;0\\-1\;0\\-1\end{smallmatrix}\right)\right).$

2. $P_{(-1,0,0)}\left(f(M)\otimes\left(\begin{smallmatrix}0\;0\;-1\\0\;-1\\0\end{smallmatrix}\right)\right)$
 $=-(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}-1\;0\;0\\0\;-1\\0\end{smallmatrix}\right)\right)+\bar{F}(M)f'\left(M\left(\begin{smallmatrix}-1\;0\;0\\-1\;0\\0\end{smallmatrix}\right)\right)$
 $-\chi_+(M)C_2(M)f'\left(M\left(\begin{smallmatrix}-1\;0\;0\\-2\;1\\0\end{smallmatrix}\right)\right).$

3. $P_{(-1,0,0)}\left(f(M)\otimes\left(\begin{smallmatrix}0\;0\;-1\\0\;0\\0\end{smallmatrix}\right)\right)$
 $=(m_{13}-m_{22}+1)(m_{13}-m_{12})f'\left(M\left(\begin{smallmatrix}-1\;0\;0\\0\;0\\0\end{smallmatrix}\right)\right)-C_2(M)f'\left(M\left(\begin{smallmatrix}-1\;0\;0\\-1\;1\\0\end{smallmatrix}\right)\right).$

Here

$$\begin{aligned}\bar{E}(M)&=C_1(M)\big\{m_{13}-m_{33}+1-\bar{C}_1(M)\big\},\\ \bar{F}(M)&=-C_2(M)\\ &\quad-\chi_+(M)\big\{(m_{13}-m_{12})(m_{22}-m_{33})+(m_{13}-m_{33}+1)\delta(M)\big\}.\end{aligned}$$

Remark 4. We may write the 3-rd coefficient $\chi_-(M)\bar{C}_1(M)$ of the formula 2-3 as $\chi_-(M)(m_{12}-m_{11})$. Similarly the 3-rd coefficient $-\chi_+(M)C_2(M)$ of 3-2 as $-\chi_+(M)(m_{11}-m_{22})(m_{23}-m_{22})$.

2.4 Proof of Theorem 1'

We can deduce it from Theorem 1 by symmetry. For given G-pattern M, we define its *dual pattern* $\hat{M}$ by

$$\hat{M} = \begin{pmatrix} -m_{33} \ -m_{23} \ -m_{13} \\ -m_{22} \ -m_{12} \\ -m_{11} \end{pmatrix}.$$

Obviously we have

$$\delta(\hat{M}) = -\delta(M), \quad \chi_+(\hat{M}) = \chi_-(M), \quad \chi_-(\hat{M}) = \chi_+(M),$$
$$C_1(\hat{M}) = \bar{C}_1(M), \quad \bar{C}_1(\hat{M}) = C_1(M), \quad C_2(\hat{M}) = C_2(M),$$

and $\widehat{M[-k]} = \hat{M}[-k]$.

We can check that Proposition 1 is self-dual with respect to this involutive mapping $M \mapsto \hat{M}$. Apply the same mapping to the whole process to deduce Theorem 1 from Proposition 1. Then we have Theorem 1'. □

Remark 5. We have an isomorphism of representations $V_{(1,1,0)} \cong V_{(0,0,-1)} \otimes V_{(1,1,1)}$ with one-dimensional representation corresponding to the trace map $tr : \mathfrak{gl}_3 \to \mathbf{C}$. Fix a non-zero element $\left(\begin{smallmatrix} 1\,1\,1 \\ 1\,1 \\ 1 \end{smallmatrix}\right)$ of $V_{(1,1,1)}$, then we have the natural identification between canonical basis of $V_{(0,0,-1)}$ and $V_{(1,1,0)}$ by

$$f(M) \otimes \left(\begin{smallmatrix} 1\,1\,1 \\ 1\,1 \\ 1 \end{smallmatrix}\right) \mapsto f\left(M \left(\begin{smallmatrix} 1\,1\,1 \\ 1\,1 \\ 1 \end{smallmatrix}\right)\right)$$

Because all the coefficients of the formulas in Theorem 1' are written in terms of differences $m_{ij} - m_{kl}$ of M, we have a similar formulas of the projectors on $V_{\mathbf{m_3}} \otimes V_{(1,1,0)}$ with completely the same coefficients as those in Theorem 1'.

2.5 The symmetric tensor product $V_{(2,0,0)}$ of the standard representation $V_{(1,0,0)}$

If we apply Theorem 1 for the special case $V_{\mathbf{m_3}} = V_{(1,0,0)}$, we have only two irreducible constituents $V_{(2,0,0)}$ and $V_{(1,1,0)}$ which occur with multiplicities one:

$$V_{(1,0,0)} \otimes V_{(1,0,0)} \cong V_{(2,0,0)} \oplus V_{(1,1,0)}.$$

The first factor $V_{(2,0,0)}$ is the symmetric tensor product of $V_{(1,0,0)}$, and the second the anti-symmetric tensor product. Here we write the correspondence between canonical basis of $V_{(1,0,0)}$ and $V_{(2,0,0)}$ explicitly.

Lemma 1 Via identification $V_{(2,0,0)}$ with $\mathrm{Sym}^2(V_{(1,0,0)})$ which is unique up to a scalar multiple, we have identifications :

$$
\begin{aligned}
\left(\begin{smallmatrix} 2\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right), \\
\left(\begin{smallmatrix} 2\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right) &= \frac{1}{2}\left\{\left(\begin{smallmatrix} 1\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\right\}, \\
\left(\begin{smallmatrix} 2\,0\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right) &= \frac{1}{2}\left\{\left(\begin{smallmatrix} 1\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\right\}, \\
\left(\begin{smallmatrix} 2\,0\,0 \\ 2\,0 \\ 0 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right), \\
\left(\begin{smallmatrix} 2\,0\,0 \\ 2\,0 \\ 1 \end{smallmatrix}\right) &= \frac{1}{2}\left\{\left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right\}, \\
\left(\begin{smallmatrix} 2\,0\,0 \\ 2\,0 \\ 2 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right),
\end{aligned}
$$

Remark 6. Similarly to the case of the standard representation, to denote the canonical basis of $V_{(2,0,0)}$ we do not write the letter 'f' before its G-pattern.

3 Tensor product with $V_{(2,0,0)}$

In this section, we want to have the irreducible decomposition of the tensor product $V_{\mathbf{m_3}} \otimes V_{(2,0,0)}$ and an explicit formula of the projectors to its irreducible components. Generically this tensor product has six irreducible components $V_{\mathbf{m_3}+\mathbf{e}_i+\mathbf{e}_j}$ $(1 \leq i, j \leq 3)$. Here $\mathbf{e}_i$ is the unit vector with unity at the i-th entry and zero at the remaining entries. Each component occurs with multiplicity one, if the weight vector $\mathbf{m_3} + \mathbf{e}_i + \mathbf{e}_j$ is dominant.

To have the projector from the total space $V_{\mathbf{m_3}} \otimes V_{(2,0,0)}$ to each irreducible component, which is unique up to scalar multiple, we firstly embed $V_{(2,0,0)}$ into the tensor product $V_{(1,0,0)} \otimes V_{(1,0,0)}$ of the two copies of the standard representation, discussed in §2.5. Roughly speaking, using the projectors of irreducible decomposition of a general simple $\mathfrak{gl}_3$-modules V with the standard representation twice, we have the projectors to the irreducible component of the tensor product $V \otimes V_{(2,0,0)}$.

More precisely we consider as follows. Put

$$W_1 = V_{\mathbf{m_3}} \otimes V_{(1,0,0)}, \text{ and } W_2 = W_1 \otimes V_{(1,0,0)} = V_{\mathbf{m_3}} \otimes V_{(1,0,0)} \otimes V_{(1,0,0)}$$

In the total space W_2, the irreducible component $V_{\mathbf{m_3}+\mathbf{e}_i+\mathbf{e}_j}$ occurs with multiplicity one if $i=j$. If $i \neq j$, it occurs with multiplicity two, or zero.

If $i \geq j$, we define a projector $Q_1 : W_2 \to V_{\mathbf{m_3}+\mathbf{e}_i+\mathbf{e}_j}$ by extending

$$Q_1(v \otimes u_1 \otimes u_2) = P_{\mathbf{e}_j}\left(P_{\mathbf{e}_i}(v \otimes u_1) \otimes u_2\right)$$

linearly. Similarly $Q_2 : W_2 \to V_{\mathbf{m_3}+\mathbf{e}_i+\mathbf{e}_j}$ by extending

$$Q_2(v \otimes u_1 \otimes u_2) = P_{\mathbf{e}_i}\left(P_{\mathbf{e}_j}(v \otimes u_1) \otimes u_2\right)$$

linearly.

When $i=j$, these two intertwining operators coincide on W_2 itself. If $i \neq j$, they gives linearly independent generators of the intertwining space

$$\dim_{\mathbf{C}} \mathrm{Hom}(W_2, V_{\mathbf{m_3}+\mathbf{e}_i+\mathbf{e}_j})$$

of dimension 2. The restrictions of Q_1 and Q_2 to the subspace $V_{\mathbf{m_3}} \otimes V_{(2,0,0)}$ in the total space W_2 are scalar multiple of each other, because of the multiplicity one.

3.1 The projectors for $V_{\mathbf{m}_3} \otimes V_{(2,0,0)}$.

In this subsection, we give an explicit formula for the projectors from the tensor product $V_{\mathbf{m}_3} \otimes V_{(2,0,0)}$ to its six irreducible components in terms of canonical basis. The proof of this is given in the next subsection. Let $\{f(M)\}$ be the canonical basis of the representation $V_{\mathbf{m}_3}$. Similarly for the standard representation, we suppress the letter 'f' before G-patterns to denote the canonical basis of $V_{(2,0,0)}$.

Theorem 2. Let $\{f'(M)\}$ be the canonical basis of the target representation.
Formula 1: The projector $P_{(2,0,0)} : V_{\mathbf{m}_3} \otimes V_{(2,0,0)} \to V_{\mathbf{m}_3+(2,0,0)}$.

1. $P_{(2,0,0)}\left(f(M) \otimes \begin{pmatrix} 2\,0\,0 \\ 0\,0 \\ 0 \end{pmatrix}\right) = f'\left(M\begin{pmatrix} 2\,0\,0 \\ 0\,0 \\ 0 \end{pmatrix}\right)$.

2. $P_{(2,0,0)}\left(f(M) \otimes \begin{pmatrix} 2\,0\,0 \\ 1\,0 \\ 0 \end{pmatrix}\right) = f'\left(M\begin{pmatrix} 2\,0\,0 \\ 1\,0 \\ 0 \end{pmatrix}\right) + \chi_-(M) f'\left(M\begin{pmatrix} 2\,0\,0 \\ 0\,1 \\ 0 \end{pmatrix}\right)$.

3. $P_{(2,0,0)}\left(f(M) \otimes \begin{pmatrix} 2\,0\,0 \\ 1\,0 \\ 1 \end{pmatrix}\right) = f'\left(M\begin{pmatrix} 2\,0\,0 \\ 1\,0 \\ 1 \end{pmatrix}\right)$.

4. $P_{(2,0,0)}\left(f(M) \otimes \begin{pmatrix} 2\,0\,0 \\ 2\,0 \\ 0 \end{pmatrix}\right) = f'\left(M\begin{pmatrix} 2\,0\,0 \\ 2\,0 \\ 0 \end{pmatrix}\right)$
$+\left\{\chi_-(M) + \chi_-^{(1)}(M)\right\} f'\left(M\begin{pmatrix} 2\,0\,0 \\ 1\,1 \\ 0 \end{pmatrix}\right) + \chi_-^{(1)}(M) f'\left(M\begin{pmatrix} 2\,0\,0 \\ 0\,2 \\ 0 \end{pmatrix}\right)$.

5. $P_{(2,0,0)}\left(f(M) \otimes \begin{pmatrix} 2\,0\,0 \\ 2\,0 \\ 1 \end{pmatrix}\right) = f'\left(M\begin{pmatrix} 2\,0\,0 \\ 2\,0 \\ 1 \end{pmatrix}\right) + \chi_-(M) f'\left(M\begin{pmatrix} 2\,0\,0 \\ 1\,1 \\ 1 \end{pmatrix}\right)$.

6. $P_{(2,0,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\2\end{smallmatrix}\right)\right)=f'\left(M\left(\begin{smallmatrix}2\,0\,0\\2\,0\\2\end{smallmatrix}\right)\right).$

Formula 2: The projector $P_{(1,1,0)}:V_{\mathbf{m}_3}\otimes V_{(2,0,0)}\to V_{\mathbf{m}_3+(1,1,0)}$.

1. $P_{(1,1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)=(m_{12}-m_{23})f'\left(M\left(\begin{smallmatrix}1\,1\,0\\0\,0\\0\end{smallmatrix}\right)\right)$
   $+\chi_+(M)C_1(M)f'\left(M\left(\begin{smallmatrix}1\,1\,0\\-1\,1\\0\end{smallmatrix}\right)\right).$

2. $P_{(1,1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)=\frac{1}{2}(2m_{12}-m_{13}-m_{23})f'\left(M\left(\begin{smallmatrix}1\,1\,0\\1\,0\\0\end{smallmatrix}\right)\right)$
   $+C_1(M)f'\left(M\left(\begin{smallmatrix}1\,1\,0\\0\,1\\0\end{smallmatrix}\right)\right).$

3. $P_{(1,1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right)=\frac{1}{2}(2m_{12}-m_{13}-m_{23})f'\left(M\left(\begin{smallmatrix}1\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right)$
   $+\frac{1}{2}\chi_+(M)\big\{C_1(M)+D(M)\big\}f'\left(M\left(\begin{smallmatrix}1\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right).$

4. $P_{(1,1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\0\end{smallmatrix}\right)\right)=-(m_{13}-m_{12})f'\left(M\left(\begin{smallmatrix}1\,1\,0\\2\,0\\0\end{smallmatrix}\right)\right)$
   $+\big\{C_1(M)-\chi_-(M)(m_{13}-m_{12})\big\}f'\left(M\left(\begin{smallmatrix}1\,1\,0\\1\,1\\0\end{smallmatrix}\right)\right)$
   $+\chi_-(M)C_1(M)f'\left(M\left(\begin{smallmatrix}1\,1\,0\\0\,2\\0\end{smallmatrix}\right)\right).$

5. $P_{(1,1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\1\end{smallmatrix}\right)\right)=-(m_{13}-m_{12})f'\left(M\left(\begin{smallmatrix}1\,1\,0\\2\,0\\1\end{smallmatrix}\right)\right)$
   $+\frac{1}{2}\big\{C_1(M)-(m_{13}-m_{12})-\chi_+(M)\delta(M)\big\}f'\left(M\left(\begin{smallmatrix}1\,1\,0\\1\,1\\1\end{smallmatrix}\right)\right).$

6. $P_{(1,1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\2\end{smallmatrix}\right)\right)=-(m_{13}-m_{12})f'\left(M\left(\begin{smallmatrix}1\,1\,0\\2\,0\\2\end{smallmatrix}\right)\right)$
   $+\chi_+(M)D(M)f'\left(M\left(\begin{smallmatrix}1\,1\,0\\1\,1\\2\end{smallmatrix}\right)\right).$

Formula 3: The projector $P_{(1,0,1)}:V_{\mathbf{m}_3}\otimes V_{(2,0,0)}\to V_{\mathbf{m}_3+(1,0,1)}$.

1. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)=(m_{12}-m_{33}+1)(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,0\\0\end{smallmatrix}\right)\right)$
   $-C_2(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\-1\,1\\0\end{smallmatrix}\right)\right).$

2. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_if'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,0\\0\end{smallmatrix}\right)[-i]\right)$
   with

$$
\begin{aligned}
c_0 &= \tfrac{1}{2}(m_{22}-m_{33})(2m_{12}-m_{13}-m_{33}+1),\\
c_1 &= \tfrac{1}{2}\big\{F(M)-C_2(M)+\chi_-(M)(m_{22}-m_{33})(m_{12}-m_{33}+1)\big\},\\
c_2 &= -\chi_-(M)C_2(M).
\end{aligned}
$$

3. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right)$
$= \frac{1}{2}(m_{22}-m_{33})(2m_{12}-m_{13}-m_{33}+1)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,0\\1\end{smallmatrix}\right)\right)$
$+\frac{1}{2}\big\{E(M)-C_2(M)\big\}f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,1\\1\end{smallmatrix}\right)\right).$

4. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{3}c_i f'\left(M\left(\begin{smallmatrix}1\,0\,1\\2\,0\\0\end{smallmatrix}\right)[-i]\right)$
with

$$\begin{aligned}
c_0 &= -(m_{13}-m_{12})(m_{22}-m_{33}),\\
c_1 &= \big\{F(M)-\chi_-^{(1)}(M)(m_{13}-m_{12})(m_{22}-m_{33})\big\},\\
c_2 &= \big\{-\chi_-(M)C_2(M)+\chi_-^{(1)}(M)F(M)\big\},\\
c_3 &= -\chi_-^{(1)}(M)C_2(M).
\end{aligned}$$

5. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}1\,0\,1\\2\,0\\1\end{smallmatrix}\right)[-i]\right)$
with

$$\begin{aligned}
c_0 &= -(m_{13}-m_{12})(m_{22}-m_{33}),\\
c_1 &= \tfrac{1}{2}\big\{E(M)+F(M)-\chi_-(M)(m_{13}-m_{12})(m_{22}-m_{33})\big\},\\
c_2 &= \tfrac{1}{2}\chi_-(M)\big\{E(M)-C_2(M)\big\}.
\end{aligned}$$

6. $P_{(1,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\2\end{smallmatrix}\right)\right)=-(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\2\,0\\2\end{smallmatrix}\right)\right)$
$+E(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,1\\2\end{smallmatrix}\right)\right).$

Formula 4: The projector $P_{(0,2,0)}: V_{\mathbf{m}_3}\otimes V_{(2,0,0)}\to V_{\mathbf{m}_3+(0,2,0)}$.

1. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\,2\,0\\0\,0\\0\end{smallmatrix}\right)[-i]\right)$
with

$$\begin{aligned}
c_0 &= (m_{12}-m_{23})(m_{12}-m_{23}-1),\\
c_1 &= C_1(M)\big\{\chi_+^{(1)}(M)(m_{12}-m_{23})+\chi_+(M)(m_{12}-m_{23}-2)\big\},\\
c_2 &= \chi_+^{(1)}(M)C_1(M)\big\{C_1(M)-1\big\}.
\end{aligned}$$

2. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\,2\,0\\1\,0\\0\end{smallmatrix}\right)[-i]\right)$
with

$$\begin{aligned}
c_0 &= -(m_{12}-m_{23})(m_{13}-m_{12}),\\
c_1 &= C_1(M)\big\{(m_{12}-m_{23}-1)-\chi_+(M)(m_{13}-m_{12})\big\},\\
c_2 &= \chi_+(M)C_1(M)\big\{C_1(M)-1\big\}.
\end{aligned}$$

3. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2&0&0\\1&0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0&2&0\\1&0\\1\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}c_0 &= -(m_{12}-m_{23})(m_{13}-m_{12}),\\ c_1 &= \chi_+(M)(m_{12}-m_{23}-1)D(M)-\chi_+^{(1)}(M)(m_{13}-m_{12})\big\{C_1(M)+1\big\},\\ c_2 &= \chi_+^{(1)}(M)C_1(M)D(M).\end{aligned}$$

4. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2&0&0\\2&0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}f'\left(M\left(\begin{smallmatrix}0&2&0\\2&0\\0\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1),\\ c_1 &= -2(m_{13}-m_{12})C_1(M),\\ c_2 &= C_1(M)\big\{C_1(M)-1\big\}.\end{aligned}$$

5. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2&0&0\\2&0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0&2&0\\2&0\\1\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1),\\ c_1 &= -(m_{13}-m_{12})\Big[\chi_I(M)\big\{D(M)+1\big\}+C_1(M)\Big],\\ c_2 &= \chi_+(M)C_1(M)D(M).\end{aligned}$$

6. $P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2&0&0\\2&0\\2\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0&2&0\\2&0\\2\end{smallmatrix}\right)[-i]\right)$

   with

$$\begin{aligned}c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1),\\ c_1 &= -(m_{13}-m_{12})\Big[\chi_+(M)D(M)+\chi_+^{(1)}(M)\big\{D(M)+2\big\}\Big],\\ c_2 &= \chi_+^{(1)}(M)D(M)\big\{D(M)+1\big\}.\end{aligned}$$

Formula 5: The projector $P_{(0,1,1)}:V_{\mathbf{m}_3}\otimes V_{(2,0,0)}\to V_{\mathbf{m}_3+(0,1,1)}$.

1. $P_{(0,1,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2&0&0\\0&0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f''\left(M\left(\begin{smallmatrix}0&1&1\\0&0\\0\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}c_0 &= (m_{12}-m_{23})(m_{12}-m_{33}+1)(m_{22}-m_{33}),\\ c_1 &= \chi_+(M)(m_{12}-m_{33}+1)(m_{22}-m_{33})C_1(M)\\ &\quad -(m_{12}-m_{23}-1)C_2(M),\\ c_2 &= -\chi_+(M)C_2(M)\big\{C_1(M)-1\big\}.\end{aligned}$$

2. $P_{(0,1,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f''\left(M\left(\begin{smallmatrix}0\,1\,1\\1\,0\\0\end{smallmatrix}\right)[-i]\right)$
   with

$$
\begin{aligned}
c_0 &= -\tfrac{1}{2}(2m_{12}-m_{23}-m_{33}+2)(m_{13}-m_{12})(m_{22}-m_{33}),\\
c_1 &= \tfrac{1}{2}(m_{12}-m_{23})F(M)\\
&\quad -\tfrac{1}{2}\bigl\{1-\chi_-(M)\bigr\}(m_{13}-m_{12})(m_{22}-m_{33})C_1(M)\\
&\quad +\tfrac{1}{2}(m_{13}-m_{12}+1)C_2(M)\\
&\quad +\tfrac{1}{2}(m_{12}-m_{33}+1)(m_{22}-m_{33})C_1(M),\\
c_2 &= -C_2(M)\bigl\{C_1(M)-1\bigr\}.
\end{aligned}
$$

3. $P_{(0,1,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\,1\,1\\1\,0\\1\end{smallmatrix}\right)[-i]\right)$
   with

$$
\begin{aligned}
c_0 &= -\tfrac{1}{2}(2m_{12}-m_{23}-m_{33}+2)(m_{13}-m_{12})(m_{22}-m_{33}),\\
c_1 &= \tfrac{1}{2}(m_{12}-m_{23})E(M)+\tfrac{1}{2}(m_{13}-m_{12}+1)C_2(M)\\
&\quad +\tfrac{1}{2}\chi_+(M)(m_{22}-m_{33})\\
&\qquad \cdot\Bigl[-(m_{13}-m_{12})\bigl\{C_1(M)+1\bigr\}+(m_{12}-m_{33}+1)D(M)\Bigr],\\
c_2 &= \tfrac{1}{2}\chi_+(M)C_2(M)\bigl\{m_{13}-m_{33}+2-C_1(M)-D(M)\bigr\}.
\end{aligned}
$$

4. $P_{(0,1,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\0\end{smallmatrix}\right)\right)=\sum_{i=0}^{3}c_i f''\left(M\left(\begin{smallmatrix}0\,1\,1\\2\,0\\0\end{smallmatrix}\Bigl[-i]\right)\right)$
   with

$$
\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33}),\\
c_1 &= -(m_{13}-m_{12})\Bigl[F(M)+(m_{22}-m_{33})\bigl\{C_1(M)+\chi_-(M)\bigr\}\Bigr],\\
c_2 &= \bigl\{C_1(M)-1+\chi_-(M)\bigr\}F(M)+\chi_-(M)(m_{13}-m_{12}+1)C_2(M),\\
c_3 &= -\chi_-(M)C_2(M)\bigl\{C_1(M)-1\bigr\}.
\end{aligned}
$$

5. $P_{(0,1,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\1\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f''\left(M\left(\begin{smallmatrix}0\,1\,1\\2\,0\\1\end{smallmatrix}\right)[-i]\right)$
   with

$$
\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33}),\\
c_1 &= -\tfrac{1}{2}(m_{13}-m_{12})\Bigl[F(M)+E(M)+(m_{22}-m_{33})\bigl\{C_1(M)+1\bigr\}\Bigr]\\
&\quad -\tfrac{1}{2}\bigl\{1-\chi_-(M)\bigr\}(m_{13}-m_{12})(m_{22}-m_{33})D(M),\\
c_2 &= \tfrac{1}{2}C_1(M)E(M)+\tfrac{1}{2}C_2(M)\bigl\{1-D(M)-\chi_-(M)\delta(M)\bigr\}.
\end{aligned}
$$

6. $P_{(0,1,1)}\left(f(M)\otimes\left(\begin{smallmatrix}2&0&0\\&2\;0&\\&2&\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0&1&1\\&2\;0&\\&2&\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33}),\\
c_1 &= -(m_{13}-m_{12})\Big[E(M)+\chi_+(M)(m_{22}-m_{33})\big\{D(M)+1\big\}\Big],\\
c_2 &= \chi_+(M)D(M)E(M).
\end{aligned}$$

Formula 6: The projector $P_{(0,0,2)}: V_{\mathbf{m}_3}\otimes V_{(2,0,0)}\to V_{\mathbf{m}_3+(0,0,2)}$.

1. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2&0&0\\&0\;0&\\&0&\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0&0&2\\&0\;0&\\&0&\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}
c_0 &= (m_{12}-m_{33}+1)(m_{12}-m_{33})(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= -2(m_{12}-m_{33})(m_{22}-m_{33})C_2(M),\\
c_2 &= C_2(M)\big\{C_1(M)-1\big\}\big\{\bar{C}_1(M)-1\big\}.
\end{aligned}$$

2. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2&0&0\\&1\;0&\\&0&\end{smallmatrix}\right)\right)=\sum_{i=0}^{3}c_i f''\left(M\left(\begin{smallmatrix}0&0&2\\&1\;0&\\&0&\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}
c_0 &= -(m_{13}-m_{12})(m_{12}-m_{33}+1)(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= (m_{22}-m_{33})\left\{(m_{12}-m_{33})F(M)+(m_{13}-m_{12})C_2\left(M\left(\begin{smallmatrix}0&0&1\\&1\;0&\\&0&\end{smallmatrix}\right)\right)\right\},\\
c_2 &= -F(M)C_2\left(M\left(\begin{smallmatrix}0&0&1\\&0\;1&\\&0&\end{smallmatrix}\right)\right)\\
&\quad -\chi_-(M)(m_{12}-m_{33}-1)(m_{22}-m_{33}+1)C_2(M),\\
c_3 &= \chi_-(M)C_2(M)\big\{C_1(M)-1\big\}\big\{\bar{C}_1(M)-1\big\}.
\end{aligned}$$

3. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2&0&0\\&1\;0&\\&1&\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0&0&2\\&1\;0&\\&1&\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}
c_0 &= -(m_{12}-m_{33}+1)(m_{13}-m_{12})(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= (m_{22}-m_{33})\\
&\qquad\cdot\Big[(m_{12}-m_{33})E(M)+(m_{13}-m_{12})\big\{C_1(M)+1\big\}\bar{C}_1(M)\Big],\\
c_2 &= -E(M)C_1(M)\big\{\bar{C}_1(M)-1\big\}.
\end{aligned}$$

4. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2\ 0\ 0\\ 2\ 0\\ 0\end{smallmatrix}\right)\right)=\sum_{i=0}^{4}c_i f''\left(M\left(\begin{smallmatrix}0\ 0\ 2\\ 2\ 0\\ 0\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= -(m_{13}-m_{12})(m_{22}-m_{33})\left\{F(M)+F\left(M\left(\begin{smallmatrix}0\ 0\ 1\\ 1\ 0\\ 0\end{smallmatrix}\right)\right)\right\},\\
c_2 &= \chi_-(M)(m_{13}-m_{12}+1)(m_{22}-m_{33}+1)C_2(M)\\
&\quad +\chi_-^{(1)}(M)(m_{13}-m_{12})(m_{22}-m_{33})\big\{C_1(M)+1\big\}\big\{\bar{C}_1(M)+1\big\}\\
&\quad +F(M)F\left(M\left(\begin{smallmatrix}0\ 0\ 1\\ 0\ 1\\ 0\end{smallmatrix}\right)\right),\\
c_3 &= -C_2(M)\left\{\chi_-^{(1)}(M)F(M)+\chi_-(M)F\left(M\left(\begin{smallmatrix}0\ 0\ 1\\ -1\ 2\\ 0\end{smallmatrix}\right)\right)\right\},\\
c_4 &= \chi_-^{(1)}(M)C_2(M)\big\{C_1(M)-1\big\}\big\{\bar{C}_1(M)-1\big\}.
\end{aligned}$$

5. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2\ 0\ 0\\ 2\ 0\\ 1\end{smallmatrix}\right)\right)=\sum_{i=0}^{3}c_i f''\left(M\left(\begin{smallmatrix}0\ 0\ 2\\ 2\ 0\\ 1\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= -(m_{13}-m_{12})(m_{22}-m_{33})\left\{F\left(M\left(\begin{smallmatrix}0\ 0\ 1\\ 1\ 0\\ 1\end{smallmatrix}\right)\right)+E(M)\right\},\\
c_2 &= E(M)F\left(M\left(\begin{smallmatrix}0\ 0\ 1\\ 0\ 1\\ 1\end{smallmatrix}\right)\right)\\
&\quad +\chi_-(M)(m_{13}-m_{12})(m_{22}-m_{33})\big\{C_1(M)+1\big\}\bar{C}_1(M),\\
c_3 &= -\chi_-(M)E(M)C_1(M)\big\{\bar{C}_1(M)-1\big\}.
\end{aligned}$$

6. $P_{(0,0,2)}\left(f(M)\otimes\left(\begin{smallmatrix}2\ 0\ 0\\ 2\ 0\\ 2\end{smallmatrix}\right)\right)=\sum_{i=0}^{2}c_i f'\left(M\left(\begin{smallmatrix}0\ 0\ 2\\ 2\ 0\\ 2\end{smallmatrix}\right)[-i]\right)$
   with

$$\begin{aligned}
c_0 &= (m_{13}-m_{12})(m_{13}-m_{12}-1)(m_{22}-m_{33})(m_{22}-m_{33}-1),\\
c_1 &= -2(m_{13}-m_{12})(m_{22}-m_{33})\big\{E(M)-\bar{C}_1(M)\big\},\\
c_2 &= E(M)\big\{\bar{C}_1(M)-1\big\}\big\{m_{13}-m_{33}-C_1(M)\big\}.
\end{aligned}$$

3.2 Proof of Formulas

In this subsection, we give a proof of the formulas 3 and 4 in the previous subsection. The rest can be proved by the similar computation.

First, we prove the formula 3. To do this, we may compute only the projector Q_2 defined in the top of this section. Let us denote by $\{f''(M)\}$

the canonical basis of the representation $V_{\mathbf{m_3}+(0,0,1)}$. Using the formula for the projectors $P_{(0,0,1)}$ in Theorem 1, we have

$$\begin{aligned}
Q_2\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right) &= P_{(1,0,0)}\left(P_{(0,0,1)}\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&= (m_{12}-m_{33}+1)(m_{22}-m_{33})P_{(1,0,0)}\left(f''\left(M\left(\begin{smallmatrix}0\,0\,1\\0\,0\\0\end{smallmatrix}\right)\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&\quad -C_2(M)P_{(1,0,0)}\left(f''\left(M\left(\begin{smallmatrix}0\,0\,1\\-1\,1\\0\end{smallmatrix}\right)\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right).
\end{aligned}$$

Then the first formula is deduced from this equation with the formula for $P_{(1,0,0)}$ in Theorem 1. Similarly, using the formulas in Theorem 1 twice, we have the equations

$$\begin{aligned}
&Q_2\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\0\end{smallmatrix}\right)\right)\\
&= -(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\2\,0\\0\end{smallmatrix}\right)\right)\\
&\quad +\left\{F(M)-\chi_-\left(M\left(\begin{smallmatrix}0\,0\,1\\1\,0\\0\end{smallmatrix}\right)\right)(m_{13}-m_{12})(m_{22}-m_{33})\right\}f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,1\\0\end{smallmatrix}\right)\right)\\
&\quad +\left\{\chi_-\left(M\left(\begin{smallmatrix}0\,0\,1\\0\,1\\0\end{smallmatrix}\right)\right)F(M)-\chi_-(M)C_2(M)\right\}f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,2\\0\end{smallmatrix}\right)\right)\\
&\quad -\chi_-(M)\chi_-\left(M\left(\begin{smallmatrix}0\,0\,1\\-1\,2\\0\end{smallmatrix}\right)\right)C_2(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,2\\0\end{smallmatrix}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&Q_2\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\2\end{smallmatrix}\right)\right)\\
&= -(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\2\,0\\2\end{smallmatrix}\right)\right)+E(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,1\\2\end{smallmatrix}\right)\right).
\end{aligned}$$

Thus we have the formulas 3-4 and 3-6, because

$$\chi_-\left(M\left(\begin{smallmatrix}0\,0\,1\\1\,0\\0\end{smallmatrix}\right)\right)=\chi_-\left(M\left(\begin{smallmatrix}0\,0\,1\\0\,1\\0\end{smallmatrix}\right)\right)=\chi_-(M)\chi_-\left(M\left(\begin{smallmatrix}0\,0\,1\\-1\,2\\0\end{smallmatrix}\right)\right)=\chi_-^{(1)}(M).$$

Two images of Q_2 in the right hand side of the equation

$$\begin{aligned}
&Q_2\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)\\
&= \frac{1}{2}\left\{Q_2\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)+Q_2\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\right\},
\end{aligned}$$

have the following expressions which are derived from Theorem 1:

$$
\begin{aligned}
&Q_2\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)\\
&\quad= (m_{12}-m_{33}+1)(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,0\\0\end{smallmatrix}\right)\right)\\
&\qquad+\left\{\chi_-\left(M\left(\begin{smallmatrix}0\,0\,1\\0\,0\\0\end{smallmatrix}\right)\right)(m_{12}-m_{33}+1)(m_{22}-m_{33})-C_2(M)\right\}\\
&\qquad\qquad\cdot f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,1\\0\end{smallmatrix}\right)\right)-\chi_-\left(M\left(\begin{smallmatrix}0\,0\,1\\-1\,1\\0\end{smallmatrix}\right)\right)C_2(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\-1\,2\\0\end{smallmatrix}\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&Q_2\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&\quad= -(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,0\\0\end{smallmatrix}\right)\right)\\
&\qquad+F(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,1\\0\end{smallmatrix}\right)\right)-\chi_-(M)C_2(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\-1\,2\\0\end{smallmatrix}\right)\right).
\end{aligned}
$$

Because $\chi_-(M)$ depends only on $\delta(M)$, the formula 3-2 holds. Similarly, the following four formulas with the identification in Lemma 1 lead to the formulas 3-3 and 3-5:

$$
\begin{aligned}
&Q_2\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right)\\
&\quad= (m_{12}-m_{33}+1)(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,0\\1\end{smallmatrix}\right)\right)-C_2(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,1\\1\end{smallmatrix}\right)\right),\\
&Q_2\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&\quad= -(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,0\\1\end{smallmatrix}\right)\right)+E(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,1\\1\end{smallmatrix}\right)\right),\\
&Q_2\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right)\\
&\quad= -(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\2\,0\\1\end{smallmatrix}\right)\right)+F(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,1\\1\end{smallmatrix}\right)\right)\\
&\qquad-\chi_-(M)C_2(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,2\\1\end{smallmatrix}\right)\right),\\
&Q_2\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)\\
&\quad= -(m_{13}-m_{12})(m_{22}-m_{33})f'\left(M\left(\begin{smallmatrix}1\,0\,1\\2\,0\\1\end{smallmatrix}\right)\right)\\
&\qquad+\{-\chi_-(M)(m_{13}-m_{12})(m_{22}-m_{33})+E(M)\}f'\left(M\left(\begin{smallmatrix}1\,0\,1\\1\,1\\1\end{smallmatrix}\right)\right)\\
&\qquad+\chi_-(M)E(M)f'\left(M\left(\begin{smallmatrix}1\,0\,1\\0\,2\\1\end{smallmatrix}\right)\right),
\end{aligned}
$$

Next we prove the formula 4. We use again the projectors in Theorem 1 twice. Similarly to the proof of the formula 3, we have the equation

$$\begin{aligned}
&P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right) = P_{(0,1,0)}\left(P_{(0,1,0)}\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&\quad= (m_{12}-m_{23})P_{(0,1,0)}\left(f''\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,0\\0\end{smallmatrix}\right)\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&\qquad+\chi_+(M)C_1(M)P_{(0,1,0)}\left(f''\left(M\left(\begin{smallmatrix}0\,1\,0\\-1\,1\\0\end{smallmatrix}\right)\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&\quad= (m_{12}-m_{23})(m_{12}-m_{23}-1)f'\left(M\left(\begin{smallmatrix}0\,2\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&\qquad+\Big\{\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,0\\0\end{smallmatrix}\right)\right)(m_{12}-m_{23})C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&\qquad\qquad+\chi_+(M)(m_{12}-m_{23}-2)C_1(M)\Big\}f'\left(M\left(\begin{smallmatrix}0\,2\,0\\-1\,1\\0\end{smallmatrix}\right)\right)\\
&\qquad+\chi_+(M)\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\-1\,1\\0\end{smallmatrix}\right)\right)C_1(M)C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\-1\,1\\0\end{smallmatrix}\right)\right)f'\left(M\left(\begin{smallmatrix}0\,2\,0\\-2\,2\\0\end{smallmatrix}\right)\right).
\end{aligned}$$

Here $\{f''(M)\}$ means the canonical basis of the representation $V_{\mathbf{m_3}+(0,1,0)}$. This equation means the formula 4-1, because of the relations

$$\begin{aligned}
\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,0\\0\end{smallmatrix}\right)\right)C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,0\\0\end{smallmatrix}\right)\right) &= \chi_+^{(1)}(M)C_1(M),\\
\chi_+(M)\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\-1\,1\\0\end{smallmatrix}\right)\right)C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\-1\,1\\0\end{smallmatrix}\right)\right) &= \chi_+^{(1)}(M)(C_1(M)-1).
\end{aligned}$$

Similarly, we can get the formulas 4-4 and 4-6 from the equations

$$\begin{aligned}
&P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\0\end{smallmatrix}\right)\right)\\
&\quad= (m_{13}-m_{12})(m_{13}-m_{12}-1)f'\left(M\left(\begin{smallmatrix}0\,2\,0\\2\,0\\0\end{smallmatrix}\right)\right)\\
&\qquad+\Big\{-(m_{13}-m_{12})C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\0\end{smallmatrix}\right)\right)-(m_{13}-m_{12})C_1(M)\Big\}f'\left(M\left(\begin{smallmatrix}0\,2\,0\\1\,1\\0\end{smallmatrix}\right)\right)\\
&\qquad+C_1(M)C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\0\end{smallmatrix}\right)\right)f'\left(M\left(\begin{smallmatrix}0\,2\,0\\0\,2\\0\end{smallmatrix}\right)\right),\\
&P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\2\end{smallmatrix}\right)\right)\\
&\quad= (m_{13}-m_{12})(m_{13}-m_{12}-1)f'\left(M\left(\begin{smallmatrix}0\,2\,0\\2\,0\\2\end{smallmatrix}\right)\right)\\
&\qquad+\Big\{-\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right)(m_{13}-m_{12})D\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right)\\
&\qquad\qquad-\chi_+(M)(m_{13}-m_{12})D(M)\Big\}f'\left(M\left(\begin{smallmatrix}0\,2\,0\\1\,1\\2\end{smallmatrix}\right)\right)\\
&\qquad+\chi_+(M)\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right)D(M)D\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right)f'\left(M\left(\begin{smallmatrix}0\,2\,0\\0\,2\\2\end{smallmatrix}\right)\right),
\end{aligned}$$

which are derived from Theorem 1 and the relations

$$
\begin{gathered}
C_1\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right)=C_1(M), \;\; C_1\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 0\,1 \\ 0 \end{smallmatrix}\right)\right)=C_1(M)-1, \\
D\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right)\right)=D(M)+2, \;\; D\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 0\,1 \\ 1 \end{smallmatrix}\right)\right)=D(M)+1, \\
\chi_+\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 1\,0 \\ 1 \end{smallmatrix}\right)\right)=\chi_+(M)\chi_+\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 0\,1 \\ 1 \end{smallmatrix}\right)\right)=\chi_+^{(1)}(M).
\end{gathered}
$$

Before giving a proof of the remainder formulas 4-2, 4-3 and 4-5, we use the following lemma.

Lemma 2. For any vectors $u_1, u_2 \in V_{(1,0,0)}$ and $1 \leq i \leq 3$, we have

$$
\begin{aligned}
P_{2\mathbf{e}_i}\left(f(M)\otimes \tfrac{1}{2}\{u_1\otimes u_2+u_2\otimes u_1\}\right) &= P_{2\mathbf{e}_i}(f(M)\otimes u_1\otimes u_2) \\
&= P_{2\mathbf{e}_i}(f(M)\otimes u_2\otimes u_1).
\end{aligned}
$$

Proof of Lemma. Let $A(V_{(1,0,0)})$ be the anti-symmetric tensor product of $V_{(1,0,0)}$ and put $W=V_{\mathbf{m}_3}\otimes A(V_{(1,0,0)})$. Then we have the irreducible decomposition

$$
W \cong V_{\mathbf{m}_3+\mathbf{e}_1+\mathbf{e}_2}\oplus V_{\mathbf{m}_3+\mathbf{e}_1+\mathbf{e}_3}\oplus V_{\mathbf{m}_3+\mathbf{e}_2+\mathbf{e}_3}.
$$

In particular, the restriction of the projector $P_{2\mathbf{e}_i}$ to the subspace W in $W_2=V_{\mathbf{m}_3}\otimes V_{(1,0,0)}\otimes V_{(1,0,0)}$ is zero for each $1\leq i\leq 3$, that is

$$
P_{2\mathbf{e}_i}\left(f(M)\otimes\{u_1\otimes u_2-u_2\otimes u_1\}\right)=0
$$

for any $u_1, u_2 \in V_{(1,0,0)}$. □

According to the above lemma and the identification in Lemma 1, we have

$$
P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix} 2\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right)=P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix} 1\,0\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\otimes\left(\begin{smallmatrix} 1\,0\,0 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\right).
$$

Theorem 1 shows that the right hand side of the above is equal to

$$
\begin{aligned}
&-(m_{13}-m_{12})(m_{12}-m_{23})f'\left(M\left(\begin{smallmatrix} 0\,2\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right) \\
&+\Big\{-\chi_+(M)\,(m_{13}-m_{12})C_1\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right) \\
&\qquad+(m_{12}-m_{23}-1)C_1(M)\Big\}f'\left(M\left(\begin{smallmatrix} 0\,2\,0 \\ 0\,1 \\ 0 \end{smallmatrix}\right)\right) \\
&+\chi_+(M)\,C_1(M)C_1\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 0\,1 \\ 0 \end{smallmatrix}\right)\right)f'\left(M\left(\begin{smallmatrix} 0\,2\,0 \\ -1\,2 \\ 0 \end{smallmatrix}\right)\right).
\end{aligned}
$$

Together with the relations

$$
C_1\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\right)=C_1(M), \;\; C_1\left(M\left(\begin{smallmatrix} 0\,1\,0 \\ 0\,1 \\ 0 \end{smallmatrix}\right)\right)=C_1(M)-1,
$$

we get the formula 4-2. Similarly the equations

$$
\begin{aligned}
&P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\1\,0\\1\end{smallmatrix}\right)\right)=P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\0\,0\\0\end{smallmatrix}\right)\right)\\
&\quad=\;-(m_{13}-m_{12})(m_{12}-m_{23})f'\left(M\left(\begin{smallmatrix}0\,2\,0\\1\,0\\1\end{smallmatrix}\right)\right)\\
&\qquad+\Big\{-\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right)(m_{13}-m_{12})C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right)\\
&\qquad\qquad+\chi_+(M)(m_{12}-m_{23}-1)D(M)\Big\}f'\left(M\left(\begin{smallmatrix}0\,2\,0\\0\,1\\1\end{smallmatrix}\right)\right)\\
&\qquad+\chi_+(M)\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right)D(M)C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right)f'\left(M\left(\begin{smallmatrix}0\,2\,0\\-1\,2\\1\end{smallmatrix}\right)\right),\\
&P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}2\,0\,0\\2\,0\\1\end{smallmatrix}\right)\right)=P_{(0,2,0)}\left(f(M)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\,0\,0\\1\,0\\0\end{smallmatrix}\right)\right)\\
&\quad=\;(m_{13}-m_{12})(m_{13}-m_{12}-1)f'\left(M\left(\begin{smallmatrix}0\,2\,0\\2\,0\\1\end{smallmatrix}\right)\right)\\
&\qquad-\Big\{(m_{13}-m_{12})C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right)+\chi_+(M)(m_{13}-m_{12})D(M)\Big\}\\
&\qquad\quad\cdot f'\left(M\left(\begin{smallmatrix}0\,2\,0\\1\,1\\1\end{smallmatrix}\right)\right)+\chi_+(M)D(M)C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right)f'\left(M\left(\begin{smallmatrix}0\,2\,0\\0\,2\\1\end{smallmatrix}\right)\right),
\end{aligned}
$$

lead to the formulas 4-3 and 4-5 with the relations

$$
\begin{gathered}
\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right)C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right)=\chi_+^{(1)}(M)(C_1(M)+1),\\
\chi_+(M)\chi_+\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right)C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right)=\chi_+^{(1)}(M)C_1(M),\\
C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\1\,0\\1\end{smallmatrix}\right)\right)=C_1(M)+\chi_+(M),\;\;\chi_+(M)C_1\left(M\left(\begin{smallmatrix}0\,1\,0\\0\,1\\1\end{smallmatrix}\right)\right)=\chi_+(M)C_1(M).
\end{gathered}
$$

$\square$

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