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Abstract

In this paper, we consider non-stationary Maxwell's equations in an anisotropic medium in the (x_1, x_2, x_3)-space where equations of the divergences of electric and magnetic flux densities are also unknown. Then we discuss an inverse problem of determining the x_3-independent components of the electric current density from observations on the plane $x_3 = 0$ over a time interval. Our main result is conditional stability in the inverse problem provided that the permittivity and the permiability are independent of x_3. The main tool is a new Carleman estimate.

1 Introduction and Main Results

Consider Maxwell's equations in an anisotropic and inhomogeneous medium:

$$\begin{cases} \partial_t \left(\varepsilon(x,t)E(x,t)\right) - \nabla \times H(x,t) + J(x,t) = 0, \quad (x,t) \in \mathbb{R}^4_+, \\ \partial_t \left(\mu(x,t)H(x,t)\right) + \nabla \times E(x,t) = 0, \quad (x,t) \in \mathbb{R}^4_+, \\ E(x,0) = H(x,0) = 0, \quad x \in \mathbb{R}^3 \end{cases} \tag{1.1}$$

where $\mathbb{R}^4_+ = \left\{(x,t) \,\middle|\, x \in \mathbb{R}^3,\ t \geq 0\right\}$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\partial_t = \frac{\partial}{\partial t}$, $\nabla\times$ denotes the rotation, $E = (E_1, E_2, E_3)^T$ and $H = (H_1, H_2, H_3)^T$ are the electric and magnetic field respectively, $\varepsilon(x,t) = (\varepsilon_{kl}(x,t))_{3\times 3}$ and $\mu(x,t) = (\mu_{kl}(x,t))_{3\times 3}$ are the permittivity tensor and the permeability tensor respectively, $J(x,t)$ is the density of the electric current. Here and henceforth $\cdot^T$ denotes the transpose of vectors or matrices under the consideration.

Here the tensors ε and μ govern the constitutive relations for the medium under consideration:

$$D = \varepsilon E: \quad \text{the electric flux density,}$$

$$B = \mu H: \quad \text{the magnetic flux density.}$$

In the anisotropic medium, ε and μ are neither necessarily scalars (e.g., in some crystals) nor diagonal matrices (cf. Kong [19], Landan-Lifshitz [20]). Throughout this paper, we assume that $\varepsilon = \varepsilon(x,t)$ and $\mu = \mu(x,t)$ are 3×3 symmetric matrices, ε, μ, $\partial_t\varepsilon$, $\partial_t\mu$ are continuous in $(x,t) \in \mathbb{R}^4_+$, and that there exists a constant $h > 0$ such that for any $\xi = (\xi_1, \xi_2, \xi_3)^T$,

$$\xi^T \varepsilon(x,t)\xi \geq h\xi^T\xi, \quad \xi^T\mu(x,t)\xi \geq h\xi^T\xi, \quad (x,t) \in \mathbb{R}^4_+. \tag{1.2}$$

Let $x = (x_1, x_2, x_3) = (x', x_3)$ with $x' = (x_1, x_2)$. In this paper, we consider

Inverse Sourse Problem

We assume that ε, μ are independent of the x_3-component and that

$$J(x,t) = R(x,t)F\left(x',t\right) \tag{1.3}$$

where $R(x,t) = (r_{kl}(x,t))_{3\times 3}$ is a given 3×3 matrix and $F\left(x',t\right)$=$(f_1\left(x',t\right), f_2\left(x',t\right), f_3\left(x',t\right))^T$. Let $\Gamma \subset \mathbb{R}^3$ be a given domain. Then determine an x', t-dependent component $F\left(x',t\right)$, $(x',t)\in\Gamma$ of current J from the observations of some components of

$$E\left(x',0,t\right), \quad H\left(x',0,t\right), \quad \left(x',t\right) \in \Gamma' : \text{ some domain.} \tag{1.4}$$

This inverse problem is concerned with the determination of properties of an antenna by components of the electric field and / or the magnetic fields on $x_3 = 0$, under the assumption that the x_3-dependence of the antenna is known.

For inverse problems of Maxwell's equations, we refer to §6 of Chapter 5 in Romanov [23], a monograph by Romanov and Kabanikhin [24], Yamamoto [28, 29]. The paper [29] proved the uniqueness in an inverse problem of determining electric source terms under some "non-degeneracy" assumption and the key is a weighted estimate called a Carleman estimate. The method in Yamamoto [29] was inspired by Bukhgeim and Klibanov [6]. For similar inverse problems for other equations, we refer to Bukhgeim [5], Imanuvilov and Yamamoto [11, 12], Isakov [14, 15], Khaĭdarov [17], Klibanov [18], Yamamoto [30].

In the formulation of the inverse problem, we take the domain Γ' of observations on the plane $x_3 = 0$ where an unknown vector-valued function F is defined (i.e., F is independent of x_3). As for this kind of formulation of inverse problems for parabolic equations, we refer to

Beznoshchenko [3, 4], Iskenderov [16], §5 of Chapter 7 in Lavrent'ev, Romanov and Shishat·skiĭ [21], §4 of Chapter 6 in Romanov [23]. For a similar inverse problem for an elliptic equation and a stationary Lamé system by Carleman estimates, we refer to Klibanov [18] and Imanuvilov and Yamamoto [13].

From the technical point of view, we will treat Maxwell's equations as a first-order symmetric system, and as for inverse problem for first-order systems, we refer to Belinskij [2], §4 of Chapter 7 in Lavrent'ev, Romanov and Shishat·skiĭ [21], Romanov [22], Chapter 5 in Romanov [23], Romanov and Belinskij [25] which assumes some extra restrictions (e.g., unknown functions are dependent only on one component of x) and reduces the inverse problem to a one dimensional problem, so that their methodology is different from ours.

To the authors' best knowledge, there are no trials for solutions to inverse problems for Maxwell's equations in anisotropic media by means of Carleman estimates.

In this paper, we will first prove a Carleman estimate for (1.1) with the weight function $\mathrm{e}^{2s\varphi}$ where

$$\varphi = \varphi(x,t) = \alpha - t - \beta|x|^2 \equiv \alpha - t - \beta\left(x_1^2 + x_2^2 + x_3^2\right). \tag{1.5}$$

By applying it, we will obtain a theorem in an inverse problem for (1.1) of determining $F(x',t)$ from observation data concerning $E(x',0,t)$ and $H(x',0,t)$ provided ε and μ are independent of x_3.

In order to state the main result, we introduce some notations. Let $\alpha > 0$ and $0 < \beta < \frac{h^2}{16\alpha}$

be suitably given. We set

$$Q_0 = \left\{(x,t) \in \mathbb{R}^4 \,|\varphi(x,t) > 0, \quad t > 0\right\},$$
$$\Gamma_0 = \left\{(x',t) \in \mathbb{R}^3 \,|\varphi(x',0,t) > 0, \quad t > 0\right\} \tag{1.6}$$

and

$$\partial_t = \frac{\partial}{\partial t}, \qquad \partial_k = \frac{\partial}{\partial x_k}, \quad k = 1,2,3,$$

where $|x'|^2 = x_1^2 + x_2^2$. Moreover, for any $U = (u_1, \dots, u_N)^T$ and $V = (v_1, \dots, v_N)^T$ $(N = 3, 6)$, we set $|U|^2 = \sum_{k=1}^N u_k^2$ and $(U,\ V)^T = (u_1, \dots, u_N, v_1, \dots, v_N)^T$. Moreover $L^2(Q_0)$, $H^1(Q_0)$, $H^2(Q_0)$, $H^2(\Gamma)$, etc. denote usual Sobolev spaces. For $n = 3, 6$, we set $\left(L^2(Q_0)\right)^n = \{U = (u_1, \dots, u_n)^T | u_k \in L^2(Q_0), k = 1, \dots, n\}$ and $\|U\|_{L^2(Q_0)} = \left(\sum_{k=1}^n \|u_k\|^2_{L^2(Q_0)}\right)^{\frac{1}{2}}$. $(L^\infty(Q_0))^9 = \{A = (a_{kl})_{3\times 3} | a_{kl} \in L^\infty(Q_0),\ k,l = 1,2,3\}$ and $\|A\|_{L^\infty(Q_0)} = \max_{1\le k,l\le 3}\left\{\|a_{kl}\|_{L^\infty(Q_0)}\right\}$. $\left(C\left(\overline{Q_0}\right)\right)^6$, $\left(H^1(Q_0)\right)^6$, etc. are similarly defined.

Now we state the main results.

Theorem 1.1 (Carleman Estimate). Let ε and μ satisfy (1.2). Furthermore assume that $\varepsilon,\ \mu,\ \partial_t\varepsilon,\ \partial_t\mu \in \left(C\left(\overline{Q_0}\right)\right)^9$ and $|\partial_t\varepsilon(x,t)|,\ |\partial_t\mu(x,t)| \le M$ for all $(x,t) \in \overline{Q_0}$ where M is a positive constant. Then there exist constants $s_0 > 1$ and $C = C(s_0,\ \alpha,\ \beta,\ h,\ M) > 0$ such that

$$\begin{aligned} &s\int_{Q_0}\left(|U|^2 + |V|^2\right) \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ &\le C\int_{Q_0}\left(|\partial_t(\varepsilon U) - \nabla \times V|^2 + |\partial_t(\mu V) + \nabla \times U|^2\right)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \end{aligned} \tag{1.7}$$

for all $s > s_0$, provided that

$$\begin{cases} U = (u_1,u_2,u_3)^T \in \left(L^2(Q_0)\right)^3, \quad V = (v_1,v_2,v_3)^T \in \left(L^2(Q_0)\right)^3, \\ (\partial_t(\varepsilon U) - \nabla\times V) \in \left(L^2(Q_0)\right)^3, \quad (\partial_t(\mu V) + \nabla \times U) \in \left(L^2(Q_0)\right)^3, \\ U\,|_{\partial Q_0} = V\,|_{\partial Q_0} = 0. \end{cases} \tag{1.8}$$

For general theories for Carleman estimates, we refer to Hörmander [9, 10], Isakov [15], but those results are for a single equation. Moreover we refer to Taylor [26]. For Carleman estimates for systems of partial differential equations, see Egorov [8], and see Cheng, Isakov, Yamamoto and Zhou [7], Imanuvilov and Yamamoto [13] especially for Lamé systems.

Here we will directly derive Carleman estimate (1.7). In the case that ε and μ are real-valued functions and the divergence conditions on B and D are assumed, we can reduce Maxwell's equations to a set of hyperbolic equations whose principal parts are decoupled, so that the general theory by [9, 10] yields relevant Carleman estimates (e.g., Yamamoto [29]). However, in this paper, we consider the anisotropic case and do not assume any conditions on div$B$ or div$D$. Moreover we do not take a weight funtion with factor $|x|^2 - \gamma t^2$. On the other hand, if we could establish a Carleman estimate with a weight function of such a form, then we would be able to prove the uniqueness of solution in some open set if it varnishes on a lateral boundary. Therefore, as the following example suggests, it is extremely difficult to establish such a Carleman estimate without divergence conditions.

Example

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let us consider

$$\nabla \times U = 0 \quad \text{in} \quad \Omega$$

with $U = (u_1, u_2, u_3)^T$. Then, in general, we cannot conclude that supp$U \subset \Omega$ implies $U \equiv 0$ in Ω. In fact, let $a \in C_0^\infty(\Omega)$ be an arbitrary real-valued function. We let $U = \nabla a$, so that $\nabla \times U = 0$ and $U \in C_0^\infty(\Omega)$. In this case, we can prove a Carleman estimate if we assume an equation of div$U$ (see Vogelsang [27]).

Theorem 1.2 (Conditional Stability). Let $\varepsilon = \varepsilon(x', t)$ and $\mu = \mu(x', t)$ be independent of x_3 and satisfy (1.2). Assume further that ε, $\mu \in \left(C^1\left(\mathbb{R}^4_+\right)\right)^9$ and $|\varepsilon(x', t)|$, $|\mu(x', t)|$, $|\partial_t\varepsilon(x', t)|$, $|\partial_t\mu(x', t)| \leq M$ for all $(x', t) \in \overline{\Gamma_0}$ where M is a positive constant. Furthermore assume that $R \in \left(C\left(\mathbb{R}^4_+\right)\right)^9$, $\partial_3 R$, $\partial_3^2 R \in \left(L^\infty_{loc}\left(\mathbb{R}^4_+\right)\right)^9$ and there exists a constant $r_0 > 0$ such that for any $\xi = (\xi_1, \xi_2, \xi_3)^T$,

$$
\begin{aligned}
&\|\partial_3 R\|_{L^\infty_{loc}(Q_0)}, \left\|\partial_3^2 R\right\|_{L^\infty_{loc}(Q_0)} \leq M, \\
&|R(x', 0, t)\xi|^2 \geq r_0 |\xi|^2, \quad (x', t) \in \overline{\Gamma_0}.
\end{aligned} \tag{1.9}
$$

Let $F \in \left(C\left(\overline{\Gamma}\right)\right)^3$, $E, H \in \left(C^1\left(\mathbb{R}^4_+\right)\right)^3 \cap \left(H^2\left(\mathbb{R}^4_+\right)\right)^3$ satisfy (1.1). Then for any given $0 < \delta < \alpha$, there exists a constant $C = C(s_0, \alpha, \beta, h, M, R, r_0, \delta) > 0$ such that

$$
\begin{aligned}
&\|F\|_{L^2(\Gamma_\delta)} \\
&\leq C\left(\|E\|_{H^2(Q_0)} + \|H\|_{H^2(Q_0)}\right)^{\frac{3\alpha - 3\delta}{3\alpha - 2\delta}} \left(\left(\|E(\cdot, 0, \cdot)\|_{H^1(\Gamma_0)} + \|H(\cdot, 0, \cdot)\|_{H^1(\Gamma_0)}\right)\right)^{\frac{\delta}{3\alpha - 2\delta}}
\end{aligned} \tag{1.10}
$$

where

$$
\Gamma_\delta = \left\{(x', t) \in \mathbb{R}^3 \left| 0 < t < \alpha - \delta - \beta |x'|^2 \right.\right\}. \tag{1.11}
$$

This theorem asserts the stability in determining F by the observations $E(\cdot, 0, \cdot)$ and $H(\cdot, 0, \cdot)$ under the condition that $H^2(Q_0)$-norms of E and H are a priori bounded. Moreover the stability is of Hölder's type whose exponent depends on the domain where we can determine F.

If an unknown vector-valued function F and the matrices R, ε and μ are special, then we can reduce the observations, that is, not all the components of $E(x', 0, t)$ and $H(x', 0, t)$ are required for the conditional stability, but in some cases, we have to take stronger norms.

Here we state only three cases among possible all cases.

Corollary 1. In addition to the assumptions in Theorem 1.2, let

$$\varepsilon(x',t) = \begin{pmatrix} \varepsilon_1(x',t) & 0 & 0 \\ 0 & \varepsilon_2(x',t) & 0 \\ 0 & 0 & \varepsilon_3(x',t) \end{pmatrix},$$

$$\mu(x',t) = \begin{pmatrix} \mu_1(x',t) & 0 & 0 \\ 0 & \mu_2(x',t) & 0 \\ 0 & 0 & \mu_3(x',t) \end{pmatrix},$$

$$R(x,t) = \begin{pmatrix} r_1(x,t) & 0 & 0 \\ 0 & r_2(x,t) & 0 \\ 0 & 0 & r_3(x,t) \end{pmatrix} \quad \text{and} \quad F(x',t) = \begin{pmatrix} f(x',t) \\ f(x',t) \\ f(x',t) \end{pmatrix}.$$

Furthermore assume that $|\partial_1\mu_3(x',t)|, |\partial_2\mu_3(x',t)| \le M$ for all $(x',t) \in \overline{\Gamma_0}$. If $|r_1(x',0,t)| + |r_2(x',0,t)| \ge r_0 > 0$, $(x',t) \in \overline{\Gamma_0}$, then (1.10) is replaced by

$$\|f\|_{L^2(\Gamma_\delta)} \le C\left(\|E\|_{H^2(Q_0)} + \|H\|_{H^2(Q_0)}\right)^{\frac{3\alpha-3\delta}{3\alpha-2\delta}} \left(\|E_1(\cdot,0,\cdot)\|_{H^2(\Gamma_0)} + \|E_2(\cdot,0,\cdot)\|_{H^2(\Gamma_0)}\right)^{\frac{\delta}{3\alpha-2\delta}}.$$

Corollary 2. In addition to the assumptions in Theorem 1.2, let

$$R(x,t) = \begin{pmatrix} r_1(x,t) & 0 & 0 \\ 0 & r_2(x,t) & 0 \\ 0 & 0 & r_3(x,t) \end{pmatrix} \quad \text{and} \quad F(x',t) = \begin{pmatrix} f(x',t) \\ f(x',t) \\ f(x',t) \end{pmatrix}.$$

If $|r_1(x',0,t)|+|r_2(x',0,t)|\ge r_0>0$, $(x',t)\in\overline{\Gamma_0}$, then (1.10) is replaced by

$$\|f\|_{L^2(\Gamma_\delta)} \le C\left(\|E\|_{H^2(Q_0)}+\|H\|_{H^2(Q_0)}\right)^{\frac{3\alpha-3\delta}{3\alpha-2\delta}}\left(\sum_{k=1}^{3}\|E_k(\cdot,0,\cdot)\|_{H^1(\Gamma_0)}+\|H_3(\cdot,0,\cdot)\|_{H^1(\Gamma_0)}\right)^{\frac{\delta}{3\alpha-2\delta}}.$$

Corollary 3. In addition to the assumptions in Theorem 1.2, let

$$\varepsilon(x',t)=\begin{pmatrix}\varepsilon_1(x',t)&0&0\\0&\varepsilon_2(x',t)&0\\0&0&\varepsilon_3(x',t)\end{pmatrix},$$

and

$$\mu(x',t)=\begin{pmatrix}\mu_1(x',t)&0&0\\0&\mu_2(x',t)&0\\0&0&\mu_3(x',t)\end{pmatrix}.$$

Furthermore assume that $|\partial_1\mu_3(x',t)|,|\partial_2\mu_3(x',t)|\le M$ for all $(x',t)\in\overline{\Gamma_0}$. Then (1.10) is replaced by

$$\|F\|_{L^2(\Gamma_\delta)} \le C\left(\|E\|_{H^2(Q_0)}+\|H\|_{H^2(Q_0)}\right)^{\frac{3\alpha-3\delta}{3\alpha-2\delta}}\left(\sum_{k=1}^{3}\|E_k(\cdot,0,\cdot)\|_{H^2(\Gamma_0)}+\sum_{k=1}^{2}\|H_k(\cdot,0,\cdot)\|_{H^2(\Gamma_0)}\right)^{\frac{\delta}{3\alpha-2\delta}}.$$

Theorem 1.1 will be proved in Section 2 and Theorem 1.2 in Section 3. The corollaries will be proved in Section 4.

2 Proof of Theorem 1.1

Let

$$P=A\partial_t+\sum_{k=1}^{3}A_k\partial_k \tag{2.1}$$

where

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}, \quad A_k = \begin{pmatrix} 0 & \Lambda_k \\ -\Lambda_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is obvious that $A^T = A$ and $A_k^T = A_k$ $(k = 1, 2, 3)$. Moreover, for any $\widetilde{U} = (\widetilde{u}_1, \ldots, \widetilde{u}_6)^T$, $\widetilde{W} = (\widetilde{w}_1, \ldots, \widetilde{w}_6)^T$ and any constant $s \geq 0$, we set

$$\|\widetilde{U}\|^2 = \int_{Q_0} \left|\widetilde{U}\right|^2 \mathrm{d}x\mathrm{d}t, \quad \|\widetilde{U}\|_s^2 = \int_{Q_0} \left|\widetilde{U}\right|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t,$$

$$\left\langle \widetilde{U}, \widetilde{W} \right\rangle = \int_{Q_0} \left(\sum_{k=1}^{6} \widetilde{u}_k \widetilde{w}_k \right) \mathrm{d}x\mathrm{d}t.$$

Proof of Theorem 1.1. By means of the mollifier and Friedrich's lemma (e.g., §1 of Chapter 17 (p.9) in Hörmander [10]), a usual density argument enables us to assume that $U,\ V \in C_0^\infty(Q_0)$ in place of regularity (1.8). Let $W = (w_1, \ldots, w_6)^T = (U,\ V)^T$. Noting (2.1), we can see that

$$PW + (\partial_t A)\, W = (\partial_t (\varepsilon U) - \nabla \times V,\ \partial_t (\mu V) + \nabla \times U)^T$$

and

$$\begin{aligned} &\int_{Q_0} \left(|\partial_t (\varepsilon U) - \nabla \times V|^2 + |\partial_t (\mu V) + \nabla \times U|^2 \right) \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ &= \int_{Q_0} |PW + (\partial_t A)\, W|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t = \|PW + (\partial_t A)\, W\|_s^2. \end{aligned} \tag{2.2}$$

The proof is inspired by Bukhgeim [5] which established a Carleman estimate for the Schödinger equation. However non-commutativity of the matrices requires us special cares.

Let $Y = (y_1, \ldots, y_6)^T = \mathrm{e}^{s\varphi} W$ and $L = \mathrm{e}^{s\varphi} P \mathrm{e}^{-s\varphi}$. It is easy to see that

$$LY = A\partial_t Y + \sum_{k=1}^{3} A_k \partial_k Y - s\left((\partial_t \varphi) A + \sum_{k=1}^{3} (\partial_k \varphi) A_k \right) Y$$

and

$$\int_{Q_0} \left(|U|^2 + |V|^2\right) \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t = \|W\|_s^2 = \|Y\|^2, \quad \|PW\|_s^2 = \|LY\|^2. \tag{2.3}$$

Then

$$\begin{aligned}
&\|LY\|^2 = \langle LY, LY \rangle \\
&= \langle LY - Y, LY - Y \rangle + 2\langle LY, Y \rangle - \langle Y, Y \rangle \\
&\geq 2\langle LY, Y \rangle - \langle Y, Y \rangle \\
&= 2\left\langle A\partial_t Y + \sum_{k=1}^{3} A_k \partial_k Y, Y \right\rangle + 2s\left\langle -\left((\partial_t \varphi) A + \sum_{k=1}^{3} (\partial_k \varphi) A_k \right) Y, Y \right\rangle - \langle Y, Y \rangle.
\end{aligned} \tag{2.4}$$

By the symmetry of A and A_k, $k = 1, 2, 3$ and $Y\mid_{\partial Q_0} = 0$, it follows that

$$\begin{aligned}
2\left\langle A\partial_t Y + \sum_{k=1}^{3} A_k \partial_k Y, Y \right\rangle &= \int_{Q_0} \left(\partial_t \left(Y^T A Y\right) + \sum_{k=1}^{3} \partial_k \left(Y^T A_k Y\right) - Y^T (\partial_t A) Y \right) \mathrm{d}x\mathrm{d}t \\
&= -\int_{Q_0} Y^T (\partial_t A) Y \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Then there exists a constant $C_1 = C_1(M)$ such that

$$\left| 2\left\langle A\partial_t Y + \sum_{k=1}^{3} A_k \partial_k Y, Y \right\rangle \right| \leq C_1 \int_{Q_0} |Y|^2 \mathrm{d}x\mathrm{d}t = C_1 \langle Y, Y \rangle. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\|LY\|^2 \geq 2s\left\langle -\left((\partial_t \varphi) A + \sum_{k=1}^{3} (\partial_k \varphi) A_k \right) Y, Y \right\rangle - (1 + C_1)\langle Y, Y \rangle. \tag{2.6}$$

Noting the definition (1.5) of φ, (1.2), $|ab| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and $\beta|x|^2 < \alpha - t$ in Q_0, we can see that

$$\langle -(\partial_t \varphi) AY, Y \rangle = \langle AY, Y \rangle \geq h \int_{Q_0} |Y|^2 \mathrm{d}x\mathrm{d}t = h\langle Y, Y \rangle$$

and

$$\begin{aligned}
&\left|\left\langle -\left(\sum_{k=1}^{3}(\partial_k\varphi)\,A_k\right)Y,Y\right\rangle\right| \\
&= \left|4\beta\int_{Q_0}(x_3y_1y_5 - x_2y_1y_6 - x_3y_2y_4 + x_1y_2y_6 + x_2y_3y_4 - x_1y_3y_5)\,\mathrm{d}x\mathrm{d}t\right| \\
&\le 4\beta\int_{Q_0}|x|\left(|y_1y_5| + |y_1y_6| + |y_2y_4| + |y_2y_6| + |y_3y_4| + |y_3y_5|\right)\mathrm{d}x\mathrm{d}t \\
&\le 4\beta\int_{Q_0}|x||Y|^2\mathrm{d}x\mathrm{d}t \\
&\le 4\sqrt{\alpha\beta}\int_{Q_0}|Y|^2\mathrm{d}x\mathrm{d}t \\
&= 4\sqrt{\alpha\beta}\,\langle Y,Y\rangle.
\end{aligned}$$

Then, by $0<\beta<\frac{h^2}{16\alpha}$, it follows that

$$\left\langle -\left((\partial_t\varphi)\,A + \sum_{k=1}^{3}(\partial_k\varphi)\,A_k\right)Y,Y\right\rangle \ge \left(h - 4\sqrt{\alpha\beta}\right)\|Y\|^2 > 0. \tag{2.7}$$

Using (2.3), (2.6) and (2.7), we can see that if $s_0>1$ is large enough, then there exists a constant $C_2= C_2(s_0,\ \alpha,\ \beta,\ h,\ M)> 0$ such that

$$s\|W\|_s^2 \le C_2\|PW\|_s^2$$

holds for all $s>s_0$. Then, noting that there exists a constant $C_3 = C_3(M)$ such that

$$\|PW\|_s^2 \le \left(\|PW + (\partial_tA)\,W\|_s + \|-(\partial_tA)\,W\|_s\right)^2 \le 2\|PW + (\partial_tA)\,W\|_s^2 + C_3\|W\|_s^2,$$

we obtain that if $s_0>1$ is large enough, then there exists a constant $C= C(s_0,\ \alpha,\ \beta,\ h,\ M)> 0$ such that

$$s\|W\|_s^2 \le C\|PW + (\partial_tA)\,W\|_s^2 \tag{2.8}$$

holds for all $s>s_0$. Thus the proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

First we set

$$Q_\delta = \left\{(x,t) \in \mathbb{R}^4 \,|\varphi(x,t) > \delta, \quad t > 0\right\}$$

and

$$\widetilde{Q} \equiv \left\{(x,t) \in \mathbb{R}^4 \,\middle|\, 0 < t < \alpha - \beta|x'|^2, \quad -\sqrt{\frac{\alpha}{\beta}} < x_3 < \sqrt{\frac{\alpha}{\beta}}\right\},$$

and recall

$$\Gamma_\delta = \left\{\left(x', t\right) \in \mathbb{R}^3 \,|\varphi(x', 0, t) > \delta, \quad t > 0\right\}.$$

For any given $0 < \delta < \alpha$, we set $\delta_1 = \frac{\delta}{3}$. We can easily verify that

$$Q_{2\delta_1} \subset Q_{\delta_1} \subset Q_0 \subset \widetilde{Q}, \qquad \Gamma_\delta = \Gamma_{3\delta_1} \subset \Gamma_{2\delta_1} \subset \Gamma_0. \tag{3.1}$$

The proof is an adjustment of the argument in [11, 12].

Proof of Theorem 1.2. In order to apply Theorem 1.1, we introduce a cut off function χ satisfying $0 \le \chi \le 1$, $\chi \in C^\infty\left(\mathbb{R}^4\right)$ and

$$\chi(x,t) = \begin{cases} 0, & (x,t) \in \widetilde{Q} \setminus Q_{\delta_1}, \\ 1, & (x,t) \in Q_{2\delta_1}. \end{cases}$$

For $s > s_0$, we set

$$U = (u_1, \ldots, u_6)^T = (E,\ H)^T \in \left(C^1\left(\overline{Q_0}\right)\right)^6 \cap \left(H^2\left(Q_0\right)\right)^6$$

and

$$Z = (z_1, \ldots, z_6)^T = \chi \mathrm{e}^{s\varphi} \partial_3 U \in \left(C\left(\overline{\widetilde{Q}}\right)\right)^6 \cap \left(H^1\left(\widetilde{Q}\right)\right)^6.$$

Then by (3.1) and the definition of χ, we can obtain

$$\begin{aligned}
\int_{\Gamma_0} |Z(x',0,t)|^2 \mathrm{d}x'\mathrm{d}t &= -\int_0^{\sqrt{\frac{\alpha}{\beta}}} \mathrm{d}x_3 \left(\partial_3 \int_{\Gamma_0} |Z(x,t)|^2 \mathrm{d}x'\mathrm{d}t\right) \\
&= -\int_0^{\sqrt{\frac{\alpha}{\beta}}} \int_{\Gamma_0} 2(\partial_3 Z \cdot Z)\mathrm{d}x\mathrm{d}t \\
&\leq \int_{\widetilde{Q}\bigcap\{x_3>0\}} \left(\frac{1}{s}|\partial_3 Z|^2 + s|Z|^2\right)\mathrm{d}x\mathrm{d}t \\
&\leq \int_{\widetilde{Q}} \left(\frac{1}{s}|\partial_3 Z|^2 + s|Z|^2\right)\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_0} \left(\frac{1}{s}|\partial_3 Z|^2 + s|Z|^2\right)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.2}$$

It is obvious that

$$\partial_3 Z = \chi \mathrm{e}^{s\varphi}\partial_3^2 U + (\partial_3\chi)\,\mathrm{e}^{s\varphi}\partial_3 U + s\,(\partial_3\varphi)\,\chi\mathrm{e}^{s\varphi}\partial_3 U. \tag{3.3}$$

Let us set $W^{(1)} \equiv \chi\partial_3 U \in \left(C\left(\overline{Q_0}\right)\right)^6 \cap \left(H^1\left(Q_0\right)\right)^6$, $\;W^{(2)} \equiv \chi\partial_3^2 U \in \left(L^2\left(Q_0\right)\right)^6$, and

$$G = G(x,t) = \begin{pmatrix} -R(x,t) \\ \\ 0 \end{pmatrix} : \quad 6\times 3 \quad \text{matrix.}$$

Then in term of (2.1), we can rewrite (1.1) as

$$PU + (\partial_t A)\,U = GF.$$

Therefore direct calculations yield

$$PW^{(1)} + (\partial_t A)\,W^{(1)} = \chi\,(\partial_3 G)\,F + \left((\partial_t\chi)\,A + \sum_{k=1}^{3}(\partial_k\chi)\,A_k\right)\partial_3 U \in \left(L^2\left(Q_0\right)\right)^6, \tag{3.4}$$

$$PW^{(2)} + (\partial_t A)\,W^{(2)} = \chi\left(\partial_3^2 G\right)F + \left((\partial_t\chi)\,A + \sum_{k=1}^{3}(\partial_k\chi)\,A_k\right)\partial_3^2 U \in \left(L^2\left(Q_0\right)\right)^6 \tag{3.5}$$

Noting that $E(x,0) = H(x,0) = 0$ and the definition of χ, we see that $W^{(1)}\,|_{\partial Q_0} = W^{(2)}\,|_{\partial Q_0} = 0$. Hence taking $s>0$ sufficiently large, we can apply Theorem 1.1 to obtain

$$s\int_{Q_0} |\chi\partial_3^2 U|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \leq C\int_{Q_0}\left|\chi\left(\partial_3^2 G\right)F + \left((\partial_t\chi)\,A + \sum_{k=1}^{3}(\partial_k\chi)\,A_k\right)\partial_3^2 U\right|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t, \tag{3.6}$$

$$s\int_{Q_0}|\chi\partial_3 U|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \le C\int_{Q_0}\left|\chi\left(\partial_3 G\right)F+\left(\left(\partial_t\chi\right)A+\sum_{k=1}^{3}\left(\partial_k\chi\right)A_k\right)\partial_3 U\right|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t. \quad (3.7)$$

Here and henceforth $C>0$ denotes generic constants which are dependent on s_0, α, β, h, M, r_0, χ and δ, but independent of $s>s_0$. By (3.2), (3.3), (3.6) and (3.7), we obtain that

$$\begin{aligned}
&\int_{\Gamma_0}|Z\left(x',0,t\right)|^2\mathrm{d}x'\mathrm{d}t\\
\le\ & C\left(\frac{1}{s}\int_{Q_0}\left|\chi e^{s\varphi}\partial_3^2 U\right|^2\mathrm{d}x\mathrm{d}t+\frac{1}{s}\int_{Q_0}\left|\left(\partial_3\chi\right)e^{s\varphi}\partial_3 U\right|^2\mathrm{d}x\mathrm{d}t+s\int_{Q_0}\left|\chi e^{s\varphi}\partial_3 U\right|^2\mathrm{d}x\mathrm{d}t\right)\\
\le\ & C\left(\int_{Q_0}\left|\chi\left(\partial_3^2 G\right)F+\left(\left(\partial_t\chi\right)A+\sum_{k=1}^{3}\left(\partial_k\chi\right)A_k\right)\partial_3^2 U\right|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\right.\\
&+\int_{Q_0}\left|\chi\left(\partial_3 G\right)F+\left(\left(\partial_t\chi\right)A+\sum_{k=1}^{3}\left(\partial_k\chi\right)A_k\right)\partial_3 U\right|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&\left.+\int_{Q_0}\left|\left(\partial_3\chi\right)e^{s\varphi}\partial_3 U\right|^2\mathrm{d}x\mathrm{d}t\right).
\end{aligned} \quad (3.8)$$

Then noting (3.1), (1.9), (3.4), (3.5) and (3.8), we obtain

$$\begin{aligned}
&\int_{\Gamma_0}|Z\left(x',0,t\right)|^2\mathrm{d}x'\mathrm{d}t\\
\le\ & C\int_{Q_0}e^{2s\varphi}\left(\left|\chi\left(\partial_3 G\right)F\right|^2+\left|\chi\left(\partial_3^2 G\right)F\right|^2\right)\mathrm{d}x\mathrm{d}t\\
&+Ce^{4s\delta_1}\int_{Q_{\delta_1}\setminus Q_{2\delta_1}}\left\{\left|\left(\partial_3\chi\right)\partial_3 U\right|^2+\left|\left(\left(\partial_t\chi\right)A+\sum_{k=1}^{3}\left(\partial_k\chi\right)A_k\right)\partial_3^2 U\right|^2\right.\\
&\left.+\left|\left(\left(\partial_t\chi\right)A+\sum_{k=1}^{3}\left(\partial_k\chi\right)A_k\right)\partial_3 U\right|^2\right\}\mathrm{d}x\mathrm{d}t \qquad (3.9)\\
\le\ & C\int_{Q_0}\left|F\left(x',t\right)\right|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t+Ce^{4s\delta_1}\Phi\\
\le\ & C\int_{Q_{2\delta_1}}\left|R\left(x',0,t\right)F\left(x',t\right)\right|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+Ce^{4s\delta_1}\int_{Q_{\delta_1}\setminus Q_{2\delta_1}}\left|F\left(x',t\right)\right|^2\mathrm{d}x\mathrm{d}t+Ce^{4s\delta_1}\Phi.
\end{aligned}$$

Here and henceforth we set

$$\Phi = \|E\|_{H^2(Q_0)}^2 + \|H\|_{H^2(Q_0)}^2. \tag{3.10}$$

Moreover, by (1.1) and the Sobolev embedding theorem (e.g., [1]), we have

$$\begin{aligned} & \int_{Q_{\delta_1} \setminus Q_{2\delta_1}} \left| F\left(x', t\right) \right|^2 \mathrm{d}x\mathrm{d}t \\ \leq\ & C \int_{Q_0} \left| R\left(x', 0, t\right) F\left(x', t\right) \right|^2 \mathrm{d}x\mathrm{d}t \leq C \int_{\Gamma_0} \left| R\left(x', 0, t\right) F\left(x', t\right) \right|^2 \mathrm{d}x'\mathrm{d}t \\ =\ & C \int_{\Gamma_0} \left| \left(-\partial_t\left(\varepsilon E\right) + \nabla \times H\right)\left(x', 0, t\right) \right|^2 \mathrm{d}x'\mathrm{d}t \\ \leq\ & C\Phi. \end{aligned} \tag{3.11}$$

It follows from (3.9) and (3.11) that

$$\int_{\Gamma_0} |Z\left(x', 0, t\right)|^2 \mathrm{d}x'\mathrm{d}t \leq C \int_{Q_{2\delta_1}} \left| R\left(x', 0, t\right) F\left(x', t\right) \right|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t + C\mathrm{e}^{4s\delta_1}\Phi. \tag{3.12}$$

Furthermore, as $s \longrightarrow \infty$, we have

$$\begin{aligned} & \int_{Q_{2\delta_1}} \left| R\left(x', 0, t\right) F\left(x', t\right) \right|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ & \leq C \int_{-\sqrt{\frac{\alpha - 2\delta_1}{\beta}}}^{\sqrt{\frac{\alpha - 2\delta_1}{\beta}}} \mathrm{e}^{-2s\beta x_3^2} \mathrm{d}x_3 \int_{\widetilde{\Gamma}_{2\delta_1}(x_3)} \left| R\left(x', 0, t\right) F\left(x', t\right) \right|^2 \mathrm{e}^{2s\varphi(x', 0, t)} \mathrm{d}x'\mathrm{d}t \\ & \leq C \int_{\Gamma_{2\delta_1}} \left| R\left(x', 0, t\right) F\left(x', t\right) \right|^2 \mathrm{e}^{2s\varphi(x', 0, t)} \mathrm{d}x'\mathrm{d}t \int_{-\sqrt{\frac{\alpha}{\beta}}}^{\sqrt{\frac{\alpha}{\beta}}} \mathrm{e}^{-2s\beta x_3^2} \mathrm{d}x_3 \\ & = O\left(\frac{1}{\sqrt{s}}\right) \int_{\Gamma_{2\delta_1}} \left| R\left(x', 0, t\right) F\left(x', t\right) \right|^2 \mathrm{e}^{2s\varphi(x', 0, t)} \mathrm{d}x'\mathrm{d}t, \end{aligned} \tag{3.13}$$

where

$$\widetilde{\Gamma}_{2\delta_1}(x_3) \equiv \left\{ \left(x', t\right) \in \mathbb{R}^3 \left| 0 < t < \alpha - 2\delta_1 - \beta\left(|x'|^2 + x_3^2\right) \right. \right\} \subset \widetilde{\Gamma}_{2\delta_1}(0) = \Gamma_{2\delta_1}.$$

Then by (3.12) and (3.13), we have

$$\begin{aligned}&\int_{\Gamma_0} |Z\left(x',0,t\right)|^2 \mathrm{d}x'\mathrm{d}t \\ &\leq O\left(\frac{1}{\sqrt{s}}\right)\int_{\Gamma_{2\delta_1}} \left|R\left(x',0,t\right)F\left(x',t\right)\right|^2 \mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t + C\mathrm{e}^{4s\delta_1}\Phi\end{aligned} \tag{3.14}$$

as $s \longrightarrow \infty$.

On the other hand, again by (1.1), we have

$$\begin{aligned}&\int_{\Gamma_{2\delta_1}} \left|R\left(x',0,t\right)F\left(x',t\right)\right|^2 \mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\ &\leq \int_{\Gamma_0} \left|\chi\left(x',0,t\right)R\left(x',0,t\right)F\left(x',t\right)\right|^2 \mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\ &= \int_{\Gamma_0} \left|\left(-\chi\partial_t\left(\varepsilon E\right)+\chi\nabla\times H\right)\left(x',0,t\right)\right|^2 \mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\ &\leq C\int_{\Gamma_0} \left\{|\left(\chi\left(\partial_3 u_5\right)\right)\left(x',0,t\right)|^2 + |\left(\chi\left(\partial_3 u_4\right)\right)\left(x',0,t\right)|^2\right\} \mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t + C\mathrm{e}^{2s\alpha}\Psi \\ &\leq C\int_{\Gamma_0} |Z\left(x',0,t\right)|^2 \mathrm{d}x'\mathrm{d}t + C\mathrm{e}^{2s\alpha}\Psi.\end{aligned} \tag{3.15}$$

Here and henceforth we set

$$\Psi = \|E(\cdot,0,\cdot)\|^2_{H^1(\Gamma_0)} + \|H(\cdot,0,\cdot)\|^2_{H^1(\Gamma_0)}. \tag{3.16}$$

By (3.14) and (3.15), we obtain

$$\begin{aligned}&\left(1-O\left(\frac{1}{\sqrt{s}}\right)\right)\int_{\Gamma_{2\delta_1}} \left|R\left(x',0,t\right)F\left(x',t\right)\right|^2 \mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\ &\leq C\mathrm{e}^{4s\delta_1}\Phi + C\mathrm{e}^{2s\alpha}\Psi\end{aligned}$$

as $s \longrightarrow \infty$. Therefore

$$\begin{aligned}
&\int_{\Gamma_{3\delta_1}} \left|R\left(x',0,t\right)F\left(x',t\right)\right|^2 \mathrm{e}^{6s\delta_1}\mathrm{d}x'\mathrm{d}t \\
&\le \int_{\Gamma_{3\delta_1}} \left|R\left(x',0,t\right)F\left(x',t\right)\right|^2 \mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\
&\le \int_{\Gamma_{2\delta_1}} \left|R\left(x',0,t\right)F\left(x',t\right)\right|^2 \mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\
&\le C\mathrm{e}^{4s\delta_1}\Phi + C\mathrm{e}^{2s\alpha}\Psi
\end{aligned}$$

as $s \longrightarrow \infty$. Then, noting (1.9), we have

$$\begin{aligned}
&\|F\|^2_{L^2\left(\Gamma_{3\delta_1}\right)} \\
&\le C\int_{\Gamma_{3\delta_1}} \left|R\left(x',0,t\right)F\left(x',t\right)\right|^2 \mathrm{d}x'\mathrm{d}t \\
&\le C\left(\mathrm{e}^{-2s\delta_1}\Phi + \mathrm{e}^{2s(\alpha-3\delta_1)}\Psi\right)
\end{aligned} \tag{3.17}$$

as $s \longrightarrow \infty$.

Let $\Psi = 0$. Then, letting $s \longrightarrow \infty$ in (3.17), we see $F\left(x',t\right) = 0$ for $\left(x',t\right) \in \Gamma_{3\delta_1}$. Hence (1.10) is true.

Next let $\Psi \neq 0$, without loss of generality, we may assume that Φ is sufficiently large, so that

$$\frac{\Phi}{\Psi} > 1.$$

We take

$$s \ge \max\left\{\frac{1}{2(\alpha-2\delta_1)}\log\frac{\Phi}{\Psi}, \quad s_0+1\right\}$$

large and fix it. Then we obtain

$$\|F\|^2_{L^2\left(\Gamma_{3\delta_1}\right)} \le C\Phi^{\frac{\alpha-3\delta_1}{\alpha-2\delta_1}}\Psi^{\frac{\delta_1}{\alpha-2\delta_1}}.$$

Noting $\delta_1 = \frac{\delta}{3}$ and the definitions (3.10) and (3.16) of Φ and Ψ, we obtain (1.10). We have finished the proof of Theorem 1.2.

4 Proof of Corollaries

Up to estimate (3.14) of $Z(x', 0, t)$, the proofs are same. According the forms of ε, μ, R, F in the corollaries, the calculations in (3.15) are changed. We will prove only Corollary 1 because the proofs of Corollaries 2 and 3 are very similar.

By (1.1), we have

$$\begin{aligned} &\partial_t (\varepsilon_1 u_1) - \partial_2 u_6 + \partial_3 u_5 = -f r_1, \\ &\partial_t (\varepsilon_2 u_2) + \partial_1 u_6 - \partial_3 u_4 = -f r_2, \\ &\partial_t (\mu_3 u_6) + \partial_1 u_2 - \partial_2 u_1 = 0. \end{aligned} \tag{4.1}$$

We set

$$a\left(x', t\right) = E_1(x', 0, t), \qquad b\left(x', t\right) = E_2(x', 0, t), \qquad \left(x', t\right) \in \Gamma_0.$$

Noting that $u_6(x', 0, 0) = 0$ and (1.2), it follows from (4.1) that

$$\begin{aligned} u_6(x', 0, t) &= \frac{1}{\mu_3(x', t)} \int_0^t \frac{\partial}{\partial \tau} \left(\mu_3\left(x', \tau\right) u_6\left(x', 0, \tau\right)\right) \mathrm{d}\tau \\ &= \frac{1}{\mu_3(x', t)} \int_0^t (\partial_2 u_1 - \partial_1 u_2)\left(x', 0, \tau\right) \mathrm{d}\tau \\ &= \frac{1}{\mu_3(x', t)} \int_0^t (\partial_2 a - \partial_1 b)\left(x', \tau\right) \mathrm{d}\tau \end{aligned}$$

and then

$$\begin{aligned} \partial_1 u_6(x', 0, t) = & \frac{1}{\mu_3(x', t)} \int_0^t \left(\partial_1 \partial_2 a - \partial_1^2 b\right)\left(x', \tau\right) \mathrm{d}\tau \\ & - \frac{\partial_1 \mu_3(x', t)}{\mu_3^2(x', t)} \int_0^t (\partial_2 a - \partial_1 b)\left(x', \tau\right) \mathrm{d}\tau, \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
\partial_2 u_6\left(x',0,t\right) \;=\; & \frac{1}{\mu_3\left(x',t\right)}\int_0^t\left(\partial_2^2 a-\partial_2\partial_1 b\right)\left(x',\tau\right)\mathrm{d}\tau \\
& -\frac{\partial_2\mu_3\left(x',t\right)}{\mu_3^2\left(x',t\right)}\int_0^t\left(\partial_2 a-\partial_1 b\right)\left(x',\tau\right)\mathrm{d}\tau.
\end{aligned}
\tag{4.3}
$$

Therefore, by (4.1)-(4.3), we obtain

$$
\begin{aligned}
& f\left(x',t\right) r_1\left(x',0,t\right) \\
& = -\partial_t\left(\varepsilon_1\left(x',t\right) u_1\left(x',0,t\right)\right)+\partial_2 u_6\left(x',0,t\right)-\partial_3 u_5\left(x',0,t\right) \\
& = -\partial_t\left(\varepsilon_1 a\right)\left(x',t\right)-\partial_3 u_5\left(x',0,t\right) \\
& +\frac{1}{\mu_3\left(x',t\right)}\int_0^t\left(\partial_2^2 a-\partial_2\partial_1 b\right)\left(x',\tau\right)\mathrm{d}\tau-\frac{\partial_2\mu_3\left(x',t\right)}{\mu_3^2\left(x',t\right)}\int_0^t\left(\partial_2 a-\partial_1 b\right)\left(x',\tau\right)\mathrm{d}\tau
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
& f\left(x',t\right) r_2\left(x',0,t\right) \\
& = -\partial_t\left(\varepsilon_2 b\right)\left(x',t\right)+\partial_3 u_4\left(x',0,t\right) \\
& -\frac{1}{\mu_3\left(x',t\right)}\int_0^t\left(\partial_1\partial_2 a-\partial_1^2 b\right)\left(x',\tau\right)\mathrm{d}\tau+\frac{\partial_1\mu_3\left(x',t\right)}{\mu_3^2\left(x',t\right)}\int_0^t\left(\partial_2 a-\partial_1 b\right)\left(x',\tau\right)\mathrm{d}\tau.
\end{aligned}
\tag{4.5}
$$

It follows from (4.4) and (4.5) that

$$
\begin{aligned}
& \int_{\Gamma_{2\delta_1}}|f\left(x',t\right)|^2|R\left(x',0,t\right)|^2\mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\
& \leq C\int_{\Gamma_0}|\chi\left(x',0,t\right) f\left(x',t\right)|^2\left(|r_1\left(x',0,t\right)|^2+|r_2\left(x',0,t\right)|^2\right)\mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\
& \leq C\int_{\Gamma_0}\left\{|\chi\left(x',0,t\right)\partial_3 u_5\left(x',0,t\right)|^2+|\chi\left(x',0,t\right)\partial_3 u_4\left(x',0,t\right)|^2\right\}\mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\
& +C\int_{\Gamma_0}|\chi\left(x',0,t\right)|^2\left\{\left(|\varepsilon_1\partial_t a|^2+|\varepsilon_2\partial_t b|^2\right)\left(x',t\right)+\left(|a\partial_t\varepsilon_1|^2+|b\partial_t\varepsilon_2|^2\right)\left(x',t\right)\right. \\
& +\frac{1}{\mu_3^2\left(x',t\right)}\left|\int_0^t\left(\partial_2^2 a-\partial_2\partial_1 b\right)\left(x',\tau\right)\mathrm{d}\tau\right|^2+\frac{1}{\mu_3^2\left(x',t\right)}\left|\int_0^t\left(\partial_1\partial_2 a-\partial_1^2 b\right)\left(x',\tau\right)\mathrm{d}\tau\right|^2 \\
& \left.+\;\frac{|\partial_1\mu_3\left(x',t\right)|^2+|\partial_2\mu_3\left(x',t\right)|^2}{\mu_3^4\left(x',t\right)}\left|\int_0^t\left(\partial_2 a-\partial_1 b\right)\left(x',\tau\right)\mathrm{d}\tau\right|^2\right\}\mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\
& \leq C\int_{\Gamma_0}|Z\left(x',0,t\right)|^2\mathrm{d}x'\mathrm{d}t+C\mathrm{e}^{2s\alpha}\Psi_1.
\end{aligned}
\tag{4.6}
$$

Here and henceforth we set

$$\Psi_1 = \|a\|^2_{H^2(\Gamma_0)} + \|b\|^2_{H^2(\Gamma_0)} = \|E_1(\cdot,0,\cdot)\|^2_{H^2(\Gamma_0)} + \|E_2(\cdot,0,\cdot)\|^2_{H^2(\Gamma_0)}.$$

Therefore, in terms of (3.14) and (4.6), we obtain

$$\begin{aligned}&\left(1 - O\left(\frac{1}{\sqrt{s}}\right)\right)\int_{\Gamma_{2\delta_1}} |f(x',t)|^2 \left(|r_1(x',0,t)|^2 + |r_2(x',0,t)|^2\right) \mathrm{e}^{2s\varphi(x',0,t)}\mathrm{d}x'\mathrm{d}t \\ &\le C\mathrm{e}^{4s\delta_1}\Phi + C\mathrm{e}^{2s\alpha}\Psi_1\end{aligned}$$

as $s \longrightarrow \infty$. Consequently we can proceed in the same way in Section 3, so that the proof of Corollary 1 is complete.

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