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The a-number stratification on the moduli space of supersingular abelian varieties

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Abstract

We study the moduli space $S_g(a)$ of principally polarized supersingular abelian varieties of dimension g with a-number a. We determine the dimension of each irreducible component of $S_g(a)$ and the number of irreducible components.

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1 Introduction

Let S_g be the moduli space of principally polarized supersingular abelian varieties over fields of characteristic $p > 0$. Recall the structure of S_g was investigated by K.-Z. Li and F. Oort in [2] rather comprehensively, improving the former results. Among others, the dimension and the number of irreducible components of S_g over the algebraically closed field $\overline{\mathbb{F}}_p$ of prime field $\mathbb{F}_p$ were calculated. Also we note that by a completely different method F. Oort had more enhanced results in [6] on the loci defined by Newton Polygon. The moduli space S_g has various stratifications: for example, the Ekedahl-Oort stratification $S_g \cap S_\varphi$ (see [5]) with elementary series φ, the stratification $S_g(a)$ by a-number defined below and the stratification $S_{g,s}$ by index introduced by K.-Z. Li ([1, p.337], also see Definition 4.4). As a matter of fact, the a-number stratification is essentially a special case of the Ekedahl-Oort stratification, i.e., the Zariski closure of $S_g(a)$ is the Zariski closure of $S_g \cap S_\varphi$ with $\varphi = (1, 2, \cdots, g-a, \cdots, g-a)$.

The main subject in this paper is to investigate the a-number stratification on S_g. In the calculation of the number of irreducible components of $S_g(a)$, we also use the stratification by index. There we shall show that different irreducible components of $S_g(a)$ have generic elements with different indices.

Given an abelian variety X over a perfect field K, we define an absolute invariant $a(X)$ of X, called a-number by

$$a(X) = \dim_K \operatorname{Hom}(\alpha_p, X),$$

where α_p is the kernel of the Frobenius map $F : \mathbb{G}_a \to \mathbb{G}_a$. The a-number stratum $S_g(a)$ is defined as a locally closed subscheme of S_g which has closed points

$$S_g(a)(K) = \{(X, \eta) \in S_g(K) \mid a(X) = a\}$$

for any perfect field K.

The main results proved in this paper are:

0. For the Zariski closure $S_g^c(a)$ of $S_g(a)$, we have $S_g^c(a) = \underset{a' \geq a}{\cup} S_g(a')$ and $S_g^c(a)$ is connected unless $a = g$;

1. The dimension of any irreducible component of $S_g(a)$ is equal to
$$\left[\frac{g^2 - a^2 + 1}{4}\right];$$

2. The number of irreducible components of $S_g(a)$ is equal to
$$\begin{cases} \binom{(g-2)/2}{(g-a-1)/2} H_g(1,p) & \text{for } g \text{ even and } a \text{ odd,} \\ \binom{(g-1)/2}{(g-a)/2} H_g(p,1) & \text{for } g, a \text{ odd,} \\ \binom{g/2-1}{(g-a)/2} H_g(p,1) + \binom{g/2-1}{(g-a)/2-1} H_g(1,p) & \text{for } g, a \text{ even,} \\ \binom{(g-1)/2-1}{(g-a-1)/2} H_g(1,p) + \binom{(g-1)/2-1}{(g-a-1)/2-1} H_g(p,1) & \text{for } g \text{ odd and } a \text{ even,} \end{cases}$$
where $H_g(p,1)$ and $H_g(1,p)$ are the class numbers of quaternion unitary groups (see [2, 4.6] and also Theorem 4.15). Here we remark that if you consider the moduli space $S_g(a)$ as a stack or the moduli space with level n structure ($n \geq 3$), the number of irreducible components is also computed by the similar formula obtained by replacing the class numbers by masses of the same quaternion unitary groups, which are explicitly calculated by mass formula.

This is a generalization of results in [2] that $S_g^c(2)$ is a divisor of S_g (see [2, Cor. 10.3]), $S_g^c(g-1)$ has dimension $[g/2]$, the number of irreducible components of $S_g^c(g-1)$ is given by the class number $H_g(1,p)$ ([2, Prop. 9.11]), and the number of irreducible components of $S_4^c(2)$ equals $H_4(p,1) + H_4(1,p)$ ([2, 9.9]).

Let us explain the outline of this paper. We start with some preliminaries on supersingular abelian varieties and Dieudonné modules in Section 2. The following Section 3 is crucial to describe each irreducible component of $S_g(a)$. After reviewing the theory of K.-Z. Li and F. Oort ([2, Section 7]), we introduce a new ingredient, i.e., "good basis" for each principally quasi-polarized supersingular Dieudonné module. Then we obtain a beautiful symmetry among coefficients which determine the actions of Frobenius and Verschiebung on such bases. These coefficients make up a parameter space which is called "period space" in this paper. Then the dimension of $S_g(a)$ is immediately calculated. Moreover we can look into the configulation of a-number stratification.

The calculation of the number of irreducible components of $S_g(a)$ is a more difficult problem. Section 4 is devoted to this. Although each subscheme of the moduli space of rigid PFTQs definded in Section 3 gives an irreducible component of the moduli space of principally quasi-polarized supersingular Dieudonné modules, it is necessary to show that different subschemes give different irreducible components. For this, we will make use of another invariant - Li's index, which can be calculated for generic elements of each virtual irreducible component.

2 Preliminaries

2.1 Dieudonné modules of supersingular abelian varieties

We fix a rational prime p and for all throughout this paper. Let K be a perfect field of characteristic p. We set

$$A_K = W(K)[F, V]/(FV - p, VF - p, Fa - a^\sigma F, Va - a^{\sigma^{-1}}V, \forall a \in W(K)).$$

Here σ is the Frobenius map on K. We denote by A the p-adic completion of A_K.

Definition 2.1. *A Dieudonné module* is a left A-module M finitely generated as $W(K)$-module. If M is free as $W(K)$-module, we call M free. Two free Dieudonné module M and N are said to be isogenous if there is an A-homomorphism from M to N with torsion cokernel. We define a-number of M as

$$a(M) = \dim_K M/(F, V)M.$$

A free Dieudonné module M is called *supersingular* (resp. *superspecial*) if M is isogenous (resp. isomorphic) to $A_{1,1}^{\oplus g}$ for some g. Here $A_{1,1} := A/(F - V)$ and g is called the genus of M.

Definition 2.2. (1) Assume $g \geq 2$. *A superspecial abelian variety over* K is an abelian variety Y over K such that there is an isomorphism between Y and E^g over algebraically closed field $\overline{K}$ with supersingular elliptic curve E. This definition does not depend on choices of E by Deligne, Ogus and Shioda (see [2, 1.6] for a stronger result).

(2) An abelian variety X over K is said to be *supersingular* if and only if there exists an isogeny from E^g to X over algebraically closed field $\overline{K}$.

By Dieudonné functor $\mathbb{D}$, we have a supersingular Dieudonné module $M := \mathbb{D}(X)$ of genus g associated with X. Then $a(X) = a(M)$ holds ([2, 5.2]).

A. Ogus proved the following important theorem, which he called supersingular Torelli's theorem ([3, Theorem 6.2]).

Theorem 2.3. *Let $\mathcal{S}_g(K)$ be the category of supersingular abelian varieties over K. Assume $g \geq 2$. The functor* $(\mathbb{D}, \mathrm{tr})$ *gives a bijection between the set of isomorphism classes of $\mathcal{S}_g(K)$ and the set of supersingular Dieudonné modules M of genus g with trace map* $\mathrm{tr} : \overset{2g}{\wedge} M \xrightarrow{\simeq} W(K)$. *Besides, for two objects X, Y of $\mathcal{S}_g(K)$, we have an isomorphism*

$$\mathrm{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathrm{Hom}_A(\mathbb{D}(Y), \mathbb{D}(X)).$$

The next lemma will be frequently used.

Lemma 2.4 (Lemma 3.1 in [1]). *For a supersingular Dieudonné module M, there are the smallest superspecial Dieudonné module $S^0(M)$ in $M \otimes$ $\mathrm{frac}\, W(K)$ containing M, and dually the biggest superspecial Dieudonné module $S_0(M)$ contained in M.*

If X has a polarization $\eta : X \to X^t$, we get the non-degenerate $W(K)$-bilinear alternative form

$$\langle\ ,\ \rangle : M \otimes_{W(K)} M \to \mathrm{frac}\, W(K),$$

which satisfies $\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma$. We call such an alternating form a quasi-polarization of M. If η is principal, then $\langle\ ,\ \rangle$ is a perfect pairing.

2.2 Polarized flag type quotient (PFTQ) and covering of moduli spaces

Recall the definition of rigid PFTQs in [2, 3.6, 6.2].

Definition 2.5. *A rigid PFTQ* of Dieudonné modules is a filtration $\{M_0 \subset M_1 \subset \cdots \subset M_{g-1}\}$ of quasi-polarized Dieudonné modules satisfying

(i) M_{g-1} is a quasi-polarized superspecial Dieudonné module such that $M_{g-1}^t \simeq F^{g-1}M_{g-1}$;

(ii) $(F,V)M_i \subset M_{i-1}$ and the rank of K-vector space M_{i-1}/M_i is i for all $0 < i \leq g-1$;

(iii) $(F,V)^i M_i \subset M^i = M_i^t$ for $0 \leq i \leq g-1$;

(iv) $M_i = M_0 + F^{g-1-i}M_{g-1}$ for $0 \leq i \leq g-1$.

The last condition is called the rigidity.

For convenience, we choose a supersingular elliptic curve E over $\mathbb{F}_p$ (see [2, 1.2] for existence of such an E). Let η be a polarization of $E^g \otimes_{\mathbb{F}_p} K$ such that $\ker(\eta) = E^g[F^{g-1}] \otimes_{\mathbb{F}_p} K$. Let S be an $\mathbb{F}_p$-scheme.

Definition 2.6. A rigid PFTQ of dimension g over S with respect to η is a series of polarized abelian varieties (Y_i, η_i) and isogenies

$$Y_{g-1} \xrightarrow{\rho_{g-1}} Y_{g-2} \xrightarrow{\rho_{g-2}} \cdots \xrightarrow{\rho_2} Y_1 \xrightarrow{\rho_1} Y_0$$

such that

(i) $Y_{g-1} = E^g \times S$ and $\eta_{g-1} = \eta \times S$;

(ii) $\ker(\rho_i)$ is an α-group of α-rank i for all $i = 1, 2, \cdots, g-1$;

(iii) $\ker(\eta_i) \subset \ker(F^{i-j} \circ V^j)$ for all $j = 1, 2, \cdots, [i/2]$;

(iv) $\ker(Y_{g-1} \to Y_i) = \ker(Y_{g-1} \to Y_0) \cap Y_{g-1}[F^{g-1-i}]$ for all i.

Let $\mathcal{N}_g$ be the moduli space of rigid PFTQs of Dieudonné modules, and $\mathcal{P}'_{g,\eta}$ be the moduli of the rigid PFTQs of dimension g with respect to η. We know that $\mathcal{N}_g$ and $\mathcal{P}'_{g,\eta}$ are isomorphic up to inseparable morphism.

Theorem 2.7 (Section 4 of [2]). *Let Λ be the set of isomorphism classes of polarizations η on E^g satisfying* $\ker(\eta) = E^g[F^{g-1}]$. *There is a canonical morphism*

$$\Psi : \coprod_{\eta \in \Lambda} \mathcal{P}'_{g,\eta} \to S_g \times \overline{\mathbb{F}}_p,$$

which is a quasi-finite surjective morphism. Moreover $\mathcal{P}'_{g,\eta}$ is nonsingular and geometically integral of dimension $\left[g^2/4\right]$ and the generic fiber over each irreducible component of S_g has a-number 1*.*

Let $\mathcal{N}_g(a)$ be the subscheme of $\mathcal{N}_g$ which parametrizes rigid PFTQs $\{M_0 \subset \cdots \subset M_{g-1}\}$ with $a(M_0) = a$, also $\mathcal{P}'_{g,\eta}(a)$ be the associated subscheme of $\mathcal{P}'_{g,\eta}$. Then we have a quasi-finite surjective morphism

$$\Psi_a : \coprod_{\eta \in \Lambda} \mathcal{P}'_{g,\eta}(a) \to S_g(a) \times \overline{\mathbb{F}}_p.$$

Therefore as far as the dimension of $S_g(a)$ is concerned, it suffices to investigate the space $\mathcal{N}_g(a)$.

3 The dimension of $S_g(a)$ and good local coordinates

3.1 Open covering of $\mathcal{N}_g(a)$

The first part of the argument in this section is almost the same as the contents of Section 7 in the book [2]. However for our purpose, basis of different type in M_{g-1} are crucial in this paper. Therefore we need to rewrite the setting of construction of moduli space of rigid PFTQs.

Given a reduced k-scheme S, let $W_S := W(\mathcal{O}_S)$ be the sheaf of Witt rings [7].

Let N be a superspecial Dieudonné module with quasi-polarization of genus g satisfying $N^t = F^{g-1}N$. Note that such a quasi-polarized superspecial Dieudonné module is uniquely determined up to isomorphism ([2, Prop. 6.1]).

Definition 3.1. *A rigid PFTQ over S* is a filtration $\{M_0 \subset M_1 \subset \cdots \subset M_{g-1}\}$ of Dieudonné modules over S such that

(i) $M_{g-1} = N \otimes W_S$;

(ii) $FM_i^{(p)} \subset M_{i-1}$, $VM_i^{(p^{-1})} \subset M_{i-1}$ and M_i/M_{i-1} is a locally free $\mathcal{O}_S$-module of rank i for $0 < i \leq g-1$;

(iii) $F^jV^{i-j}M_i^{(p^{2j-i})} \subset M_i^t =: M^i$ for $0 \leq i \leq g-1$ and $0 \leq j \leq [i/2]$;

(iv) $M_i = M_0 + F^{g-1-i}M_{g-1}$ for $0 \leq i \leq g-1$;.

Let $\tilde{\Phi}$ be the set of H-basis $\Theta = \{x_0, \cdots, x_{g-1}\}$ of the skelton

$$\tilde{N} = \{x \in N \mid (F-V)x = 0\}$$

such that for all i and j,

$$\langle x_i, F^g x_{g-1-j}\rangle = \delta_{ij}\varepsilon$$

and

$$\langle x_i, F^{g-1}x_j\rangle = 0.$$

Here ε is a Teichmüller lifting in $W(\mathbb{F}_{p^2})$ satisfying $\varepsilon = -\varepsilon^\sigma$ and δ_{ij} is the Kronecker's delta. We note that the way to choose bases of $\tilde{M}_{g-1}$ here is different from that in [2, Section 7].

We denote by Φ the set of representatives of $\tilde{\Phi}$ modulo p. Then we see $\sharp\Phi < \infty$.

Definition 3.2. For given $\Theta = \{x_0, x_1, \cdots, x_{g-1}\}$, We denote by U^Θ the open subscheme of $\mathcal{N}_g$ consisting of rigid PFTQs $\{M_0 \subset \cdots \subset M_{g-1}\}$ with basis Θ of M_{g-1} such that M_0 has a basis of following type:

$$w_i = \sum_{j \geq i} \alpha_{ij} F^i x_j$$

with $\alpha_{ij} \in A$ and $\alpha_{ii} = 1$.

Then we get an open covering $\coprod_{\Theta\in\Phi} U^{\Theta} \to \mathcal{N}_g$.

Let U_m^{Θ} be the category of $\{M'_0 \subset \cdots \subset M'_{g-3}; M_m \subset \cdots \subset M_{g-1}\}$ with several properties (see [2, 7.6] for the precise definition). Here M'_0 is supposed to have a basis

$$w'_i = \sum_{j=i}^{g-2} \alpha'_{ij} F^i x_j \quad (\alpha'_{ij} \in A, \alpha'_{ii} = 1) \tag{1}$$

for $i = 1, \cdots, g-2$ as in Definition 3.2. Let t_m be the truncation morphism from U_{m-1}^{Θ} to U_m^{Θ}. We calculate the local chart of $\mathcal{N}_g$ inductively by

$$\mathcal{N}_g \supset U^{\Theta} = U_0^{\Theta} \to U_1^{\Theta} \to \cdots \to U_{g-1}^{\Theta} \subset \mathcal{N}_{g-2}.$$

Here U_{g-1}^{Θ} is the open subscheme $U^{\Theta'}$ of $\mathcal{N}_{g-2}$ for basis $\Theta' = \{Fx_1, \cdots, Fx_{g-2}\}$.

K.-Z. Li and F. Oort proved ([2, 7.11]):

Lemma 3.3. *Fix v_m be an element of M_m with x_0-coefficient 1. Let us write*

$$v_m = x_0 + \sum_{i=1}^{g-1} \zeta_i x_i$$

and

$$(F - V)v_m \mod Ax_{g-1} - \sum_{j<g-m} \lambda_j w'_j = \sum_{j \geq g-m} \mu_j F^{g-m} x_j.$$

Then the set of the K-valued points of the fiber of t_m of $\{M'_0 \subset \cdots \subset M'_{g-3}; M_m \subset \cdots \subset M_{g-1}\} \in U_m^{\Theta}(K)$ is bijective to the set of

$$v = v_m + \sum_{j=g-m}^{g-1} \beta_j F^{g-m-1} x_j \in M_m$$

modulo $F^{g-m}M_{g-1}$ (i.e. it is determined by $\overline{\beta}_j$) satisfying

$$\overline{\beta}_j^{p^2} - \overline{\beta}_j = \alpha'_{g-m,j}\overline{\tau} - \overline{\mu}_j$$

for $m > 2$ and $j < g-1$ and a equation in $\overline{\beta}_{g-1}$ coming from the condition

$$\begin{cases} \langle v, p^{(m-2)/2} F v \rangle \subset W(K) & \textit{for even } m \\ \langle v, p^{(m-3)/2} F^2 v \rangle \subset W(K) & \textit{for odd } m \geq 3. \end{cases}$$

Moreover, every equation gives Artin-Schreier extension.

From now, we start some new materials in this paper.

Lemma 3.4. *Let $M_0 \subset \cdots \subset M_{g-1}$ be a rigid PFTQ over a perfect field K. We have equalities as sets:*

$$\begin{aligned} M^i \cap A < x_i, x_{i+1}, \cdots, x_{g-1} > &= W(K)[F] < w_i, \cdots, w_{g-1} > \\ &= A < w_i, \cdots, w_{g-1} > . \end{aligned}$$

Proof. The first equality implies the second one, because the first term is an A-module. Since $M^i = M_0 \cap F^i M_{g-1}$, the first term contains the middle term obviously. We shall prove the first term is contained in the middle term by induction. For $i = g-1$, this is obvious. Assume that it holds for $i+1$. Put $M(i) = A < x_i, x_{i+1}, \cdots, x_{g-1} >$. Since $M^i \cap M(i)/M^{i+1} \cap M(i) = K < \overline{w}_i >$, we get

$$M^i \cap M(i) = W(K) < w_i > + M^{i+1} \cap M(i).$$

Take an element v of $M^i \cap M(i)$. Then $v = aw_i + m$ with $a \in W(K)$ and $m \in M^{i+1} \cap M(i)$. There is $b \in W(K)F$ such that $m = bw_i + m'$ with $m' \in M^{i+1} \cap M(i+1)$. By the hypothesis of induction, we have $m' \in W(K)[F] < w_{i+1}, \cdots, w_{g-1} >$, which implies $v \in W(K)[F] < w_i, \cdots, w_{g-1} >$. $\square$

Let us write

$$w_i = \sum_{j=i}^{g-1} \sum_{k=i}^{j-1} \beta_{ij}^{(k)} F^k x_j \tag{2}$$

and define $\tau_{ij} \in A$ by

$$(F - V)w_i = \tau_{i,i+1} w_{i+1} + \cdots + \tau_{i,g-1} w_{g-1}.$$

By Lemma 3.4, we may assume $\tau_{ij} \in W(K)[F]$.

Recall that a principally quasi-polarized supersingular Dieudonné module M does not uniquely determine $\{w_i\}$ nor therefore $\{\beta_{ij}^{(k)}\}$. In fact, Lemma 3.3 says only that the class $\overline{\beta}_{0j}^{(k)}$ are inductively determined. We have to note that the equations and their solutions at each step depend on choices of liftings of already known data $\overline{\beta}_{ij}^{(k)}$ $(i \geq 1)$. The next aim is to find good liftings of $\overline{\beta}_{ij}^{(k)}$ and therefore *good basis* $\{w_i\}$ of M.

Lemma 3.5. *For given a principally quasi-polarized supersingular Dieudonné module M, we can take a basis $w_0, \cdots, w_{g-1}$ as (2) such that we have*

$$\langle w_i, F w_{g-1-j} \rangle = \delta_{ij} \varepsilon, \quad \langle w_i, w_j \rangle = 0.$$

for all i and j.

Proof. We prove this by induction on g. Assume w_i' $(i = 1, \cdots, g-2)$ satisfy $\langle w_i', F w_{g-1-j}' \rangle = \delta_{ij}\varepsilon$ $\langle w_i', F w_{g-1-j}' \rangle = \delta_{ij}\varepsilon$. Then as w_i, we can take an element of the form:

$$w_i = w_i' + \sum_{k=i}^{g-2} \beta_{i,g-1}^{(k)} F^k x_{g-1}$$

for $i = 1, 2, \cdots, g-2$. By the hypothesis of induction, it follows that

$$\langle w_i, F w_{g-1-j} \rangle = \delta_{ij} \varepsilon, \qquad \langle w_i, w_{g-1-j} \rangle = 0$$

for $1 \leq i, j \leq g-2$. Since $w_{g-1} = F^{g-1} x_{g-1}$, we get

$$\langle w_i, F w_{g-1} \rangle = \delta_{i0} \varepsilon, \qquad \langle w_i, w_{g-1} \rangle = 0.$$

Now we have to find appropriate $\beta_{0,j}^{(k)}$ and $\beta_{i,g-1}^{(k)}$. Assume we have already determined $\beta_{0,j}^{(k')}$ for all $k' < k$. Since $\overline{\beta}_{0,j}^{(k)}$ is nothing but β_j for $m = g-k-1$ in Lemma 3.3, we take a Teichmüler lifting $\beta_{0,j}^{(k)}$ of solution $\overline{\beta}_j$ for $k \leq g-2$. Next for $k < g-2$, we choose liftings $\beta_{i,g-1}^{(k)}$ of $\overline{\beta}_{i,g-1}^{(k)}$ which are automatically determined by the already known coefficient $\beta_{ij}^{(k)}$. Finally we determine $\beta_{g-1-j,g-1}^{(g-2)}$ so that

$$\langle w_0, Fw_{g-1-j}\rangle = \delta_{0j}\varepsilon \qquad \langle w_0, w_{g-1-j}\rangle = 0 \tag{3}$$

for all j. Since the equation in $\overline{\beta}_{g-1-j,g-1}^{(g-2)}$, which is equivalent to the equation:

$$\overline{\langle w_0, w_{g-1-j}\rangle} = 0$$

in $p^{-1}W/W$, has a solution for each $1 \leq j \leq g-1$, there exists a lifting $\beta_{g-1-j,g-1}^{(g-2)}$ satisfying the equations (3). □

From now on, we assume that a basis $\{w_0, \cdots, w_{g-1}\}$ of M satisfies Lemma 3.5. By the lemma above, it follows that $\tau_{ij} \in W(K)$, since $\langle (F-V)w_i, w_{g-1-j}\rangle = 0$ for all i and j. Here we note that $\langle w_i, Vw_{g-1-j}\rangle = \delta_{ij}\varepsilon$.

Lemma 3.6. *The following symmetry holds:*

$$\tau_{ij} = \tau_{g-1-j,g-1-i}.$$

Proof. It follows from the straightforward calculation:

$$\begin{aligned}
\varepsilon\tau_{ij} &= \langle (F-V)w_i, Fw_{g-1-j}\rangle \\
&= -\langle Vw_i, Fw_{g-1-j}\rangle \\
&= \langle Fw_{g-1-j}, Vw_i\rangle \\
&= \langle (F-V)w_{g-1-j}, Vw_i\rangle \\
&= \langle (F-V)w_{g-1-j}, Fw_i\rangle \\
&= \varepsilon\tau_{g-1-j,g-1-i}
\end{aligned}$$

for all $1 \leq i, j \leq g-1$. □

Put $T = (\overline{\tau}_{ij})_{0\leq i,j\leq g-1}$ with $\overline{\tau}_{ij} = 0$ $(i \geq j)$.

Lemma 3.7. *We have* $a(M_0) = a$ *if and only if* $\operatorname{rk} T = g - a$.

Proof. It follows from

$$\begin{aligned}
M_0/(F,V)M_0 &= \frac{W(K)[F] < w_0, \cdots, w_{g-1} >}{(F,V)W(K)[F] < w_0, \cdots, w_{g-1} >} \\
&= \frac{W(K)[F] < w_0, \cdots, w_{g-1} >}{W(K)[F] < Fw_0, \cdots, Fw_{g-1}, (F-V)w_0, \cdots, (F-V)w_{g-1} >} \\
&= \frac{W(K) < w_0, \cdots, w_{g-1} >}{W(K) < pw_0, \cdots, pw_{g-1}, \{\sum_{j>i}\tau_{ij}w_j\} >} \\
&\simeq \operatorname{coker} T.
\end{aligned}$$

□

3.2 Investigation of "period domains"

In this subsection, we investigate the variety ∇_g which the matrix $T = (\overline{\tau}_{ij})$ belongs to. We may call this "period domain":

$$\nabla_g(K) = \{M \in \mathfrak{n}(K) \mid {}^t(Mw) = Mw\}$$

with $w = (\delta_{i,g-1-j})_{i,j}$. Here $\mathfrak{n}(K)$ is the set of strict uppertriangular $g \times g$ matrices with K-coefficients. We also define the subvariety $\nabla_{g,a}$ of ∇_g by

$$\nabla_{g,a}(K) = \{M \in \mathfrak{n}(K) \mid {}^t(Mw) = Mw,\ \operatorname{rk} M = g-a\}.$$

By Lemma 3.6 and 3.7, the matrix T is in $\nabla_{g,a}(K)$. Therefore we have a natural morphism $U^{\Theta}(a) \to \nabla_{g,a}$, which is étale by Lemma 3.3. Now we can show:

Proposition 3.8. *The morphism $U^{\Theta}(a) \to \nabla_{g,a}$ is étale and surjective.*

Proof. It suffices to show the surjectivity of the morphism $U^{\Theta} \to \nabla_g$. For $T = (\overline{\tau}_{ij}) \in \nabla_g$, we introduce a number by

$$m(T) := \min_{0 \le j \le g-1} \{j \mid \overline{\tau}_{0,k} = 0 \text{ for any } k > j\}.$$

Then we prove our Proposition by double induction on g and $m(T)$. The initial step $m(T) = 0$ of the inner induction is shown by the following.
Claim 1. The locus

$$\{T' = (\overline{\tau}'_{ij}) \in \nabla_g \mid \overline{\tau}'_{0,i} = 0,\ \overline{\tau}'_{i,g-1} = 0\ (i = 0, \cdots, g-1)\}$$

is contained in the image of $U^{\Theta} \to \nabla_g$.
Proof of Claim 1. Deleting the top and the bottom rows and the first and the last columns from T', we obtain $T'_{\mathrm{red}} \in \nabla_{g-2}$. By the hypothesis of induction on g, for $T'_{\mathrm{red}} = (\overline{\tau}_{ij})_{1 \le i,j \le g-2}$, there is a rigid PFTQ $\{M'_0 \subset \cdots \subset M'_{g-3}\}$ such that M'_0 is a Dieudonné module of genus $g-2$ with basis $w'_1, \cdots, w'_{g-2}$ of the form (1) satisfying

$$(F - V)w'_i = \tau'_{i,i+1}w'_{i+1} + \cdots + \tau'_{i,g-2}w'_{g-2} \quad (i = 1, \cdots, g-2).$$

Here M'_{g-3} is the superspecial Dieudonné module generated by $Fx_1, \cdots, Fx_{g-2}$. By the natural inclusion from M'_{g-3} to $M_{g-1} = A < x_0, x_1, \cdots, x_{g-2}, x_{g-1} >$, we regard w'_i as an element, say w_i, of M_{g-1}. Let us put $w_0 = x_0$ and $w_{g-1} = F^{g-1}x_{g-1}$. Then the Deudonné module M_0 generated by $w_0, \cdots, w_{g-1}$ is sent to T'. This completes the proof of Claim 1.

Now we may assume that our Proposition is true for T such that $m(T) = m-1$. It suffices to show the next claim under this assumption:
Claim 2. Given T with $m(T) = m$, there exists a rigid PFTQ $\{M_0 \subset \cdots \subset M_{g-1}\}$ in U^{Θ} which is sent to T.
Proof of Claim 2. For T satisfying $m(T) = m$, we define an auxiliary $T' = (\tau'_{ij})$ from $T = (\overline{\tau}_{ij})$ by $\tau'_{ij} = \tau_{ij}$ for $(i,j) \neq (0,m), (m,0)$ and $\tau'_{0m} = \tau'_{m0} = 0$. Then we have $m(T') = m-1$. By the hypothesis of induction on $m(T)$, there is a rigid PTFQ $\{M_0 \subset \cdots \subset M_{g-1}\}$ in U^{Θ} which is sent to T'. Namely there is a good basis $w'_0, \cdots, w'_{g-1}$ of M_0 satisfying

$$(F - V)w'_i = \tau'_{i,i+1}w_{i+1} + \cdots + \tau'_{i,g-1}w_{g-1}$$

for all $i = 0, \cdots, g-1$.

Applying Lemma 3.3 for w'_0 as v_{g-m}, we can construct a principally quasi-polarized super-special Dieudonné module with basis $w_0, \cdots, w_{g-1}$ such that $(F-V)w_i = \tau_{i,i+1}w_{i+1} + \cdots + \tau_{i,g-1}w_{g-1}$, which is mapped to the original T. $\square$

From now on, we investigate the structure of $\nabla_{g,a}$. Let $M = (a_{ij})_{0 \le i,j \le g-1}$ be an element of $\nabla_{g,a}(K)$. If we write $M = (A_{ij})_{1 \le i \le j \le t+1}$, it means the unique block expression

$$M = \begin{pmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1t+1} \\ 0 & 0 & A_{23} & \cdots & A_{2t+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{tt+1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{4}$$

where

$$A_{kk+1} = \begin{pmatrix} * & * \\ a_{i_k,j_k} & * \end{pmatrix} \tag{5}$$

with element a_{i_k,j_k} of $K^\times$. Since above t is determined by M, we denote it by t_M. Obviously $t_M \le g-a$.

For each $M = (A_{ij})$, in $\{0, 1, \cdots, g-1\} \times \{0, 1, \cdots, g-1\}$ we associate the subset

$$S_M = \left\{ (i_k, j_k) \;\middle|\; a_{i_k,j_k} \ne 0 \text{ in } A_{kk+1} = \begin{pmatrix} * & * \\ a_{i_k,j_k} & * \end{pmatrix} \right\}.$$

Proposition 3.9. *Let $r = g-a$. Every irreducible component of $\nabla_{g,g-r}$ is of either of the following types:*

(i) For odd $g+r$, the Zariski closure $\nabla^c_{g,g-r}(i_1, \cdots, i_r)$ in $\nabla_{g,g-r}$ of

$$\nabla_{g,g-r}(i_1, \cdots, i_r) := \{M \in \nabla_{g,g-r} \mid S_M = \{(i_k, j_k) \mid k = 1, \cdots, r\}\} \tag{6}$$

with $j_k = i_k + 1$. Here $\nabla_{g,g-r}(i_1, \cdots, i_r)$ is defined for each sequence $0 \le i_1 < i_2 < \cdots < i_r < g-1$ satisfying the condition $i_k + i_{r+1-k} = g-2$ for any $k = 1, \cdots, r$.

(ii) For even g and even r,

(a) $\nabla^c_{g,g-r}(i_1, \cdots, i_r)$ defined in the same way as (i), or

(b) the Zariski closure $\nabla^c_{g,g-r}(i_1, \cdots, i_{r-1})$ in $\nabla_{g,g-r}$ of

$$\nabla_{g,g-r}(i_1, \cdots, i_{r-1}) := \{M \in \nabla_{g,g-r} \mid S_M = \{(i_k, j_k) \mid k = 1, \cdots, r-1\}\} \tag{7}$$

with $j_k = i_k + 1$. Here $0 \le i_1 < i_2 < \cdots < i_{r-1} < g-1$ is any sequence satisfying the condition $i_k + i_{r-k} = g-2$ (note $i_{r/2} = g/2 - 1$).

(iii) For odd g and odd r,

(a) the Zariski closure $\nabla^c_{g,g-r}(i_1,\cdots,i_r)$ *in* $\nabla_{g,g-r}$ *of the set* $\nabla_{g,g-r}(i_1,\cdots,i_r)$*:*

$$\{M\in\nabla_{g,g-r}\mid S_M=\{(i_k,j_k)\mid k=1,\cdots,r\}\} \tag{8}$$

with $j_k=i_k+1$ *for* $k\neq(r+1)/2$ *and* $j_{\frac{r+1}{2}}=i_{\frac{r+1}{2}}+2$*. Here* $0\le i_1<i_2<\cdots<i_r<g-1$ *is any sequence satisfying* $i_k+i_{r+1-k}=g-2$ $(\forall k\neq(r+1)/2)$ *with* $i_{(r+1)/2}=(g-3)/2$*, or*

(b) the Zariski closure $\nabla^c_{g,g-r}(i_1,\cdots,i_{r-1})$ *in* $\nabla_{g,g-r}$ *of*

$$\nabla_{g,g-r}(i_1,\cdots,i_{r-1}):=\{M\in\nabla_{g,g-r}\mid S_M=\{(i_k,j_k)\mid k=1,\cdots,r-1\}\} \tag{9}$$

with $j_k=i_k+1$*. Here* $0\le i_1<i_2<\cdots<i_{r-1}<g-1$ $(r\ge 3)$ *is any sequence satisfying* $i_k+i_{r-k}=g-2$ *with* $i_{(r-1)/2}=(g-3)/2$ *and* $i_{(r+1)/2}=(g-1)/2$*.*

Before the proof, we give some examples of elements M of

$$\nabla_{g,g-r}(i_1,\cdots,i_r)\quad\text{or}\quad\nabla_{g,g-r}(i_1,\cdots,i_{r-1})$$

to help the understanding of the reader:

Example 3.10.

(i) $g=6, r=3; i_1=0, i_2=2, i_3=4$,

$$\begin{pmatrix}0&a_{01}&a_{02}&a_{03}&a_{04}&a_{05}\\0&0&0&a_{13}&a_{13}^2/a_{23}&a_{04}\\0&0&0&a_{23}&a_{13}&a_{03}\\0&0&0&0&0&a_{02}\\0&0&0&0&0&a_{01}\\0&0&0&0&0&0\end{pmatrix}$$

(ii-b) $g=6, r=4; i_1=0, i_2=2, i_3=4$,

$$\begin{pmatrix}0&a_{01}&a_{02}&a_{03}&a_{04}&a_{05}\\0&0&0&a_{13}&a_{14}&a_{04}\\0&0&0&a_{23}&a_{13}&a_{03}\\0&0&0&0&0&a_{02}\\0&0&0&0&0&a_{01}\\0&0&0&0&0&0\end{pmatrix}$$

(iii-a) $g=5, r=3; i_1=0, i_2=1, i_3=3$,

$$\begin{pmatrix}0&a_{01}&a_{02}&a_{03}&a_{04}\\0&0&0&a_{13}&a_{03}\\0&0&0&0&a_{02}\\0&0&0&0&a_{01}\\0&0&0&0&0\end{pmatrix}$$

(iii-b) $g=5, r=3; i_1=1, i_2=2$,

$$\begin{pmatrix}0&0&a_{02}&a_{03}&a_{04}\\0&0&a_{12}&a_{13}&a_{03}\\0&0&0&a_{12}&a_{02}\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

with $a_{ij}\in K$.

Proof. First we show that every variety $\nabla^c_{g,g-r}(i_1,\cdots,i_{r'})$ ($r'=r$ or $r-1$) defined above is irreducible. Obviously it suffices to show the following.
Claim 1. $\nabla_{g,g-r}(i_1,\cdots,i_{r'})$ is irreducible.
Proof of Claim 1. This can be shown by induction on r. For $r=0$, $\nabla_{g,g}$ consists of one point 0.

For $r=1$, we investigate $\nabla_{g,g-1}(i_1)$ with $i_1=[(g-2)/2]$. Let j_1 be i_1+1 for even g and i_1+2 for odd g. Any element $(a_{ij})_{0\le i,j\le g-1}$ of $\nabla_{g,g-1}(i_1)$ is written as

$$a_{ij}=b_ib_{g-1-j}/b_{i_1}\qquad\text{for } i\le i_1 \text{ and } j\ge j_1$$

for some $b_0, \cdots, b_{i_1} \in K$ with $b_{i_1} \neq 0$ and

$$a_{ij} = 0 \qquad \text{for other } i, j.$$

Hence we have

$$\nabla_{g,g-1}(i_1) \simeq \mathbb{G}_m \times \mathbb{A}^{[g/2]-1},$$

which is irreducible.

For $r = 2$, there are two cases:

(1) $\nabla_{g,g-2}(i_1, i_2)$ with $i_1 + i_2 = g - 2$ (the cases (i) and (ii-a) in Proposition 3.9) or

(2) $\nabla_{g,g-2}(i_1)$ with $i_1 = (g-2)/2$ (the case (ii-b) in Proposition 3.9).

Since the irreducibility in the case (2) will be shown simultaneously in the argument for general r below, we restrict ourselves to the case (1). Any element $(a_{ij})_{0 \le i,j \le g-1}$ of $\nabla_{g,g-2}(i_1, i_2)$ is written as

$$\begin{array}{rll} a_{ij} = & b_i b_{j-1}/b_{i_1} & \text{for } i \le i_1 \text{ and } j_1 \le j < j_2, \\ a_{ij} = & b_i b_{j-1}/b_{i_1} + b_{g-1-j} b_{g-1-i-1}/b_{i_1} & \text{for } i \le i_1 \text{ and } j \ge j_2, \\ a_{ij} = & b_{g-1-j} b_{g-1-i-1}/b_{i_1} & \text{for } i_1 < i \le i_2 \text{ and } j \ge j_2 \end{array}$$

with $j_1 = i_1 + 1$ and $j_2 = i_2 + 1$ for some $b_0, \cdots, b_{g-2} \in K$ with $b_{i_1} \neq 0$ and

$$a_{ij} = 0 \qquad \text{for other } i, j.$$

Hence we have

$$\nabla_{g,g-2}(i_1, i_2) \simeq \mathbb{G}_m \times \mathbb{A}^{g-2},$$

which is irreducible.

Let us show the irreduciblity for general r. For any element M of $\nabla_{g,g-r}(i_1, \cdots, i_{r'})$, there is a unique pair (N, N') such that

$$M = N + N'$$

with

$$N \in \begin{cases} \nabla_{g,g-2}(i_1, i_{r'}) & \text{for } r' \neq 1, \\ \nabla_{g,g-1}(i_1) & \text{for } r' = 1 \end{cases}$$

and N' has zero $(i_1, *)$, $(*, j_1)$, $(i_{r'}, *)$ and $(*, j_{r'})$ entries for $* = 0, 1, \cdots, g-1$. Then N' can be regarded as an element of

$$\begin{cases} \nabla_{g-2,g-r}(i_2 - 1, \cdots, i_{r'-1} - 1) & \text{for } r' \neq 1, \\ \nabla_{g-1,g-r}(i_1 - 1) & \text{for } r' = 1. \end{cases}$$

The fact that we have a unique decomposition $M = N + N'$ implies that

$$\nabla_{g,g-r}(i_1, \cdots, i_{r'}) = \begin{cases} \nabla_{g-2,g-r}(i_2 - 1, \cdots, i_{r'-1} - 1) \times \nabla_{g,g-2}(i_1, i_{r'}) & \text{for } r' \neq 1, \\ \nabla_{g-1,g-r}(i_1 - 1) \times \nabla_{g,g-1}(i_1) & \text{for } r' = 1. \end{cases} \tag{10}$$

By the hypothesis of induction, $\nabla_{g-2,g-r}(i_2 - 1, \cdots, i_{r'-1} - 1)$ for $r' \neq 1$ and $\nabla_{g-1,g-r}(i_1 - 1)$ for $r' = 1$ are irreducible. Also $\nabla_{g,g-2}(i_1, i_{r'})$ for $r' \neq 1$ and $\nabla_{g,g-1}(i_1)$ for $r' = 1$ is irreducible. Hence $\nabla_{g,g-r}(i_1, \cdots, i_{r'})$ is irreducible. Claim 1 is proved.

The next lemma completes the proof of our proposition. □

Lemma 3.11. *For every $N \in \nabla_{g,a}$, there exists a sequence of elements*

$$N^{(1)}_{s_1}\ (s_1 \in K),\ N^{(2)}_{s_1,s_2}\ (s_1 \in K^\times, s_2 \in K),\ \cdots,\ N^{(m)}_{s_1,\cdots s_m}\ (s_1,\cdots,s_{m-1} \in K^\times, s_m \in K)$$

in $\nabla_{g,a}$ satisfying specialization conditions

$$N = N^{(1)}_0,\ N^{(1)}_{s_1} = N^{(2)}_{s_1,0},\ N^{(2)}_{s_1,s_2} = N^{(3)}_{s_1,s_2,0},\ \cdots,\ N^{(m-1)}_{s_1,\cdots,s_{m-1}} = N^{(m)}_{s_1,\cdots,s_{m-1},0}$$

and $N^{(m)}_{s_1,\cdots,s_m} \in \nabla_{g,g-r}(i_1,\cdots,i_r)$ or $\nabla_{g,g-r}(i_1,\cdots,i_{r-1})$ for all $s_1,\cdots,s_m \in K^\times$. In short, $N^{(m)}_{s_1,\cdots,s_m},\cdots,N^{(1)}_{s_1}$ is a sequence of specializations to N.

Proof. Let N be an element of $\nabla_{g,g-r}$ with block expression (A_{ij}) as in (4) with

$$A_{k,k+1} = \begin{pmatrix} * & * \\ a_{l_k,m_k} & * \end{pmatrix} \qquad (a_{l_k,m_k} \neq 0,\ k = 1,2,\cdots,t_N).$$

For N, we associate an element μ_N of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ defined by

$$\mu_N = \left(t_N\ ,\ \sum_k m_k - l_k - 1 \right).$$

We define an order on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. For two elements $\mu_1 = (t_1, d_1)$ and $\mu_2 = (t_2, d_2)$ in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, we denote by $\mu_1 < \mu_2$ when $t_1 < t_2$, or $t_1 = t_2, d_1 > d_2$.

We have the following equivalences

$$\mu_N = (r,0) \Leftrightarrow N \in \nabla_{g,g-r}(i_1,\cdots,i_r)$$

for odd $g + r$,

$$\begin{array}{lcl} \mu_N = (r,0) & \Leftrightarrow & N \in \nabla_{g,g-r}(i_1,\cdots,i_r), \\ \mu_N = (r-1,0) & \Leftrightarrow & N \in \nabla_{g,g-r}(i_1,\cdots,i_{r-1}) \end{array}$$

for even g, r and

$$\begin{array}{lcl} \mu_N = (r,1) & \Leftrightarrow & N \in \nabla_{g,g-r}(i_1,\cdots,i_r), \\ \mu_N = (r-1,0) \text{ and (A)} & \Leftrightarrow & N \in \nabla_{g,g-r}(i_1,\cdots,i_{r-1}) \end{array}$$

for odd g, r. Here (A) is the condition

(A): N does not have generalization N_s with $\mu_{N_s} = (r,1)$ for all $s \in K^\times$.

Hence it suffices to show the following.

Claim 2. For given $N \in \nabla_{g,g-r}$, we can construct a generalization N_s such that $\mu_{N_s} > \mu_N$ $(s \in K^\times)$ unless

$$\begin{cases} \mu_N = (r,0) & \text{for odd } g+r, \\ \mu_N = (r,0) \text{ or } (r-1,0) & \text{for even } g,r, \\ \mu_N = (r,1) \text{ or } (r-1,0) & \text{for odd } g,r. \end{cases}$$

Proof of Claim 2. We will show this by induction on r. Since N has to be 0 for $r = 0$, there is nothing to prove in this case. Let us take $N \in \nabla_g$ with rank $r > 0$.

For $t_N \neq 1$ and $m_1 - l_1 - 1 \geq 1$, let N_s be the matrix obtained by adding the column vector

$$^t(a_{0,m_1}s, \cdots, a_{l_1,m_1}s, 0, \cdots, 0)$$

to (l_1+1)-th column vector and by adding further the row vector

$$(0, \cdots, 0, a_{l_1,m_1}s, \cdots, a_{0,m_1}s)$$

to $(g-1-l_1)$-th row vector. Then it follows that $N_s \in \nabla_{g,a}$ for all $s \in K$. Since N_s satisfies $m_1 - l_1 - 1 = 0$ for all $s \in K^\times$, we have $\mu_{N_s} > \mu_N$.

For $t_N = 1$ and $m_1 - l_1 - 1 \geq 1$ for even g or $m_1 - l_1 - 1 \geq 2$ for odd g, we construct a generalization N_s which has $l_1 = g/2 - 1, m_1 = g/2$ for even g or $l_1 = (g-1)/2 - 1, m_1 = (g+1)/2$ for odd g for $s \in K^\times$. Indeed we define N_s by adding s^2 to $(g/2-1, g/2)$-th resp. $((g-1)/2-1, (g+1)/2)$-th entry and by adding

$$^t(b_0 s, \cdots, b_{l_1}s, 0, \cdots, 0)$$

to $g/2$-th resp. $(g-1)/2$-th column vector and by adding

$$(0, \cdots, 0, b_{l_1}s, \cdots, b_0 s)$$

to $(g/2-1)$-th resp. $(g-3)/2$-th row vector where b_j $(j = 0, \cdots, l_1)$ are uniquely determined by the equations $b_{l_1}b_j = a_{j,m_1}$ $(j = 0, \cdots, l_1)$. Then since N_s $(s \in K^\times)$ have the desired l_1 and m_1, it follows that $\mu_{N_s} > \mu_N$ for all $s \in K^\times$.

The remaining problem is to show this Claim 2 for N with

$$\begin{cases} m_1 - l_1 - 1 = 0 & \text{for } t_N \neq 1 \text{ or for } t_N = 1 \text{ and odd } g, \\ m_1 - l_1 - 1 = 1 & \text{for } t_N = 1 \text{ and even } g. \end{cases} \tag{11}$$

In general, for given N we have a decomposition

$$N = N_1 + N' \tag{12}$$

such that N_1 has

$$S_{N_1} = \{(l_1, m_1), (g-1-m_1, g-1-l_1)\}$$

and N_1 is of rank 2 for $t_N \neq 1$ or of rank 1 for $t_N = 1$ and N' has zero l_1-th and $(g-1-l_1)$-th row vectors and zero m_1-th and $(g-1-m_1)$-th column vectors. We note that such N_1 and therefore N' are uniquely determined. Then the rank of N' is equal to $r-2$ for $t_N \neq 1$ or $r-1$ for $t_N = 1$. We can regard N' as a $(g-2)\times(g-2)$-matrix resp. $(g-1)\times(g-1)$-matrix by deleting the l_1-th and the $(g-1-l_1)$-th row vectors and the m_1-th and the $(g-1-m_1)$-th column vectors.

When $t_N = 1$, $t_{N'} = 1$ and $m_1 - l_1 - 1 = 1$ (this is a special case of the latter case of (11)), we can construct a further generalization. Let $N = N_1 + N'$ and $N' = N'_1 + N''$ be the decompositions as in (12) for N and N' respectively. By a similar method as above, for the matrix $N_2 := N_1 + N_1'$ of rank 2, we can construct a genelarization with the same rank 2 of the form

$$\begin{pmatrix} 0 & t_{1n} & t_{2n} & * & * \\ 0 & \vdots & \vdots & * & * \\ 0 & t_{12} & t_{22} & \cdots & t_{2n} \\ 0 & t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with} \quad \mathrm{rk}\begin{pmatrix} t_{12} & t_{22} \\ t_{11} & t_{12} \end{pmatrix} = 2, \tag{13}$$

where t_{11} is the $((g-1)/2-1, (g+1)/2)$-th entry for even g. Hence we may assume $N_2 := N_1 + N_1'$ has such a form. Then we have a generalization $N_{2,s}$ of N_2 defined by

$$\begin{pmatrix} 0 & sx_n & t_{1n} & t_{2n} & * & * \\ 0 & \vdots & \vdots & \vdots & * & * \\ 0 & sx_2 & t_{12} & t_{22} & \cdots & t_{2n} \\ 0 & s & t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & 0 & s & sx_2 & \cdots & sx_n \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where x_2 is a solution of $t_{11}x_2^2 - 2t_{12}x_2 + t_{22} = 0$ and $x_i = (t_{12} - t_{11}x_2)^{-1}(t_{2i} - t_{1i}x_2)$ for all $i = 2, \cdots, n$. Let us put $N_s = N_{2,s} + N''$, then it follows that $\mu_{N_s} = (2, 0) > \mu_N = (1, 1)$ for $s \in K^\times$.

Lastly we have to settle the other case of (11). This is included in the following case.

(B): the three conditions $t_N = 1$, $t_{N'} = 1$ and $m_1 - l_1 - 1 = 1$ do not occur simultaneously.

We utilize the unique decomposition $N = N_1 + N'$ obtained in (12) again. By the hypothesis of induction, we may assume that

$$N' \in \begin{cases} \nabla_{g-2,g-r}(i_1', \cdots, i_{r''}') & (r'' = r-2 \text{ or } r-3) \quad \text{for } t_N \neq 1, \\ \nabla_{g-1,g-r}(i_1'', \cdots, i_{r'''}'') & (r''' = r-1 \text{ or } r-2) \quad \text{for } t_N = 1. \end{cases} \tag{14}$$

Indeed, otherwise taking a generalization N_s' which gives an element of $\nabla_{g-2,g-r}(i_1', \cdots, i_{r''}')$ or $\nabla_{g-1,g-r}(i_1'', \cdots, i_{r'''}'')$ for each $s \in K^\times$, we define a generalization N_s of N by $N_s := N_1 + N_s'$. Then it follows that $\mu_{N_s} > \mu_N$ $(s \in K^\times)$.

The conditions (11), (14) and (B) imply that N is in $\nabla_{g,g-r}(i_1, \cdots, i_{r'})$ $(r' = r$ or $r-1)$. This completes the proof of our Claim 2, therefore Lemma 3.11, and Proposition 3.9. □

Corollary 3.12. *Let us denote by $J_{g,a}$ the set of irreducible components of $\nabla_{g,a}$. Then we have*

$$\sharp J_{g,a} = \begin{cases} \begin{pmatrix} (g-2)/2 \\ (g-a-1)/2 \end{pmatrix} & \text{if } g \text{ is even and } a \text{ is odd}, \\ \begin{pmatrix} [g/2] \\ [(g-a)/2] \end{pmatrix} & \text{otherwise.} \end{cases}$$

Proof. (i) For even g and odd a, the number of $J_{g,a}$ is the number of choices of $i_1, \cdots, i_{(r-1)/2}$ in $\{0, \cdots, (g-2)/2 - 1\}$.

For odd g and odd a, the number of $J_{g,a}$ is the number of choices of $i_1, \cdots, i_{r/2}$ in $\{0, \cdots, (g-1)/2 - 1\}$.

(ii) For even g and even a, the number of $J_{g,a}$ is the sum of the number of choices of $i_1, \cdots, i_{r/2}$ in $\{0, \cdots, (g-2)/2-1\}$ and the number of choices of $i_1, \cdots, i_{(r-2)/2}$ in $\{0, \cdots, (g-2)/2-1\}$.

(iii) For odd g and even a, the number of $J_{g,a}$ is the number of choices of $i_1, \cdots, i_{(r-1)/2}$ in $\{0, \cdots, (g-1)/2-1\}$. $\square$

Corollary 3.13. *Let $\nabla^c_{g,a}$ be the Zariski closure of $\nabla_{g,a}$ in ∇_g. Then we have*

$$\nabla^c_{g,a} = \bigcup_{a' \geq a} \nabla_{g,a'}.$$

Moreover $\nabla^c_{g,a}$ is connected.

Proof. The connectivity of $\nabla^c_{g,a}$ follows from the fact that any component of $\nabla^c_{g,a}$ obviously contains the locus consisting of $T = (a_{ij})$ with $a_{ij} = 0$ unless $i = 0, j = g-1$.

For the first statement, it suffices to show that $\nabla_{g,a+1}$ is in $\nabla^c_{g,a}$.

For any element N of $\nabla_{g,a+1}$, there is a generalization N_s ($s \in \overline{K}$) such that $N = N_0$ and $N_s \in \nabla_{g,a+1}(i'_1, \cdots, i'_{r''})$ ($r'' = g-a-1$ or $g-a-2$) for $s \neq 0$, by the proof of Proposition 3.9.

Hence we may assume N is in $\nabla_{g,a+1}(i'_1, \cdots, i'_{r''})$ ($r'' = g-a-1$ or $g-a-2$). Let us construct a generalization of such an element N to a certain line in $\nabla_{g,a}$.

Case 1. When $r'' = g-a-1$: Let i be an integer such that $0 \leq i \leq g-2$ and $i \neq i'_k$ ($k = 1, \cdots, r''$). We define a generalization of N_s ($s \in \overline{K}$) by $N + M_s$ where $M_s = (a_{kl})$ with

$$a_{kl} = s\delta_{ik}\delta_{g-1-i,l}.$$

Then N_s is in $\nabla_{g,a}$ for $s \neq 0$.

Case 1. When $r'' = g-a-2$: Let us denote by v_i the i-th row vector in N. Since the rank of N is $g-a-1$, the row vectors in N are generated by $v_{i'_1}, \cdots, v_{i'_{r''}}$ and another vector. Then N can be written as $N = N_1 + N_2$ such that any row vector in N_1 is a linear combination of $v_{i'_1}, \cdots, v_{i'_{r''}}$ and N_2 has zero i'_k-th row vectors and zero j'_k-th vectors for all $0 \leq k \leq r''$ (j'_k are determined by i'_k as in Proposition 3.9). Then N_2 is a matrix of rank 1 of the form (a_{kl}) with

$$a_{kl} = b_k b_{g-1-l}/b_i \qquad \text{for } k \leq i \text{ and } l \geq g-1-i$$

for some $b_0, \cdots, b_i \in \overline{K}$ with $b_i \neq 0$ and

$$a_{kl} = 0 \qquad \text{for other } k, l,$$

for a certain integer $i \neq i'_1, \cdots, i'_{r''}$. In particular N_2 can be regarded as an element of $\nabla_{g,g-1}$. We define a generalization $N_{2,s} = (c_{kl})$ ($s \in \overline{K}$) of N_2 by

$$c_{k,g-2-i} = b_k s \quad (0 \leq \forall k \leq i), \qquad c_{i+1,l} = b_{g-1-l}s \quad (g-1-i \leq \forall l \leq g-1)$$

and $c_{kl} = a_{kl}$ for other k, l. Note $N_{2,s}$ are of rank 2 for all $s \neq 0$. Then $N_s = N_1 + N_{2,s}$ is a desired genelarization. $\square$

Proposition 3.14. *Any irreducible component of $\nabla_{g,a}$ has dimension*

$$\left[\frac{g^2 - a^2 + 1}{4}\right].$$

Proof. As Proposition 3.9, we set $r = g - a$. It suffices to show that all $\nabla_{g,a}(i_1, \cdots, i_{r'})$ ($r' = r$ or $r-1$) in Proposition 3.9 have the same dimension $\left[(g^2 - a^2 + 1)/4\right]$. We prove this by induction on $r = g - a$.

In the first part of the proof of Proposition 3.9, we have already shown that

$$\nabla_{g,g-1}(i_1) \simeq \mathbb{G}_m \times \mathbb{A}^{[g/2]-1}$$

with $i_1 = [(g-2)/2]$ and

$$\nabla_{g,g-2}(i_1, i_2) \simeq \mathbb{G}_m \times \mathbb{A}^{g-2}$$

for all $i_1 + i_2 = g - 2$. Hence the dimension formulas in these cases follow immediately.

In general cases, by the equation (10), we have

$$\dim \nabla_{g,a}(i_1, \cdots, i_{r'}) = \begin{cases} \dim \nabla_{g,g-2}(i_1, i_{r'}) + \dim \nabla_{g-2,a}(i_2 - 1, \cdots, i_{r'-1} - 1) & \text{for} \quad r' \neq 1, \\ \dim \nabla_{g,g-1}(i_1) + \dim \nabla_{g-1,a}(i_1 - 1) & \text{for} \quad r' = 1. \end{cases}$$

The straightforward calculations:

$$\left[\frac{g^2 - (g-2)^2 + 1}{4}\right] + \left[\frac{(g-2)^2 - a^2 + 1}{4}\right] = \left[\frac{g^2 - a^2 + 1}{4}\right]$$

and

$$\left[\frac{g^2 - (g-1)^2 + 1}{4}\right] + \left[\frac{(g-1)^2 - a^2 + 1}{4}\right] = \left[\frac{g^2 - a^2 + 1}{4}\right],$$

show this proposition. □

3.3 Conclusions of this section

Let $\mathcal{A}_g$ be the coarse moduli space over $\mathbb{Z}$ of principally polarized abelian varieties. By the fact that the set of supersingular points is closed in $\mathcal{A}_g \otimes \mathbb{F}_p$, giving the reduced structure to the locus, we have the closed subscheme S_g in $\mathcal{A}_g \otimes \mathbb{F}_p$. We denote by $S_g(a)$ the locally closed subscheme in S_g parametrizing principally polarized supersingular abelian varieties with a-number a.

Theorem 3.15. *(0) Let $S_g^c(a)$ be the Zariski closure of $S_g(a)$ in S_g. Then we have*

$$S_g^c(a) = \bigcup_{a' \geq a} S_g(a').$$

Moreover $S_g^c(a)$ is connected unless $a = g$.

(1) Evey irreducible component of $S_g(a)$ has dimension $\left[\frac{g^2 - a^2 + 1}{4}\right]$.

Proof. All of the statements except the connectivity have already been proved by Corollary 3.13 and Proposition 3.14.

The connectivity of $S_g^c(a)$ follows from [5, Theorem 1.1] and Corollary 3.13. In fact, any irreducible component of the locus L in [5] can be interpreted as the locus with $T = (\overline{\tau}_{ij})$ ($\tau_{ij} \neq 0$ only for $i = 0, j = g-1$) in the moduli space $\mathcal{N}_g$ of rigid PFTQs for a certain basis $(x_1, \cdots, x_{g-1}) \in \Phi$. The one-dimensional locus consisting of such T is obviously contained in the Zariski closure of $\nabla_{g,a}(i_1, \cdots, i_{r'})$ ($r' = r$ or $r - 1$) in ∇_g for any $a (\neq g)$. Conversely, any irreducible component of $S_g^c(a)$ contains an irreducible component of L by the above interpretation. Hence the connectivity of L ([5, Theorem 1.1]) implies the connectivity of $S_g^c(a)$. □

4 The number of irreducible components of $S_g(a)$

4.1 Reformulation of the problem

Let $D_g(a)$ be the moduli space of principally quasi-polarized supersingular Dieudonné modules with a-number a. Let $I_{g,a}$ be the set of irreducible components of $D_g(a)$. It has already been known that $\sharp I_{g,1} = \sharp I_{g,g-1} = 1$ and $\sharp I_{4,2} = 2$ ([2, 9.9]).

If we denote by $D_g(a,x)$ the irreducible component of $D_g(a)$ corresponding to $x \in I_{g,a}$, we have the decomposition with irreducible components of $D_g(a)$:

$$D_g(a) = \bigcup_{x \in I_{g,a}} D_g(a,x).$$

Lemma 4.1. *Let x be an element of $I_{g,a}$. Then there exists an open subscheme U of $D_g(a,x)$ such that there is a quasi-polarized superspecial Dieudonné module N such that we have an isomorphism as quasi-polarized Dieudonné modules between N and $S^0(M)$ for any $M \in U$,*

The proof of this lemma will be given after Corollary 4.14 with explicit formula of N.

There is a natural quasi-finite surjection $f : S_g(a) \to D_g(a)$. We shall investigate the irreducible components in $S_g(a,x) := f^{-1}D_g(a,x)$ for each x. This is done by investigation of polarizations on superspecial abelian varieties, which is a global problem.

Definition 4.2. Let x be an element of $I_{g,a}$ and N the quasi-polarized superspecial Dieudonné module given in the above lemma. We denote by Λ_x the finite set consisting of polarizations η on E^g such that $\ker \eta$ is a p-group satisfying

$$\mathbb{D}(\ker \eta) \simeq N/N^t.$$

Then we have:

Proposition 4.3. *The cardinal number of irreducible components of $S_g(a)$ is equal to*

$$\sum_{x \in I_{g,a}} \sharp \Lambda_x.$$

Proof. Let x be an element of $I_{g,a}$ and W be one of irreducible components of $S_g(a)$ mapped to x by the natural map from $S_g(a)$ to $D_g(a)$. Then there is an irreducible component $\tilde{W}$ of $\mathcal{P}'_{g,\eta}(a)$ for some η such that there is a quasi-finite surjective morphism from $\tilde{W}$ to W. Recall we have a purely inseparable morphism from $\mathcal{P}'_{g,\eta}(a)$ to $\mathcal{N}_g(a)$. Let $\tilde{W}'$ be the corresponding irreducible component of $\mathcal{N}_g(a)$.

We will show there are Λ_x irreducible components in $S_g(a)$ for each $x \in I_{g,a}$. First we show the next claim.

Claim. For $x \in I_{g,a}$, let N be the quasi-polarized superspecial Dieudonné module given in Lemma 4.1. Then we can find an embedding ι from N to M_{g-1} as quasi-polarized Dieudonné modules such that for any generic element $M_0 \subset \cdots \subset M_{g-1}$ in $\tilde{W}'$, we have $S^0(M_0) = \iota(N) \subset M_{g-1}$. Here $S^0(M_0)$ is the smallest superspecial Dieudonné module in M_{g-1} containing M_0.

Proof of Claim. Recall there are only finite number of quasi-polarized Dieudonné submodule N' of M_{g-1} which is isomorphic to N. Indeed since N and M_{g-1} are superspecial, giving an embedding from N to M_{g-1} is equivalent to giving an embedding from the skelton $\tilde{N}$ to the skelton $\tilde{M}_{g-1}$. The inclusion $F^{g-1}\tilde{M}_{g-1} \subset \tilde{N}' \subset \tilde{M}_{g-1}$ implies there are only finite possibilities.

Note $\tilde{M}_{g-1}/F^{g-1}\tilde{M}_{g-1}$ is a finite set. Hence by the irreduciblity of $\tilde{W}'$ and uniqueness of $S^0(M_0)$, there exists a dense open subscheme U of W' such that any point $\{M_0 \subset \cdots \subset M_{g-1}\}$ of U have the same $S^0(M_0)$ in M_{g-1}.

By this Claim, for given polarization η' in Λ_x, we have an irreducible subscheme $\tilde{W}''$ in $\mathcal{N}_g(a)$ generically consisting of rigid PFTQs $M_0 \subset \cdots \subset M_{g-1}$ satisfying $S^0(M_0) = \iota(N) \subset M_{g-1}$ for fixed $\iota(N)$ and also the associated subscheme W'' of $\mathcal{P}'_{g,\eta}(a)$ where η is the pull back of η' by the isogeny from E^g to E^g corresponding with the embedding from N to M_{g-1}. We note that we can take the same submodule N of M_{g-1} for all $\eta' \in \Lambda_x$, since the way to embed principally quasi-polarized supersingular Dieudonné modules to rigid PFTQs does not depend on choices of $\eta' \in \Lambda_x$. Recall that W'' generically consists of isogenies of polarized supersingular abelian varieties

$$(E^g, \eta) \to (Y_{g-2}, \eta_{g-2}) \to \cdots \to (Y_0, \eta_0)$$

which factors as $(Y_{g-1}, \eta) \to (E^g, \eta') \to (Y_0, \eta_0)$ and the isogenies $(E^g, \eta') \to (Y_0, \eta_0)$ are the minimal isogenies defined in [2, Lemma 1.8]. Then the image of W'' in $S_g(a)$ gives an irreducible component of $S_g(a)$. By the uniqueness of minimal isogeny ([2, Lemma 1.8]), another polarization in Λ_x gives a different irreducible component. □

The aim of the rest of this section is to relate $I_{g,a}$ with a set of Li's indices and to show that $\sharp\Lambda_x$ equals a certain class number of the quaternion unitary group over $\mathbb{Q}$ with similitude:

$$G = \{g \in GL_g(B) \mid g^t\overline{g} = \lambda(g)1_g, \lambda(g) \in \mathbb{Q}\}$$

with $B = \mathrm{End}(E) \otimes \mathbb{Q} (\simeq \mathbb{Q}_{\infty,p})$.

4.2 Investigation of index

We can determine the set $I_{g,a}$ by using index introduced by K.-Z. Li ([1, p. 337]). The purpose of this subsection is to show Theorem 4.13. First let us recall the definition of index.

Definition 4.4. (1) A sequence of integers $s = (s_1, \cdots, s_{g-1})$ is called *an index* if $0 \le s_1 \le \cdots \le s_{g-1} < g$, and $s_k < s_{k+1}$ unless $s_{k+1} = 0$.

(2) For two indices $s = (s_k)$ and $t = (t_k)$, the notation $s \prec t$ means that $s_k \le t_k$ for all k.

(3) Let $s = (s_1, \cdots, s_{g-1})$ be an index. We say that a supersingular Dieudonné module M *has index* s if we have

$$\dim_K V_k(M) = s_k$$

for all $k = 1, 2, \cdots, g-1$ with

$$V_k(M) = \frac{M + F^{g-1-k}S^0(M)}{M + F^{g-k}S^0(M)}.$$

(4) We denote by $S_{g,s}$ the locally closed subset consisting of principally polarized supersingular abelian varieties X whose Dieudonné module $\mathbb{D}(X)$ has index s.

Remark 4.5. By Lemma 1.9 in [1], we have $S_g = \coprod_s S_{g,s}$.

There are a few elementary results:

Lemma 4.6. *Let M be a principally quasi-polarized supersingular Dieudonné module with a-number a. Then it follows that $F^{g-a}S^0(M) \subset M$.*

Proof. If M is not superspecial, we know that $a(M) < a((F,V)M)$. Therefore $(F,V)^{g-a}M$ is superspecial. Since $M \subset F^{-g+a}(F,V)^{g-a}M$, we have $S^0(M) \subset F^{-g+a}(F,V)^{g-a}M$ by the minimality of $S^0(M)$. Hence the inclusion $F^{g-a}S^0(M) \subset (F,V)^{g-a}M \subset M$ holds. □

Corollary 4.7. *We have $s_1 > 0$ if and only if $a(M) = 1$. And if $s_1 > 0$, then $s_1 = 1$.*

Proof. By definition, we have $V_1(M) \neq 0$ if and only if $F^{g-2}S^0(M) \not\subset M$, which is equivalent to $a(M) = 1$ by the above lemma. The second statement follows from the definition of index. □

The next lemma is the first step of the proof for Therem 4.13.

Lemma 4.8. *Let M be a principally quasi-polarized supersingular Dieudonné module with basis $w_0, \cdots, w_{g-1}$ and $T = (\tau_{ij})$ a lifting of the associated element of ∇_g as in the previous section. By using the $g \times g$-matrix*

$$L_{n+1}(T) := \sum_{l=0}^{n} \left\{ \sum_{0 \le p_0 < p_1 < \cdots < p_l \le n} (-1)^{n-l-p_0} \binom{n}{p_0} T^{\sigma^{2p_0-n}} T^{\sigma^{2p_1-n-1}} \cdots T^{\sigma^{2p_l-n-l}} \right\} F^{n-l}$$

with coefficient in $W(K)[F]$, we have

$$\begin{pmatrix} (F-V)^{n+1}w_0 \\ \vdots \\ (F-V)^{n+1}w_{g-1} \end{pmatrix} = L_{n+1}(T) \begin{pmatrix} w_0 \\ \vdots \\ w_{g-1} \end{pmatrix}. \tag{15}$$

Proof. We show this by induction of n. For $n = 0$, we have $F - V = T$ by definition.

Let $P_{n-l}(T, n+1)$ be the F^{n-l} coefficient of $(F-V)^{n+1}$. By the equation

$$\begin{aligned}
(F-V)^{n+1} &= (F-V)(F-V)^n \\
&= (F-V)\sum_{l=0}^{n-1} P_{n-1-l}(T,n)F^{n-1-l} \\
&= \sum_{l=0}^{n-1} P_{n-1-l}(T,n)^{\sigma}F^{n-l} - P_{n-1-l}(T,n)^{\sigma^{-1}}F^{n-l-1}V \\
&= \sum_{l=0}^{n-1}(P_{n-1-l}(T,n)^{\sigma} - P_{n-1-l}(T,n)^{\sigma^{-1}})F^{n-l} + P_{n-1-l}(T,n)^{\sigma^{-1}}F^{n-l-1}T \\
&= \sum_{l=0}^{n}(P_{n-l}(T,n)^{\sigma^{-1}}T^{\sigma^{n-l}} + P_{n-l-1}(T,n)^{\sigma} - P_{n-l-1}(T,n)^{\sigma^{-1}})F^{n-l}
\end{aligned}$$

with $P_n(T,n) = 0$ and $P_{-1}(T,n) = 0$, we have

$$P_{n-l}(T,n+1) = P_{n-l}(T,n)^{\sigma^{-1}}T^{\sigma^{n-l}} + P_{n-l-1}(T,n)^{\sigma} - P_{n-l-1}(T,n)^{\sigma^{-1}}.$$

By using this equality and the hypothesis of induction, we get

$$
\begin{aligned}
P_{n-l}(T,n+1) = & \sum_{0\le p_0<p_1<\cdots<p_{l-1}\le n-1} (-1)^{n-l-p_0}\binom{n-1}{p_0}\left(\prod_{j=0}^{l-1} T^{\sigma^{2p_j-n-j}}\right) T^{\sigma^{n-l}} \\
& + \sum_{0\le p_0<p_1<\cdots<p_l\le n-1} (-1)^{n-1-l-p_0}\binom{n-1}{p_0}\prod_{j=0}^{l} T^{\sigma^{2p_j-n-j+2}} \\
& - \sum_{0\le p_0<p_1<\cdots<p_l\le n-1} (-1)^{n-1-l-p_0}\binom{n-1}{p_0}\prod_{j=0}^{l} T^{\sigma^{2p_j-n-j}} \\
= & \sum_{0\le p_0<p_1<\cdots<p_{l-1}<p_l=n} (-1)^{n-l-p_0}\binom{n-1}{p_0}\prod_{j=0}^{l} T^{\sigma^{2p_j-n-j}} \\
& + \sum_{1\le p_0<p_1<\cdots<p_l\le n} (-1)^{n-l-p_0}\binom{n-1}{p_0-1}\prod_{j=0}^{l} T^{\sigma^{2p_j-n-j}} \\
& + \sum_{0\le p_0<p_1<\cdots<p_l\le n-1} (-1)^{n-l-p_0}\binom{n-1}{p_0}\prod_{j=0}^{l} T^{\sigma^{2p_j-n-j}} \\
= & \sum_{0\le p_0<p_1<\cdots<p_l\le n} (-1)^{n-l-p_0}\binom{n}{p_0}\prod_{j=0}^{l} T^{\sigma^{2p_j-n-j}}.
\end{aligned}
$$

□

Let

$$
L_{n+1}(T,m+1) := \sum_{l=m}^{n}\left\{\sum_{0\le p_0<p_1<\cdots<p_l\le n} (-1)^{n-l-p_0}\binom{n}{p_0}\prod_{j=0}^{l} T^{\sigma^{2p_j-n-j}}\right\} F^{n-l}.
$$

Lemma 4.9. *Let M be a principally quasi-polarized supersingular Dieudonné module. Then $V_k(M)$ is generated by the classes of the entries of*

$$
F^{-1}L_{g-k}(T,g-k)^t(w_0,\cdots,w_{g-1}),\ \cdots\ ,\ F^{-k}L_{g-1}(T,g-k)^t(w_0,\cdots,w_{g-1}).
$$

Proof. By the equality $S^0(M)=F^{-g+1}(F,V)^{g-1}M$, we see that $F^{g-1-k}S^0(M)$ is generated over $W(K)[F]$ by $F^{g-1-k-n}(F-V)^n w_j$ for any $n=0,\cdots,g-1$ and for any j. Since $F^{g-1-k-n}(F-V)^n w_j$ is contained in M for $n\le g-1-k$, it follows that V_k is generated by the classes of

$$
F^{-1}(F-V)^{g-k}w_j,\cdots,F^{-k}(F-V)^{g-1}w_j
$$

for all $0\le j\le g-1$. Lemma 4.8 implies that V_k is generated by the classes of the entries of

$$
F^{-1}L_{g-k}(T)^t(w_0,\cdots,w_{g-1}),\ \cdots\ ,\ F^{-k}L_{g-1}(T)^t(w_0,\cdots,w_{g-1}).
$$

Since the contribution of the terms with $0\le l\le g-k-2$ in $F^{g-k-n-2}L_{n+1}(T)^t(w_0,\cdots,w_{g-1})$:

$$
\sum_{l=0}^{n}\left\{\sum_{0\le p_0<p_1<\cdots<p_l\le n} (-1)^{n-l-p_0}\binom{n}{p_0}\prod_{j=0}^{l} T^{\sigma^{2p_j-2n-j+g-k-2}}\right\} F^{g-k-2-l}
$$

is in M, we have this lemma. □

From now, we shall treat polynomials in $a_{ij}^{\sigma^n}$ (a_{ij}: entries of T and $n \in \mathbb{Z}$). Let the degree of $a_{ij}^{\sigma^n}$ be p^n even for $n < 0$. Such a polynomial in $a_{ij}^{\sigma^n}$ ($n \in \mathbb{Z}$) is called a semi-polynomial in T.

The next is a key lemma for the proof of Theorem 4.13 below.

Lemma 4.10. *The $g \times g$ matrices*

$$F^{-1}L_{g-k}(T, g-k), \ \cdots \ , \ F^{-k}L_{g-1}(T, g-k)$$

contain terms which have the lowest degrees at each coefficient of F^{-l}:

$$\begin{aligned}
&\{U_{g-k}^{\sigma^{-1}}F^{-1}\},\\
&\{-U_{g-k}^{\sigma^{-2}}F^{-1}, U_{g-k+1}^{\sigma^{-2}}F^{-2}\},\\
&\qquad\vdots\\
&\{(-1)^{k-1}U_{g-k}^{\sigma^{-k}}F^{-1}, \cdots, U_{g-1}^{\sigma^{-k}}F^{-k}\}
\end{aligned}$$

respectively with

$$\begin{aligned}
U_{g-k} :=& T^{\sigma^{-g+k+1}}T^{\sigma^{-g+k+2}}\cdots T^{\sigma^{-1}}T\\
U_{g-k+1} :=& T^{\sigma^{-g+k}}T^{\sigma^{-g+k+1}}T^{\sigma^{-g+k+2}}\cdots T^{\sigma^{-1}}T\\
&\vdots\\
U_{g-1} :=& T^{\sigma^{-g+2}}T^{\sigma^{-g+3}}T^{\sigma^{-g+4}}T^{\sigma^{-g+5}}\cdots T^{\sigma^{-1}}T.
\end{aligned}$$

Proof. The terms are nothing but the terms corresponding to the lowest $p_0, \cdots, p_l$. The straightforward calculation show this lemma. $\square$

We need two more lemmas.

Lemma 4.11. *For a lifting T of an element of $\nabla_{g,g-r}(i_1, \cdots, i_{r'})$ ($r' = r$ or $r-1$) in Proposition 3.9, we have*

$$S_{L_{n+1}(T,g-k)} = S_{U_{g-k}} = \{(i_l, j_{l+g-1-k}) | l = 1, \cdots, r' - (g-1-k)\}$$

where $j_l = i_l + 1$ unless (iii-a) $l = (r+1)/2$ and $j_l = i_l + 2$ for (iii-a) $l = (r+1)/2$.

Proof. This follows obviously from definition of $\nabla_{g,g-r}(i_1, \cdots, i_{r'})$ in Proposition 3.9. $\square$

Lemma 4.12. *(1) Let l, m, n be positive integers with $l, m > n$. In the affine space $\mathbb{A}^l$, the locus*

$$\{(a_1, \cdots, a_l) \in \mathbb{A}^l | \operatorname{rk} J(a_1, \cdots, a_l; m) \le n\}$$

is a proper closed subset in $\mathbb{A}^l$ with

$$J(a_1, \cdots, a_l; m) := \begin{pmatrix} a_1 & \cdots & a_l \\ a_1^{\sigma} & \cdots & a_l^{\sigma} \\ \vdots & & \vdots \\ a_1^{\sigma^{m-1}} & \cdots & a_l^{\sigma^{m-1}} \end{pmatrix}.$$

(2) We have

$$\det J(a_1, \cdots, a_l; l) = (-1)^{l(l-1)/2} \prod_{1 \le i \le l} \prod_{\lambda_{i+1}, \cdots, \lambda_l \in \mathbb{F}_p} (a_i + \lambda_{i+1} a_{i+1} + \cdots + \lambda_l a_l) \tag{16}$$

Proof. Obviously (1) follows from (2). The equation (16) is very similar to [1, Lemma 1.4]. It is clear that each factor of the right hand side devides the left hand side. In order to determine the sign, it suffices to compare the coefficients of $a_1^{p^{l-1}}$ of the both sides. By induction, we can show that the sign is equal to $(-1)^{l(l-1)/2}$. □

Theorem 4.13. *Any generic element M associated with an element of*

$$\nabla_{g,g-r}(i_1, \cdots, i_{r'}) \quad (r' = r \text{ or } r - 1)$$

has index

$$s := (0, \cdots, 0, i_1 + 1, i_2 + 1, \cdots, i_{r'} + 1),$$

i.e., $s_k = i_{r'-(g-1-k)} + 1$ with $i_j = -1$ $(j \le 0)$.

Proof. Let s_M be the index of M. By Lemma 4.9 and 4.11, we have $s_M \prec s$ for any M. Let us show that $s_M \succ s$ for generic M.

Since $g - j_{g-k} = i_{r'-(g-1-k)} + 1$, it suffices to show that $g - j_{g-k}$ elements

$$\begin{array}{c} F^{-1} w_{j_{g-k}}, \cdots, F^{-1} w_{j_{g-k+1}-1} \\ F^{-2} w_{j_{g-k+1}}, \cdots, F^{-2} w_{j_{g-k+2}-1} \\ \vdots \\ F^{g-k-1-r'} w_{j_{r'}}, \cdots, F^{g-k-1-r'} w_{g-1} \end{array} \tag{17}$$

give linearly independent classes of $V_k(M)$ for generic M.

From now on, we show this by induction of k. For $k = 1$, by Corollary 4.7 we have $\dim V_1(M) = s_1 = 1$ if $a(M) = 1$ and $\dim V_1(M) = s_1 = 0$ otherwise, since $a(M) = 1$ is equivalent to $r = g - 1$. If $a(M) = 1$, then $F^{-1} w_{g-1}$ generates the one dimensional K-vector space $V_1(M)$ (note $r' = g - 1$ and $j_{r'} = g - 1$).

Let the first entry of $F^{-l-1} L_{g-k-l}(T, g-k)^t (w_0, \cdots, w_{g-1})$ be

$$\alpha_{l, j_{g-k}} w_{j_{g-k}} + \alpha_{l, j_{g-k}+1} w_{j_{g-k}+1}, \cdots, \alpha_{l, g-1} w_{g-1}.$$

We denote by $\overline{\alpha}_{lm}$ the w_m-coefficient of the above element modulo $M + F^{g-k} S^0(M)$. By the hypothesis of induction, we know that $M + F^{g-k} S^0(M)$ is generated by

$$\begin{array}{c} F^{-1} w_{j_{g-k+1}}, \cdots, F^{-1} w_{j_{g-k+2}-1} \\ \vdots \\ F^{g-k-r'} w_{j_{r'}}, \cdots, F^{g-k-r'} w_{g-1} \end{array}$$

for generic M. Hence we can regard $\overline{\alpha}_{lm}$ as an element of K.

It suffices to show the matrix $\overline{\alpha} = (\overline{\alpha}_{lm})$ has rank $g - j_{g-k}$ for generic M. Let $u_{g-k-l+1}$ be the first raw vector of $U_{g-k-1+l}$ modulo $(j_{g-k+l}, \cdots, g-1)$-th entries. Let us consider the $k \times (g - j_{g-k})$ matrix

$$u = \begin{pmatrix} u_{g-k}^{\sigma^{-1}} & 0 & 0 & \cdots & 0 \\ u_{g-k}^{\sigma^{-2}} & u_{g-k+1}^{\sigma^{-2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ u_{g-k}^{\sigma^{g-k-r'}} & u_{g-k+1}^{\sigma^{g-k-r'}} & u_{g-k+2}^{\sigma^{g-k-r'}} & \cdots & 0 \\ u_{g-k}^{\sigma^{g-1-k-r'}} & u_{g-k+1}^{\sigma^{g-1-k-r'}} & u_{g-k+2}^{\sigma^{g-1-k-r'}} & \cdots & u_{r'}^{\sigma^{g-1-k-r'}} \\ \vdots & \vdots & \vdots & & \vdots \\ u_{g-k}^{\sigma^{-k}} & u_{g-k+1}^{\sigma^{-k}} & u_{g-k+2}^{\sigma^{-k}} & \cdots & u_{r'}^{\sigma^{-k}} \end{pmatrix}$$

By Lemma 4.10, each (i,j)-th entry of u is equal, up to sign, to the part with the lowest degree of the (i,j)-th entry of the matrix $\overline{\alpha}$ as the semi-polynomials in T. Moreover any minor of u is up to sign given by the part with the lowest degree of the associated minor of $\overline{\alpha}$. Hence we have only to show that u has rank $g - j_{g-k}$. Because the fact that a minor of u is not identically zero implies that the associated minor of $\overline{\alpha}$ is not identically zero.

Let u' be the matrix (u'_{ij}) which is defined by $u'_{ij} = u_{i+k-g+j_{g-k},j}$ for $j_l - j_{g-k} + 1 \le i, j \le j_{l+1} - j_{g-k}$ ($l = g-k, \cdots, r'$ with $j_{r'+1} := g$) and $u'_{ij} = 0$ for the other i and j. Here we note that $j_l \ge l$ for all l. The determinant of u' is the minor with the lowest degree as a semi-polynomial in T among minors of u with size $g - j_{g-k}$. Since entries of u_l are algebraically independent of each other, for each $l = g-k, \cdots, g-1$, the determinant of u' is generically non-zero by Lemma 4.12 (1). Then u and therefore $\overline{\alpha}$ have rank $g - j_{g-k}$ generically. $\square$

Corollary 4.14. *The number of $I_{g,a}$ is*

$$\begin{cases} \begin{pmatrix} (g-2)/2 \\ (g-1-a)/2 \end{pmatrix} & \textit{if } g \textit{ is even and } a \textit{ is odd}, \\ \begin{pmatrix} [g/2] \\ [(g-a)/2] \end{pmatrix} & \textit{otherwise}. \end{cases}$$

Proof. Corollary 3.12 and Theorem 4.13 imply $\sharp I_{g,a} = \sharp J_{g,a}$. $\square$

4.3 Main results and their proofs

First we show Lemma 4.1.

Proof of Lemma 4.1. For $M \in D_g(a)$ with good basis $w_0, \cdots, w_{g-1}$ satisfying Theorem 4.13, let us put

$$\begin{aligned} N &= W(K)[F] < w_0, \cdots, w_{j_1-1}, F^{-1}w_{j_1}, \cdots, F^{-1}w_{j_2-1}, \cdots, F^{-r'}w_{j_{r'}}, \cdots, F^{-r'}w_{g-1} > \\ &= A < w_0, \cdots, w_{j_1-1}, F^{-1}w_{j_1}, \cdots, F^{-1}w_{j_2-1}, \cdots, F^{-r'}w_{j_{r'}}, \cdots, F^{-r'}w_{g-1} > . \end{aligned}$$

Then N is a quasi-polarized superspecial Dieudonné module, since $a(M) = \dim N/(F,V)N = \dim N/FN = g$ and $N = S^0(M)$ for generic M by the proof of Theorem 4.13. By using Lemma 3.5, we can determine the quasi-polarization on N. In fact the quasi-polarization on

N is characterized by $N^t = F^{r'}N$ unless (iii-a). In the case of (iii-a), we have an orthogonal decomposition $N = N_1 \oplus N_2$ such that $N_1^t = F^{r'}N_1, N_2^t = F^{r'-1}N_2$ and $N_2 \simeq A_{1,1}$.

Since such a quasi-polarized superspecial Dieudonné module as N is uniquely determined up to isomorphism (see [2, Proposition 6.1]), we obtain Lemma 4.1. $\square$

Proposition 4.15. *Let x be the element of $I_{g,a}$ associated with index $(0, \cdots, 0, i_1{+}1, \cdots, i_{r'}{+}1)$ with $r' = r$ or $r - 1$. The sequence $(i_1, \cdots, i_{r'})$ is of either of types listed in Proposition 3.9. Then $\sharp\Lambda_x$ is equal to*

$$\begin{cases} H_g(p,1) & \text{for } r' \text{ even}, \\ H_g(1,p) & \text{for } r' \text{ odd}, \end{cases}$$

where $H_g(p,1)$ is the class number of G with genus 1_g at prime spot p (principal genus), $H_g(1,p)$ is the class number with genus $\mathrm{diag}(A, \cdots, A, B)$ at p with $A = \begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix}$ and $B = A$ for even g and (p) for odd g (non-principal genus).

Proof. For given $\eta \in \Lambda_x$, it follows that $\ker(\eta) = \ker(F^{r'})$ unless (iii-a) in Proposition 3.9 by the proof of Lemma 4.1. First let us investigate the cases other than (iii-a). For even r', by applying Corollary 4.8 (i) in [2] we see that the number of Λ_x is given by $H_g(p,1)$. When r' is odd, g has to be even by the classification of Proposition 3.9. Then we have $\deg \eta = p^{2(ng+g/2)}$ with $r' = 2n+1$. Hence we can apply Corollary 4.8 (ii) in [2] to this case and we obtain $\sharp\Lambda_x = H_g(1,p)$.

In the case of (iii-a), we have $\ker(F^{r'}) \supset \ker(\eta)$ and $\deg \eta = p^{r'g-1} = p^{2(ng+g-[(g+1)/2])}$ with $r' = 2n+1$. This follows from the proof of Lemma 4.1. Then Corollary 4.8 (iii) in [2] implies that $\sharp\Lambda_x$ is equal to $H_g(1,p)$. $\square$

By putting together Theorem 4.13, Proposition 4.3 and 4.15, we have the final result:

Theorem 4.16. *The cardinal number of irreducible components of $S_g(a)$ is equal to*

$$\begin{cases} \binom{(g-2)/2}{(g-a-1)/2} H_g(1,p) & \text{for } g \text{ even and } a \text{ odd}, \\ \binom{(g-1)/2}{(g-a)/2} H_g(p,1) & \text{for } g, a \text{ odd}, \\ \binom{g/2-1}{(g-a)/2} H_g(p,1) + \binom{g/2-1}{(g-a)/2-1} H_g(1,p) & \text{for } g, a \text{ even}, \\ \binom{(g-1)/2-1}{(g-a-1)/2} H_g(1,p) + \binom{(g-1)/2-1}{(g-a-1)/2-1} H_g(p,1) & \text{for } g \text{ odd and } a \text{ even}. \end{cases}$$

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