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by

Mourad Choulli and Masahiro Yamamoto

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

CONDITIONAL STABILITY IN DETERMINING A HEAT SOURCE

[1] Mourad Choulli and [2]Masahiro Yamamoto

[1] Department of Mathematics, Université de Metz
57045 Metz Cedex 1, France
choulli@poncelet.sciences.univ-metz.fr
[2]Department of Mathematical Sciences, The University of Tokyo
3-8-1 Komaba, Meguro, Tokyo 153 Japan
myama@ms.u-tokyo.ac.jp

Abstract. We establish the uniqueness and conditional stability in determining a heat source term from boundary measurements which are started after some time. The key is analyticity of solutions in the time and we apply the maximum principle for analytic functions.

§1. Introduction.

We consider an initial/boundary value problem for the the two-dimensional heat equation :

$$
\left\{
\begin{aligned}
&\frac{\partial u}{\partial t}(x_1, x_2, t) = \Delta u(x_1, x_2, t)\\
&\qquad\qquad + \sigma(t) f(x_1, x_2), \quad (x_1, x_2) \in \Omega,\ 0 < t < T,\\
&u(x_1, x_2, 0) = 0, \qquad (x_1, x_2) \in \Omega,\\
&\frac{\partial u}{\partial n}(x_1, x_2, t) = 0, \qquad (x_1, x_2) \in \partial\Omega,\ 0 < t < T.
\end{aligned}
\right.
\tag{1.1}
$$

Here $\Omega \subset \mathbb{R}^2$ is the rectangle: $\Omega = (0,1) \times (0,1)$ and $x = (x_1, x_2) \in \mathbb{R}^2$, Δ is the Laplacian. The non-homogeneous term $\sigma(t) f(x_1, x_2)$ is considered as a heat source, and in the case of $\sigma(t) = e^{-\lambda t}$ with $\lambda > 0$, system (1.1) describes a heat process where a radioactive isotope with the decay rate λ supplies heat, whose spatial density is given by $f(x_1, x_2)$, $(x_1, x_2) \in \Omega$. Especially in the case of $\sigma(t) = e^{-\lambda t}$,

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system (1.1) corresponds to a simple thermal model of the earth where the high temperature inside results from the decay of a radioactive isotope (e.g., Lavrentiev, Romanov and Vasiliev [6]).

For $f \in L^2(\Omega)$ and $\sigma \in C[0,T]$, there exists a unique mild solution $u \in C([0,T];L^2(\Omega))$ to (1.1), and we denote it by $u = u(f)$ (e.g., Pazy [10], Tanabe [11]) provided that σ is fixed.

For a given $\nu > 0$, our problem is determination of f from

$$u(f)(x_1,0,t) = h(x_1,t), \qquad 0 < x_1 < 1,\ \nu < t < T,$$

and we here discuss the uniqueness and the stability.

In the case of $\nu = 0$, in other words, if we can observe the temperature from the initial time $t = 0$, then we refer to Lavrentiev, Romanov and Vasiliev [6] for the uniqueness, to Cannon [2], Yamamoto [12], [13] for the conditional stability under a priori bound of f with Sobolev norm of higher order.

The purpose of this paper is to discuss the uniqueness and the conditional stability in the case of $\nu > 0$. In particular, when we relate the determination problem of a heat source with a problem in geophysics proposed by Tikhonov (see Lavrentiev, Romanov and Vasiliev [6], pp.49-50), it is natural to assume that $\nu > 0$, that is, the observation should be started after some time passing.

The key is the analyticity of solutions in t, and our main result on the conditional stability is proved by means of the result in Yamamoto [12], [13] and the maximum principle for analytic fucntions.

§2. Main results.

Throughout this paper, f and σ are real-valued and so $u(f)(x_1, x_2, t)$ is real for $(x_1, x_2) \in \overline{\Omega}$ and $0 \le t \le T$.

We define an operator A in $L^2(\Omega)$ by

$$(Au)(x) = -\Delta u(x), \qquad x = (x_1, x_2) \in \Omega$$

$$\mathcal{D}(A) = \{\, u \in H^2(\Omega);\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0\,\}. \tag{2.1}$$

Let $\|\cdot\|$ denote the norm in $L^2(\Omega)$, and $H^\alpha(\Omega)$, $\alpha > 0$, denote a usual Sobolev space (e.g., Adams [1]). We fix $\gamma > 0$. Then we can define a fractional power $(A+\gamma)^\alpha$, $\alpha \in \mathbb{R}$ (e.g., Pazy [10], Tanabe [11]) and

$$\mathcal{D}((A+\gamma)^\alpha) = H^{2\alpha}(\Omega), \quad 0 < \alpha < \frac{3}{4} \tag{2.2}$$

and there exists a constant $C_1 > 0$ such that

$$C_1^{-1}\|u\|_{H^{2\alpha}(\Omega)} \le \|(A+\gamma)^\alpha u\| \le C_1\|u\|_{H^{2\alpha}(\Omega)}, \qquad u \in \mathcal{D}((A+\gamma)^\alpha) \tag{2.3}$$

(e.g., Fujiwara [4]). We set

$$S_\theta = \{z \in \mathbb{C}; 0 < |z| < T, |\arg z| < \theta\} \tag{2.4}$$

with $\theta \in (0, \frac{\pi}{2}]$.

Now we are ready to state our main results.

Theorem 1 (Uniqueness). *Let us assume*

$$\sigma \in C^1[0,T] \text{ is extended analytically to } S_\theta \text{ with some } \theta \in (0, \frac{\pi}{2}], \quad \sigma(0) \ne 0,$$
$$\sup_{z\in S_\theta}\left|\frac{d^k\sigma}{dz^k}(z)\right| < \infty, \quad k = 0, 1 \tag{2.5}$$

and let $0 < \nu < T$ *be given arbitrarily. Moreover* $\Gamma = \partial\Omega \cap \{x; |x - x_0| < \rho\}$ *with some* $x_0 \in \mathbb{R}^2$ *and* $\rho > 0$*, and we assume that* $\Gamma \neq \emptyset$*. Then* $u(f)(x_1, x_2, t) = 0$*,* $(x_1, x_2) \in \Gamma$*,* $\nu < t < T$ *with* $f \in L^2(\Omega)$ *implies* $f(x_1, x_2) = 0$*,* $(x_1, x_2) \in \Omega$*.*

Theorem 2 (Conditional Stability). *Let* $M > 0$ *and*

$$1 > \alpha > \frac{1}{4}, \tag{2.6}$$

and we set

$$\mathcal{U}_{M,\alpha} = \{f \in \mathcal{D}((A + \gamma)^\alpha); \|(A + \gamma)^\alpha f\| \le M\}. \tag{2.7}$$

We assume (2.5). Let $0 < \nu < T$ *be given arbitrarily. Then for* $0 < \kappa < \alpha$*, there exists a constant* $C > 0$ *depending on* σ*,* M*,* α *and* κ *such that*

$$\|f\| \le C \left(\frac{1}{\log\log\left(\frac{1}{\|u(f)(\cdot,0,\cdot)\|_{L^2((\nu,T)\times(0,1))}}\right)} \right)^\kappa \tag{2.8}$$

for $f \in \mathcal{U}_{M,\alpha}$*.*

Here we set

$$\|u(f)(\cdot, 0, \cdot)\|_{L^2((\nu,T)\times(0,1))} = \left(\int_\nu^T \int_0^1 |u(f)(x_1, 0, t)|^2 dx_1 dt \right)^{\frac{1}{2}}.$$

By the regularity property for the parabolic equation, we cannot expect any continuity of the map $u(f)(\cdot, 0, \cdot) \longmapsto f$ from $L^2((\nu, T) \times (0, 1))$ to $L^2(\Omega)$ if for unknown f, we do not assume a priori bound such as $\mathcal{U}_{M,\alpha}$. In the case of $\nu = 0$, a single logarithmic stability is proved in Yamamoto [12]. More precisely,

Theorem 0. *([12]) Let* $M > 0$*,* $\alpha > 0$ *and* $T > 0$ *be given arbitrarily, and* $\sigma \in C^1[0, T]$ *such that* $\sigma(0) \neq 0$*. Then for* $0 < \kappa < \alpha$*, there exists a constant* $C > 0$ *depending on* σ*,* M*,* α *and* κ *such that*

$$\|f\| \le C \left(\frac{1}{\log\left(\frac{1}{\|u(f)(\cdot,0,\cdot)\|_{H^1(0,T;L^2(0,1))}}\right)} \right)^\kappa$$

for $f \in \mathcal{U}_{M,\alpha}$.

Here we set

$$\|u(f)(\cdot,0,\cdot)\|_{H^1(0,T;L^2(0,1))} = \left(\int_0^T \int_0^1 |u(f)(x_1,0,t)|^2 + \left|\frac{\partial u(f)}{\partial t}(x_1,0,t)\right|^2 dx_1 dt\right)^{\frac{1}{2}}.$$

In this paper, we take the rectangle as Ω, but we can obtain similar results for a general domain.

§3. Analyticity of solutions.

We show

Lemma 1. *Let (2.5) be satisfied and let $0 < \alpha < 1$. We can extend $(A+\gamma)^\alpha u(f)(\cdot,\cdot,t)$ analytically to a map $S_\theta \longrightarrow L^2(\Omega)$. Moreover there exists a constant $C_2 = C_2(\alpha,T,\theta) > 0$ such that*

$$\|(A+\gamma)^\alpha u(f)(\cdot,\cdot,z)\| \le C_2\|f\|, \quad z \in S_\theta. \tag{3.1}$$

Proof. Since $-A$ generates an analytic semigroup in $\mathrm{Re}\, z > 0$ (e.g., [10], [11]), we can define $U(z)$ by

$$\begin{aligned} U(z) &= \int_{0z} (A+\gamma)^\alpha e^{-(z-s)A}\sigma(s) f ds \\ &= \int_{0z} (A+\gamma)^\alpha e^{-sA}\sigma(z-s) f ds \end{aligned} \tag{3.2}$$

where the integration is done along the segment in $\mathbb{C}$ connecting 0 and $z \in \mathbb{C}$. We will prove that $U(z)$ is differentiable in $z \in S_\theta \subset \mathbb{C}$. We set

$$U_\epsilon(z) = \int_{\epsilon z} (A+\gamma)^\alpha e^{-sA}\sigma(z-s) f ds \tag{3.3}$$

for small $\epsilon > 0$. First we note

$$\|(A+\gamma)^\alpha e^{-\eta A}\| \le C_3|\eta|^{-\alpha}, \quad \eta \in S_\theta \tag{3.4}$$

(e.g., [10], [11], Hayden and Massey [5], p.429). Therefore $U_\epsilon(z)$ is well-defined in $L^2(\Omega)$ for all $z \in S_\theta$. Since

$$\frac{dU_\epsilon}{dz}(z) = (A+\gamma)^\alpha e^{-zA}\sigma(0)f + \int_{\epsilon z}(A+\gamma)^\alpha e^{-sA}\frac{d\sigma}{dz}(z-s)f ds,$$

we see that $U_\epsilon : S_\theta \longrightarrow L^2(\Omega)$ is analytic. Furthermore, by (3.4), we have

$$\begin{aligned}
&\left\|\int_{0\epsilon}(A+\gamma)^\alpha e^{-sA}\sigma(z-s)f ds\right\| = \left\|\int_0^\epsilon (A+\gamma)^\alpha e^{-\eta zA}\sigma(z-\eta z)zf d\eta\right\| \\
&\le \int_0^\epsilon \|(A+\gamma)^\alpha e^{-\eta zA}d\eta|z|\|\sigma\|_{L^\infty(S_\theta)}\|f\| \\
&\le C_3\int_0^\epsilon \eta^{-\alpha}d\eta|z|^{1-\alpha}\|\sigma\|_{L^\infty(S_\theta)}\|f\| = C_3\frac{\epsilon^{1-\alpha}}{1-\alpha}|z|^{1-\alpha}\|\sigma\|_{L^\infty(S_\theta)}\|f\|.
\end{aligned} \tag{3.5}$$

The right hand side tends to 0 as $\epsilon \downarrow 0$, so that

$$\lim_{\epsilon\downarrow 0}\left\|\int_{0\epsilon}(A+\gamma)^\alpha e^{-sA}\sigma(z-s)f ds\right\| = 0$$

in $L^2(\Omega)$ and the convergence is uniform for z in any compact subset in S_θ. Since $U_\epsilon : S_\theta \longrightarrow L^2(\Omega)$ is analytic, we see that $U : S_\theta \longrightarrow L^2(\Omega)$ is analytic. Finally we can prove (3.1) similarly to (3.5).

Next, taking $\alpha \in \left(\frac{1}{4}, 1\right)$ in Lemma 1, by the trace theorem, we can easily derive

Lemma 2. *We can extend $u(f)(\cdot,\cdot,t)$ analytically to a map $S_\theta \longrightarrow L^2(\partial\Omega)$ and*

$$\|u(f)(\cdot,\cdot,z)\|_{L^2(\partial\Omega)} \le C_2\|f\|, \quad z \in S_\theta.$$

§4. Proof of Theorem 1.

By Lemma 2, $u(f)(x_1,x_2,t) = 0$ for $(x_1,x_2) \in \Gamma$ and $\nu < t < T$ implies

$$u(f)(x_1,x_2,t) = 0, \qquad (x_1,x_2) \in \Gamma,\ 0 < t < T. \tag{4.1}$$

On the other hand, by the variation of constants, we have

$$u(f)(x_1,x_2,t) = \int_0^t \sigma(t-s)w(x_1,x_2,s)ds, \quad (x_1,x_2) \in \overline{\Omega},\ 0<t<T, \tag{4.2}$$

where $w(x_1,x_2,t)$ satisfies

$$\begin{cases} \dfrac{\partial w}{\partial t}(x_1,x_2,t) = \Delta w(x_1,x_2,t), \quad (x_1,x_2) \in \Omega,\ 0<t<T, \\ w(x_1,x_2,0) = f(x_1,x_2), \qquad (x_1,x_2) \in \Omega, \\ \dfrac{\partial w}{\partial n}(x_1,x_2,t) = 0, \qquad (x_1,x_2) \in \partial\Omega,\ 0<t<T. \end{cases} \tag{4.3}$$

In fact, by direct calculations, we can verify that the right hand side of (4.2) satisfies (1.1). Apply (4.1) in (4.2), and we obtain

$$\int_0^t \sigma(t-s)w(x_1,x_2,s)ds = 0, \quad (x_1,x_2) \in \Gamma,\ 0<t<T. \tag{4.4}$$

Therefore we can differentiate the both hand sides of (4.4) in t, and we obtain

$$\begin{gathered} \sigma(0)w(x_1,x_2,t) + \int_0^t \frac{d\sigma}{dt}(t-s)w(x_1,x_2,s)ds = 0, \\ (x_1,x_2) \in \Gamma,\ 0<t<T. \end{gathered} \tag{4.5}$$

For $f \in L^2(\Omega)$, in general, $w|_\Gamma$ is not in $L^2(\Gamma \times (0,T))$, and for verifying $w=0$ on $\Gamma \times (0,T)$, we will proceed as follows. Let us fix $\frac{1}{2} < \beta < 1$. Then

$$\widehat{w} \equiv t^\beta w|_\Gamma \in C([0,T]; H^{\frac{1}{2}}(\Gamma)). \tag{4.6}$$

In fact, let $F(t) = t^\beta e^{-tA} f$. Then

$$\begin{aligned} &(A+\gamma)^{\frac{1}{2}}(F(t+h) - F(t)) \\ =&((t+h)^\beta - t^\beta)(A+\gamma)^{\frac{1}{2}} e^{-(t+h)A} f \\ +&t^\beta (A+\gamma)^{\frac{1}{2}} e^{-tA}(e^{-hA} f - f), \end{aligned}$$

and

$$\|(A+\gamma)^{\frac{1}{2}}(F(t+h)-F(t))\| \le C_4(h^{\beta-1/2}+\|e^{-hA}f-f\|)$$

by (3.4). Therefore

$$\lim_{h\to 0}\|(A+\gamma)^{\frac{1}{2}}(F(t+h)-F(t))\| = 0$$

for $t\in[0,T]$. Since $\mathcal{D}((A+\gamma)^{\frac{1}{2}}) = H^1(\Omega)$, the trace theorem implies (4.5).

Setting

$$k(t,s) = \frac{1}{\sigma(0)}t^\beta s^{-\beta}\frac{d\sigma}{dt}(t-s),$$

we see that (4.5) is equivalent to

$$\widehat{w}(t) + \int_0^t k(t,s)\widehat{w}(s)ds = 0, \qquad 0\le t\le T. \tag{4.7}$$

For any $\phi\in L^2(\Gamma)$, we set $v(t) = (\widehat{w}(t),\phi)_{L^2(\Gamma)}$. Then by (4.6), we see that $v\in C[0,T]$, and equality (4.7) yields

$$v(t) = -\int_0^t k(t,s)v(s)ds, \qquad 0\le t\le T. \tag{4.8}$$

Set $M_1 = \|v\|_{C[0,T]}$. Since $|k(t,s)|\le C_5t^\beta s^{-\beta}$, by (4.8), we obtain

$$|v(t)| \le M_1C_5\frac{t}{1-\beta}, \qquad 0\le t\le T. \tag{4.9}$$

Apply (4.9) at the right hand side of (4.8), we have

$$|v(t)| \le \frac{M_1C_5^2t^2}{(1-\beta)(2-\beta)}, \qquad 0\le t\le T.$$

Continuing this argument, for $n\in\mathbb{N}$, we see

$$\begin{aligned}|v(t)| &\le \frac{M_1C_5^nt^n}{(1-\beta)(2-\beta)\cdots(n-\beta)}\\ &\le \frac{M_1(C_5T)^n}{(1-\beta)(2-\beta)\cdots(n-\beta)}, \qquad 0\le t\le T.\end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{(C_5T)^n}{(1-\beta)(2-\beta)\cdots(n-\beta)} = 0,$$

we have $v(t) = (\widehat{w}(t), \phi)_{L^2(\Gamma)} = 0$, $0 \le t \le T$. Since $\phi \in L^2(\Gamma)$ is arbitrary, this implies that $\widehat{w}(t) = t^\beta w_{|\Gamma} = 0$, $0 \le t \le T$. Thus $w_{|\Gamma} = 0$ on $\Gamma \times (0, T)$.

Therefore application of the unique continuation theorem (e.g., Mizohata [8], [9]) to (4.3) yields $w(x_1, x_2, t) = 0$, $(x_1, x_2) \in \Omega$, $0 < t < T$. For the application of the unique continuation, we note that $w \in C^2(\Omega \times (0, T))$, by the smoothing property in (4.3) (e.g., [10]). Consequently $f(x_1, x_2) = w(x_1, x_2, 0) = 0$, $(x_1, x_2) \in \Omega$, follows. Thus the proof of Theorem 1 is complete.

§5. Proof of Theorem 2.

Without loss of generality, we may assume that $\|u(f)(\cdot, 0, \cdot)\|_{L^2((\nu,T)\times(0,1))}$ is sufficiently small for $M > 0$ and S_θ. We divide the proof into the following four steps.

Let us choose $\beta > 0$ such that $\frac{1}{4} < \beta < \alpha$ and $\alpha - \beta$ is sufficiently small.

First Step. We show: There exists a constant $C_6 > 0$ such that

$$\|(A+\gamma)^\beta u(f)\|_{C^1([0,T];L^2(\Omega))} \le C_6 \|(A+\gamma)^\beta f\|, \quad f \in \mathcal{D}((A+\gamma)^\beta). \tag{5.1}$$

Proof of (5.1). We have

$$u(f)(\cdot, t) = \int_0^t e^{-(t-s)A} \sigma(s) f ds, \quad t > 0, \tag{5.2}$$

so that

$$(A+\gamma)^\beta u(f)(\cdot, t) = \int_0^t e^{-(t-s)A} \sigma(s) (A+\gamma)^\beta f ds, \quad t > 0.$$

We set $g = (A+\gamma)^\beta f \in L^2(\Omega)$. Then

$$
\begin{aligned}
&(A+\gamma)^\beta u(f)(\cdot,t) \\
&= \int_0^t e^{-(t-s)A}(\sigma(s)-\sigma(t))g\,ds + \left(\int_0^t e^{-sA}ds\right)\sigma(t)g \\
&\equiv S_1(t) + S_2(t). \qquad (5.3)
\end{aligned}
$$

Since $\sigma \in C^1[0,T]$ and (3.4) with $\alpha = 1$, we have

$$\frac{dS_1}{dt}(t) = \int_0^t -Ae^{-(t-s)A}(\sigma(s)-\sigma(t))g\,ds - \int_0^t e^{-(t-s)A}\frac{d\sigma}{dt}(t)g\,ds$$

and

$$\|S_1\|_{C^1([0,T];L^2(\Omega))} \le C_6'\|g\|.$$

Next, since

$$\frac{dS_2}{dt}(t) = e^{-tA}\sigma(t)g + \int_0^t e^{-sA}ds\frac{d\sigma}{dt}(t)g, \quad 0 < t < T,$$

we see $\|S_2\|_{C^1([0,T];L^2(\Omega))} \le C_6'\|g\|$. Thus (5.1) is seen.

Second we will prove

$$\|(A+\gamma)^\beta u(f)\|_{C^2([0,T];L^2(\Omega))} \le C_7\|(A+\gamma)^{\beta+1}f\|, \quad f \in \mathcal{D}((A+\gamma)^{\beta+1}). \qquad (5.4)$$

Proof of (5.4). By (5.2) and $f \in \mathcal{D}((A+\gamma)^{\beta+1})$, we have

$$\frac{du(f)}{dt}(t) = -\int_0^t e^{-(t-s)A}\sigma(s)Af\,ds + \sigma(t)f,$$

so that

$$
\begin{aligned}
&(A+\gamma)^\beta\frac{du(f)}{dt}(t) = -\int_0^t e^{-(t-s)A}\sigma(s)(A+\gamma)^\beta Af\,ds \\
&+\sigma(t)(A+\gamma)^\beta f, \qquad 0 \le t \le T. \qquad (5.5)
\end{aligned}
$$

We apply an argument similar to (5.3) to the first term at the right hand side of (5.5) and use (2.5) in the second term, and the proof of (5.4) is complete.

Second Step. We recall that

$$\frac{1}{4} < \alpha < 1. \tag{5.6}$$

Define an operator K by

$$Kf = (A+\gamma)^\beta u(f).$$

Therefore noting that $C^2([0,T];L^2(\Omega)) \subset H^2(0,T;L^2(\Omega))$ and $C^1([0,T];L^2(\Omega)) \subset H^1(0,T;L^2(\Omega))$, we see from (5.1) and (5.4) that

$$K : \mathcal{D}((A+\gamma)^{\beta+1}) \longrightarrow H^2(0,T;L^2(\Omega))$$

and

$$K : \mathcal{D}((A+\gamma)^{\beta}) \longrightarrow H^1(0,T;L^2(\Omega))$$

are bounded. Therefore by the interpolation theorem (e.g., Lions and Magenes [7, Vol.I, p.27]), we have

$$K : [\mathcal{D}((A+\gamma)^{\beta+1}), \mathcal{D}((A+\gamma)^{\beta})]_{1+\beta-\alpha} \longrightarrow [H^2(0,T;L^2(\Omega)), H^1(0,T;L^2(\Omega))]_{1+\beta-\alpha}$$

is a bounded operator. Here by (5.6) we note that $0 < \frac{5}{4} - \alpha < 1$. Again by the interpolation theorem: Theorem 6.1 (Vol.I, p.28) and (2.7) (Vol.II, p.8) in [7], we see

$$[H^2(0,T;L^2(\Omega)), H^1(0,T;L^2(\Omega))]_{1+\beta-\alpha} = H^{\alpha-\beta+1}(0,T;L^2(\Omega)).$$

On the other hand, we have

$$[\mathcal{D}((A+\gamma)^{\beta+1}), \mathcal{D}((A+\gamma)^{\beta})]_{1+\beta-\alpha} = \mathcal{D}((A+\gamma)^{\alpha})$$

(e.g., [7, Vol.I]).

Recalling that $\frac{1}{4} < \beta < \alpha < 1$, we note that $\alpha - \beta > 0$. Consequently, by the trace theorem and $\beta > \frac{1}{4}$, we obtain

$$\|u(f)(\cdot,0,\cdot)\|_{H^{1+\alpha-\beta}(0,T;L^2(0,1))} \le C_8 \|(A+\gamma)^\alpha f\|, \quad f \in \mathcal{D}((A+\gamma)^\alpha). \tag{5.7}$$

Third Step. Let $\alpha > \frac{1}{4}$ and $f \in \mathcal{U}_{M,\alpha}$. Then we will prove

$$\|f\| \le C\left(\frac{1}{\log\left(\frac{1}{\|u(f)(\cdot,0,\cdot)\|_{L^2((0,T)\times(0,1))}}\right)}\right)^{\kappa}. \tag{5.8}$$

Since

$$[H^{1+\alpha-\beta}(0,T;L^2(0,1)), L^2(0,T;L^2(0,1))]_{\frac{\alpha-\beta}{1+\alpha-\beta}} = H^1(0,T;L^2(0,1))$$

(e.g., Proposition 2.1 (p.7) in [7], Vol. II), we have

$$\begin{aligned}&\|u(f)(\cdot,0,\cdot)\|_{H^1(0,T;L^2(0,1))}\\ \le& C_9\|u(f)(\cdot,0,\cdot)\|^{\frac{1}{1+\alpha-\beta}}_{H^{1+\alpha-\beta}(0,T;L^2(0,1))}\|u(f)(\cdot,0,\cdot)\|^{\frac{\alpha-\beta}{1+\alpha-\beta}}_{L^2(0,T;L^2(0,1))}\end{aligned}$$

(e.g., Proposition 2.3 (p.19) in [7], Vol I). Therefore

$$\begin{aligned}&\|u(f)(\cdot,0,\cdot)\|_{H^1(0,T;L^2(0,1))}\\ \le& C_9\left(C_8\|(A+\gamma)^{\alpha}f\|\right)^{\frac{1}{1+\alpha-\beta}}\|u(f)(\cdot,0,\cdot)\|^{\frac{\alpha-\beta}{1+\alpha-\beta}}_{L^2(0,T;L^2(0,1))}\\ \le& C_9(M)\|u(f)(\cdot,0,\cdot)\|^{\frac{\alpha-\beta}{1+\alpha-\beta}}_{L^2(0,T;L^2(0,1))} \end{aligned}\tag{5.9}$$

by $f \in \mathcal{U}_{M,\alpha}$ and (5.7). Application of (5.9) in Theorem 0 yields (5.8).

Fourth Step. We set

$$\phi(z) = \int_0^1 u(f)(x_1,0,z)^2 dx_1, \qquad z \in S_\theta. \tag{5.10}$$

Here we notice that $u(f)(x_1,0,z)^2$ is not necessarily non-negative for $z \notin \mathbb{R}$, but $u(f)(x_1,0,t)^2 \ge 0$ for $0 \le t \le T$, because f is real-valued and so is $u(f)(x_1,0,t)$.

By Lemma 2 and $f \in \mathcal{U}_{M,\alpha}$, we see that

$$\phi \text{ is analytic in } S_\theta \tag{5.11}$$

and

$$|\phi(z)| \le C_{10}M, \qquad z \in S_\theta. \tag{5.12}$$

Here $C_{10} > 0$ depends only on α and T, θ. In view of (5.11) and (5.12), we can apply the method by the harmonic measure for estimating analytic functions (e.g., Corollary 10.6.1 (p.126) in Cannon [3]), so that there exists $\lambda \in (0,1)$ depending on S_θ, ν and T such that

$$|\phi(t)| \le C_{11}M\epsilon^{C_{11}t^{1/\lambda}}, \quad 0 \le t \le \nu \tag{5.13}$$

where we set

$$\epsilon = \sup_{\nu \le t \le T} |\phi(t)| = \|u(f)(\cdot,0,\cdot)\|^2_{L^\infty(\nu,T;L^2(0,1))}.$$

Since $u(f)(x_1,0,t) \in \mathbb{R}$ for $0 \le x_1 \le 1$ and $\nu \le t \le T$, by the Sobolev embedding theorem and the interpolation inequality (e.g., Theorem 6.1 (Vol.I, p.28) and (2.7) (Vol.II, p.8) in Lions and Magenes [7]), we have

$$\begin{aligned}\epsilon \le\, & C'_{12}\|u(f)(\cdot,0,\cdot)\|^2_{H^1(\nu,T;L^2(0,1))}\\ \le\, & C_{12}\|u(f)(\cdot,0,\cdot)\|^{\frac{2}{1+\alpha-\beta}}_{H^{1+\alpha-\beta}(\nu,T;L^2(0,1))}\|u(f)(\cdot,0,\cdot)\|^{\frac{2(\alpha-\beta)}{1+\alpha-\beta}}_{L^2((\nu,T)\times(0,1))}.\end{aligned}$$

In view of (5.7), we obtain

$$\epsilon \le C_{13}\left(C_8\|(A+\gamma)^\alpha f\|\right)^{\frac{2}{1+\alpha-\beta}}\|u(f)(\cdot,0,\cdot)\|^{\frac{2(\alpha-\beta)}{1+\alpha-\beta}}_{L^2((\nu,T)\times(0,1))}. \tag{5.14}$$

Therefore, by (5.13) and (5.14), we see

$$
\begin{aligned}
\|u(f)(\cdot,0,\cdot)\|^2_{L^2((0,\nu)\times(0,1))} &= \int_0^\nu |\phi(t)|dt \le C_{11}M\int_0^\nu \epsilon^{C_{11}t^{1/\lambda}}dt\\
=&C_{11}M\lambda\int_0^{\nu^{1/\lambda}} s^{\lambda-1}\exp\left(-(C_{11}\log\frac{1}{\epsilon})s\right)ds\\
\le&C_{11}M\lambda\int_0^{\infty} s^{\lambda-1}\exp\left(-(C_{11}\log\frac{1}{\epsilon})s\right)ds\\
=&C_{11}M\lambda\times\frac{\Gamma(\lambda)}{C_{11}^\lambda}\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\\
\le&\frac{C_{14}}{\left(\log\left(\frac{1}{\|u(f)(\cdot,0,\cdot)\|_{L^2((\nu,T)\times(0,1))}}\right)\right)^\lambda}
\end{aligned}
$$

where we note that $\|u(f)(\cdot,0,\cdot)\|_{L^2((\nu,T)\times(0,1))}$ is assumed to be sufficienttly small for S_θ and ν. Therefore we can take constants $C_{15}>0$ and $\chi>0$ depending only on α, θ, T, ν such that

$$
\begin{aligned}
&\|u(f)(\cdot,0,\cdot)\|_{L^2((0,T)\times(0,1))}\\
\le&\frac{C_{15}}{\left(\log\left(\frac{1}{\|u(f)(\cdot,0,\cdot)\|_{L^2((\nu,T)\times(0,1))}}\right)\right)^\chi}.
\end{aligned}
\tag{5.15}
$$

Application of (5.15) in (5.8) yields the conclusion (2.8). Thus the proof of Theorem 2 is complete.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012