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Uniqueness in the inverse scattering problem within polygonal obstacles by a single incoming wave

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UNIQUENESS IN THE INVERSE SCATTERING PROBLEM WITHIN POLYGONAL OBSTACLES BY A SINGLE INCOMING WAVE

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ABSTRACT. We consider a two dimensional inverse scattering problem of determining an obstacle by the far field pattern. We establish the uniqueness in the inverse problem in the sound-soft case within a class of polygonal domains, by a single incoming plane wave. The key is the analyticity of the solution of the scattering problem and reflection of solutions.

1. INTRODUCTION

Let $D \subset \mathbb{R}^2$ be a bounded domain and $k \in \mathbb{R}$. For $x \in \mathbb{R}^2$, we set $r = |x|$. We consider a scattering problem with sound-soft obstacle:

$$\Delta u + k^2 u = 0 \qquad in \quad \mathbb{R}^2 \setminus cl(D) \tag{1.1}$$

$$u = 0 \qquad on \quad \partial D \tag{1.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r}u^S(x) - iku^S(x)\right) = 0. \tag{1.3}$$

Henceforth $cl(D)$ denotes the closure of a domain D, and we set $i = \sqrt{-1}$, $d \in S^1 \equiv \{x \in \mathbb{R}^2;\ |x| = 1\}$ and

$$u^S(x) = u(x) - e^{ikx\cdot d},$$

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which is called the scattered field, while u is called the total field. We consider $d \in S^1$ and $k \in \mathbb{R}$ respectively as the direction of the incoming plane wave (i.e., $e^{ikx\cdot d}$) and the wave number given by the medium in $\mathbb{R}^2 \setminus cl(D)$.

Condition (1.3) is the Sommerfeld radiation condition. Under suitable conditions on D, for $k \in \mathbb{R}$ and $d \in S^1$, there exists a unique H^1-solution $u(x) = u(D)(x)$ to (1.1) - (1.3), and we can define the far field pattern $u_\infty(D)\left(\frac{x}{r}\right)$:

$$u^S(D)(x) = \frac{e^{ikr}}{\sqrt{r}}\left\{u_\infty(D)\left(\frac{x}{r}\right) + O\left(\frac{1}{r}\right)\right\} \quad \text{as } r \longrightarrow \infty. \tag{1.4}$$

The scattering problem is physically important and there are vast references. Here we refer, for example, to Cakoni, Colton and Monk [2], Colton, Coyle and Monk [3], Colton and Kress [4], Kirsch [8], Kress and Tran [10], Potthast [15] and the references therein. In this paper, we mainly consider

Inverse scattering problem: Determine D from the far field pattern $u_\infty(D)$ for given k and d (possibly by changing them).

This inverse problem is also physically significant and has been studied by many authors. We refer only to Colton and Kress [4], Potthast [15] as books on this topic and also to Isakov [6], [7].

The first basic topic for this inverse problem is the uniqueness: Does

$$u_\infty(D_1)(x) = u_\infty(D_2)(x), \qquad |x| = 1 \tag{1.5}$$

(for possible several d and k) imply $D_1 = D_2$?

There is a classical uniqueness result within smooth D_1, D_2 if (1.5) holds for an infinite number of $d \in S^1$, which is proved based on Schiffer's idea (see Lax and Phillips [11]). For the proof, see Theorem 5.1 in Colton and Kress [4] for example. Also see Kirsch and Kress [9].

For the uniqueness by means of a finite number of $d \in S^1$, see Colton and Sleeman [5], Theorem 5.2 in [4]. Moreover the uniqueness is known with *a single* d, provided that D_1, D_2 are contained in a ball of radius ρ such that $k\rho < \pi$. See Corollary 5.3 in [4] and [5]. Moreover Rondi [16] proves the uniqueness in determining many scatterers by a finite number of incoming plane waves.

An important open problem is the uniqueness in the inverse scattering problem with *a single* (d, k). This problem is interesting from the theoretical point of view, because the far field patterns with many d are overdetermining data for determination of D and we can expect the uniqueness with a single far field pattern. Moreover the formulation with a single (d,k) is helpful for justification of numerical reconstruction of D, because one can usually use far field patterns observed by taking a single or a finite number of d.

The purpose of this paper is to give a positive answer to the uniqueness within polygonal (but not necessarily convex) obstacles. For an inverse scattering problem with polygonal obstacles, we refer to Ari and Firth [1] for example.

This paper is composed of five sections:

§1. Introduction

§2. Main result

§3. Key lemmata

§4. Proof of the main result

§5. Concluding remarks.

2. Main result

Let $k \in \mathbb{R}$ and $d \in S^1$ be arbitrarily fixed. Henceforth, for $P, Q \in \mathbb{R}^2$, we understand that $\overline{PQ}$ is an open segment (not including the end points P and Q). Moreover for a polygonal domain D and $P \in \partial D$, $Q \notin cl(D)$ such that $\overline{PQ} \in \mathbb{R}^2 \setminus cl(D)$, by $\angle(\overline{PQ}, \partial D)$ we denote the least angle among the two angles in $\mathbb{R}^2 \setminus cl(D)$ formed by $\overline{PQ}$ and ∂D. By a polygonal domain D, we mean that ∂D is composed of a finite number of segments.

Definition 2.1. Let $D \subset \mathbb{R}^2$ be a bounded polygonal domain. Let ℓ-points $P_1, ..., P_\ell$, $\ell \ge 2$, satisfy the following conditions (i) - (iv) (Figure 1):

(i) $P_1, ..., P_\ell \in \partial D$.

For $1 \le j \le \ell$, we set

$$\theta_j = \begin{cases} \text{the exterior angle of } D \text{ at } P_j, & \text{if } P_j \text{ is a vertex of a polygon } D, \\ \pi, & \text{otherwise.} \end{cases}$$

(ii) $\overline{P_j P_{j+1}} \subset \mathbb{R}^2 \setminus cl(D)$ for $1 \le j \le \ell$.

(iii) $\angle(\overline{P_{j-1}P_j}, \partial D) = \angle(\overline{P_j P_{j+1}}, \partial D)$, $1 \le j \le \ell$, if $\overline{P_{j-1}P_j}$ does not bisect θ_j at P_j.

(iv) For $1 \le j \le \ell$, we have

$$\frac{\theta_j}{\angle(\overline{P_{j-1}P_j}, \partial D)} \in \mathbb{Q}.$$

Here we set $P_0 = P_\ell$ and $P_{\ell+1} = P_1$ and

$$TR(D : P_1, ..., P_\ell) = \begin{cases} \text{a closed broken line } P_1 \to P_2 \to \cdots \to P_\ell \to P_1 \\ \text{if } \overline{P_1 P_\ell} \text{ does not bisect } \theta_1 \text{ at } P_1, \\ \text{a non-closed broken line } P_1 \to P_2 \to \cdots \to P_\ell, \text{ otherwise.} \end{cases}$$

We call $TR(D : P_1, ..., P_\ell)$ a *trapped ray of D with rational angles.*

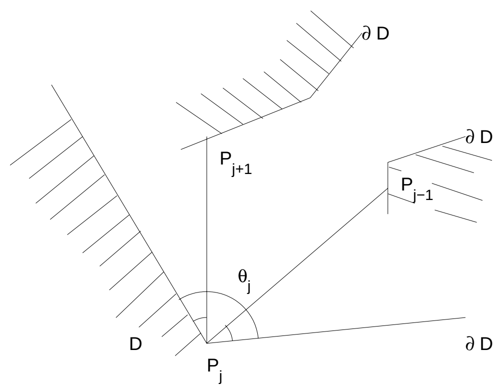

Figure 1

By $TR(D)$, we denote the sum of all the trapped rays of D with rational angles. If $TR(D) \ne \emptyset$, then we call D trapping with rational angles.

In other words, if $TR(D) = \emptyset$, then there are no rays in $\mathbb{R}^2 \setminus cl(D)$ which go out to ∞ after finite times reflecting on ∂D subject to physical law (iii) with stricter constraint (iv) for angles of incidence. It is easily seen that if D is a convex polygon, then $TR(D) = \emptyset$.

Example 1. *Let $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (1,1)$, $A_4 = \left(\frac{2}{3}, 1\right)$, $A_5 = \left(\frac{2}{3}, \frac{1}{2}\right)$, $A_6 = \left(\frac{1}{3}, \frac{1}{2}\right)$, $A_7 = \left(\frac{1}{3}, 1\right)$, $A_8 = (0,1)$, and D_1 be the non-convex polygon with the vertices $A_1, ..., A_8$ (Figure 2). Then, for any $t \in \left(\frac{1}{2}, 1\right)$, the segment $\left\{(s,t); \frac{1}{3} < s < \frac{2}{3}\right\}$ is a trapped ray of D with rational angle (i.e., $\frac{\pi}{2}$) and so*

$$TR(D_1) = \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(\frac{1}{2}, 1\right).$$

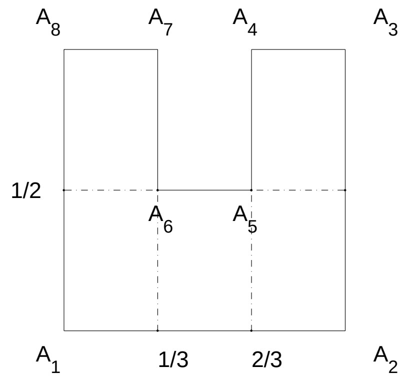

Figure 2

Henceforth $u \in H^1_{loc}(\mathbb{R}^2 \setminus cl(D))$ means that $u \in H^1((\mathbb{R}^2 \setminus cl(D)) \cap \{|x| < \rho\})$ for any $\rho > 0$.

Noting that ∂D is Lipschitz continuous, we can prove (e.g., McLean [12]) that there exists a unique solution $u(D) \in H^1_{loc}(\mathbb{R}^2 \setminus cl(D))$ to (1.1) - (1.3).

We can state our main result:

Theorem 2.2. *Let $k \in \mathbb{R}$ and $d \in S^1$ be arbitrarily fixed and let*

$$\partial D_1 \cap TR(D_2) = \emptyset \quad \text{and} \quad \partial D_2 \cap TR(D_1) = \emptyset. \tag{2.1}$$

Then $u_\infty(D_1)(x) = u_\infty(D_2)(x)$, $|x| = 1$, implies $D_1 = D_2$.

In particular, if $TR(D_1) = TR(D_2) = \emptyset$, then $u_\infty(D_1)(x) = u_\infty(D_2)(x)$, $|x| = 1$, implies $D_1 = D_2$. As such one case, we can show

Corollary 2.3. *Let D_1 and D_2 be star-shaped polygons. Then $u_\infty(D_1)(x) = u_\infty(D_2)(x)$, $|x| = 1$, implies $D_1 = D_2$.*

This corollary is seen because if D is a star-shaped domain, then $TR(D) = \emptyset$ (e.g., Proposition 3.1 (p.157) in [11]).

By the definition, the break of condition (2.1) happens rarely. However we do not know the uniqueness if (2.1) does not hold. In fact, we have the following trapping D_1, D_2 where our proof in Section 4 does not work.

Example 2. *Let us form D_1, D_2 as follows.*

(1) We take a square $A_1A_2A_3A_4$. For convenience, we set $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (1,1)$, $A_4 = (0,1)$.

(2) In the interior of the square $A_1A_2A_3A_4$*, we take a regular triangle* $B_1B_2B_3$ *(i.e., the lengths of the sides are equal). Here we choose vertices* B_1*,* B_2*,* B_3 *such that* $B_1 \to B_2 \to B_3$ *is counterclockwise and that* $\overline{B_1B_2} \parallel \overline{A_1A_2}$*.*

(3) Take the midpoints P_1 *and* P_2 *of the sides* $\overline{B_1B_3}$ *and* $\overline{B_2B_3}$ *respectively.*

(4) Take a point Q_1 *on the segment* $\overline{B_3P_2}$ *arbitrarily.*

(5) Take two points Q_2*,* Q_3 *on the side* $\overline{A_2A_3}$ *such that* $\overline{B_3Q_3} \parallel \overline{A_1A_2}$ *and* $\overline{Q_1Q_2} \parallel \overline{A_1A_2}$*.*

(6) By D_1 *we denote the interior bounded by the closed broken line* $A_1A_2Q_2Q_1B_2B_1B_3Q_3A_3A_4$ *(which is a non-convex polygon with those vertices). By* D_2 *we denote the interior bounded by the closed broken line* $A_1A_2Q_2Q_1P_2P_1B_3Q_3A_3A_4$ *(Figure 3).*

Then D_1 *is trapping with rational angles. In fact, let* P_3 *be the midpoint of the side* $\overline{B_1B_2}$*. For* D_1*, we can see that* $P_1P_2P_3$ *satisfies conditions (i) - (iv), and we have* $TR(D_1) \cap \partial D_2 \supset \overline{P_1P_2} \neq \emptyset$*, that is, condition* (2.1) *does not hold. In this example, we note that* $TR(D_1 : P_1, P_2, P_3)$ *is a closed broken line* $P_1 \to P_2 \to P_3 \to P_1$*. For these* D_1 *and* D_2*, our proof in Section 4 does not work.*

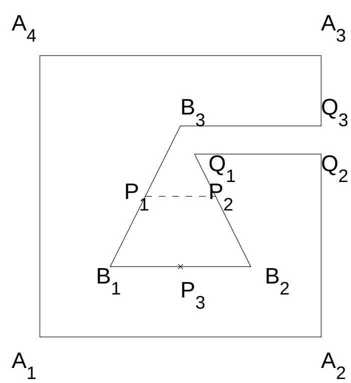

Figure 3

On the other hand, even though condition (2.1) is not satisfied, our argument may sometimes work.

Example 3. *Let us choose the same points* A_j*,* $1 \le j \le 8$ *as in Example 1, and for* $0 < b < \frac{1}{2}$*, let us set* $A_5'' = \left(\frac{2}{3}, b\right)$ *and* $A_6'' = \left(\frac{1}{3}, b\right)$*. By* D_2 *we denote the non-convex polygon with the vertices* $A_1, A_2, A_3, A_4, A_5'', A_6'', A_7, A_8$ *(Figure 4). Then* $TR(D_2) \cap \partial D_1 = \left\{\left(s, \frac{1}{2}\right); \frac{1}{3} < s < \frac{2}{3}\right\}$*, that is, (2.1) is not true. However, as is seen in the proof in Section 4 (in particular, by Lemmata 3 and 5), we see that* $u_\infty(D_1) \equiv u_\infty(D_2)$ *yields* $D_1 = D_2$*.*

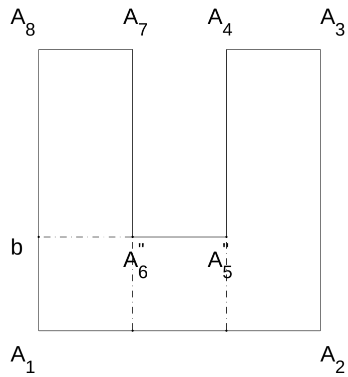

Figure 4

3. Key lemmata

For the proof, we will show key lemmata. Henceforth $\triangle ABC$ denotes the interior of the triangle ABC.

Lemma 1. *Let $E \subset \mathbb{R}^2$ be a domain and let $v \in H^1_{loc}(E)$ satisfy $\Delta v + k^2 v = 0$ in E where $k \in \mathbb{R}$. Let $L_0 \subset L \subset E$ be two segments. If $v = 0$ on L_0, then $v = 0$ on L.*

Proof. Since v satisfies the Helmholtz equation, the function v is real analytic in E (e.g., [4]). Therefore $v|_L$ is an analytic function in one variable, so that the lemma follows. $\square$

Lemma 2. *Let $A = (\varepsilon, 0)$, $O = (0,0)$, $B = (\varepsilon \cos\theta, \varepsilon \sin\theta)$, $E = \{x \in \mathbb{R}^2; 0 < arg x < \theta, |x| < \varepsilon\}$ for $\varepsilon > 0$ and $0 < \theta < 2\pi$. We take $P \in E$ and set $\varphi = \angle AOP$. We assume that*

$$\frac{\varphi}{\theta} \notin \mathbb{Q}. \tag{3.1}$$

Let $v \in H^1(E)$ satisfy

$$\Delta v + k^2 v = 0 \qquad in\ E \tag{3.2}$$

$$v = 0 \qquad on\ \overline{OA} \cup \overline{OB} \tag{3.3}$$

and

$$v = 0 \qquad on\ \overline{OP}. \tag{3.4}$$

Then $v = 0$ *in* E.

Remark 1. Assumption (3.1) is essential. For example, take $A = (1, 0)$, $P = (0, 1)$, $B = (-1, 0)$, $k = 0$. Then, setting $v(x) = r^2 \sin 2\alpha$ for $x = (r\cos\alpha, r\sin\alpha)$, we see that $\Delta v = 0$ and $v = 0$ on $\overline{OA} \cup \overline{OB} \cup \overline{OP}$, but $v \not\equiv 0$ in E.

Proof: For the proof, we will show

Lemma 3. *Let* K *be the symmetric transformation in* $\mathbb{R}^2$ *with respect to the straight line* OA. *Let* $v \in H^1(\triangle OAB)$ *satisfy*

$$v = 0 \qquad \text{on } \overline{OA} \tag{3.5}$$

and

$$\Delta v + k^2 v = 0 \qquad \text{in } \triangle OAB. \tag{3.6}$$

We set

$$V(x_1, x_2) = \begin{cases} v(x_1, x_2), & (x_1, x_2) \in \triangle OAB \\ -v(K(x_1, x_2)), & (x_1, x_2) \in K(\triangle OAB). \end{cases}$$

Then

$$v \in H^1(\triangle OAB \cup K(\triangle OAB) \cup \overline{OA}) \tag{3.7}$$

and

$$\Delta V + k^2 V = 0 \quad \text{in } \triangle OAB \cup K(\triangle OAB) \cup \overline{OA}. \tag{3.8}$$

In particular, if v *satisfies* $\Delta v + k^2 v = 0$ *in* $\triangle OAB \cup K(\triangle OAB) \cup \overline{OA}$ *and* $v = 0$ *on* $\overline{OA} \cup \overline{OB}$, *then* $v = 0$ *on* $K(\overline{OB})$.

Proof. Without loss of generality, we may take $A = (a, 0)$, $O = (0, 0)$, $B = (b_1, b_2)$ where $a > 0$. We set $\Omega = \triangle OAB$. Then $K(x_1, x_2) = (x_1, -x_2)$, and

$$V(x_1, x_2) = \begin{cases} v(x_1, x_2), & (x_1, x_2) \in \Omega, \\ -v(x_1, -x_2), & (x_1, x_2) \in K\Omega. \end{cases}$$

In view of (3.5), we can directly verify

$$(\partial_2 V)(x_1, x_2) = \begin{cases} \partial_2 v(x_1, x_2), & (x_1, x_2) \in \Omega, \\ \partial_2 v(x_1, -x_2), & (x_1, x_2) \in K\Omega, \end{cases} \tag{3.9}$$

$$(\partial_1 V)(x_1,x_2) = \begin{cases} \partial_1 v(x_1,x_2), & (x_1,x_2)\in\Omega, \\ -\partial_1 v(x_1,-x_2), & (x_1,x_2)\in K\Omega \end{cases} \tag{3.10}$$

Here and henceforth we set $\partial_1 = \frac{\partial}{\partial x_1}$ and $\partial_2 = \frac{\partial}{\partial x_2}$. Therefore (3.7) follows.

Next we have to prove (3.8). For this, let $\psi \in C_0^\infty(\Omega \cup \overline{OA} \cup K\Omega)$. Then we can write $\psi(x_1,x_2) = \psi_1(x_1,x_2) + \psi_2(x_1,x_2)$ where $\psi_1(x_1,-x_2) = \psi_1(x_1,x_2)$ and $\psi_2(x_1,-x_2) = -\psi_2(x_1,x_2)$ for $(x_1,x_2)\in\Omega$. That is, ψ_1 and ψ_2 are even and odd in x_2 respectively. Hence $\partial_1\psi_1$ is even in x_2, while $\partial_1\psi_2$ is odd. Consequently, by (3.10), we have

$$\begin{aligned} &\int_{\Omega\cup\overline{OA}\cup K\Omega} (\partial_1 V)(\partial_1\psi)dx_1dx_2 \\ &= \int_{\Omega\cup\overline{OA}\cup K\Omega} \{(\partial_1 V)(\partial_1\psi_1) + (\partial_1 V)(\partial_1\psi_2)\}dx_1dx_2 \\ &= 2\int_\Omega (\partial_1 v)(\partial_1\psi_2)dx_1dx_2, \end{aligned} \tag{3.11}$$

because $(\partial_1 V)(\partial_1\psi_1)$ is odd in x_2 and $\Omega\cup\overline{OA}\cup K\Omega$ is symmetric with respect to $\overline{OA}$. Since ψ_2 is odd in x_2, we have $\psi_2(x_1,0)=0$, so that

$$\psi_2 \in H_0^1(\Omega). \tag{3.12}$$

Next, by (3.9), we have

$$\int_{\Omega\cup\overline{OA}\cup K\Omega} (\partial_2 V)(\partial_2\psi)dx_1dx_2 = 2\int_\Omega (\partial_2 v)(\partial_2\psi_2)dx_1dx_2. \tag{3.13}$$

By (3.11) - (3.13), noting (3.6) in the H^1-sense, we obtain

$$\begin{aligned} &\int_{\Omega\cup\overline{OA}\cup K\Omega} \nabla V\cdot\nabla\psi dx_1dx_2 = 2\int_\Omega \nabla v\cdot\nabla\psi_2 dx_1dx_2 \\ &= 2k^2\int_\Omega v\psi_2 dx_1dx_2 = k^2\int_{\Omega\cup\overline{OA}\cup K\Omega} V\psi dx_1dx_2 \end{aligned}$$

for all $\psi\in C_0^\infty(\Omega\cup\overline{OA}\cup K\Omega)$. This means (3.8).

We prove the final statement. Both V and v satisfy (3.8) and $V=v$ in $\triangle OAB$. Therefore the classical unique continuation yields $V=v$ in $\triangle OAB\cup K(\triangle OAB)\cup\overline{OA}$. By the definition of V and $v=0$ on $\overline{OB}$, we see that $V=0$ on $K(\overline{OB})$, so that $v=0$ on $K(\overline{OB})$ follows. Thus the proof of Lemma 3 is complete.

□

Now we will complete the proof of Lemma 2. We consider the following procedures.

(i) We choose $m_0 \in \mathbb{N} \cup \{0\}$ and $0 \le \varphi_1 < \varphi$ such that

$$\theta = m_0\varphi + \varphi_1.$$

Then, by (3.1), $\varphi_1 > 0$.

(ii) We choose $m_1 \in \mathbb{N} \cup \{0\}$ and $0 \le \varphi_2 < \varphi_1$ such that

$$\theta = m_1\varphi_1 + \varphi_2.$$

Similarly (3.1) implies that $\varphi_2 > 0$. In fact, if $\varphi_2 = 0$, then $\frac{\varphi}{\theta} = \frac{m_1 - 1}{m_0 m_1} \in \mathbb{Q}$, which is impossible.

Continuing these procedures. we define a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ such that

$$\varphi > \varphi_1 > \varphi_2 > \cdots > 0. \tag{3.14}$$

Set

$$A_j = (\varepsilon \cos \varphi_{2j}, \varepsilon \sin \varphi_{2j}), \qquad j \in \mathbb{N}.$$

We will prove

$$v = 0 \qquad \text{on } \overline{OA_j}. \tag{3.15}$$

In fact, by the procedure (i) and Lemma 3, we see that $v(x) = 0$ if arg $x = \theta - \varphi_1$ and $|x| < \varepsilon$. Therefore with $v = 0$ on $\overline{OB}$, we repeat application of Lemma 3 to see that $v(x) = 0$ if $x \in E$ satisfies arg $x = \theta - 2\varphi_1$ or $, \cdots, = \theta - m_1\varphi_1 \equiv \varphi_2$. Therefore (3.15) holds for $j = 1$. Then, setting $\varphi = \varphi_2$ in (i), we start procedures (i) and (ii), so that we see (3.15) for $j = 2$ in terms of Lemma 3. Continuing the argument, we complete the proof of (3.15).

Moreover we can prove

$$\lim_{j\to\infty} \varphi_j = 0. \tag{3.16}$$

Assume contrarily. By (3.14), we see that

$$\lim_{j\to\infty} \varphi_j = \varphi_\infty > 0. \tag{3.17}$$

By the procedures, we have

$$\theta = m_j\varphi_j + \varphi_{j+1}, \qquad j \ge 2, \in \mathbb{N}, \tag{3.18}$$

which implies that $\lim_{j\to\infty} m_j\varphi_j$ exists. Hence (3.17) implies $\lim_{j\to\infty} m_j = m_\infty \in \mathbb{N}$. Consequently, for large $N \in \mathbb{N}$, if $j \ge N$, then $m_j = m_\infty$, and (3.18) yields

$$m_\infty\varphi_j + \varphi_{j+1} = m_\infty\varphi_{j+1} + \varphi_{j+2}$$

for any $j \ge N$. This is impossible because $\varphi_j > \varphi_{j+1}$ and $\varphi_{j+1} > \varphi_{j+2}$. Therefore φ_∞ must be zero and the proof of (3.16) is complete.

Let us return to (3.15). Again application of Lemma 3 yields

$$v(x) = 0 \quad \text{if } x \in E \text{ satisfies } \arg\, x = j\varphi_{2\ell} \text{ for } j, \ell \in \mathbb{N}. \tag{3.19}$$

By means of (3.16), the set

$$\bigcup_{j,\ell\in\mathbb{N}} \{x \in E;\, arg\, x = j\varphi_{2\ell}\}$$

is dense in E. By (3.19) and the continuity of v, we obtain $v = 0$ in E. Thus the proof of Lemma 2 is complete.

We conclude this section with an algebraic lemma.

Lemma 4. *Let the sector E, the points A, B, O be defined as in Lemma 2, and let $P \in E$ and $\varphi = \angle AOP$. Let $v \in H^1(E)$ satisfy* (3.2) - (3.4). *We assume that*

$$\frac{\varphi}{\theta} \in \mathbb{Q}. \tag{3.20}$$

Then there exists a points $Q \in E$ such that

$$\angle AOP = \angle BOQ \tag{3.21}$$

and

$$v = 0 \qquad \text{on } \overline{OQ}. \tag{3.22}$$

Here $Q = P$ may happen.

Proof. By (3.20), we can set

$$\varphi = \frac{n}{m}\theta,$$

where $m, n \in \mathbb{N}$, $1 \le n \le m-1$ and the greatest common divisor of m and n is one. We will prove that there exist points $\widetilde{Q}, \widetilde{R} \in E$ such that

$$\angle AO\widetilde{Q} = \frac{\widetilde{n}}{m}\theta, \qquad \angle AO\widetilde{R} = \frac{\widetilde{n}+1}{m}\theta \quad \text{with some } \widetilde{n} \in \mathbb{N},$$

$$v = 0 \quad \text{on } \overline{O\widetilde{Q}} \cup \overline{O\widetilde{R}}. \tag{3.23}$$

The proof of (3.23) will be done by the well-known Euclidean algorithm for determining the greatest common divisor of two natural numbers. For completeness, we will give the proof.

First Step. Let

$$m = nq + r_0 \tag{3.24}$$

where $q, r_0 \in \mathbb{N}$ and $0 \le r_0 \le n-1$. Since the greatest common divisor of m and n is one, we have $1 \le r_0 \le n-1$. We take $A_1, B_1 \in E$ such that $\angle AOA_1 = \frac{(n-1)q}{m}\theta$ and $\angle AOB_1 = \frac{nq}{m}\theta$. Then $\angle BOB_1 = \frac{r_0}{m}\theta$. By Lemma 3, we have $v = 0$ on $\overline{OA_1} \cup \overline{OB_1}$. Since $v = 0$ on $\overline{OB}$, we again apply Lemma 3 in $\triangle OBB_1$, so that $v = 0$ on $\overline{OP_1}$, where P_1 is in the sector OA_1B_1 and $\angle B_1OP_1 = \frac{r_0}{m}\theta$ (Figure 5).

We consider the sector OA_1B_1 and $\overline{OP_1}$. We note that $\angle A_1OB_1 = \frac{n}{m}\theta$ and $\angle B_1OP_1 = \frac{r_0}{m}\theta$. Let $r_0 = 1$. Then, by $\angle AOB_1 = \frac{nq}{m}\theta$ and $\angle B_1OP_1 = \frac{1}{m}\theta$, we can set $\widetilde{Q} = P_1$ and $\widetilde{R} = B_1$, so that the verification of (3.23) is complete. Therefore we may assume that $r_0 \ge 2$.

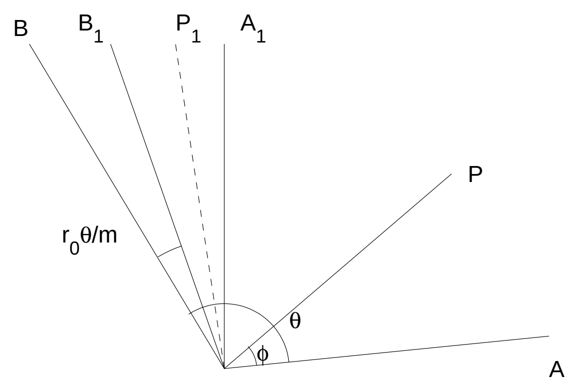

Figure 5

Second Step. Let

$$n = r_0 q_0 + r_1 \tag{3.25}$$

where $q_0, r_1 \in \mathbb{N}$ and $0 \le r_1 \le r_0 - 1$. Since the greatest common divisor of m and n is one, by (3.24), we see that $1 \le r_1 \le r_0 - 1$. In fact, if $r_1 = 0$, then $r_0 \ge 2$, and m, n have a common divisor $r_0 \ge 2$, which is impossible.

Taking $\overline{OB_1}$ as starting line and repeating the clockwise rotations of $\overline{OP_1}$ with angle $\frac{r_0}{m}\theta$, we define points $A_2, B_2 \in E$ such that $\angle B_1OB_2 = \frac{(q_0-1)r_0}{m}\theta$ and $\angle B_1OA_2 = \frac{q_0r_0}{m}\theta$ (Figure 6). Then $\angle A_2OA_1 = \frac{r_1}{m}\theta$. By Lemma 3, we have $v = 0$ on $\overline{OA_2} \cup \overline{OB_2}$. Moreover, applying Lemma 3 in $\triangle OA_1A_2$, we obtain $v = 0$ on $\overline{OP_2}$ where P_2 is in the sector OA_2B_2 and $\angle A_2OP_2 = \frac{r_1}{m}\theta$. Thus we obtain points A_2, B_2, P_2 in the sector OA_1B_1 such that P_2 is in the sector OA_2B_2, $v = 0$ on $\overline{OA_2} \cup \overline{OB_2} \cup \overline{OP_2}$ and $\angle A_2OB_2 = \frac{r_0}{m}\theta$, $\angle A_2OP_2 = \frac{r_1}{m}\theta$. We note that $1 \le r_1 \le r_0 - 1$. If $r_1 = 1$, then similarly to the argument in First Step, we have already completed the proof of (3.23). Therefore we may assume that $2 \le r_1 \le r_0 - 1$.

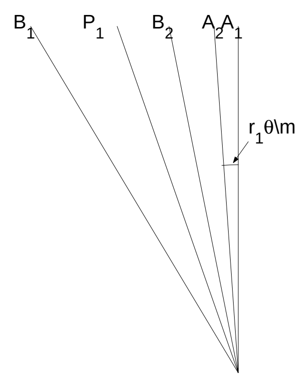

Figure 6

Third Step. Let

$$r_0 = r_1q_1 + r_2, \tag{3.26}$$

where $q_1, r_1 \in \mathbb{N}$ and $0 \le r_2 \le r_1 - 1$. Since the greatest common divisor of m and n is one and $r_1 \ge 2$, in terms of (3.24) – (3.26), we have $1 \le r_2 \le r_1 - 1$ similarly for (3.25).

Now we can take the same procedure in First Step by regarding $\overline{OA_2}, \overline{OP_2}, \overline{OB_2}$ respectively $\overline{OA}, \overline{OP}, \overline{OB}$, so that we obtain three points B_3, P_3, A_3 in the sector OA_2B_2 such that P_3 is in the sector OA_3B_3, and $v = 0$ on $\overline{OA_3} \cup \overline{OB_3} \cup \overline{OP_3}$, $\angle A_3OB_3 = \frac{r_1}{m}\theta$, $\angle B_3OP_3 = \frac{r_2}{m}\theta$. If $r_2 = 1$, then we can complete the proof of (3.23). If $r_2 \ge 2$, then we can continue the procedure in First Step and Second

Step until we will find $\widetilde{Q}$ and $\widetilde{R}$ satisfying (3.23). Thus the proof of (3.23) is complete.

Now we will finish the proof of Lemma 4. Repeat application of Lemma 3 in $\triangle O\widetilde{Q}\widetilde{R}$ both counterclockwise and clockwise as long as the resulting segments are in the sector OAB, and we see that $v = 0$ on $\overline{OP^j}$, $1 \le j \le m-1$, where $P^j \in E$ and $\angle AOP^j = \frac{j}{m}\theta$. Setting $Q = P^{m-n}$, we see that Q satisfies (3.21) and (3.22).

Thus the proof of Lemma 4 is complete.

□

4. Proof of Theorem 2.2

First we show

Lemma 5. *Let* $u(D) \in H^1_{loc}(\mathbb{R}^2 \setminus cl(D))$ *satisfy* (1.1) - (1.3). *Then*

(i) There does not exist an open subset $E \subset \mathbb{R}^2 \setminus cl(D)$ *where* $u(D) = 0$ *in* E.

(ii) There does not exist an infinite half straight line $L \subset \mathbb{R}^2 \setminus cl(D)$ *where* $u(D) = 0$ *on* L.

Proof. If $u(D) = 0$ in an open subset $E \subset \mathbb{R}^2 \setminus cl(D)$, then $u(D) = 0$ in $\mathbb{R}^2 \setminus cl(D)$ by the classical unique continuation. Therefore part (i) follows from (ii), and so it suffices to prove part (ii).

Assume contrarily that there exists such a line L. Without loss of generality, we may assume that there exists $a \in \mathbb{R}$ such that $L = \{(x_1, ax_1); x_1 > 0\}$. Then (1.4) yields

$$
\begin{aligned}
0 &= u(D)(x_1, ax_1) \\
&= \frac{\exp(ikx_1\sqrt{1+a^2})}{(1+a^2)^{\frac{1}{4}}x_1^{\frac{1}{2}}}\left\{u_\infty(D)\left(\frac{1}{x_1\sqrt{1+a^2}}(x_1, ax_1)\right) + O\left(\frac{1}{x_1\sqrt{1+a^2}}\right)\right\} \\
&\quad + e^{ikbx_1}
\end{aligned}
$$

as $x_1 \longrightarrow \infty$. Here we set

$$
b = \begin{pmatrix} 1 \\ a \end{pmatrix} \cdot d \in \mathbb{R}.
$$

The first term at the right hand side converges to 0 as $x_1 \longrightarrow \infty$, so that $\lim_{x_1\to\infty} e^{ikbx_1} = 0$, which is a contradiction by $b \in \mathbb{R}$. Thus the proof of Lemma 5 is complete.

□

Now we proceed to the proof of Theorem 2.2. Assume contrarily that $D_1 \neq D_2$. For simplicity, we set

$$u_j = u(D_j), \qquad j = 1, 2.$$

By the Rellich theorem (e.g., Lemma 2.11 in [4]), we see from $u_\infty(D_1) \equiv u_\infty(D_2)$ that

$$u_1 = u_2 \qquad \text{in } \mathbb{R}^2 \setminus cl(D_1 \cup D_2) \tag{4.1}$$

(e.g., Theorem 2.13 in [4]).

For completing the proof of the theorem, in terms of Lemmata 1 and 5, it suffices to find a (finite) segment L such that

(a) $L \subset \mathbb{R}^2 \setminus cl(D_1)$, L is extended to ∞ in $\mathbb{R}^2 \setminus cl(D_1)$ and $u_1 = 0$ on L,

or

(b) $L \subset \mathbb{R}^2 \setminus cl(D_2)$, L is extended to ∞ in $\mathbb{R}^2 \setminus cl(D_2)$ and $u_2 = 0$ on L,

or

(c) there exists an open set ω where $u_1 = 0$ or $u_2 = 0$ in ω.

We separately consider the two cases: $\partial D_1 \cap \partial D_2 \neq \emptyset$ and $\partial D_1 \cap \partial D_2 = \emptyset$.

Case A: $\partial D_1 \cap \partial D_2 \neq \emptyset$

First Step: By translating and rotating and exchanging D_1 and D_2 if necessary, without loss of generality, we may take $O = (0,0)$, $A = (\varepsilon, 0)$ and $B = (\varepsilon \cos\theta, \varepsilon \sin\theta)$ with $\varepsilon > 0$ and $\theta \in (0, 2\pi)$ such that

$$E = \{x; 0 < arg\, x < \theta,\ |x| < \varepsilon\} \subset \mathbb{R}^2 \setminus cl(D_1),$$

$\overline{OA} \cup \overline{OB} \subset \partial D_1$ and there exists $P \in E$ satisfying $\overline{OP} \subset \partial D_2$. We set $\varphi = \angle AOP$ (Figure 7).

First we assume

$$\frac{\varphi}{\theta} \notin \mathbb{Q}. \tag{4.2}$$

In view of (4.1) and (1.2), we have $u_1 = 0$ on $\overline{OA} \cup \overline{OB} \cup \overline{OP}$, so that we can apply Lemma 2 to u_1, and we obtain $u_1 = 0$ in E. This is a contradiction by Lemma 5, and so (4.2) is impossible.

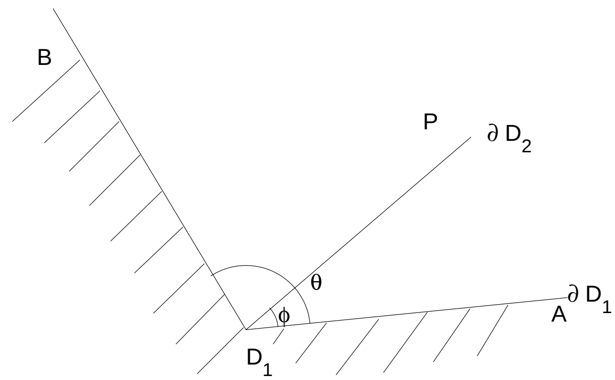

Figure 7

Second Step: Next let us assume

$$\frac{\varphi}{\theta} \in \mathbb{Q}. \tag{4.3}$$

By Lemma 4, we can choose $P'' \in E$ such that

$$\angle AOP = \angle BOP'',$$

$$\overline{OP''} \subset \mathbb{R}^2 \setminus cl(D_1)$$

and

$$u_1 = 0 \qquad \text{on } \overline{OP''}.$$

First let either the infinite half line OP or OP'' can reach ∞ in $\mathbb{R}^2 \setminus cl(D_1)$. Then we have been already led to a contradiction by Lemma 5.

Second let both the infinite half lines OP and OP'' intersect ∂D_1. We assume that $P \neq P''$, because the proof is same in the case of $P = P''$. The intersection points of OP and OP'' respectively with ∂D_1 which are nearest to O, are denoted by O_1 and O_1''. Near O_1 and O_1'', we take points $A_1, B_1, A_1'', B_1'' \in \partial D_1$ such that O_1 is between A_1 and B_1, O_1'' is between A_1'' and B_1'' and $\overline{A_1B_1} \cap \overline{A_1''B_1''}$ has no interior points (Figure 8). We set

$$\theta_1 = \begin{cases} \pi, & \text{if } O_1 \text{ is not a vertex of } D_1, \\ \text{the exterior vertex angle of } D_1, & \text{if } O_1 \text{ is a vertex,} \end{cases}$$

$$\varphi_1 = \min\{\angle(\overline{O_1A_1}, \overline{O_1P}), \angle(\overline{O_1B_1}, \overline{O_1P})\}$$

and

$$\theta_1'' = \begin{cases} \pi, & \text{if } O_1'' \text{ is not a vertex of } D_1, \\ \text{the exterior vertex angle of } D_1, & \text{if } O_1'' \text{ is a vertex,} \end{cases}$$

$$\varphi_1'' = \min\{\angle(\overline{O_1''A_1''}, \overline{O_1''P''}), \angle(\overline{O_1''B_1''}, \overline{O_1''P''}).$$

Here and henceforth, for example, $\angle(\overline{O_1A_1}, \overline{O_1P})$ means the angle formed by $\overline{O_1A_1}$ and $\overline{O_1P}$ in $\mathbb{R}^2 \setminus cl(D_1)$.

Without loss of generality, we may assume that

$$\angle(\overline{O_1A_1}, \overline{O_1P}) \le \angle(\overline{O_1B_1}, \overline{O_1P}), \quad \angle(\overline{O_1''A_1''}, \overline{O_1''P''}) \le \angle(\overline{O_1''B_1''}, \overline{O_1''P''}).$$

Let

$$\frac{\varphi_1}{\theta_1} \notin \mathbb{Q} \quad \text{or} \quad \frac{\varphi_1''}{\theta_1''} \notin \mathbb{Q}.$$

Then the application of Lemma 2 in the sector $O_1A_1B_1$ or $O_1''A_1''B_1''$ in $\mathbb{R}^2 \setminus cl(D_1)$, yields $u_1 = 0$ in some open set of $\mathbb{R}^2 \setminus cl(D_1)$. This is impossible by Lemma 5.

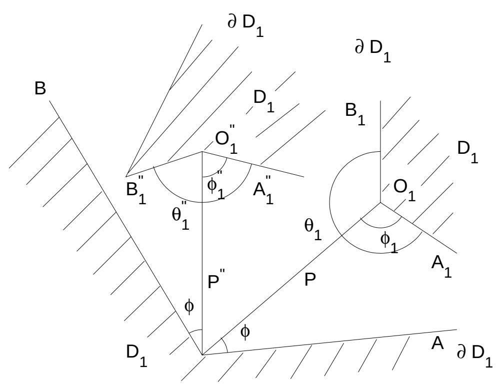

Figure 8

Third Step: By the second step, it must hold that

$$\frac{\varphi_1}{\theta_1} \in \mathbb{Q} \quad \text{and} \quad \frac{\varphi_1''}{\theta_1''} \in \mathbb{Q}.$$

Then, by Lemma 4, we can take a point P_1 in the sector $A_1O_1B_1$ in $\mathbb{R}^2 \setminus cl(D_1)$ with vertex angle θ_1 such that $\overline{O_1P_1} \subset \mathbb{R}^2 \setminus cl(D_1)$, $\angle(\overline{O_1P_1}, \overline{O_1B_1}) = \angle(\overline{O_1P}, \overline{O_1A_1}) = \varphi_1$ and $u_1 = 0$ on $\overline{O_1P_1}$ (Figure 9). Then we will repeat the argument in Second Step as many times as possible. Consequently we have the following alternatives:

(i) The above procedures terminate finite times: We have a trapped ray $TR(D_1 : P_1, ..., P_\ell)$ with rational angles and $\overline{P_1P_2} \in \partial D_2$.

(ii) The above procedures continue infinitely: There exist two infinite sets of points $\{P_j\}_{j\in\mathbb{N}}$ and $\{Q_j\}_{j\in\mathbb{N}}$ such that

$$P_j, Q_j \in \partial D_1,\ \overline{P_jQ_j} \in \mathbb{R}^2 \setminus cl(D_1), \quad u_1 = 0 \quad \text{on } \overline{P_jQ_j}, \quad j \in \mathbb{N}. \tag{4.4}$$

By assumption (2.1) of Theorem, case (i) cannot happen.

Next we will prove that case (ii) is also excluded. Since the length $|\partial D_1|$ of the curve ∂D_1 is finite and $P_m \neq P_n$, $Q_m \neq Q_n$ if $m \neq n$, we can choose subsequences $\{P_j\}_{j\in\mathbb{N}}$ and $\{Q_j\}_{j\in\mathbb{N}}$, which are denoted by the same letters, such that

$$\lim_{j\to\infty} P_j = P_\infty, \quad \lim_{j\to\infty} Q_j = Q_\infty. \tag{4.5}$$

Without loss of generality, by further taking a subsequence of $\{P_j\}_{j\in\mathbb{N}}$, we may assume that

$$P_j,\ j \in \mathbb{N}, \text{ are located at one side of } P_\infty. \tag{4.6}$$

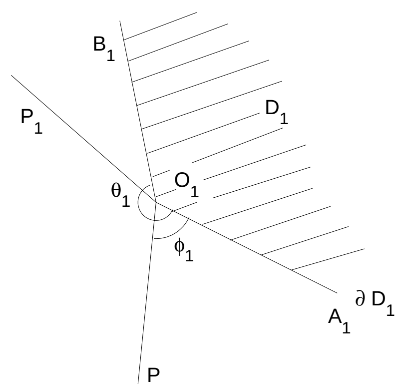

Figure 9

We note that the corresponding Q_j, $j \in \mathbb{N}$, are not necessarily located at one side of Q_∞ and that $\overline{P_jP_{j+1}} \subset \partial D_1$ for all large j.

We will prove

Lemma 6. *There exist domains Ω_j, $j \in \mathbb{N}$, such that*

$$\lim_{j\to\infty} |\Omega_j| = 0, \tag{4.7}$$

where $|\Omega_j|$ denotes the area of Ω_j, and

$$\begin{cases} \Delta u_1 + k^2 u_1 = 0 & \text{in } \Omega_j, \\ u_1 = 0 & \text{on } \partial\Omega_j. \end{cases} \tag{4.8}$$

Moreover Ω_j *is a triangle, a quadrilateral or a pentagon (not necessarily convex).*

Proof. We will prove the lemma separately in the following two cases:

(a) Neither P_∞ nor Q_∞ is a vertex of D_1.

(b) At least one, say Q_∞ of P_∞ and Q_∞ is a vertex of D_1.

Case (a): For all large j, we see that $\overline{Q_jQ_{j+1}} \subset \partial D_1$. We have the two subcases:

Case (a)-(i): The four points $P_j, P_{j+1}, Q_{j+1}, Q_j$ form a quadrilateral with this order (Figure 10).

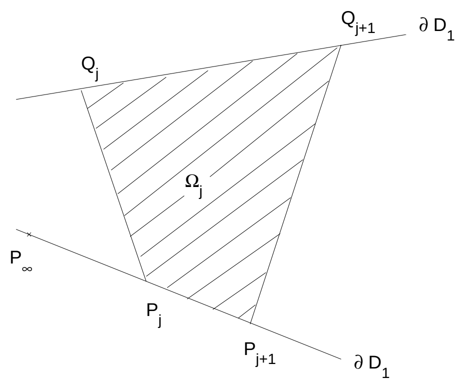

Figure 10

Case (a)-(ii): The four points $P_j, P_{j+1}, Q_j, Q_{j+1}$ form a quadrilateral with this order. Let R_j be the intersection point of the two segments $\overline{P_jQ_j}$ and $\overline{P_{j+1}Q_{j+1}}$ (Figure 11).

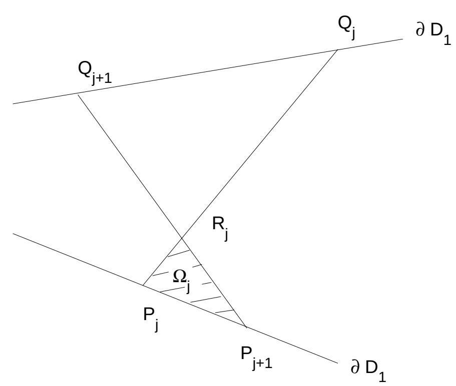

Figure 11

Let

$$\Omega_j = \begin{cases} \text{the interior of the quadrilateral } P_jP_{j+1}Q_{j+1}Q_j \text{ in case (a)-(i)}, \\ \text{the interior of the triangle } P_jP_{j+1}R_j \text{ in case (a)-(ii)}. \end{cases}$$

Then $\partial\Omega_j \subset \partial D_1 \cup \overline{P_jQ_j} \cup \overline{P_{j+1}Q_{j+1}}$, so that (4.8) is seen by (1.2) and (4.4). Moreover, setting $\rho = \max_{x,y\in\partial D_1}|x-y|$, we see that Ω_j is contained in a sum of two triangles whose bases are $\overline{P_jP_{j+1}}$ and $\overline{Q_jQ_{j+1}}$ and heights are at most ρ. Therefore

$$|\Omega_j| \le \frac{\rho}{2}(|\overline{P_jP_{j+1}}| + |\overline{Q_jQ_{j+1}}|). \tag{4.9}$$

Case (b): We have

Case (b) - (i): The two points Q_j and Q_{j+1} are located at the different sides of Q_∞ (Figure 12). Then we see that $\overline{Q_jQ_\infty} \subset \partial D_1$ for all large j.

Case (b) - (ii): The two points Q_j and Q_{j+1} are located at the same side of Q_∞. Then we see that $\overline{Q_jQ_{j+1}} \subset \partial D_1$ for all large j.

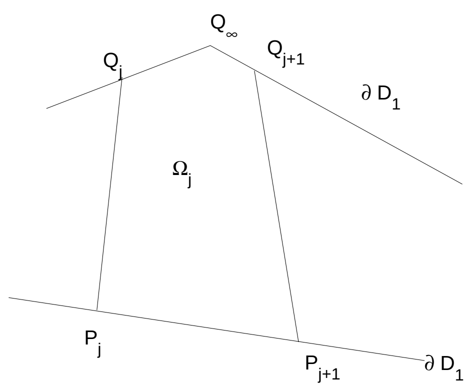

Figure 12

In case (b) - (ii), in the same way as in case (a), we can construct Ω_j satisfying (4.8). In case (b) - (i), let

$$\Omega_j = \begin{cases} \text{the interior of the pentagon with vertices } P_j, P_{j+1}, Q_j, Q_{j+1}, Q_\infty, \\ \quad \text{if } \overline{P_jQ_j} \text{ and } \overline{P_{j+1}Q_{j+1}} \text{ do not intersect (Figure 12)}, \\ \text{the interior of the triangle with vertices } P_j, P_{j+1}, R_j, \\ \quad \text{if } \overline{P_jQ_j} \text{ and } \overline{P_{j+1}Q_{j+1}} \text{ intersect at } R_j \text{ (Figire 13)}. \end{cases}$$

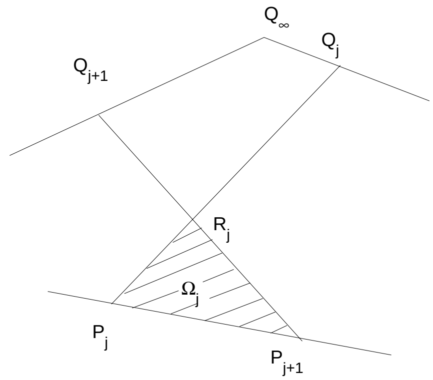

Figure 13

Then $\partial\Omega_j \subset \partial D_1 \cup \overline{P_jP_{j+1}} \cup \overline{Q_jQ_{j+1}}$, so that we obtain (4.8). Moreover Ω_j is divided into three triangles with bases $\overline{Q_jQ_\infty}$, $\overline{Q_{j+1}Q_\infty}$ and $\overline{P_jP_{j+1}}$ in the former case, while Ω_j is a triangle with the base $\overline{P_jP_{j+1}}$ in the latter case. Consequently

$$|\Omega_j| \le \frac{\rho}{2}(|\overline{P_jP_{j+1}}| + |\overline{Q_jQ_\infty}| + |\overline{Q_{j+1}Q_\infty}|). \tag{4.10}$$

Hence, in both cases (a) and (b), we see (4.7) and (4.8) in terms of (4.5), (4.9) and (4.10). Thus the proof of Lemma 6 is complete.

□

Now we will complete the proof that case (ii) leads to a contradiction. Equation (4.8) implies that k^2 is an eigenvalue of $-\Delta$ in Ω_j with the homogeneous Dirichlet boundary condition for $j \ge 1$. Let $\lambda_1(D)$ denote the first eigenvalue (i.e., the least eigenvalue) of $-\Delta$ in a domain D with the homogeneous Dirichlet boundary condition. Then, by a theorem by Rayleigh-Faber-Krahn (e.g., Nehari [13], p.18 in Pólya and Szegö [14]), we have

$$\lambda_1(\Omega_j) \ge \frac{\pi\nu_0^2}{|\Omega_j|},$$

where ν_0 is the first positive root of the Bessel function $J_0(t)$. Therefore

$$k^2 \ge \frac{\pi\nu_0^2}{|\Omega_j|}$$

for any $j \ge 1$, which is a contradiction by (4.7) in letting $j \longrightarrow \infty$. Consequently case (ii) cannot happen. Thus Case (A): $\partial D_1 \cap \partial D_2 \ne \emptyset$, is impossible.

Case (B): $\partial D_1 \cap \partial D_2 = \emptyset$

Then we have either $cl(D_1) \subset D_2$, $cl(D_2) \subset D_1$, $cl(D_1) \subset \mathbb{R}^2 \setminus cl(D_2)$ or $cl(D_2) \subset \mathbb{R}^2 \setminus cl(D_1)$. Without loss of generality, we may assume that $cl(D_1) \subset D_2$ or $cl(D_1) \subset \mathbb{R}^2 \setminus cl(D_2)$. In the former case, we readily see that one side of D_2 can be extended to ∞ in $\mathbb{R}^2 \setminus cl(D_1)$, and this case cannot happen in terms of Lemma 5. In the latter case, we have the alternatives (a) and (b):

(a) there exist vertices A_1, A_2 of D_2 such that the infinite half line A_1A_2 is in $\mathbb{R}^2 \setminus cl(D_1)$.

(b) The extended lines of any sides of D_2 intersect ∂D_1.

In case (a), we directly derive a contradiction by Lemma 5. In case (b), we can take points O, P and Q such that $O \in \partial D_1$, $\overline{PQ} \in \partial D_2$ and O, P, Q are on the same line.

By (4.1) and Lemma 1, we have $u_1 = 0$ on $\overline{OP}$. Therefore, similarly to Case A, we can reach a contradiction. Thus the proof of Theorem 2.2 is complete.

5. Concluding Remarks

(I) As is seen from Example 3 in Section 2, our assumption (2.1) for the uniqueness is not optimal. In fact, in Third Step of the proof in the case of $\partial D_1 \cap \partial D_2 \neq \emptyset$, we have chosen only one zero level line $\overline{O_1P_1}$ of u_1 for forming a trapped ray, although we can produce more zero level lines of u_1. As a consequence, in general, we have restricted possibilities of gaining zero level lines which are in $\mathbb{R}^2 \setminus cl(D_1)$ and extended to ∞. Therefore our choice of a single zero level line may be not able to lead to a contradiction in some cases. However in Example 2 in Section 2, a single choice is a unique possibility.

(II) In this paper, for simplicity of the proof, we consider a single obstacle D. We can prove the uniqueness in the same manner in the case where D is a sum of a finite number of polygonal domains $D^1, ..., D^N$ such that $D^m \cap D^n$, $m \neq n$, has no interior points.

(III) We can similarly discuss the sound-hard obstacle (i.e., (1.1), (1.3) and $\frac{\partial u}{\partial \nu} = 0$ on ∂D). Moreover our argument works in the three dimensional case.

(IV) The argument works in a similar inverse obstacle scattering problem for the Maxwell equations with real analytic permittivity and permeablity.

(V) With certain restrictions, we can similarly discuss the case where ∂D is piecewise analytic. For example, let

$$U = \Big\{ D;\ D \text{ is a convex bounded domain, and there exist}$$
$$N \geq 2, \in \mathbb{N} \text{ and analytic curves } \gamma_j,\ 1 \leq j \leq N \text{ such that } \partial D = \bigcup_{j=1}^{N} \gamma_j$$
$$\text{and } \gamma_j \text{ can be extended in } \mathbb{R}^2 \setminus cl(D) \text{ analytically to } \infty. \Big\}$$

Then we can prove the uniqueness within U.

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