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RECOVERY OF A SUPPORT OF A SOURCE TERM IN AN ELLIPTIC EQUATION

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Abstract. This paper deals with an inverse problem of determining the shape and location of inhomogeneity D in an elliptic equation with a Neumann boundary condition $\frac{\partial u}{\partial \nu} = g$ on $\partial\Omega$

$$-\Delta u(x) + p\chi_D(x)u(x) = 0, \quad x \in \Omega,$$

where $\overline{D} \subset \Omega$ and χ_D is the characteristic function of a subdomain D. We prove the global uniqueness in this inverse problem by a single measurement of Dirichlet data u on $\partial\Omega$.

1. Introduction

We consider the Neumann problem

$$\begin{cases} -\Delta u + p\chi_D u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where p is a positive constant, χ_D is the characteristic function of a subdomain D and ν is the unit outward normal vector to $\partial\Omega$. We assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary C^2 and D is a subdomain compactly contained in Ω with Lipschitz boundary ∂D. Since there exists a unique solution $u \in H^1(\Omega)$ of (1.1) for a given domain D and $g \in H^{-\frac{1}{2}}(\partial\Omega)$, the Neumann-to-Dirichlet map $\Lambda_D : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ can be defined by

$$\Lambda_D(g) := u|_{\partial\Omega}. \tag{1.2}$$

The inverse problem in this paper is to identify the unknown domain D by a single boundary measurement $(g, \Lambda_D(g))$ on $\partial\Omega$.

This type of elliptic equation appears in the determination of the metal-to-semiconductor contact and its resistivity of electric devices. Let us briefly explain

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a metal-to-semiconductor contact according to [9, 10]. Every semiconductor device has metal-to-semiconductor contacts and such contacts form a barrier for electrons and holes. In particular, a large mismatch between the Fermi energy of the metal and the semiconductor can result in a high-resistance rectifying contact. In fabrication of semiconductor devices, therefore, it is important to control and reduce the surface states at the interface between the metal and the semiconductor, i.e., to obtain accurate values of contact resistivity and the physical parameters that govern the interfacial contact resistance. Thus there has been many researches which analyze these contacts as well as the contact resistivity between a (thin) metal layer and a (thin) semiconductor (e.g., [9, 10]). Since the depth of each layer is very thin compared with the length and width, [10] approximated the actual three-dimensional problem to the two-dimentional modelling. Then current density g is applied into a side of semiconductor layer Ω (called the diffusion layer) and the corresponding electric potential u satisfies an elliptic equation

$$\begin{cases} -\Delta u + p\chi_D(u-v) = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on} \quad \partial\Omega, \end{cases}$$

where $p = \frac{R_s}{\rho_c}$ (R_s : the sheet resistance of the semiconductor layer and ρ_c : the contact resistivity), D is the metal-to-semiconductor contact and v is the metal layer potential corresponding to the applied density g. Since the metal layer is usually much more conductive than the semiconductor layer, the metal layer potential v is essentially constant. This constant metal potential v is set at zero and hence two-dimentional main equation (1.1) is obtained.

There are extensive studies concerning our inverse problem (e.g. [3, 5, 12, 13, 14]). In [3], a uniqueness result within a one-parameter monotone family from a one-point boundary measurement of the potential was obtained. Moreover [14] provides a global uniqueness result and a reconstruction scheme within the class of two- or three-dimensional balls from a single boundary measurement. Furthermore Kim and Yamamoto [13] shows a proposition about some non-existence of an H^2-solution to a Cauchy problem of the Laplace equation and can prove the global uniqueness within convex hulls of general polygons by using the non-existence proposition. In the same paper, however, one example is shown, which says that the non-existence proposition does not work in the case where ∂D is smooth. From this standpoint, [12] proposes one global method for our inverse problem. The method is to convert the related two- or three-dimensional partial differential equation into an ordinary differential equation and [12] succeeds partially in proving the global uniqueness within domains with smooth boundary. More detailedly speaking, [12] proves the global uniqeness within families of D's with some seperation property or symmetry. The families can contain many kinds of subdomain D's, for example, balls and concentric ellipses, but is rather restrictive. The purpose of this paper is to relax the restrictions in [12] and to present more improved uniqueness results. Furthermore we deal with the global uniqueness within convex domains. We add that even within convex domains the global uniqueness is referred as an open problem in Isakov [8] (Problem 4.7.2 on p 104).

As for related inverse problems of determining piecewise continuous coefficient $\gamma = \gamma(x)$ in $\nabla \cdot (\gamma \nabla u) = 0$ in Ω, we can refer to [1, 2, 6, 15, 16, 17]. Inverse problems of the determination of the potential q in the Schrödinger equation $-\Delta u + qu = 0$ in Ω is another related significant problem. Our inverse problem is concerned with the determination of shapes of domains and is of a character similar to the classical inverse source problem or the inverse gravimetry where we are required to determine a domain D in $-\Delta u + k\chi_D = 0$ by a single measurement of an exterior potential, where k is a nonzero constant. Moreover the gravimetry problem has the uniqueness result in classes of star-shaped domains with respect to their centres of gravity or x_n-direction convex domains (e.g., [7]). To convert integrals in the interior of domains into integrals on their boundaries, in addition to integration by parts, [7] used an observation that if v is harmonic, then so is $x \cdot \nabla v + 3v$. Because of the potential u in the source term $p\chi_D(x)u(x)$, however, this method cannot be applied to our inverse problem.

The outline of this paper is as follows. In Section 2, we are going just to touch the case in which domains have common-contact boundary portion. In Section 3, we introduce some orthogonality relation, which plays an essential role in proving our remaining main theorems. Next we prove two uniqueness results Theorem 3.2 and Theorem 3.3, which generalize the previous results in [12]. Finally, in Section 4, we prove the uniqueness within convex domains with some reflection condition which will be defined in the same section.

2. Common-Contact Case

Throughout this paper we assume that $g \in C^{0,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$, $g \geq 0$ and $g \not\equiv 0$ on $\partial\Omega$, and that the domains D under consideration are bounded, simply connected and compactly contained in Ω. If u is the solution to (1.1) corresponding to D and g, then it is well known (e.g., [4, 11]) that

$$u \in C^1(\Omega) \cap H^2(\Omega). \tag{2.1}$$

Also by the maximum principle and Hopf's lemma, we see that

$$u > 0 \quad \text{in} \quad \Omega \setminus D \quad \text{and} \quad u \geq 0, \not\equiv 0 \quad \text{in} \quad D. \tag{2.2}$$

Definition 2.1. *Let D and E be simply connected and compactly contained domains in Ω. We say that D and E satisfy* **the i-contact condition** *if the sets $\Omega \setminus (\overline{D \cup E})$, $D \cap E$ are connected, $(\partial D \cap \partial E) \cap int(\overline{D \cup E}) = \emptyset$ and there exists a nonempty hypersurface which belongs to the boundaries of both $\Omega \setminus (\overline{D \cup E})$ and $D \cap E$.*

For instance, if D and E are star-shaped domains with respect to the origin so that $\partial D \cap \partial E$ contains a nonempty relatively open portion, then D and E satisfy

the i-contact condition. The following theorem states the global uniqueness within domains satisfying **the i-contact condition**.

Theorem 2.2. *Assume that D_1 and D_2 satisfy* **the i-contact condition**. *Then $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$ implies that $D_1 = D_2$.*

Proof. Let u_j, $j = 1$ and 2, be the solution to (1.1) corresponding to the domain D_j. Setting $y = u_1 - u_2$ in Ω, then y is a harmonic function in $\Omega \setminus (\overline{D_1 \cup D_2})$ with $y = \frac{\partial y}{\partial \nu} = 0$ on $\partial\Omega$. Since D_1 and D_2 satisfy **the i-contact condition**, $\Omega \setminus (\overline{D_1 \cup D_2})$ and $D_1 \cap D_2$ are connected. It follows from the unique continuation that

$$y \equiv 0 \qquad \text{in} \quad \Omega \setminus (\overline{D_1 \cup D_2}). \tag{2.3}$$

Let S be a nonempty hypersurface which belongs to the boundaries of both $\Omega \setminus (\overline{D_1 \cup D_2})$ and $D_1 \cap D_2$. Then (2.1) and (2.3) imply that

$$y = |\nabla y| = 0 \qquad \text{on} \quad S. \tag{2.4}$$

Since $D_1 \cap D_2$ is connected and $\Delta y = py$ in $D_1 \cap D_2$, (2.4) and the unique continuation say that

$$y \equiv 0 \qquad \text{in} \quad D_1 \cap D_2. \tag{2.5}$$

Assume contrarily that $D_1 \neq D_2$. Then either $D_1 \setminus \overline{D_2} \neq \emptyset$ or $D_2 \setminus \overline{D_1} \neq \emptyset$. Renumbering, if necessary, we may assume that $D_1 \setminus \overline{D_2} \neq \emptyset$. By (2.3) and (2.5), we have

$$y = |\nabla y| = 0 \qquad \text{on} \quad \partial(D_1 \setminus \overline{D_2}). \tag{2.6}$$

Thus (1.1) and (2.2) yield

$$0 < p \int_{D_1 \setminus \overline{D_2}} u_1 \, dx = \int_{D_1 \setminus \overline{D_2}} \Delta y \, dx = \int_{\partial(D_1 \setminus \overline{D_2})} \frac{\partial y}{\partial \nu} \, d\sigma = 0,$$

which is a contradiction. Hence $D_1 = D_2$, so our proof of Theorem 2.2 is complete. □

3. Orthogonality Relation And Symmetric Case

For a moment we consider the gravimetry problem which has a similar form to the governing equation of our inverse problem, that is,

$$-\Delta u + k\chi_D = 0 \qquad \text{in} \quad \mathbb{R}^2 \qquad \text{and} \qquad \lim_{|x| \to \infty} u(x) = O(\ln |x|), \tag{3.1}$$

where k is a positive (or negative) constant. For details, we can refer to [7]. Because there is no fear of confusion, we denote the solution to (3.1) corresponding to D_j, $j = 1$ and 2, by u_j and $u_1 - u_2$ by y. Let Ω_0 be a bounded domain with

$(\overline{D_1 \cup D_2}) \subset \Omega_0$. If $\nabla u_1 = \nabla u_2$ on $\partial\Omega_0$ and $\Omega_0 \setminus (\overline{D_1 \cup D_2})$ is connected, then for any $v \in H^1(\Omega_0)$ we can apply the Green theorem to get

$$0 = \int_{D_1 \cup D_2} (\Delta y)v - y(\Delta v)\ dx. \tag{3.2}$$

In the gravimetry problem, an harmonic function is taken as a test function v and we have

$$0 = \int_{D_1 \cup D_2} (\Delta y)v\ dx. \tag{3.3}$$

And since $\Delta y = k$ in $D_1 \setminus \overline{D_2}$, $\Delta y = -k$ in $D_2 \setminus \overline{D_1}$ and $\Delta y = 0$ in $D_1 \cap D_2$, we can obtain an orthogonality relation

$$\int_{D_1 \setminus \overline{D_2}} v\ dx = \int_{D_2 \setminus \overline{D_1}} v\ dx, \tag{3.4}$$

on which the uniqueness of a domain in the gravimetry problem is based.

However, if an harmonic function is taken as a test function v in our inverse problem, then (3.3) can be changed as follows

$$0 = \int_{D_1 \setminus \overline{D_2}} u_1 v\ dx + \int_{D_1 \cap D_2} yv\ dx - \int_{D_2 \setminus \overline{D_1}} u_2 v\ dx. \tag{3.5}$$

Here, unfortunately, we have no information about y on $D_1 \cap D_2$, and so (3.5) is not of help to us. The following lemma gives an alternative for our inverse problem to (3.4) and our remaining theorems are essentially based on this lemma.

Lemma 3.1. *Let D_j, $j = 1$ and 2, be a simply connected and compactly contained domain in Ω and let u_j the solution to (1.1) corresponding to D_j. Assume that $\Omega \setminus (\overline{D_1 \cup D_2})$ is simply connected and let Ω_0 be a bounded domain with $(\overline{D_1 \cup D_2}) \subset \Omega_0$. If $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$, then for any function $v \in H^1(\Omega_0)$ satisfying an elliptic equation $\Delta v = pv$ in Ω_0 the following orthogonality relation holds*

$$\int_{D_1 \setminus \overline{D_2}} u_2 v\ dx = \int_{D_2 \setminus \overline{D_1}} u_1 v\ dx.$$

Proof. By (1.1), (2.3) and (3.2), we obtain

$$\begin{aligned} 0 &= p \textstyle\int_{D_1 \setminus \overline{D_2}} u_1 v\ dx + p \int_{D_1 \cap D_2} yv\ dx - p \int_{D_2 \setminus \overline{D_1}} u_2 v\ dx - \int_{D_1 \cup D_2} y(\Delta v)\ dx \\ &= p \textstyle\int_{D_1 \cup D_2} yv\ dx + p \int_{D_1 \setminus \overline{D_2}} u_2 v\ dx - p \int_{D_2 \setminus \overline{D_1}} u_1 v\ dx - \int_{D_1 \cup D_2} y(\Delta v)\ dx. \end{aligned} \tag{3.6}$$

Since $\Delta v = pv$ in $D_1 \cup D_2$, hence we have

$$0 = \int_{D_1 \setminus \overline{D_2}} u_2 v\ dx - \int_{D_2 \setminus \overline{D_1}} u_1 v\ dx,$$

which completes our proof of Lemma 3.1. □

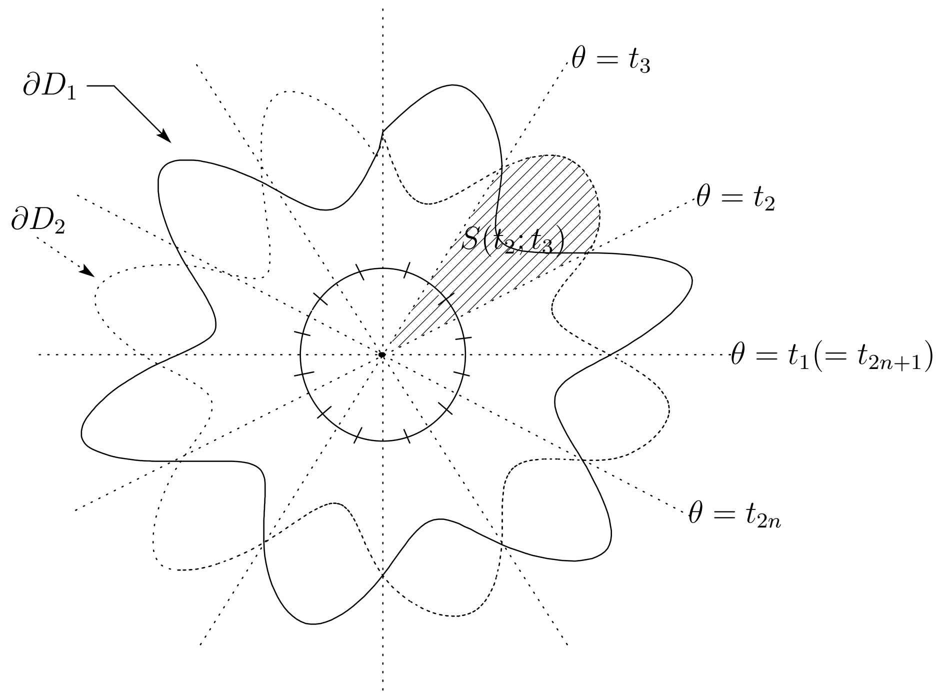

FIGURE 1. Symmetric case

Now we are ready to state the next theorems, which are derived from Lemma 3.1.

Theorem 3.2. *Let D_j, $j = 1$ and 2, be a star-shaped domain with respect to the origin so that either $D_1 = D_2$ or if $D_1 \neq D_2$, then there exists a partition $0 \leq t_1 < t_2 < \cdots < t_{2n} < 2\pi$, $n \geq 1$, of the interval $[0, 2\pi]$ such that*

$$\begin{array}{ll} (i) & t_2 - t_1 = \cdots = t_{2n+1} - t_{2n} \quad (mod\ 2\pi), \\ (ii) & S(t_{2k-1}; t_{2k}) \subset D_1 \quad and \quad S(t_{2k}; t_{2k+1}) \subset D_2 \quad for \quad k = 1, \cdots, n, \\ (iii) & (\overline{D_1 \cup D_2}) = \cup_{i=1}^{2n} \overline{S(t_i; t_{i+1})}, \end{array} \tag{3.7}$$

where $S(t_i; t_{i+1}) := (D_1 \cup D_2) \cap \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 | \theta \in (t_i; t_{i+1})$ and $r > 0\}$ and $2n + 1$ is interpreted as 1. If $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$, then $D_1 = D_2$.

Proof. Suppose that $D_1 \neq D_2$. Taking a suitable rotation, if necessary, we may assume that

$$t_1 = 0.$$

Let us take an open disk B containing $\overline{D_1 \cup D_2}$ and centered at the origin, and let v the unique solution of the Dirichlet problem to an elliptic equation

$$\begin{cases} \Delta v = pv & \text{in} \quad B \\ v(r\cos\theta, r\sin\theta) = \sin n\theta & \text{on} \quad \partial B. \end{cases} \tag{3.8}$$

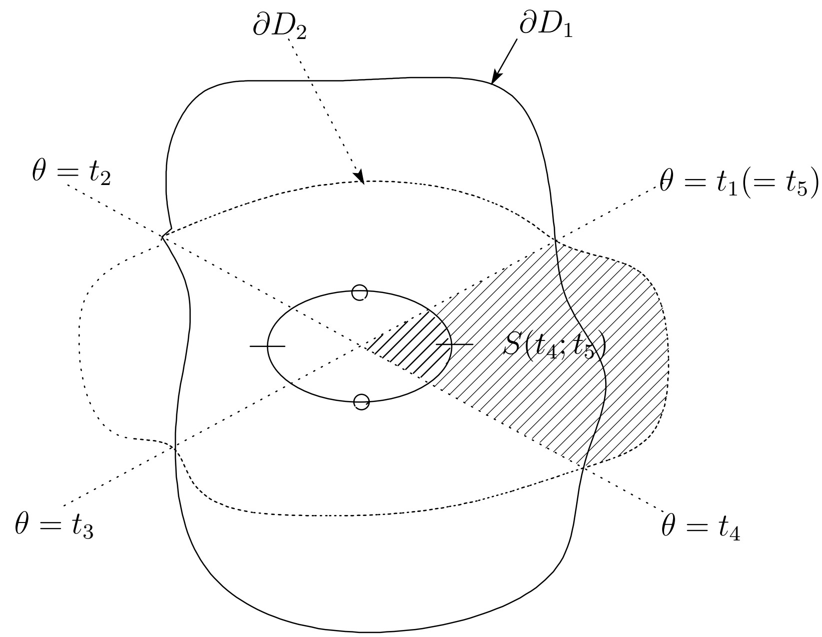

FIGURE 2. Non-symmetric case ($n = 2$)

It follows from the symmetry of the equation (3.8) and the maximum principle that

$$\begin{aligned} & v > 0 && \text{in} \quad S(t_{2k-1}; t_{2k}) \subset D_1, && \\ \text{and} \quad & v < 0 && \text{in} \quad S(t_{2k}; t_{2k+1}) \subset D_2 && \text{for } k = 1, \cdots, n. \end{aligned} \tag{3.9}$$

Therefore, by (2.2), (3.9) and Lemma 3.1, we have

$$0 < \int_{D_1 \setminus \overline{D_2}} u_2 v \, dx = \int_{D_2 \setminus \overline{D_1}} u_1 v \, dx < 0,$$

which is a contradiction. Thus the proof of Theorem 3.2 is complete. □

We add that Theorem 2.1 in [12] deals with the case where $n = 1$, so Theorem 3.2 can be thought of as a generalization of it. The next theorem tells us, in particular if $n = 2$, that the first condition of (3.7) can be relaxed. See Figure 2.

Theorem 3.3. *Let D_j, $j = 1$ and 2, be a star-shaped domain with respect to the origin so that either $D_1 = D_2$ or if $D_1 \neq D_2$, then there exists a partition $0 \leq t_1 < t_2 < t_3 < t_4 < 2\pi$ of the interval $[0, 2\pi]$ such that*

(i) $t_2 - t_1 = t_4 - t_3$ *and* $t_3 - t_2 = t_1 - t_4 \pmod{2\pi}$
(ii) $S(t_1; t_2) \cup S(t_3; t_4) \subset D_1$ *and* $S(t_2; t_3) \cup S(t_4; t_1) \subset D_2$
(iii) $\overline{(D_1 \cup D_2)} = \cup_{i=1}^{4} \overline{S(t_i; t_{i+1})}$.

If $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$, then $D_1 = D_2$.

Proof. Suppose that $D_1 \neq D_2$. Taking a suitable rotation, if necessary, we may assume that

$$t_1 = 0.$$

If $t_2 = \frac{\pi}{2}$, then $t_3 - t_2 = \frac{\pi}{2}$, and so it follows from Theorem 3.2 that $D_1 = D_2$. So we may assume that

$$t_2 \neq \frac{\pi}{2} \qquad \text{and} \qquad 0 < t_2 < \pi.$$

To orthogonalize two independent lines $\{x_2 = 0\}$ and $\{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 | \theta = t_2, t_4 \text{ and } r \geq 0\}$, we introduce a transformation Ψ from the x_1x_2-plane into the $\eta_1\eta_2-$plane

$$\Psi(x_1, x_2) := ((\tan t_2)x_1 - x_2, |\sec t_2| x_2).$$

Then the transformation Ψ maps the line $\{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 | \theta = t_2, t_4, \text{ and } r \geq 0\}$ in the x_1x_2-plane onto the line $\{\eta_1 = 0\}$ in the $\eta_1\eta_2-$plane, and the line$\{x_2 = 0\}$ onto the line $\{\eta_2 = 0\}$. Defining $Y(\eta_1, \eta_2) := y \circ \Psi^{-1}(\eta_1, \eta_2)$ in $\Psi(\overline{D_1 \cup D_2})$, we see that the function Y satisfies

$$\Delta_\eta Y - \frac{2}{|\sec t_2|}\partial_{\eta_1}\partial_{\eta_2} Y = \frac{1}{\sec^2 t_2}(\partial^2_{x_1} y + \partial^2_{x_2} y) \circ \Psi^{-1} \quad \text{in} \quad \Psi(\overline{D_1 \cup D_2}) \tag{3.10}$$

and

$$Y = |\nabla Y| = 0 \qquad \text{on} \qquad \Psi(\partial(D_1 \cup D_2)), \tag{3.11}$$

where $\Delta_\eta = \partial^2_{\eta_1} + \partial^2_{\eta_2}$. Let us take an open disk containing $\overline{\Psi(D_1 \cup D_2)}$ and centered at the origin, and let V the unique solution of the Dirichlet problem in an elliptic equation

$$\begin{cases} \Delta_\eta V - \frac{2}{|\sec t_2|}\partial_{\eta_1}\partial_{\eta_2} V = \frac{p}{\sec^2 t_2} V & \text{in} \quad B \\ V(\eta_1, \eta_2) = V(r\cos\theta, r\sin\theta) = \sin 2\theta & \text{on} \quad \partial B. \end{cases} \tag{3.12}$$

The symmetry of the equation (3.12) and the maximum principle imply that

$$\begin{aligned} V > 0 \quad & \text{in} \quad \Psi(S(0; t_2)) \cup \Psi(S(t_3; t_4)), \\ \text{and} \quad V < 0 \quad & \text{in} \quad \Psi(S(t_2; t_3)) \cup \Psi(S(t_4; 2\pi)). \end{aligned} \tag{3.13}$$

Setting $U_j = u_j \circ \Psi^{-1}$, $j = 1$ and 2, in $\Psi(\overline{D_1 \cup D_2})$, then (3.10), (3.11), (3.13) and the Green theorem we have

$$\begin{aligned}
& 0 \\
= \quad & \tfrac{p}{\sec^2 t_2} \int_{\Psi^{-1}(D_1\setminus\overline{D_2})} U_1 V \, d\eta + \tfrac{p}{\sec^2 t_2} \int_{\Psi^{-1}(D_1\cap D_2)} Y V \, d\eta - \tfrac{p}{\sec^2 t_2} \int_{\Psi^{-1}(D_2\setminus\overline{D_1})} U_2 V \, d\eta \\
& + \tfrac{2}{|\sec t_2|} \int_{\Psi^{-1}(D_1\cup D_2)} (\partial_{\eta_1}\partial_{\eta_2} Y) V \, d\eta - \int_{\Psi^{-1}(D_1\cup D_2)} (\Delta_\eta Y) V \, d\eta \\
= \quad & \tfrac{p}{\sec^2 t_2} \int_{\Psi^{-1}(D_1\setminus\overline{D_2})} U_2 V \, d\eta - \tfrac{p}{\sec^2 t_2} \int_{\Psi^{-1}(D_2\setminus\overline{D_1})} U_1 V \, d\eta + \tfrac{p}{\sec^2 t_2} \int_{\Psi^{-1}(D_1\cup D_2)} Y V \, d\eta \\
& + \tfrac{2}{|\sec t_2|} \int_{\Psi^{-1}(D_1\cup D_2)} Y (\partial_{\eta_1}\partial_{\eta_2} V) \, d\eta - \int_{\Psi^{-1}(D_1\cup D_2)} Y (\Delta_\eta V) \, d\eta \\
= \quad & \tfrac{p}{\sec^2 t_2} \int_{\Psi^{-1}(D_1\setminus\overline{D_2})} U_2 V \, d\eta - \tfrac{p}{\sec^2 t_2} \int_{\Psi^{-1}(D_2\setminus\overline{D_1})} U_1 V \, d\eta.
\end{aligned} \tag{3.14}$$

However (3.13) implies that the right hand side of (3.14) is strictry positive, which is a contradiction. Therefore we can conclude that $D_1 = D_2$. $\square$

Theorem 2.2 in [12] deals also with the case where $n = 2$, yet it requires a strong restriction on domains that there exist two pairs of two parallel lines seperating $D_1 \setminus \overline{D_2}$ from $D_2 \setminus \overline{D_1}$. So this restriction has, for example, ellipses have the common center for uniqueness. Through Theorem 3.3, we can obtain the global uniqueness within general ellipses. From this standpoint, Theorem 3.3 can be considered as a generalization of Theorem 2.2 in [12].

4. Convex Case

In this section, we consider convex domains D, E and regard them as star-shaped domains with respect to the origin. Since any convex bounded domains $D, E \subset \mathbb{R}^2$ with $D \cap E \neq \emptyset$ can be regarded as star-shaped domains with respect to any point in $D \cap E$, our assumption is resonable. That is, ∂D is assumed to be represented by the continuous closed curve $\alpha : [0, 2\pi] \to \partial D$;

$$\alpha(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta) \qquad \text{and} \qquad 0 < r(\theta) \qquad \text{for all} \quad \theta \in [0, 2\pi].$$

Now let us introduce some reflection condition on convex domains.

Definition 4.1. *For convex domains D and E in $\mathbb{R}^2$ containing the origin with boundaries, respectively, $\alpha(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$ and $\beta(\theta) = (\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)$, they are said to satisfy* **the inward reflection condition** *if $\alpha(\theta_1) = \beta(\theta_1)$, $\alpha(\theta_2) = \beta(\theta_2)$ for $0 \le \theta_1 < \theta_2 \le 2\pi$ and $r(\theta) \ge \rho(\theta)$ (or $r(\theta) \le \rho(\theta)$) for all $\theta \in (\theta_1, \theta_2)$, then the reflection of $\alpha(\theta_1, \theta_2) := \{\alpha(\theta) | \theta \in (\theta_1, \theta_2)\}$ (or $\beta(\theta_1, \theta_2)$) with respect to the line $L(\theta_1, \theta_2)$ passing through $\alpha(\theta_1)$ and $\alpha(\theta_2)$ is contained in $D \cap E$.*

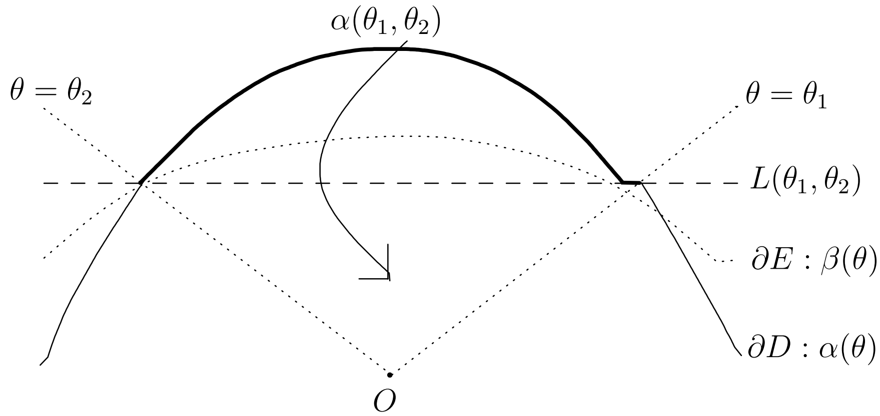

FIGURE 3. The inward reflection condition

The next theorem shows the global uniqueness result of our inverse problem within convex domains satisfying **the inward reflection condition**.

Theorem 4.2. *Let D_j, $j = 1$ and 2, be a convex subdomain of Ω containing the origin with boundary $\alpha_j(\theta) = (r_j(\theta)\cos\theta, r_j(\theta)\sin\theta)$. If D_1 and D_2 satisfy* **the inward reflection condition**, *then $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$ yields $D_1 = D_2$.*

Proof. Suppose that $D_1 \neq D_2$. It is easy from Lemma 3.1 to see that

$$D_1 \setminus \overline{D_2} \neq \emptyset \qquad \text{and} \qquad D_2 \setminus \overline{D_1} \neq \emptyset.$$

Then since D_1 and D_2 satisfy **the inward reflection condition**, there exists a partition $0 \leq t_1 < t_2 < \cdots < t_{2n} < 2\pi$, $n \geq 1$, of the interval $[0, 2\pi]$ such that

$$\begin{array}{llll} (i) & \alpha_1(t_i) = \alpha_2(t_i) & \text{for} \quad i = 1, \cdots, 2n & \\ (ii) & r_1(\theta) \geq r_2(\theta) & , \theta \in (t_{2k-1}, t_{2k}) & \\ \text{and} & r_1(\theta) \leq r_2(\theta) & , \theta \in (t_{2k-1}, t_{2k}) & \text{for} \quad k = 1, \cdots, n, \end{array}$$

where $2n+1$ is interpreted as 1. Taking a suitable rotation, if necessary, we may assume that

$$t_1 = 0.$$

Let us fix a sufficiently small $\epsilon > 0$ and set

$$\begin{array}{rl} & \Gamma_k^1 := L(t_{2k-1} + \epsilon, t_{2k} - \epsilon) \cap D_1 \\ \text{and} & \Gamma_k^2 := L(t_{2k} + \epsilon, t_{2k+1} - \epsilon) \cap D_2 \qquad \text{for} \quad k = 1, \cdots, n. \end{array}$$

Let us take a convex domain D_0 with smooth boundary so that

$$\begin{array}{ll} (i) & \cup_{k=1}^n (\Gamma_k^1 \cup \Gamma_k^2) \subset \partial D_0, \\ (ii) & \partial D_0 \setminus \cup_{k=1}^n (\Gamma_k^1 \cup \Gamma_k^2) \subset \Omega \setminus (D_1 \cup D_2). \end{array}$$

Let us fix any positive $\lambda \in \mathbb{R}$ and let v be a solution in $H^1(D_0)$ of the elliptic equation

$$\begin{cases} \Delta v = pv & \text{in} \quad D_0, \\ v = (-1)^{j+1}\lambda & \text{on} \quad \Gamma_k^j, \quad j = 1,2 \text{ and } k = 1, \cdots, n, \\ |v| \le \lambda & \text{on} \quad \partial D_0, \\ v \in C^{2,\alpha}(\partial D_0) & \text{for some} \quad 0 < \alpha < 1. \end{cases} \tag{4.1}$$

Since ∂D_0 is smooth, we see (e.g., [4, 11]) that

$$v \in C^2(\overline{D_0}). \tag{4.2}$$

To extend the function v, we need to introduce some notaions. For $j = 1$, 2 and $k = 1, \cdots, n$, let us denote by D_k^j the subdomain of D_j with $\partial D_k^j = \alpha_j(t_{m(j,k)} + \epsilon, t_{m(j,k)+1} - \epsilon) \cup \Gamma_k^j$ and by n_k^j the unit outward normal vector to $\Gamma_k^j \subset \partial D_0$, where $m(j,k)$ is $2k-1$ if $j = 1$ and $2k$ if $j = 2$. We are going to extend the function v by

$$v_E(x) = \begin{cases} v(x), \qquad \text{if} \quad x \in \overline{D_0} \\ -v(x_k^j - |x - x_k^j| n_k^j) + 2\lambda(-1)^{j+1} \cosh \sqrt{p}|x - x_k^j|, \\ \text{ if } x \in D_k^j \text{ for } j = 1,2 \text{ and } k = 1, \cdots, n, \end{cases} \tag{4.3}$$

where x_k^j is the orthogonal projection of $x \in D_k^j$ onto Γ_k^j for $j = 1$, 2 and $k = 1, \cdots, n$. Since D_1 and D_2 satisfy **the inward reflection condition**, if ϵ is sufficiently small, then v_E is well-defined. If $x \in \Gamma_k^j$ for $j = 1$, 2 and $k = 1, \cdots, n$, then $x = x_k^j$, and so by (4.1) we have

$$\begin{aligned} & \lim_{y \to x \text{ and } y \in D_k^j} [-v(y_k^j - |y - y_k^j| n_k^j) + 2\lambda(-1)^{j+1} \cosh \sqrt{p}|y - y_k^j|] \\ = \; & -v(x) + 2\lambda(-1)^{j+1} \\ = \; & v(x). \end{aligned} \tag{4.4}$$

Therefore (4.2) - (4.4) imply that

$$v_E \in C(\overline{D_1 \cup D_2}). \tag{4.5}$$

Since $|v| \le \lambda$ on ∂D_0, the maximum principle says that

$$|v| \le \lambda \qquad \text{in} \quad \overline{D_0}. \tag{4.6}$$

For $k = 1, \cdots, n$, by (4.3) and (4.6) we obtain for any $x \in D_k^1$

$$\begin{aligned} v_E(x) \; & \ge -\lambda + 2\lambda \cosh \sqrt{p}|x - x_k^1| \\ & \ge \lambda \exp(-\sqrt{p}|x - x_k^1|) \\ & > 0, \end{aligned} \tag{4.7}$$

and for any $x \in D_k^2$

$$\begin{aligned} v_E(x) \; & \le \lambda - 2\lambda \cosh \sqrt{p}|x - x_k^2| \\ & \le -\lambda \exp(-\sqrt{p}|x - x_k^2|) \\ & < 0. \end{aligned} \tag{4.8}$$

Next I claim that for $j = 1$, 2 and $k = 1, \cdots, n$ we have

$$\left.\frac{\partial v_E}{\partial n_k^j}\right|_+ (x) = \left.\frac{\partial v_E}{\partial n_k^j}\right|_- (x) \qquad , \quad x \in \Gamma_k^j, \tag{4.9}$$

where $\left.\frac{\partial v_E}{\partial n_k^j}\right|_+ (x) = \lim_{h\downarrow 0} \frac{v_E(x+hn_k^j)-v_E(x)}{h}$ and $\left.\frac{\partial v_E}{\partial n_k^j}\right|_- (x) = \lim_{h\downarrow 0} \frac{v_E(x)-v_E(x-hn_k^j)}{h}$. In fact, since $x - hn_k^j \in D_0$ and $v_E = v$ on $\overline{D_0}$, we have

$$\left.\frac{\partial v_E}{\partial n_k^j}\right|_- (x) = \lim_{h\downarrow 0} \frac{v(x) - v(x - hn_k^j)}{h} = \frac{\partial v}{\partial n_k^j}(x) \qquad , \quad x \in \Gamma_k^j. \tag{4.10}$$

On the other hand, the direct calculation shows that for $x \in \Gamma_k^j$

$$\begin{aligned} \left.\tfrac{\partial v_E}{\partial n_k^j}\right|_+ (x) &= \lim_{h\downarrow 0} \tfrac{v_E(x+hn_k^j)-v_E(x)}{h} \\ &= \lim_{h\downarrow 0} \tfrac{-v(x-hn_k^j)+2\lambda(-1)^{j+1}\cosh\sqrt{p}h - v(x)}{h} \\ &= \lim_{h\downarrow 0} \tfrac{v(x)-v(x-hn_k^j)}{h} + \lim_{h\downarrow 0} \tfrac{2\lambda(-1)^{j+1}\cosh\sqrt{p}h - 2v(x)}{h} \\ &= \lim_{h\downarrow 0} \tfrac{v(x)-v(x-hn_k^j)}{h} \\ &= \tfrac{\partial v}{\partial n_k^j}(x). \end{aligned} \tag{4.11}$$

Thus by (4.10) and (4.11), (4.9) has been proved. Finally I will prove that $\Delta v_E = pv_E$ in $(D_1 \cup D_2)$. Since $v_E = v$ on $\overline{D_0}$, it suffices to show that

$$\Delta v_E = pv_E \qquad \text{in} \quad D_k^j \qquad \text{for } j = 1, 2 \text{ and } k = 1, \cdots, n. \tag{4.12}$$

Let us fix any $j = 1, 2$ and $k = 1, \cdots, n$. Taking a suitable rotation and translation, if necessary, we may assume that

$$n_k^j = (1, 0) \qquad \text{and} \qquad \Gamma_k^j \subset \{x_1 = 0\}.$$

Then for $x = (x_1, x_2) \in D_k^j$ we have

$$\begin{aligned} v_E(x) &= -v((0, x_2) - (x_1, 0)) + 2\lambda(-1)^{i+1} \cosh\sqrt{p}x_1 \\ &= -v(-x_1, x_2) + 2\lambda(-1)^{i+1} \cosh\sqrt{p}x_1, \end{aligned}$$

so

$$\begin{aligned} \Delta v_E(x) &= -\Delta v(-x_1, x_2) + 2p\lambda(-1)^{i+1} \cosh\sqrt{p}x_1 \\ &= -pv(-x_1, x_2) + 2p\lambda(-1)^{i+1} \cosh\sqrt{p}x_1 \\ &= pv_E(x). \end{aligned}$$

Since j and k are chosen arbitrily, the proof of (4.12) is complete. Therefore by Lemma 3.1 we have

$$\int_{D_1\setminus\overline{D_2}} u_2 v_E \, dx = \int_{D_2\setminus\overline{D_1}} u_1 v_E \, dx \tag{4.13}$$

and

$$\sum_{k=1}^{n}\int_{D_k^1} u_2 v_E \, dx + \int_{(D_1\setminus\overline{D_2})\cap D_0} u_2 v_E \, dx = \sum_{k=1}^{n}\int_{D_k^2} u_1 v_E \, dx + \int_{(D_2\setminus\overline{D_1})\cap D_0} u_1 v_E \, dx. \tag{4.14}$$

Since $|(D_1 \setminus \overline{D_2}) \cap D_0| + |(D_2 \setminus \overline{D_1}) \cap D_0| = O(\epsilon)$ and $v_E \le \lambda$, (4.7) and (4.8) say that the left hand side of (4.14) is strictly positive and the right hand side of (4.14) is strictly negative, which is a contradiction. Thus the proof of Theorem 4.2 is complete. □

Unfortunately, Theorem 4.2 does not say the global uniqueness within general convex domains. Theorem 3.2, 3.3 and 4.2, however, cover pretty large portion of the class of convex domains. The measure $|D_1 \setminus \overline{D_2}| + |D_2 \setminus \overline{D_1}|$ is smaller as n is larger.

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