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The Johnson homomorphism and the third rational cohomology group of the Torelli group

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Abstract

To study the structure of the Torelli group, the Johnson homomorphism and the representation theory of the symplectic group are essential tools. Using them, we give a lower bound for the dimension of the third rational cohomology group and a new approach to the non-triviality of characteristic classes of surface bundles on the Torelli group.

1 Introduction

Let Σ_g be a closed oriented surface of genus $g \geq 3$ and let $\mathcal{M}_g$ be its mapping class group, namely it is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g. $\mathcal{M}_g$ acts on the first homology group of Σ_g and it gives the classical representation

$$\mathcal{M}_g \longrightarrow \mathrm{Sp}(2g, \mathbb{Z})$$

with the kernel $\mathcal{I}_g$ called the Torelli group, which is the main object of this paper.

To clarify the structure of $\mathcal{I}_g$, Johnson defined a surjective homomorphism

$$\tau : \mathcal{I}_g \longrightarrow U$$

in [Jo1], where U is a certain free abelian group. This homomorphism is now called the (first) Johnson homomorphism, and many properties of $\mathcal{I}_g$ were found using this.

In this paper, we use this homomorphism to investigate the cohomological structure of $\mathcal{I}_g$. In particular, we pay attention to the homomorphism

$$\tau^* : \Lambda^n U_{\mathbb{Q}} \longrightarrow H^n(\mathcal{I}_g, \mathbb{Q})$$

where $U_{\mathbb{Q}} = U \otimes_{\mathbb{Z}} \mathbb{Q}$. $n = 0$ is the trivial case. The case of $n = 1$ was studied by Johnson in [Jo2] where he showed that τ^* is an isomorphism. The case of $n = 2$ was settled by Hain in [Ha] using the representation theory of the symplectic group. Now we treat the case of $n = 3$ using the same method as Hain. To each irreducible component of $\Lambda^3 U_{\mathbb{Q}}$ except one, we determine whether it survives in $H^3(\mathcal{I}_g, \mathbb{Q})$ or not. Then we show that there exists an interesting relationship between the non-triviality of the remained irreducible component and that of certain characteristic classes of surface bundles whose monodromy groups are contained in the Torelli group.

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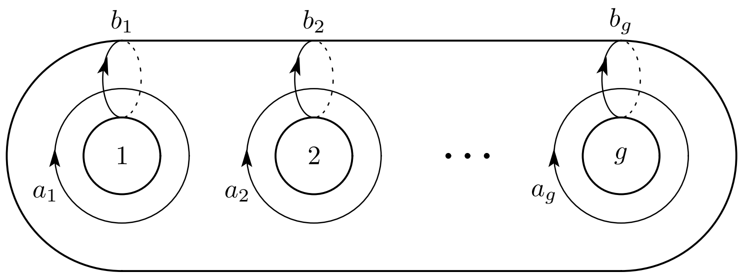
2 Summary of previous results

First, we prepare some notations. Σ_g is a closed oriented surface of genus $g \geq 3$. Let $H_1(\Sigma_g)$ be its first homology group with coefficients in $\mathbb{Z}$. By the Poincaré duality, we can take a symplectic basis of $H_1(\Sigma_g)$ with respect to the intersection form $\mu : H_1(\Sigma_g) \otimes H_1(\Sigma_g) \to \mathbb{Z}$ which is a non-degenerate bilinear form. Now we fix a symplectic basis $\langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle$ as follows.



The Poincaré duality also says that we can identify $H_1(\Sigma_g)$ with its dual module $\mathrm{Hom}\,(H_1(\Sigma_g), \mathbb{Z}) = H^1(\Sigma_g)$, the first cohomology group of Σ_g with coefficients in $\mathbb{Z}$. In this identification, a_i (resp. b_i) $\in H_1(\Sigma_g)$ corresponds to $-b_i^*$ (resp. a_i^*) $\in H^1(\Sigma_g)$ where $\langle a_1^*, \ldots, a_g^*, b_1^*, \ldots, b_g^* \rangle$ is the dual basis of $H^1(\Sigma_g)$. We use the same symbol H for these identified abelian groups. We denote $H \otimes_{\mathbb{Z}} \mathbb{Q}$ by $H_{\mathbb{Q}}$ and similarly $U \otimes_{\mathbb{Z}} \mathbb{Q}$ by $U_{\mathbb{Q}}$ where U is an abelian group.

As mentioned above, $\mathcal{M}_g$ is the mapping class group of Σ_g and $\mathcal{I}_g$ is its Torelli group. We also use the pointed mapping class group $\mathcal{M}_{g,*}$. It is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g which fix the basepoint $* \in \Sigma_g$ where these isotopies fix $*$ at every level. We denote the corresponding Torelli group by $\mathcal{I}_{g,*}$.

Next we introduce the Johnson homomorphism. In [Jo1], Johnson defined a homomorphism

$$\tau : \mathcal{I}_g \longrightarrow \Lambda^3 H/H$$

and he showed that τ is surjective. Here H is considered as a subgroup of $\Lambda^3 H$ by the injection

$$H \hookrightarrow \Lambda^3 H \quad \left(x \mapsto x \wedge \left(\sum_{i=1}^{g} a_i \wedge b_i \right) \right).$$

¿From now on we denote $\Lambda^3 H/H$ by U. U is a free abelian group of rank $\binom{2g}{3} - 2g$.

In the pointed case, the Johnson homomorphism has $\Lambda^3 H$ as its target group and we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{I}_{g,*} & \xrightarrow{\tau} & \Lambda^3 H \\ \downarrow & & \downarrow \\ \mathcal{I}_g & \xrightarrow{\tau} & U \end{array}$$

where the first vertical map is the homomorphism of forgetting the basepoint $*$ and the second one is the natural projection. We use the same symbol τ for the above two versions of Johnson homomorphisms but any confusion will not occur.

One important property of the Johnson homomorphism is that τ is an $\mathcal{M}_g$-equivariant homomorphism, where $\mathcal{M}_g$ acts on $\mathcal{I}_g$ by the conjugation and acts on U by the way induced from the action of $\mathcal{M}_g$ on H. In the pointed case there exist similar properties.

Applying the functor $H^*(\,\cdot\,,\mathbb{Q})$ to $\tau:\mathcal{I}_g\to U$, we obtain

$$\tau^*: H^*(U,\mathbb{Q})\cong \Lambda^* U_{\mathbb{Q}} \longrightarrow H^*(\mathcal{I}_g,\mathbb{Q})$$

which is an $\mathcal{M}_g$-homomorphism. We now consider the kernel of τ^*. $\operatorname{Ker}\tau^*$ is a priori an $\mathcal{M}_g$-subspace of $\Lambda^* U_{\mathbb{Q}}$. In the whole of $\mathcal{M}_g$, $\mathcal{I}_g$ acts on U trivially so that $\operatorname{Ker}\tau^*$ becomes an $\mathrm{Sp}(2g,\mathbb{Z})$-submodule. We can say more by the next lemma.

Lemma 2.1 $\operatorname{Ker}\tau^*$ *is actually an* $\mathrm{Sp}(2g,\mathbb{Q})$*-submodule.*

For the proof of this lemma, see [AN]. This lemma enables us to use the representation theory of the symplectic group for the determination of $\operatorname{Ker}\tau^*$. We summarize notations and general facts about it in section 4.

In cases of low degrees, τ^* have already been studied as follows. The case of $*=0$ is trivial. The case of $*=1$ was settled by Johnson. He showed the following theorem.

Theorem 2.2 ([Jo2]) $\tau^*: U_{\mathbb{Q}}\to H^1(\mathcal{I}_g,\mathbb{Q})$ *is an isomorphism.*

This theorem implies that in the setting of the rational cohomology, the Johnson homomorphism measures the gap between $\mathcal{I}_g$ and its abelianization.

The case of $*=2$ was settled by Hain in [Ha]. First, we see the irreducible decomposition of $\Lambda^2 U_{\mathbb{Q}}$.

Lemma 2.3 *If $g\ge 3$, then the irreducible decomposition of $\Lambda^2 U_{\mathbb{Q}}$ is given by*

$$\Lambda^2 U_{\mathbb{Q}} = \begin{cases} [0]+[2^2]+[1^2]+[2^21^2]+[1^4]+[1^6] & (g\ge 6)\\ [0]+[2^2]+[1^2]+[2^21^2]+[1^4] & (g=5)\\ [0]+[2^2]+[1^2]+[2^21^2] & (g=4)\\ [0]+[2^2] & (g=3). \end{cases}$$

As mentioned in [Ha], the irreducible decomposition stabilizes for sufficiently large g. In this case the stability range is $g\ge 6$. With respect to the above decomposition, Hain showed the following theorem.

Theorem 2.4 ([Ha]) *For all $g\ge 3$,* $\operatorname{Ker}\tau^* = [0]+[2^2]$.

As an immediate corollary of this theorem, we see that when $g=3$, τ^* is the 0-map for all degrees greater than 1.

3 Main results

In this paper, we treat the case of $* = 3$. First, we need to know the irreducible decomposition of $\Lambda^3 U_{\mathbb{Q}}$. In this case the stability range is given by $g \geq 9$.

Lemma 3.1 *The irreducible decomposition of $\Lambda^3 U_{\mathbb{Q}}$ is given by the following table, where numbers indicate multiplicities of the decomposition and numbers in brackets indicate ones of* $\mathrm{Image}\,(\cup : U_{\mathbb{Q}} \otimes ([2^2] + [0]) \to \Lambda^3 U_{\mathbb{Q}})$ *(i.e. they are in the kernel of τ^* as mentioned later).*

	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$	$g=8$	$g \geq 9$
$[3^2 1^3]$			1	1	1	1	1
$[3^2 1]$		1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
$[3 2^3]$		1	1	1	1	1	1
$[321^2]$		1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
$[32]$	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
$[2^3 1^3]$				1	1	1	1
$[2^3 1]$			1	1	1	1	1
$[2^2 1^5]$					1	1	1
$[2^2 1^3]$			1 (1)	2 (1)	2 (1)	2 (1)	2 (1)
$[2^2 1]$		1 (1)	2 (1)	2 (1)	2 (1)	2 (1)	2 (1)
$[21^5]$				1	1	1	1
$[21^3]$		1 (1)	2 (1)	2 (1)	2 (1)	2 (1)	2 (1)
$[21]$		1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
$[1^9]$							1
$[1^7]$						1	1
$[1^5]$			1	1	2	2	2
$[1^3]$	1 (1)	2 (2)	2 (2)	3 (2)	3 (2)	3 (2)	3 (2)
$[1]$			1	1	1	1	1

Proof. By a direct calculation using a computer. $\square$

Now we mention the main results of this paper.

Theorem 3.2 *For all $g \geq 9$,* $\mathrm{Ker}\,\tau^*$ *contains the direct sum*

$$[3^2 1] + [321^2] + [32] + [2^2 1^3] + [2^2 1] + [21^3] + [21] + 2[1^3]$$

which is equal to $\mathrm{Image}\,(\cup : U_{\mathbb{Q}} \otimes ([2^2] + [0]) \to \Lambda^3 U_{\mathbb{Q}})$. *Moreover, one of the following two possibilities holds:*

a) $\mathrm{Ker}\,\tau^*$ *is actually equal to it.*

b) $\mathrm{Ker}\,\tau^*$ *is equal to the direct sum of it with one more summand* [1].

The determination of $\mathrm{Ker}\,\tau^*$ is equivalent to that of $\mathrm{Image}\,\tau^*$ so that this theorem gives a lower bound for the dimension of the third rational cohomology of the Torelli group. In section 5, we also calculate the cases of lower genera but here we omit the details. With respect to the summand [1], we can relate it with characteristic classes of surface bundles defined in [Mo1] and [Mu] as follows.

Theorem 3.3 *For all $g \geq 5$,*

$$\tau^*([1]) = \{0\} \subset H^3(\mathcal{I}_g, \mathbb{Q}) \iff e_2 - (2-2g)e^2 = 0 \in H^4(\mathcal{I}_{g,*}, \mathbb{Q})$$

where e is the Euler class and e_2 is the second Morita-Mumford class.

The above condition is compatible with the pull-back of the universal Σ_g-bundle. Therefore comparing the result of Morita in [Mo2], we obtain the next corollary.

Corollary 3.4 *For every amenable group G and every group homomorphism $f : G \to \mathcal{I}_g$,*

$$f^*\tau^*([1]) = \{0\} \subset H^3(G, \mathbb{Q}).$$

In the proof of Theorem 3.2, we construct some abelian cycles to find summands which survive in $H^3(\mathcal{I}_g, \mathbb{Q})$. This corollary implies that on any abelian cycle the summand $\tau^*([1])$ is equal to 0.

4 Use of the representation theory of the symplectic group

In this section, we summarize some notations and basic facts concerning the representation theory of $\mathrm{Sp}(2g, \mathbb{Q})$ from [FH],[Ha] and [Mo5]. Let $\mathfrak{sp}(2g, \mathbb{C})$ be the Lie algebra of $\mathrm{Sp}(2g, \mathbb{C})$. By the general theory of the representation, we know that representations of $\mathrm{Sp}(2g, \mathbb{C})$ and that of $\mathfrak{sp}(2g, \mathbb{C})$ are the same and their common irreducible representations are parameterized by Young diagrams whose number of rows are less than or equal to g. We use the notation in [Mo5] to describe Young diagrams. These representations are all rational representations defined over $\mathbb{Q}$ so that we can consider them as irreducible representations of $\mathrm{Sp}(2g, \mathbb{Q})$ and $\mathfrak{sp}(2g, \mathbb{Q})$. For example $H_\mathbb{Q}$ is the fundamental representation denoted by [1]. We fix its symplectic basis $\langle a_1, \ldots, a_g, b_1, \ldots, b_g\rangle$ with respect to the intersection form $\mu : H_\mathbb{Q} \otimes H_\mathbb{Q} \to \mathbb{Q}$ as before. Notice that any representation is realized as some $\mathfrak{sp}(2g, \mathbb{Q})$-submodule of appropriate tensors of $H_\mathbb{Q}$. Another important example is $U_\mathbb{Q} = [1^3]$ so that we can write that $\Lambda^3 H_\mathbb{Q} = [1] + [1^3]$ (where $+$ is the sum in the representation ring of $\mathrm{Sp}(2g, \mathbb{Q})$).

For later use, we define the following elements $X_{i,j}, Y_{i,j}$ $(i \neq j)$ and U_i of $\mathfrak{sp}(2g, \mathbb{Q})$ characterized by

$$\begin{array}{rclrcl} X_{i,j}(a_k) & = & \delta_{jk}a_i, & X_{i,j}(b_k) & = & -\delta_{ik}b_j \\ Y_{i,j}(a_k) & = & 0, & Y_{i,j}(b_k) & = & \delta_{ik}a_j + \delta_{jk}a_i \\ U_i(a_k) & = & 0, & U_i(b_k) & = & \delta_{ik}a_i \end{array}$$

where δ_{ij} is the Kronecker's delta. We can easily check that $X_{i,j}, Y_{i,j}, U_i$ are certainly elements of $\mathfrak{sp}(2g, \mathbb{Q})$.

To $\mathfrak{sp}(2g, \mathbb{Q})$-modules $\Lambda^* H_\mathbb{Q}$ (where $\Lambda^0 H_\mathbb{Q} = \mathbb{Q}$ is the trivial representation), we have the contraction homomorphisms $C_k : \Lambda^k H_\mathbb{Q} \to \Lambda^{k-2} H_\mathbb{Q}$ $(k = 2, 3, \cdots)$ given by

$$C_k(x_1 \wedge \cdots \wedge x_k) = \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} \mu(x_i, x_j) x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_k.$$

As is well known, the kernel of C_k is the irreducible $\mathfrak{sp}(2g,\mathbb{Q})$-module corresponding to the Young diagram $[1^k]$. We especially pay attention to the case of $k=3$. Then we have an exact sequence

$$0 \longrightarrow U_{\mathbb{Q}} \xrightarrow{i} \Lambda^3 H_{\mathbb{Q}} \xrightarrow{C_3} H_{\mathbb{Q}} \longrightarrow 0.$$

Here, we define a map $q : \Lambda^3 H_{\mathbb{Q}} \to \Lambda^3 H_{\mathbb{Q}}$ by

$$q(\xi) = \xi - \frac{1}{g-1} C_3(\xi) \wedge \omega \quad (\xi \in \Lambda^3 H_{\mathbb{Q}}),$$

where $\omega = \sum_{i=1}^g a_i \wedge b_i$ is the symplectic class. Then we can easily see that $\operatorname{Image} q = \operatorname{Ker} C_3$ so that we have an explicit description of the direct sum decomposition

$$\Lambda^3 H_{\mathbb{Q}} = U_{\mathbb{Q}} \oplus H_{\mathbb{Q}}$$

given by the correspondence $\xi \mapsto (q(\xi), \frac{1}{g-1} C_3(\xi) \wedge \omega)$.

Now we define

$$\begin{aligned} p_{ij} &= q(a_i \wedge a_j \wedge b_j) = a_i \wedge a_j \wedge b_j - \tfrac{1}{g-1} a_i \wedge \omega, \\ q_{ij} &= q(b_i \wedge a_j \wedge b_j) = b_i \wedge a_j \wedge b_j - \tfrac{1}{g-1} b_i \wedge \omega. \end{aligned}$$

It is easily checked that there are 2g relations

$$\sum_{j \neq i} p_{ij} = 0, \quad \sum_{j \neq i} q_{ij} = 0 \quad (i = 1, \cdots g).$$

Lemma 4.1 *$U_{\mathbb{Q}}$ is the vector space generated by the following elements*

$$\begin{array}{lll} a_i \wedge a_j \wedge a_k, & b_i \wedge b_j \wedge b_k & (i<j<k) \\ a_i \wedge a_j \wedge b_k, & b_i \wedge b_j \wedge a_k & (i<j,\ k \neq i,j) \\ p_{ij},\ q_{ij} & (i \neq j) & \end{array}$$

and $2g$ *relations of* $\sum_{j\neq i} p_{ij} = 0$, $\sum_{j \neq i} q_{ij} = 0$ $(i = 1, 2, \cdots, g)$ *represent a complete system of linear relations among them.*

Proof. It is easy to see that above elements are certainly in $U_{\mathbb{Q}}$. Then the result follows from a count of dimensions. □

U can be considered as a lattice in $U_{\mathbb{Q}}$. Namely, U is isomorphic to the $\mathrm{Sp}(2g,\mathbb{Z})$-submodule of $U_{\mathbb{Q}}$ generated by above generators.

We often use following three types of $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphisms. Here V is some $\mathrm{Sp}(2g,\mathbb{Q})$-vector space and v_i are elements of V.

1) The *canonical inclusion* $i_V^n : \Lambda^n V \hookrightarrow \otimes^n V$ is given by

$$i_V^n (v_1 \wedge v_2 \wedge \cdots \wedge v_n) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\ v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$$

where $\mathfrak{S}_n$ is the symmetric group of degree n.

2) The *multiplication* $multi.: \Lambda^m V \otimes \Lambda^n V \to \Lambda^{m+n} V$ is given by

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge v_{m+2} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_{m+n}.$$

We define the multiplication $multi.: \Lambda^m(\Lambda^n V) \to \Lambda^{mn} V$ similarly.

3) Using 1) and the canonical projection $\otimes^2 V \to \Lambda^2 V$ given by $v_1 \otimes v_2 \mapsto v_1 \wedge v_2$, we also define the inclusion $j_V : \Lambda^3 V \hookrightarrow V \otimes \Lambda^2 V$ given by

$$j_V(v_1 \wedge v_2 \wedge v_3) = v_1 \otimes (v_2 \wedge v_3) + v_2 \otimes (v_3 \wedge v_1) + v_3 \otimes (v_1 \wedge v_2).$$

For the sake of simplicity, we denote $a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_n}$ by $A_{i_1 i_2 \cdots i_n}$. Some indices may be negative, then we interpret them by changing a into b. For example, A_{1-2-34} stands for $a_1 \wedge b_2 \wedge b_3 \wedge a_4$.

5 Proof of the main results

Now we prove Theorem 3.2 and 3.3. In sections 5.1 and 5.2, all vector spaces are $\mathrm{Sp}(2g, \mathbb{Q})$-modules and all homomorphisms are $\mathrm{Sp}(2g, \mathbb{Q})$-equivariant so that we omit the symbol "$\mathrm{Sp}(2g, \mathbb{Q})$-".

5.1 Summands which are in the kernel

Due to Hain's results, we can obtain some summands in $\mathrm{Ker}\, \tau^*$ by taking the cup product with ones in degree 2. We now determine this part, namely we calculate the image of the following homomorphism

$$\cup : H^1(U; \mathbb{Q}) \otimes \mathrm{Ker}\left(H^2(U; \mathbb{Q}) \xrightarrow{\tau^*} H^2(\mathcal{I}_g; \mathbb{Q})\right) \longrightarrow H^3(U; \mathbb{Q}).$$

In terms of $\mathfrak{sp}(2g, \mathbb{Q})$-modules, we determine the image of

$$\wedge : [1^3] \otimes ([2^2] + [0]) \longrightarrow \Lambda^3 [1^3]$$

where $[2^2] + [0]$ is in $\Lambda^2 [1^3]$ and the homomorphism $\wedge$ is given by taking the wedge product.

Lemma 5.1 *If $g \geq 3$, the irreducible decomposition of $[1^3] \otimes ([2^2] + [0])$ is given by*

$$\begin{cases} 2[1^3] + [32] + [21] + [3^2 1] + [31^2] + [21^3] + [2^2 1] + [321^2] + [2^2 1^3] & (g \geq 5) \\ 2[1^3] + [32] + [21] + [3^2 1] + [31^2] + [21^3] + [2^2 1] + [321^2] & (g = 4) \\ 2[1^3] + [32] + [21] + [3^2 1] + [31^2] & (g = 3). \end{cases}$$

With respect to the above decomposition, we prove the following proposition (this fact for the stable range is mentioned in [Mo5] without proof).

Proposition 5.2 *For $g \geq 3$, the irreducible decomposition of* $\mathrm{Image}\left(\wedge : [1^3] \otimes ([2^2] + [0]) \to \Lambda^3 [1^3]\right)$ *is given by*

$$\begin{cases} 2[1^3] + [32] + [21] + [3^2 1] + [21^3] + [2^2 1] + [321^2] + [2^2 1^3] & (g \geq 5) \\ 2[1^3] + [32] + [21] + [3^2 1] + [21^3] + [2^2 1] + [321^2] & (g = 4) \\ [1^3] + [32] & (g = 3). \end{cases}$$

Proof. It is easy to see that the highest weight vector $v_{[2^2]}$ of $[2^2] \subset \Lambda^2 U_{\mathbb{Q}}$ is

$$\sum_{i=3}^{g} A_{12i} \wedge A_{12-i}$$

and the highest weight vector $v_{[0]}$ of $[0] \subset \Lambda^2 U_{\mathbb{Q}}$ is

$$\sum_{1 \leq i<j<k \leq g} A_{ijk} \wedge A_{-i-j-k} - \sum_{\substack{1 \leq i<j \leq g \\ 1 \leq k \leq g,\, k \neq i,j}} A_{ij-k} \wedge A_{-i-jk} + \sum_{\substack{1 \leq i,j \leq g \\ i \neq j}} p_{ij} \wedge q_{ij}.$$

Using $v_{[2^2]}$ and $v_{[0]}$, we construct some vectors in Image$\wedge$ and decompose them to each irreducible component. To do so we need a lot of $\mathrm{Sp}(2g,\mathbb{Q})$-modules and $\mathrm{Sp}(2g,\mathbb{Q})$-homomorphisms and we summarize them in the following diagram

$$\begin{tikzcd}
\Lambda^3[1^3] = \Lambda^3 U_{\mathbb{Q}} \arrow[r, "\iota"] & \Lambda^3(\Lambda^3 H_{\mathbb{Q}}) \arrow[dl, "f_1"'] \arrow[d, "g_1"] & \\
(\Lambda^3 H_{\mathbb{Q}})^{\otimes 3} \arrow[d, "f_2"'] & \Lambda^3 H_{\mathbb{Q}} \otimes \Lambda^2(\Lambda^3 H_{\mathbb{Q}}) \arrow[d, "g_2"'] & \\
(H_{\mathbb{Q}})^{\otimes 3} \otimes (\Lambda^3 H_{\mathbb{Q}})^{\otimes 2} \arrow[d, "f_3"'] & \Lambda^3 H_{\mathbb{Q}} \otimes \Lambda^6 H_{\mathbb{Q}} \arrow[d, "g_3"'] & \\
H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} \arrow[d, "f_4"'] & \Lambda^3 H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} \arrow[d, "g_4"'] \arrow[r, "h_1"] & H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} \arrow[d, "h_2"] \\
H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} \arrow[d, "f_5"'] & \Lambda^3 H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \arrow[d, "g_5"'] & \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^5 H_{\mathbb{Q}} \arrow[d, "h_3"] \\
H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} & H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \arrow[d, "g_6"'] & \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^3 H_{\mathbb{Q}} \arrow[d, "h_4"] \\
& H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} & \Lambda^2 H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}
\end{tikzcd}$$

where ι is the inclusion and homomorphisms f_i, g_i and h_i are as follows.

$$\begin{cases}
f_1 = i^3_{\Lambda^3 H_{\mathbb{Q}}} \\
f_2 = i^3_{H_{\mathbb{Q}}} \otimes 1 \otimes 1 \\
f_3(x_1 \otimes x_2 \otimes x_3 \otimes A_{ijk} \otimes A_{lmn}) = x_1 \otimes (x_2 \wedge A_{ijk}) \otimes (x_3 \wedge A_{lmn}) \\
f_4 = 1 \otimes C_4 \otimes 1 \\
f_5 = 1 \otimes 1 \otimes C_2
\end{cases}$$

$$\begin{cases}
g_1 = j_{\Lambda^3 H_{\mathbb{Q}}} \\
g_2 = 1 \otimes multi. \\
g_3 = 1 \otimes C_6 \\
g_4 = 1 \otimes C_4 \\
g_5 = j_{H_{\mathbb{Q}}} \otimes 1 \\
g_6 = 1 \otimes multi.
\end{cases}$$

$$\begin{cases}
h_1 = j_{H_{\mathbb{Q}}} \otimes 1 \\
h_2(x \otimes A_{ij} \otimes A_{klmn}) = A_{ij} \otimes (x \wedge A_{klmn}) \\
h_3 = 1 \otimes C_5 \\
h_4 = 1 \otimes C_3
\end{cases}$$

$[3^21]$: Taking the wedge product of $v_{[2^2]}$ and A_{123}, we obtain

$$\sum_{i=3}^{g} A_{12i} \wedge A_{12-i} \wedge A_{123} \in \Lambda^3 U_{\mathbb{Q}}.$$

When $g \geq 4$, this is the non-zero highest weight vector of $[3^21]$. Hence we see that Image $\wedge$ contains the summand $[3^21]$.

$[321^2]$: Take the wedge product of $v_{[2^2]}$ and A_{134}. Then

$$\begin{aligned}
& v_{[2^2]} \wedge A_{134} = \sum_{i=3}^{g} A_{12i} \wedge A_{12-i} \wedge A_{134} \\
& \xmapsto{f_1 \circ \iota} \sum_{i=3}^{g} A_{12i} \otimes A_{12-i} \otimes A_{134} - \sum_{i=3}^{g} A_{12i} \otimes A_{134} \otimes A_{12-i} \\
& \quad + \sum_{i=3}^{g} A_{12-i} \otimes A_{134} \otimes A_{12i} - \sum_{i=3}^{g} A_{12-i} \otimes A_{12i} \otimes A_{134} \\
& \quad + \sum_{i=3}^{g} A_{134} \otimes A_{12i} \otimes A_{12-i} - \sum_{i=3}^{g} A_{134} \otimes A_{12-i} \otimes A_{12i} \\
& \xmapsto{f_3 \circ f_2} -\sum_{i=3}^{g} (a_1 \otimes A_{i12-i} \otimes A_{2134}) - \sum_{i=3}^{g} (a_1 \otimes A_{2134} \otimes A_{i12-i}) \\
& \quad + \sum_{i=3}^{g} (a_1 \otimes A_{2134} \otimes A_{-i12i}) + \sum_{i=3}^{g} (a_1 \otimes A_{-i12i} \otimes A_{2134}) \\
& \quad + \sum_{i=3}^{g} (a_1 \otimes A_{312i} \otimes A_{412-i} - a_1 \otimes A_{412i} \otimes A_{312-i}) \\
& \quad - \sum_{i=3}^{g} (a_1 \otimes A_{312-i} \otimes A_{412i} - a_1 \otimes A_{412-i} \otimes A_{312i}) \\
& \xmapsto{f_4} 2(g-1) a_1 \otimes A_{12} \otimes A_{1234}.
\end{aligned}$$

Hence we obtain

$$f_4 \circ f_3 \circ f_2 \circ f_1 \circ \iota(v_{[2^2]} \wedge A_{134}) = 2(g-1) a_1 \otimes A_{12} \otimes A_{1234}.$$

This is the highest weight vector of $[321^2]$ so that Image $\wedge$ contains the summand $[321^2]$ for $g \geq 4$.

$[32]$: Take the wedge product of $v_{[2^2]}$ and $p_{13} - p_{12} = A_{13-3} - A_{12-2}$. Then

$$v_{[2^2]} \wedge (p_{13} - p_{12}) = \sum_{i=3}^{g} A_{12i} \wedge A_{12-i} \wedge (p_{13} - p_{12})$$

$$
\begin{aligned}
&\overset{f_1\circ\iota}{\longmapsto} \sum_{i=3}^{g} A_{12i}\otimes A_{12-i}\otimes(p_{13}-p_{12}) - \sum_{i=3}^{g} A_{12i}\otimes(p_{13}-p_{12})\otimes A_{12-i}\\
&\quad + \sum_{i=3}^{g} A_{12-i}\otimes(p_{13}-p_{12})\otimes A_{12i} - \sum_{i=3}^{g} A_{12-i}\otimes A_{12i}\otimes(p_{13}-p_{12})\\
&\quad + \sum_{i=3}^{g} (p_{13}-p_{12})\otimes A_{12i}\otimes A_{12-i} - \sum_{i=3}^{g} (p_{13}-p_{12})\otimes A_{12-i}\otimes A_{12i}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{f_3\circ f_2}{\longmapsto} -\sum_{i=3}^{g} a_1\otimes A_{i12-i}\otimes(a_2\wedge(p_{13}-p_{12})) - \sum_{i=3}^{g} a_1\otimes(a_2\wedge(p_{13}-p_{12}))\otimes A_{i12-i}\\
&\quad + \sum_{i=3}^{g} a_1\otimes(a_2\wedge(p_{13}-p_{12}))\otimes A_{-i12i} + \sum_{i=3}^{g} a_1\otimes A_{-i12i}\otimes(a_2\wedge(p_{13}-p_{12}))\\
&\quad + \sum_{i=3}^{g} a_1\otimes A_{312i}\otimes A_{-312-i} - \sum_{i=3}^{g} a_1\otimes A_{-312i}\otimes A_{312-i}\\
&\quad - \sum_{i=3}^{g} a_1\otimes A_{312-i}\otimes A_{-312i} + \sum_{i=3}^{g} a_1\otimes A_{-312-i}\otimes A_{312i}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{f_5\circ f_4}{\longmapsto} -a_1\otimes(g-2)A_{12}\otimes(-A_{12}) - a_1\otimes(-A_{12})\otimes(g-2)A_{12}\\
&\quad +a_1\otimes(-A_{12})\otimes(2-g)A_{12} + a_1\otimes(2-g)A_{12}\otimes(-A_{12})\\
&\quad -a_1\otimes(-A_{12})\otimes A_{12} - a_1\otimes A_{12}\otimes(-A_{12})\\
&\quad = (4g-6)a_1\otimes A_{12}\otimes A_{12}.
\end{aligned}
$$

Hence we obtain

$$f_5\circ f_4\circ f_3\circ f_2\circ f_1\circ\iota(v_{[2^2]}\wedge(p_{13}-p_{12})) = (4g-6)a_1\otimes A_{12}\otimes A_{12}.$$

This is the highest weight vector of [32] so that Image $\wedge$ contains the summand [32] for $g \geq 3$.

$[2^21^3]$: Take the wedge product of $v_{[2^2]}$ and A_{345}. By the similar calculation, we obtain

$$h_2\circ h_1\circ g_3\circ g_2\circ g_1\circ\iota(v_{[2^2]}\wedge A_{345}) = -3A_{12}\otimes A_{12345}.$$

This is the highest weight vector of $[2^21^3]$ so that Image $\wedge$ contains the summand $[2^21^3]$ for $g \geq 5$.

$[2^21]$: Take the wedge product of $v_{[2^2]}$ and $p_{34}-p_{32} = A_{34-4}-A_{32-2}$. Then we obtain

$$h_3\circ h_2\circ h_1\circ g_3\circ g_2\circ g_1\circ\iota(v_{[2^2]}\wedge(p_{34}-p_{32})) = (2g-9)A_{12}\otimes A_{123}.$$

This is the highest weight vector of $[2^21]$ so that Image $\wedge$ contains the summand $[2^21]$ for $g \geq 4$.

[21]: Take the wedge product of $v_{[2^2]}$ and $q_{21}-q_{23}=A_{-21-1}-A_{-23-3}$. Then we obtain

$$h_4\circ h_3\circ h_2\circ h_1\circ g_3\circ g_2\circ g_1\circ\iota(v_{[2^2]}\wedge(q_{21}-q_{23}))=-8(g-3)A_{12}\otimes a_1.$$

This is the highest weight vector of [21] so that Image $\wedge$ contains the summand [21] for $g\geq 4$.

[21^3]: Take the wedge product of $v_{[2^2]}$ and A_{-234}. Then we obtain

$$g_6\circ g_5\circ g_4\circ g_3\circ g_2\circ g_1\circ\iota(v_{[2^2]}\wedge A_{-234})=4a_1\otimes A_{1234}.$$

This is the highest weight vector of [21^3] so that Image $\wedge$ contains the summand [21^3] for $g\geq 4$.

[1^3]: We now claim that Image $\wedge$ has the summand [1^3] whose multiplicity is 2 for $g\geq 4$ and 1 for $g=3$. To show it, consider following two homomorphisms

$$\Lambda^3 U_{\mathbb{Q}}\xrightarrow{g_4\circ\ldots\circ g_1\circ\iota}\Lambda^3 H_{\mathbb{Q}}\otimes\Lambda^2 H_{\mathbb{Q}}\xrightarrow{1\otimes C_2}\Lambda^3 H_{\mathbb{Q}},$$

$$\Lambda^3 U_{\mathbb{Q}}\xrightarrow{g_4\circ\ldots\circ g_1\circ\iota}\Lambda^3 H_{\mathbb{Q}}\otimes\Lambda^2 H_{\mathbb{Q}}\xrightarrow{multi.}\Lambda^5 H_{\mathbb{Q}}\xrightarrow{C_5}\Lambda^3 H_{\mathbb{Q}}.$$

We denote the former map by F and the latter one by G. By the similar calculation, we obtain

$$\begin{array}{rcl}F(v_{[2^2]}\wedge A_{-1-23}) & = & 6A_{123},\\ G(v_{[2^2]}\wedge A_{-1-23}) & = & -4(g-3)A_{123}.\end{array}$$

On the other hand, we prepare another vector $v_{[0]}\wedge A_{123}$. Then

$$\begin{aligned}
& v_{[0]}\wedge A_{123}\\
\overset{g_1\circ\iota}{\longmapsto}\; & \sum_{1\leq i<j<k\leq g} A_{ijk}\otimes A_{-i-j-k}\wedge A_{123}+\sum_{1\leq i<j<k\leq g} A_{-i-j-k}\otimes A_{123}\wedge A_{ijk}\\
& +\sum_{1\leq i<j<k\leq g} A_{123}\otimes A_{ijk}\wedge A_{-i-j-k}\\
& +\sum_{\substack{1\leq i<j\leq g\\ 1\leq k\leq g,\,k\neq i,j}} A_{ij-k}\otimes A_{-i-jk}\wedge A_{123}+\sum_{\substack{1\leq i<j\leq g\\ 1\leq k\leq g,\,k\neq i,j}} A_{-i-jk}\otimes A_{123}\wedge A_{ij-k}\\
& +\sum_{\substack{1\leq i<j\leq g\\ 1\leq k\leq g,\,k\neq i,j}} A_{123}\otimes A_{ij-k}\wedge A_{-i-jk}\\
& +\sum_{\substack{1\leq i,j\leq g\\ i\neq j}}(p_{ij}\otimes q_{ij}\wedge A_{123}+q_{ij}\otimes A_{123}\wedge p_{ij}+A_{123}\otimes p_{ij}\wedge q_{ij})\\
\overset{g_2}{\longmapsto}\; & \sum_{1\leq i<j<k\leq g}(A_{ijk}\otimes A_{-i-j-k123}+A_{-i-j-k}\otimes A_{123ijk}+A_{123}\otimes A_{ijk-i-j-k})
\end{aligned}$$

$$+\sum_{\substack{1\le i<j\le g\\ 1\le k\le g,\,k\neq i,j}}(A_{ij-k}\otimes A_{-i-jk123}+A_{-i-jk}\otimes A_{123ij-k}+A_{123}\otimes A_{ij-k-i-jk})$$
$$+\sum_{\substack{1\le i,j\le g\\ i\neq j}}(p_{ij}\otimes q_{ij}\wedge A_{123}+q_{ij}\otimes A_{123}\wedge p_{ij}+A_{123}\otimes p_{ij}\wedge q_{ij})$$

$$\overset{g_4\circ g_3}{\longmapsto} A_{123}\otimes 2(A_{1-1}+A_{2-2}+A_{3-3})$$
$$+\sum_{k=4}^{g}\{A_{13k}\otimes(-2A_{2-k})+A_{12k}\otimes 2A_{3-k}+A_{23k}\otimes 2A_{1-k}\}$$
$$+\sum_{1\le i<j<k\le g}A_{123}\otimes\{-2(A_{i-i}+A_{j-j}+A_{k-k})\}$$
$$-\sum_{k=4}^{g}\{A_{12-k}\otimes 2A_{3k}+A_{13-k}\otimes(-2A_{2k})+A_{23-k}\otimes 2A_{1k}\}$$
$$-\sum_{\substack{1\le i<j\le g\\ 1\le k\le g,\,k\neq i,j}}A_{123}\otimes 2(A_{i-i}+A_{j-j}+A_{k-k})$$
$$+2(p_{12}+p_{13})\otimes A_{23}-2(p_{21}+p_{23})\otimes A_{13}+2(p_{31}+p_{32})\otimes A_{12}$$
$$-A_{123}\otimes 6(g-2)\omega.$$

Hence, we obtain

$$\begin{array}{rcl} F(v_{[0]}\wedge A_{123}) &=& -2(2g^3-3g^2-2g-3)A_{123},\\ G(v_{[0]}\wedge A_{123}) &=& \frac{-2(g-3)(2g^3-5g^2+7g+2)}{g-1}A_{123}. \end{array}$$

It is easily checked that $v_{[2^2]}\wedge A_{-1-23}$ and $v_{[0]}\wedge A_{123}$ have $\mathrm{Sp}(2g,\mathbb{Q})$-linearly independent images in $[1^3]$ when $g\ge 4$. Therefore our claim follows.

¿From above computations, the proposition follows. $\square$

5.2 Summands which are not in the kernel

Considering the dual homomorphism

$$\tau_*: H_3(\mathcal{I}_g,\mathbb{Q})\longrightarrow H_3(U,\mathbb{Q})\cong\Lambda^3U_{\mathbb{Q}}.$$

of $\tau^*:\Lambda^3U_{\mathbb{Q}}\to H^3(\mathcal{I}_g,\mathbb{Q})$, we can obtain summands which are not in $\mathrm{Ker}\,\tau^*$ by decomposing the image of τ_* into each irreducible component. To find elements in $H_3(\mathcal{I}_g,\mathbb{Q})$, we construct abelian cycles, which are fundamental cycles of abelian subgroups of $\mathcal{I}_g$.

To treat third homology groups of abelian groups, we need the following lemma where we denote the basis elements $(1,0,0),(0,1,0),(0,0,1)$ of $\mathbb{Z}^3$ by x_1,x_2,x_3, respectively.

Lemma 5.3 *Let A be an abelian group and $f:\mathbb{Z}^3\to A$ be a group homomorphism. Then the image of $1\in\mathbb{Z}\cong H_3(\mathbb{Z}^3)$ in $H_3(A)\cong\Lambda^3A$ is given by*

$$f(x_1)\wedge f(x_2)\wedge f(x_3).$$

Proof. The cycle which represents the image of $1 \in H_3(\mathbb{Z}^3)$ in the third cycle group of A is given by

$$\sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma)\ [f(x_{\sigma(1)})|f(x_{\sigma(2)})|f(x_{\sigma(3)})]$$

where this cycle is written in the form using the standard complex. Then the result follows from the definition of the Pontryagin product. □

We can find some abelian cycles by choosing simple closed curves along which Dehn-twists are done. Recall that we embedded U into $U_{\mathbb{Q}}$ in the following lemma.

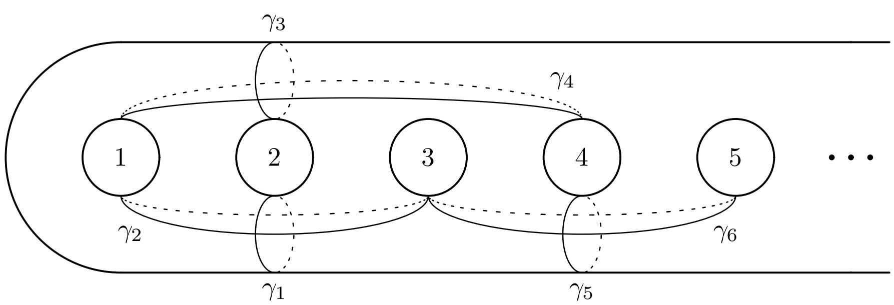
Lemma 5.4 *There exist some abelian cycles whose homology classes mapped into* $\Lambda^3 U$ *by* τ_* *are given by*

$$\begin{aligned} w_1 &= A_{123} \wedge A_{124} \wedge A_{345} \quad (g \geq 5), \\ w_2 &= p_{21} \wedge \left(\sum_{i=1}^{2} p_{3i} \right) \wedge \left(\sum_{i=1}^{3} p_{4i} \right) \quad (g \geq 5), \\ w_3 &= p_{12} \wedge p_{34} \wedge p_{56} \quad (g \geq 6). \end{aligned}$$

Namely, they are in $\mathrm{Image}\,(\tau_* : H_3(\mathcal{I}_g) \cong \Lambda^3 U \to H_3(U))$.

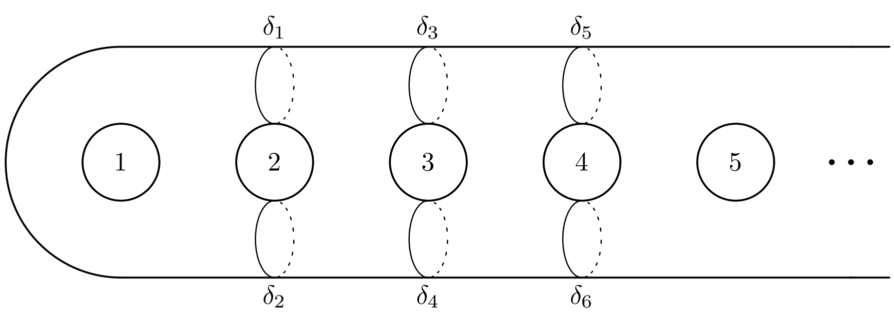
Proof. Construct homomorphisms $f_i : \mathbb{Z}^3 \to \mathcal{I}_g$ as follows, where T_c is the isotopy class of the Dehn-twist along the simple closed curve c.

f_1: $f_1(x_1) = [T_{\gamma_1}, T_{\gamma_2}]$, $f_1(x_2) = [T_{\gamma_3}, T_{\gamma_4}]$, $f_1(x_3) = [T_{\gamma_5}, T_{\gamma_6}]$

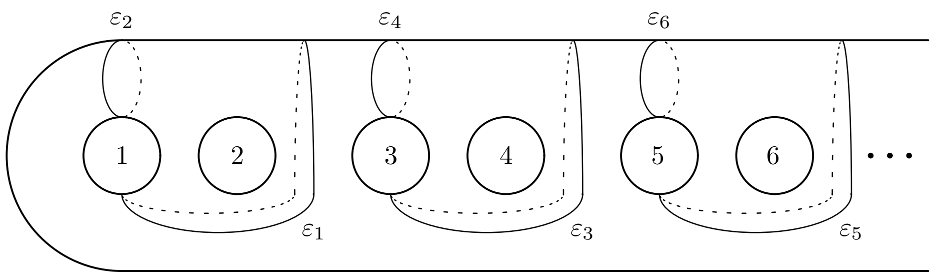


where $[T_{\gamma_i}, T_{\gamma_j}] = T_{\gamma_i} T_{\gamma_j} T_{\gamma_i}^{-1} T_{\gamma_j}^{-1}$.

f_2: $f_2(x_1) = T_{\delta_1} T_{\delta_2}^{-1}$, $f_2(x_2) = T_{\delta_3} T_{\delta_4}^{-1}$, $f_2(x_3) = T_{\delta_5} T_{\delta_6}^{-1}$



f_3: $\quad f_3(x_1) = T_{\varepsilon_1} T_{\varepsilon_2}^{-1}$, $f_3(x_2) = T_{\varepsilon_3} T_{\varepsilon_4}^{-1}$, $f_3(x_3) = T_{\varepsilon_5} T_{\varepsilon_6}^{-1}$



Since curves are disjoint from each other, f_i are well-defined homomorphisms. By Lemma 5.3, we see that $\tau_* \circ f_i$ define the homology classes $\tilde{w}_i$ given by

$$\begin{aligned}
\tilde{w}_1 &= B_{123} \wedge B_{124} \wedge B_{345}, \\
\tilde{w}_2 &= q_{21} \wedge \left(\sum_{i=1}^{2} q_{3i} \right) \wedge \left(\sum_{i=1}^{3} q_{4i} \right), \\
\tilde{w}_3 &= q_{12} \wedge q_{34} \wedge q_{56}.
\end{aligned}$$

Since τ is $\mathcal{M}_g$-equivariant, the lemma follows. $\square$

Under the above preparation, we prove Theorem 3.2. We now assume $g \geq 5$. As in the previous subsection, we summarize $\mathrm{Sp}(2g, \mathbb{Q})$-modules and $\mathrm{Sp}(2g, \mathbb{Q})$-homomorphisms in the following diagran

$$\begin{tikzcd}
\Lambda^3[1^3] = \Lambda^3 U_{\mathbb{Q}} \arrow[r, "\iota"] & \Lambda^3(\Lambda^3 H_{\mathbb{Q}}) \arrow[dl, "g_1"'] \arrow[d, "f_1"] & \\
\Lambda^3 H_{\mathbb{Q}} \otimes \Lambda^2(\Lambda^3 H_{\mathbb{Q}}) \arrow[dd, "g_2"] & (\Lambda^3 H_{\mathbb{Q}})^{\otimes 3} \arrow[r, "f_3 \circ f_2"] \arrow[d, "f_6"] & H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} \\
 & (H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}})^{\otimes 2} \otimes \Lambda^3 H_{\mathbb{Q}} \arrow[r, "f_7"] & \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^5 H_{\mathbb{Q}} \\
\Lambda^3 H_{\mathbb{Q}} \otimes \Lambda^6 H_{\mathbb{Q}} \arrow[r, "g_7"] \arrow[d, "g_3"] & H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^6 H_{\mathbb{Q}} \arrow[r, "g_8"] & \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^7 H_{\mathbb{Q}} \\
\Lambda^3 H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} \arrow[r, "g_9"] \arrow[d, "g_4"] & H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^4 H_{\mathbb{Q}} \arrow[r, "g_{10}"] \arrow[dr, "g_{11}"] & \Lambda^2 H_{\mathbb{Q}} \otimes \Lambda^5 H_{\mathbb{Q}} \\
\Lambda^3 H_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}} \arrow[r, "g_{12}"] \arrow[d, "g_{13}"] & \Lambda^5 H_{\mathbb{Q}} & H_{\mathbb{Q}} \otimes \Lambda^6 H_{\mathbb{Q}} \\
\Lambda^3 H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \arrow[r, "g_{14}"] & \Lambda^4 H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} &
\end{tikzcd}$$

where new homomorphisms are given by

$$\begin{cases} f_6 = j_{H_{\mathbb{Q}}} \otimes j_{H_{\mathbb{Q}}} \otimes 1 \\ f_7(x_1 \otimes A_{kl} \otimes x_2 \otimes A_{mn} \otimes A_{pqr}) = A_{kl} \otimes A_{mn} \otimes (x_1 \wedge x_2 \wedge A_{pqr}) \end{cases}$$

$$\begin{cases} g_7 = j_{H_\mathbb{Q}} \otimes 1 \\ g_8(x \otimes A_{kl} \otimes A_{mnopqr}) = A_{kl} \otimes (x \wedge A_{mnopqr}) \\ g_9 = j_{H_\mathbb{Q}} \otimes 1 \\ g_{10}(x \otimes A_{kl} \otimes A_{pqrs}) = A_{kl} \otimes (x \wedge A_{pqrs}) \\ g_{11} = 1 \otimes multi. \\ g_{12} = multi. \\ g_{13} = 1 \otimes i^2_{H_\mathbb{Q}} \\ g_{14} = multi. \otimes 1. \end{cases}$$

$[32^3]$: From the direct calculation, we have

$$X_{1,5}w_1 = A_{123} \wedge A_{124} \wedge A_{134}$$

$$\begin{aligned} &\xmapsto{f_1 \circ \iota} A_{123} \otimes A_{124} \otimes A_{134} - A_{123} \otimes A_{134} \otimes A_{124} + A_{124} \otimes A_{134} \otimes A_{123} \\ &\qquad -A_{124} \otimes A_{123} \otimes A_{134} + A_{134} \otimes A_{123} \otimes A_{124} - A_{134} \otimes A_{124} \otimes A_{123} \end{aligned}$$

$$\xmapsto{f_3 \circ f_2} 6a_1 \otimes A_{1234} \otimes A_{1234}.$$

Hence we obtain

$$f_3 \circ f_2 \circ f_1 \circ \iota(X_{1,5}w_1) = 6A_{12} \otimes A_{1234} \otimes A_{1234}.$$

This is the highest weight vector of $[32^3]$ so that $[32^3]$ is not in $\operatorname{Ker} \tau^*$ for $g \geq 5$.

$[3^2 1^3]$: By the similar calculation, we have

$$f_7 \circ f_6 \circ f_1 \circ \iota(X_{2,4}X_{1,3}w_1) = 6A_{12} \otimes A_{12} \otimes A_{12345}.$$

This is the highest weight vector of $[3^2 1^3]$ so that $[3^2 1^3]$ is not in $\operatorname{Ker} \tau^*$ for $g \geq 5$.

As for $[1^9], [1^7], [1^3]$ and $[1^5]$, we calculate images of w_3 by following maps

$$\Lambda^3 U_\mathbb{Q} \xrightarrow{\iota} \Lambda^3(\Lambda^3 H_\mathbb{Q}) \xrightarrow{multi.} \Lambda^9 H_\mathbb{Q} \xrightarrow{C_9} \Lambda^7 H_\mathbb{Q} \xrightarrow{C_7} \Lambda^5 H_\mathbb{Q} \xrightarrow{C_5} \Lambda^3 H_\mathbb{Q}.$$

beforehand. Then we obtain

$$w_3 \overset{multi. \circ \iota}{\longmapsto} p_{12} \wedge p_{34} \wedge p_{56}$$

$$\begin{aligned} &\xmapsto{C_9} 6(g-1)^{-1} A_{135} \wedge (A_{2-24-4} + A_{2-26-6} + A_{4-46-6}) \\ &-10(g-1)^{-2} A_{135} \wedge (A_{2-2} + A_{4-4} + A_{6-6}) \wedge \omega \\ &+12(g-1)^{-3} A_{135} \wedge \omega^2 \end{aligned}$$

$$\begin{aligned} &\xmapsto{C_7} 2(g+19)(g-1)^{-2} A_{135} \wedge (A_{2-2} + A_{4-4} + A_{6-6}) \\ &-6(g+11)(g-1)^{-3} A_{135} \wedge \omega \end{aligned}$$

$$\xmapsto{C_5} 12(5g+7)(g-1)^{-3} A_{135}.$$

$[1^9]$: From the above calculation, we have

$$multi.\circ\iota(Y_{6,9}Y_{4,8}Y_{2,7}w_3) = A_{123456789}.$$

This is the highest weight vector of $[1^9]$ so that $[1^9]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 9$.

$[1^7]$: Similarly we have

$$C_9 \circ multi.\circ\iota(X_{6,8}Y_{4,7}Y_{2,8}w_3) = 6(g-1)^{-1}A_{1234567}.$$

This is the highest weight vector of $[1^7]$ so that $[1^7]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 8$.

$[1^3]$: We have

$$C_5 \circ C_7 \circ C_9 \circ multi.\circ\iota(X_{2,5}w_3) = -(60g+84)(g-1)^{-3}A_{123}.$$

This is the highest weight vector of $[1^3]$ so that $[1^3]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 3$.

Similar calculations can be applied to the other components except [1]. But we now omit the details.

$[2^3 1^3]$: We have

$$g_2 \circ g_1 \circ \iota(X_{1,6}Y_{1,6}Y_{2,6}Y_{3,5}w_2) = -A_{123} \otimes A_{123456}.$$

This is the highest weight vector of $[2^3 1^3]$ so that $[2^3 1^3]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 6$.

$[2^3 1]$: We have

$$g_3 \circ g_2 \circ g_1 \circ \iota(X_{2,4}X_{1,5}X_{4,5}Y_{1,5}Y_{3,5}w_2) = -3(g-1)^{-1}(g-3)A_{123} \otimes A_{1234}.$$

This is the highest weight vector of $[2^3 1]$ so that $[2^3 1]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 5$.

$[2^2 1]$: We have

$$g_4 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{2,4}X_{1,5}X_{1,3}Y_{3,5}w_2) = 2(g-1)^{-1}(g-3)(g+1)A_{123} \otimes A_{12}.$$

This is the highest weight vector of $[2^2 1]$ so that $[2^2 1]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 5$.

$[2^2 1^5]$: We have

$$g_8 \circ g_7 \circ g_2 \circ g_1 \circ \iota(X_{1,3}Y_{1,7}Y_{2,6}Y_{3,5}w_2) = -A_{12} \otimes A_{1234567}.$$

This is the highest weight vector of $[2^2 1^5]$ so that $[2^2 1^5]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 7$.

$[2^2 1^3]$: We have

$$g_{10} \circ g_9 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{1,3}X_{1,6}Y_{2,6}Y_{3,5}w_2) = 3(g-1)^{-1}A_{12} \otimes A_{12345}.$$

This is the highest weight vector of $[2^2 1^3]$ so that $[2^2 1^3]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 6$.

$[21^5]$: We have

$$g_{11} \circ g_9 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{1,3}Y_{1,6}Y_{3,5}w_2) = 3(g-1)^{-1}(g-3)a_1 \otimes A_{123456}.$$

This is the highest weight vector of $[21^5]$ so that $[21^5]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 6$.

$[21^3]$: We have

$$g_{14} \circ g_{13} \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{1,5}X_{1,3}Y_{3,5}w_2) = 2(g-1)^{-2}(g-3)(g+1)A_{1234} \otimes a_1.$$

This is the highest weight vector of $[21^3]$ so that $[21^3]$ is not in $\operatorname{Ker}\tau^*$ for $g \geq 5$.

$[1^5]$: We now claim that $\operatorname{Image}\tau_* \otimes \mathbb{Q}$ has $[1^5]$ whose multiplicity is actually 2 when $g \geq 7$ and 1 when $g = 5, 6$. It is easy to see that

$$\begin{array}{rcl} C_7 \circ C_9 \circ multi. \circ \iota(Y_{1,5}w_2) & = & -22(g-1)^{-2}(g-5)(g-6)A_{12345}, \\ g_{12} \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ \iota(Y_{1,5}w_2) & = & 2(g-1)^{-2}(g+1)(4g-15)A_{12345}. \end{array}$$

On the other hand, we prepare another vector $X_{4,7}Y_{2,7}w_3$ which is defined when $g \geq 7$. Then

$$\begin{array}{rcl} C_7 \circ C_9 \circ multi. \circ \iota(X_{4,7}Y_{2,7}w_3) & = & -2(g-1)^{-2}(g+19)A_{12345}, \\ g_{12} \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ \iota(X_{4,7}Y_{2,7}w_3) & = & -2(g-1)^{-2}(g+1)A_{12345}. \end{array}$$

It is easily checked that $Y_{1,5}w_2$ and $X_{4,7}Y_{2,7}w_3$ have $\mathrm{Sp}(2g,\mathbb{Q})$-linearly independent images in $[1^5]$ when $g \geq 7$. Therefore our claim follows.

¿From above computations, we have determined the image of τ^* except the summand $[1]$. $\square$

5.3 The summand of $[1]$ and characteristic classes of surface bundles on the pointed Torelli group

In this subsection, we treat the remaining summand of $[1]$. As the result, we see that this summand embodies one of characteristic classes of surface bundles.

Let $v_{[1]}$ be the highest weight vector of $[1]$ in $\Lambda^3 U_{\mathbb{Q}}$. By the universal coefficient theorem, $v_{[1]}$ can be considered as an element of $\operatorname{Hom}(H_3(U), \mathbb{Q})$. Then this homomorphism factors through $H_{\mathbb{Q}}$ as follows.

$$\begin{tikzcd} H_3 U \cong \Lambda^3 U \arrow[r] & \Lambda^3 U_{\mathbb{Q}} \arrow[r] \arrow[d, "p"'] & \mathbb{Q} \\ & H_{\mathbb{Q}} \arrow[ur, "b_1^*"'] & \end{tikzcd}$$

where the vertical map p is an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant projection $\Lambda^3 U_{\mathbb{Q}} \to [1] = H_{\mathbb{Q}}$. Then

$$\tau^*(v_{[1]}) = 0 \in H^3(\mathcal{I}_g,\mathbb{Q}) \iff \tau^*(p) = p\circ\tau_* = 0 \in \mathrm{Hom}\,(H_3(\mathcal{I}_g), H_{\mathbb{Q}}).$$

Since $\mathcal{I}_g$ acts on $H_{\mathbb{Q}}$ trivially, we see that $\tau^*(p) \in \mathrm{Hom}\,(H_3(\mathcal{I}_g), H_{\mathbb{Q}}) \cong H^3(\mathcal{I}_g, H_{\mathbb{Q}})$. The following result admits us to consider $\tau^*(p)$ to be an element of $H^4(\mathcal{I}_{g,*},\mathbb{Q})$.

Proposition 5.5 ([Mo1],[KM]) *The cohomology group $H^*(\mathcal{I}_{g,*},\mathbb{Q})$ has a canonical decomposition of*

$$H^*(\mathcal{I}_{g,*},\mathbb{Q}) \cong H^*(\mathcal{I}_g,\mathbb{Q}) \oplus H^{*-1}(\mathcal{I}_g, H_{\mathbb{Q}}) \oplus H^{*-2}(\mathcal{I}_g,\mathbb{Q}).$$

Explicitly, the inclusion $H^{*-1}(\mathcal{I}_g, H_{\mathbb{Q}}) \hookrightarrow H^*(\mathcal{I}_{g,*},\mathbb{Q})$ is given by the composition map

$$H^{*-1}(\mathcal{I}_g, H_{\mathbb{Q}}) \to H^{*-1}(\mathcal{I}_{g,*}, H_{\mathbb{Q}}) \xrightarrow{\cup\chi} H^*(\mathcal{I}_{g,*}, H_{\mathbb{Q}}\otimes H_{\mathbb{Q}}) \xrightarrow{\mu_*} H^*(\mathcal{I}_{g,*},\mathbb{Q})$$

where $\chi \in H^1(\mathcal{I}_{g,*}, H_{\mathbb{Q}}) \cong \mathrm{Hom}\,(H_1(\mathcal{I}_{g,*}), H_{\mathbb{Q}})$ is the $\mathcal{M}_{g,*}$-equivariant homomorphism which is the composition of the Johnson homomorphism $\tau : H_1(\mathcal{I}_{g,*}) \to \Lambda^3 H_{\mathbb{Q}}$ and the contraction $C_3 : \Lambda^3 H_{\mathbb{Q}} \to H_{\mathbb{Q}}$ and the last map is applying the intersection form μ to coefficients in $H_{\mathbb{Q}}\otimes H_{\mathbb{Q}}$.

¿From this result, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}\,(\Lambda^3 U_{\mathbb{Q}}, H_{\mathbb{Q}}) & \xrightarrow{\tau^*} & \mathrm{Hom}\,(H_3\mathcal{I}_g, H_{\mathbb{Q}}) \\
\| & & \| \\
\Lambda^3 U_{\mathbb{Q}}\otimes H_{\mathbb{Q}} & & H^3(\mathcal{I}_g, H_{\mathbb{Q}}) \\
\downarrow & & \downarrow \\
\Lambda^4(\Lambda^3 H_{\mathbb{Q}}) & \xrightarrow{\tau^*} & H^4(\mathcal{I}_{g,*},\mathbb{Q})
\end{array}$$

where the left vertical map is the canonical inclusion with respect to the decomposition

$$\begin{aligned}
&\Lambda^4(\Lambda^3 H_{\mathbb{Q}}) \\
&\quad = (\Lambda^4 U_{\mathbb{Q}}) \oplus (\Lambda^3 U_{\mathbb{Q}}\otimes H_{\mathbb{Q}}) \oplus (\Lambda^2 U_{\mathbb{Q}}\otimes \Lambda^2 H_{\mathbb{Q}}) \oplus (U_{\mathbb{Q}}\otimes \Lambda^3 H_{\mathbb{Q}}) \oplus (\Lambda^4 H_{\mathbb{Q}}).
\end{aligned}$$

Here p is an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphism so that p can be considered to be an $\mathrm{Sp}(2g,\mathbb{Q})$-invariant vector as an element of the $\mathrm{Sp}(2g,\mathbb{Q})$-module $\mathrm{Hom}\,(\Lambda^3 U_{\mathbb{Q}}, H_{\mathbb{Q}})$. Now we use the following commutative diagram

$$\begin{array}{ccc}
(\Lambda^4(\Lambda^3 H_{\mathbb{Q}}))^{\mathrm{Sp}} & \xrightarrow{\rho_1^*} & H^4(\mathcal{M}_{g,*},\mathbb{Q}) \\
\downarrow & & \downarrow \\
\Lambda^4(\Lambda^3 H_{\mathbb{Q}}) & \xrightarrow{\tau^*} & H^4(\mathcal{I}_{g,*},\mathbb{Q})
\end{array}$$

constructed by Morita [Mo4] where ρ_1 is the extended Johnson homomorphism defined in [Mo3]. Then we see that $p \in \Lambda^4(\Lambda^3 H_{\mathbb{Q}})$ comes from the invariant part

$$(\Lambda^3 U_{\mathbb{Q}}\otimes H_{\mathbb{Q}})^{\mathrm{Sp}} \subset (\Lambda^4(\Lambda^3 H_{\mathbb{Q}}))^{\mathrm{Sp}}$$

which is 1-dimensional for $g \geq 5$. The top horizontal map ρ_1^* in the above diagram is completely determined by Kawazumi and Morita in [KM] where they calculated the images of basis elements described by trivalent graphs. Hence we need to write the invariant vector of $(\Lambda^3 U_{\mathbb{Q}} \otimes H_{\mathbb{Q}})^{\mathrm{Sp}}$ using them. By the method described in [Mo6], we obtain an equality

$$\begin{aligned} c \cdot p &= 4(g-1)^3\Lambda_3 + 12(g-1)^2\Lambda_4 + 2(g+5)(g-1)\Lambda_5 \\ &\quad +2(g-1)^2\Lambda_7 + (7g-1)\Lambda_8 \end{aligned}$$

where c is some non-zero scalar and Λ_i $(i = 1, 2, \cdots, 8)$ are trivalent graphs defined in [KM]. We may assume that $c = 1$. Then

$$\begin{aligned} \rho_1^*(p) &= 2(g-1)(2g+1)(g+1)e_2 + 4(g-1)^2(2g+1)(g+1)e^2 \\ &\quad +(2g+1)(g+1)e_1^2 + (g-1)(2g+1)(g+1)ee_1. \end{aligned}$$

Considering that e_1 vanishes on $H^*(\mathcal{I}_{g,*}, \mathbb{Q})$, we obtain

$$\tau^*(p) = 2(g-1)(2g+1)(g+1)\{e_2 - (2-2g)e^2\}.$$

Therefore we can say that

$$\tau^*(v_{[1]}) = 0 \in H^3(\mathcal{I}_g, \mathbb{Q}) \iff e_2 - (2-2g)e^2 = 0 \in H^4(\mathcal{I}_{g,*}, \mathbb{Q}).$$

This completes the proof of Theorem 3.3. □

Using the well-known fact about the realization of homology classes, we see that the problem is reduced to the case of three dimensional groups, which are fundamental groups of some three dimensional closed manifolds. This gives a new approach to the non-triviality problem of characteristic classes of surface bundles on the Torelli group. Notice that the same argument is valid for summands [1] of higher degrees in $\Lambda^* U_{\mathbb{Q}}$

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