UTMS 2003–19 April 18, 2003

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CARLEMAN ESTIMATE FOR A STATIONARY ISOTROPIC LAMÉ SYSTEM AND THE APPLICATIONS

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Abstract. For the isotropic stationary Lamé system with variable coefficients equipped with Dirichlet or surface stress boundary condition, we obtain a Carleman estimate such that (i) the right hand side is estimated in a weighted L^2-space and (ii) the estimate includes nonhomogeneous surface displacement or surface stress. Using this estimate we establish the conditional stability in Sobolev's norm of the displacement by means of measurements in an arbitrary subdomain or measurements of surface displacement and stress on an arbitrary subboundary. Finally by the Carleman estimate, we prove the uniqueness and conditional stability for an inverse problem of determining a source term by a single interior measurement.

§1. Introduction.

We consider the stationary isotropic Lamé system with variable coefficients:

$$(Pu)(x) \equiv \mu(x)\Delta u(x) + (\lambda(x)+\mu(x))\nabla(\operatorname{div} u(x))$$

$$+(\operatorname{div} u(x))\nabla\lambda(x) + (\nabla u(x) + (\nabla u(x))^T)\nabla\mu(x), \quad x \in \Omega. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain and the boundary $\partial\Omega$ is of class C^3, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $u(x) = (u_1(x), ..., u_n(x))^T$, $\nabla\lambda(x) = \left(\frac{\partial\lambda}{\partial x_1}, ..., \frac{\partial\lambda}{\partial x_n}\right)^T$, $\operatorname{div} u = \sum_{j=1}^n \frac{\partial u_j}{\partial x_j}$, $\nabla u(x) = \left(\frac{\partial u_j}{\partial x_k}\right)_{1\le j,k\le n}$ and $\cdot^T$ denotes the transposes of matrices under

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX consideration. Moreover λ and μ are the Lamé coefficients in the isotropic medium and we assume

$$\lambda, \mu \in C^3(\overline{\Omega}), \quad \mu(x) > 0,\ n\lambda(x) + 2\mu(x) > 0,\ \lambda(x) + \mu(x) > 0, \quad x \in \overline{\Omega}. \tag{1.2}$$

The unique continuation property for (1.1) is one important problem from the theoretical point of view. Here by the unique continuation, we mean: Does a solution u to $Pu = 0$ vanish identically in Ω if u equals zero in some non-empty open subset of Ω? As for a single elliptic equation, the unique continuation has been well understood (e.g., Hörmander [9]). On the other hand, for systems of elliptic equations, the unique continuation is more difficult and we can refer to Egorov [8] and Zuily [25] for a general theory.

However the isotropic Lamé system requires a proper consideration for obtaining the unique continuation. As an original paper establishing the unique continuation for the stationary Lamé system, we refer to Weck [22], which is based on a special structure of the Lamé system that the system is written as an elliptic system with u and $\operatorname{div} u$ whose principal parts are Δ (i.e., a weakly coupled elliptic system). Later Ang, Ikehata, Trong and Yamamoto [1] and Dehman and Robbiano [7] discussed the unique continuation for the stationary Lamé system. We further refer to Weck [23].

The fundamental tool for proving the unique continuation is a Carleman estimate which is a weighted L^2-estimate of a solution u to $Pu = f$ with a given $f = (f_1, ..., f_n)^T$. In Weck [22] (and [23]), after decoupling by introducing an extra component $\operatorname{div} u$, a Carleman estimate is established, but as direct consequence of the introduction of $\operatorname{div} u$, the Carleman estimate requires the weighted L^2-norm of $\operatorname{div} f$ as well as f. In [7], a Carleman estimate was proved with weighted L^2-norm of f, provided that $\lambda, \mu \in C^\infty(\overline{\Omega})$. In [7], although it is remarked that the C^∞-regularity can be relaxed, the proof is done for the case of $\lambda, \mu \in C^\infty(\overline{\Omega})$ and a Carleman estimate is established locally in x for the functions with a compact support. See also [2] for a Carleman estimate for the Lamé system with a large parameter.

The first main purpose of this paper is to establish a Carleman estimate for the stationary Lamé operator P in the case where

(1) the right hand side is estimated in $L^2(\Omega)$ (unlike in [1], [22], [23]).

(2) $\lambda, \mu \in C^3(\overline{\Omega})$, and a solution u is not assumed to have a compact support. (unlike in [7]).

Next, taking advantage of the above-mentioned first point, we apply our Carleman estimate to conditional stability in a Cauchy problem for the Lamé system and an inverse problem of determining source terms. For example, in the inverse problem, when we use a Carleman estimate involved with the L^2-norm of also $\operatorname{div} f$, extra derivatives of an unknown source function are produced and determination of a source term is more complicated.

Here in the Cauchy problem, we are required to determine u in Ω (or some subdomain) by the observations of u and the surface stress on a part of $\partial\Omega$, and the Cauchy problem is essential for estimating the state of the medium by boundary measurements, and it is known to be ill-posed in the sense that solutions may change tremendously after small perturbations of boundary measurements. The physical significance of the inverse source problem is easily understood because in practise, we often need to determine an unknown source which has caused the current stationary elastic field.

The paper is composed of 4 sections and two appendices. In Section 2, we will prove the Carleman estimate. In Section 3, we will establish the conditional stability. In Section 4, we discuss an inverse problem of determining a source term which is independent of the x_n-component, by some interior observations.

§2. Carleman estimate for the stationary Lamé operator.

We consider the stationary Lamé system

$$Pu = f \qquad \text{in } \Omega. \tag{2.1}$$

We recall that P is defined by (1.1). To (2.1), we attach a boundary condition:

$$Bu = g \qquad \text{on } \partial\Omega, \tag{2.2}$$

where $u = (u_1, \ldots, u_n)^T$, $f = (f_1, \ldots, f_n)^T$, $g = (g_1, \ldots, g_n)^T$ are vector-valued functions, $\Omega \subset \mathbb{R}^n$ is a bounded domain $\partial\Omega \subset C^3$ and B is an operator of boundary condition.

Henceforth $W_2^\ell(\Omega)$, $W_2^\ell(\partial\Omega)$ denote the Sobolev spaces. For the boundary operator B, we consider the following two cases.

the Dirichlet boundary condition:

$$Bu \equiv u|_{\partial\Omega} = g, \tag{2.3}$$

the surface stress condition:

$$Bu \equiv \sum_{j=1}^n \nu_j \sigma_{jk}|_{\partial\Omega} = \sigma(u)\nu = g, \tag{2.4}$$

where $\nu = (\nu_1, \dots, \nu_n)^T$ is the outward unit normal vector to $\partial\Omega$, $\delta_{jk} = 0$ if $j \neq k$, $\delta_{jj} = 1$, and we set $\sigma(u) = (\sigma_{jk}(u))_{1 \le j,k \le n}$,

$$\sigma_{jk}(u)(x) = \lambda(x)(\operatorname{div} u)\delta_{jk} + \mu(x)\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right). \tag{2.5}$$

The goal of this section is to obtain the Carleman estimate for system (2.1) with boundary condition (2.3) or (2.4). Let ω be an arbitrary subdomain of Ω. In order to construct a weight function, we need the following lemma.

Lemma 1. *There exists a function $\psi \in C^3(\overline{\Omega})$ such that*

$$\psi|_{\partial\Omega} = 0, \quad \psi > 0 \text{ in } \Omega, \quad |\nabla\psi(x)| > 0 \text{ on } \overline{\Omega \setminus \omega}. \tag{2.6}$$

For the proof, we can refer to Imanuvilov [10].

Example. Let $\Omega = \{x; |x| < R\}$ and $0 \in \omega$. Then, setting $\psi(x) = R^2 - |x|^2$, we see that this ψ satisfies (2.6).

Using the function $\psi(x)$, we construct the weight function

$$\varphi(x) = e^{\tau\psi(x)}, \tag{2.7}$$

where $\tau > 1$.

We are ready to state our first main result, a Carleman estimate.

Theorem 1. *Let us set $\iota = \frac{1}{2}$ for (2.3) and $\iota = \frac{3}{2}$ for (2.4). Let $f \in (L^2(\Omega))^n, g \in (W_2^{\iota}(\partial\Omega))^n$, the function φ be given by (2.7) and $u \in (L^2(\Omega))^n$ be a solution to problem (2.1) and (2.2). Then there exists $\widehat{\tau} > 1$ such that for any $\tau > \widehat{\tau}$, there exists $s_0(\tau) > 0$ such that*

$$\begin{aligned} \|u\|_Y^2 &\triangleq \int_\Omega \left(\frac{1}{(s\varphi)^2} \sum_{j,k=1}^n \left| \frac{\partial^2 u}{\partial x_j \partial x_k} \right|^2 + \tau^2|\nabla u|^2 + s^2\tau^4\varphi^2|u|^2 \right) e^{2s\varphi} dx \\ &\le C_1 \left(\int_\Omega |f|^2 e^{2s\varphi} dx + \|g\|_X^2 + \int_\omega (\tau^2|\nabla u|^2 + s^2\tau^4\varphi^2|u|^2) e^{2s\varphi} dx \right), \quad \forall s \ge s_0(\tau), \end{aligned} \tag{2.8}$$

where the constant $C_1 > 0$ is independent of s, τ and we set

$$\|\cdot\|_X = \frac{1}{\sqrt{s}}\|\cdot\|_{W_2^{\frac{3}{2}}(\partial\Omega)} + \|\cdot\|_{W_2^{\frac{1}{2}}(\partial\Omega)} \quad \textit{in case (2.3)}$$

and

$$\|\cdot\|_X = \frac{1}{\sqrt{s}}\|\cdot\|_{W_2^{\frac{1}{2}}(\partial\Omega)} + \|\cdot\|_{L^2(\partial\Omega)} \quad \textit{in case (2.4).}$$

Remark. Estimate (2.8) is a Carleman estimate with boundary value g, while the Carleman estimate in [7] is for functions with compact supports in Ω.

Proof.

Henceforth $B_\delta(\widehat{x})$ denotes the ball with radius $\delta > 0$ and the centre $\widehat{x}$.

First we note that it suffices to establish Carleman estimate (2.8) locally. That is, it is sufficient to prove (2.8) for u satisfying $\operatorname{supp} u \subset B_\delta(\widehat{x})$ where $\delta > 0$ is sufficiently small. Without loss of generality, we may below assume that $\widehat{x} = 0$. Let us prove that this localization is possible. Since $\overline{\Omega}$ is a compact set, from any covering $\cup_{y\in\overline{\Omega}} B_\delta(y)$ of Ω, one can take a finite subcovering $\cup_{j=1}^N B_\delta(y_j)$. Next we consider the partition of the unity which corresponds to this subcovering :

$$\sum_{j=1}^N e_j(x) = 1, \quad e_j \in C_0^\infty(B_\delta(y_j)).$$

Set $u_j(x) = u(x)e_j(x)$. Henceforth $[P, Q]$ denotes the commutator of the two operators: $[P, Q] = PQ - QP$. Then

$$Lu_j = e_j f + [L, e_j]u \quad \text{in}\,\Omega, \quad Bu_j|_{\partial\Omega} = e_j g - [e_j, B]u, \quad \operatorname{supp} u_j \subset B_\delta(y_j).$$

Since estimate (2.8) is assumed to be already established for a solutions of (2.1) and (2.2) with $\operatorname{supp} u \subset B_\delta(y_j)$, we have

$$\|u\|_Y^2 \le C_1 \sum_{j=1}^N \|u_j\|_Y^2 \le C_2 \sum_{j=1}^N \Bigg(\int_\omega (\tau^2|\nabla u_j|^2 + s^2\tau^4\varphi^2|u_j|^2)e^{2s\varphi}dx$$
$$+ \int_\Omega |[L, e_j]u|^2 e^{2s\varphi}dx + \int_\Omega |e_j f|^2 e^{2s\varphi}dx + \|g\|_X^2 + \|[e_j, B]u\|_X^2 \Bigg)$$
$$\le C_3 \Bigg(\int_\omega (\tau^2|\nabla u|^2 + s^2\tau^4\varphi^2|u|^2)e^{2s\varphi}dx$$
$$+ \int_\Omega (|\nabla u|^2 + |u|^2)e^{2s\varphi}dx + \int_\Omega |f|^2 e^{2s\varphi}dx + \|g\|_X^2 \Bigg), \quad \forall s \ge s_0(\tau).$$

Since the constant C_3 is independent of s, τ, we obtain (2.8) if we take the parameter $\widehat{\tau}$ sufficiently large.

Now we may assume that the support of the function u is concentrated near some small neighbourhood of zero. Moreover, using the fact that for any orthogonal matrix $\mathcal{O}$, the function $\mathcal{O}u(\mathcal{O}^{-1}x)$ is also a solution to problem (2.1) - (2.2), we may assume that

$$\frac{\partial\psi}{\partial x_n}(0) = \left(0, \cdots, 0, \frac{\partial\psi}{\partial x_n}(0)\right) \quad \text{with } \frac{\partial\psi}{\partial x_n}(0) \ne 0.$$

Hence by means of a smooth function ℓ, the boundary $\partial\Omega$ is locally given by an equation $x_n - \ell(x') = 0$, and $x \in \Omega$ implies $x_n - \ell(x') > 0$, where $x' = (x_1, \dots, x_{n-1})$. After the change of variables $y_j = x_j$ for $j \in \{1, \dots, n-1\}$ and $y_n = x_n - \ell(x')$, the neighbourhood under consideration is transformed to an open set $\mathcal{G}$ in $\mathbb{R}^{n-1} \times (0,1)$ and problem (2.1) - (2.2) is rewritten by

$$P(y, D_y)u = f, \qquad y_n > 0, \tag{2.9}$$

and

$$\widetilde{B}u|_{y_n=0} = g, \quad u|_{y_n=1} = \nabla u|_{y_n=1} = 0. \tag{2.10}$$

The principal symbol of the operator P is given (see e.g., [24]):

$$p(y,\xi)=-(\lambda+\mu)(\widetilde{\xi}+G\xi_n)^T(\widetilde{\xi}+G\xi_n)-\mu|\widetilde{\xi}+G\xi_n|^2E_n,$$

where $G(y)=\left(-\frac{\partial \ell}{\partial x_1},...,-\frac{\partial \ell}{\partial x_{n-1}},1\right)$, $\widetilde{\xi}=(\xi',0)$, $\xi'=(\xi_1,\dots,\xi_{n-1})$, $y'=(y_1,\dots,y_{n-1})$, E_n is the $n\times n$ unit matrix. The principal symbol of the operator $\widetilde{B}$ in the case of stress boundary condition (2.4) is given by the formula

$$\lambda G^T(\widetilde{\xi}+G\xi_n)+\mu G\cdot(\widetilde{\xi}+G\xi_n)E_n+\mu(\widetilde{\xi}+G\xi_n)^TG.$$

We set $U=(U_1,U_2)$, where

$$U_1(x)=\int_{\mathbb{R}^{n-1}}(1+|\xi'|^2)^{\frac12}\widehat{u}(\xi',x_n)e^{i<y',\xi'>}d\xi'\equiv\Lambda u$$

and $U_2(x)=D_{y_n}u$. Here and henceforth we set $i=\sqrt{-1}$, $<y',\xi'>=\sum_{j=1}^{n-1}y_j\xi_j$, $D_{y_j}=\frac{1}{i}\frac{\partial}{\partial y_j}$ and

$$\widehat{u}(\xi',x_n)=(2\pi)^{-\frac{n-1}{2}}\int_{\mathbb{R}^{n-1}}u(x',x_n)e^{-i<x',\xi'>}dx'.$$

In the new notations, problem (2.9) - (2.10) can be written in the form

$$D_{y_n}U=M(y,D_{y'})U+\widetilde{f}\quad\text{in } \mathcal{G}=\mathbb{R}^{n-1}\times[0,1],\quad B(y,D_{y'})U(y)|_{y_n=0}=g,\tag{2.11}$$

where $\widetilde{f}=(0,f)^T$. Here $M(y,D_{y'})$ is the $(2n)\times(2n)$ matrix pseudodifferential operator whose principal symbol $M_1(y,\xi')$ is given by [24]:

$$M_1(y,\xi')=\begin{pmatrix}0 & \Lambda_1E_n\\ A^{-1}M_{21}\Lambda_1^{-1} & A^{-1}M_{22}\end{pmatrix},$$

where

$$M_{21}(y,\xi')=-\mu|\xi'|^2E_n-(\lambda+\mu)\widetilde{\xi}^T\widetilde{\xi},\quad M_{22}(y,\xi')=-(\lambda+\mu)(\widetilde{\xi}^TG+G^T\widetilde{\xi})-2\mu(\widetilde{\xi},G)E_n,$$

and

$$A(y,\xi') = (\lambda+\mu)G^TG + \mu|G|^2E_n, \qquad \Lambda_1 = |\xi'|, \quad \widetilde{\xi} = (\xi_1,\dots,\xi_{n-1},0).$$

For the Dirichlet boundary condition, we have $B(y,D_{y'}) = (E_n, 0)$, and for the stress boundary condition $B(y,D_{y'}) = (B_1\Lambda^{-1}, B_2)(y,D_{y'})$, where the principal symbols of B_1 and B_2 are $B_2(y',\xi') = A$ and $B_1(y,\xi') = \lambda G^T\widetilde{\xi} + \mu\widetilde{\xi}^TG + \mu(G,\widetilde{\xi})E_n$.

It is known ([24]) that all the eigenvalues $\alpha = \alpha(y,\xi')$ of the matrix $M_1(x,\xi')$ satisfy the equation $|\widetilde{\xi} + G\alpha| = 0$. Hence we have two eigenvalues

$$\alpha_\pm(y,\xi') = -\frac{(\widetilde{\xi}\cdot G)}{|G|^2} \pm \sqrt{-\frac{|\xi'|^2}{|G|^2} + \frac{(\widetilde{\xi}\cdot G)^2}{|G|^4}}. \tag{2.12}$$

Note that

$$\left| -\frac{|\xi'|^2}{|G|^2} + \frac{(\widetilde{\xi}\cdot G)^2}{|G|^4} \right| \ne 0 \tag{2.13}$$

for $|\xi'| = 1$. Therefore $\alpha_\pm \in C^1(\mathbb{R}^{n-1}\times\mathbb{R}^1_+\times S^{n-1})$. Here and henceforth $S^{n-1} = \{\xi' \in \mathbb{R}^{n-1}; |\xi'| = 1\}$. Next we find the eigenvectors of the matrix M_1. If $b^\pm = (b_1^\pm, b_2^\pm)$, $b_1^\pm, b_2^\pm \in \mathbb{R}^n$, is the eigenvector of the matrix M_1 which corresponds to the eigenvalue $\alpha_\pm$, then $\alpha_\pm b_1^\pm = |\xi'|b_2^\pm$. Hence it suffices to determine only the first n coordinates of the eigenvector. The equation for them is $M_{21}b_1^\pm + \alpha_\pm M_{22}b_1^\pm = \alpha_\pm^2 A b_1^\pm$. After short calculations, this equation can be written in the form

$$(\widetilde{\xi} + \alpha_\pm G)^T(\widetilde{\xi} + \alpha_\pm G)b_1^\pm = 0. \tag{2.14}$$

One solution for this system is the vector

$$b_1^\pm = \frac{\widetilde{\xi} + \alpha_\pm G}{|\widetilde{\xi} + \alpha_\pm G|},$$

where $|\cdot|$ denotes the norm in the complex space $\mathbb{C}^n$. Then

$$w_1^\pm = \left(\frac{\widetilde{\xi}+\alpha_\pm G}{|\widetilde{\xi}+\alpha_\pm G|}, \frac{\alpha_\pm}{|\xi'|}\frac{\widetilde{\xi}+\alpha_\pm G}{|\widetilde{\xi}+\alpha_\pm G|}\right) \tag{2.15}$$

are eigenvectors of the matrix M_1. The set of solutions of equation (2.14) is the space of vectors orthogonal to $(\widetilde{\xi}+\alpha_\pm G)$ with respect to the scalar product in $\mathbb{R}^n$. If $\{b_1(x,\xi),\dots,b_{n-1}(x,\xi)\}$ is a basis in this space, then all the eigenvectors of the matrix M_1 are given by formulae $w_k^\pm = \left(b_k(x,\xi), \frac{\alpha_\pm}{|\xi'|}b_k(x,\xi)\right)$, $k\in\{1,\dots,n-1\}$. Let us show that the Jordan form of the matrix M_1 has two Jordan blocks of the size 2×2 and one $(2n-4)$-dimensional Jordan block.

Really let us consider the system

$$(M_1-\alpha_\pm)\eta^\pm = w_1^\pm,$$

where $\eta^\pm = (\eta_1^\pm,\eta_2^\pm)$. From the first n equations, we obtain

$$\eta_2^\pm = \frac{1}{|\xi'|}(\alpha_\pm\eta_1^\pm + b_1^\pm). \tag{2.16}$$

The remaining n equations can be written as

$$A^{-1}M_{21}|\xi'|^{-1}\eta_1^\pm + A^{-1}M_{22}\eta_2^\pm - \alpha_\pm\eta_2^\pm = \alpha_\pm\frac{b_1^\pm}{|\xi'|}. \tag{2.17}$$

Using (2.16), one can transform (2.17) to

$$-(\lambda+\mu)(\widetilde{\xi}+\alpha_\pm G)^T(\widetilde{\xi}+\alpha_\pm G)\eta_1^\pm = 2A\alpha_\pm b_1^\pm - M_{22}b_1^\pm.$$

Since

$$(2\alpha_\pm A - M_{22})b_1^\pm = (\lambda+3\mu)\frac{(G\cdot(\widetilde{\xi}+\alpha_\pm G))}{|\widetilde{\xi}+\alpha_\pm G|}(\widetilde{\xi}+\alpha_\pm G),$$

one can take $\eta_1^\pm$ as

$$\eta_1^\pm = -\frac{\lambda+3\mu}{\lambda+\mu}\frac{|\xi'|}{|\widetilde{\xi}+\alpha_\pm G|}G, \quad \forall\ |\xi'|=1, \tag{2.18}$$

$$\eta_2^{\pm} = \frac{1}{|\xi'|}\left(-\alpha_{\pm}\frac{\lambda+3\mu}{\lambda+\mu}\frac{G|\xi'|}{|\widetilde{\xi}+\alpha_{\pm}G|} + \frac{(\widetilde{\xi}+\alpha_{\pm}G)}{|\widetilde{\xi}+\alpha_{\pm}G|}\right). \tag{2.19}$$

Now we have to show that the set $\{w_1^{\pm}, \dots, w_{n-1}^{\pm}, \eta^{\pm}\}$ forms a basis in $\mathbb{R}^{2n}$.

First let us show that the vectors $\{w_1^{\pm}, \dots, w_{n-1}^{\pm}\}$ are linearly independent. Our proof is by contradiction. Assume that these vectors are linearly dependent. Then there exist two vectors $v_{\pm} = \sum_{j=1}^{n-1} c_j^{\pm} w_j^{\pm}$ such that $v_+ = v_-$ and at least one of the constants $c_1^+, ..., c_{n-1}^+, c_1^-, ..., c_{n-1}^-$ is not zero. This means

$$\sum_{j=1}^{n-1} c_j^+ w_j^+ = \sum_{j=1}^{n-1} c_j^- w_j^-$$

and

$$\alpha_+ \sum_{j=1}^{n-1} c_j^+ w_j^+ = \alpha_- \sum_{j=1}^{n-1} c_j^- w_j^-.$$

Therefore $(\alpha_+ - \alpha_-)\sum_{j=1}^{n-1} c_j^{\pm} w_j^{\pm} = 0$. However this is impossible because the set $\{b_1^{\pm}, \dots, b_{n-1}^{\pm}\}$ is chosen as a basis in $\mathbb{R}^{n-1}$ and $\alpha_+ - \alpha_- \neq 0$. Now assume that $\eta^+ = c\eta^- + \sum_{j=1}^{n-1}(c_j^+ w_j^+ + c_j^- w_j^-)$. Then $cM_1\eta^- + \sum_{j=1}^{n-1}(\alpha_+ c_j^+ w_j^+ + \alpha_- c_j^- w_j^-) = c\alpha_+\eta^- + \alpha_+ \sum_{j=1}^{n-1}(c_j^+ w_j^+ + c_j^- w_j^-) + w_1^+$. This is equivalent to

$$cw_1^- + \alpha_- \sum_{j=1}^{n-1} c_j^- w_j^- = c(\alpha_+ - \alpha_-)\eta^- + \alpha_+ \sum_{j=1}^{n-1} c_j^- w_j^- + w_1^+.$$

Since $(\alpha_+ - \alpha_-) \neq 0$, we have $c\eta^- + \sum_{j=1}^{n-1} c_j^- w_j^- + \frac{w_1^+ - cw_1^-}{\alpha_+ - \alpha_-} = 0$. If $c = 0$, then $w_1^+ = -(\alpha_+ - \alpha_-)(\sum_{j=1}^{n-1} c_j^- w_j^-)$, but this is impossible because the sets $\{w_j^+\}$ and $\{w_j^-\}$ are linearly independent. Hence $\eta^- = Cw_1^+ + \sum_{j=1}^{n-1} C_j^- w_j^-$. Therefore $M_1\eta^- - \alpha_-\eta^- = C\alpha_+ w_1^+ - C\alpha_- w_1^+ = w_1^-$ or $w_1^- = C(\alpha_+ - \alpha_-)w_1^+$. This is impossible by (2.12).

Now we fix the new basis $\mathcal{B} = \{b_1^-, \eta^-, b_2^-, \dots, b_{n-1}^-, b_1^+, \eta^+, b_2^+, \dots, b_{n-1}^+\}$. Let $Q(y, \xi')$ be the matrix of transformation from the new basis to the original one. We have $J(y,\xi') = Q^{-1}(y,\xi')M_1(y,\xi')Q(y,\xi')$, where

$$J(y,\xi') = \begin{pmatrix} J_1(y,\xi') & 0 \\ 0 & J_2(y,\xi') \end{pmatrix} \tag{2.20}$$

where

$$J_1(y,\xi') = \begin{pmatrix} \alpha_-(y,\xi') & |\xi'| & 0 & \cdots & 0 & 0 \\ 0 & \alpha_-(y,\xi) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \\ 0 & 0 & 0 & \cdots & \alpha_-(y,\xi') & 0 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_-(y,\xi') \end{pmatrix}$$

and

$$J_2(y,\xi') = \begin{pmatrix} \alpha_+(y,\xi') & |\xi'| & 0 & \cdots & 0 & 0 \\ 0 & \alpha_+(y,\xi) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \\ 0 & 0 & 0 & \cdots & \alpha_+(y,\xi') & 0 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_+(y,\xi') \end{pmatrix}.$$

Now we extend the matrices $Q(y,\xi')$ and $Q^{-1}(y,\xi')$ for all $|\xi'| \ge 1$ by setting $Q(y,\xi') = Q\left(y, \frac{\xi'}{|\xi'|}\right)$ and $Q^{-1}(y,\xi') = Q^{-1}\left(y, \frac{\xi'}{|\xi'|}\right)$. Also we extend $Q(y,\xi')$ and $Q^{-1}(y,\xi')$ up to smooth matrices for $|\xi'| \le 1$. Denote $V = (V_1, V_2) = Q^{-1}(y, D_{y'})U$. Then $U = SV + K(y_n)U$, where S is the parametrix for the elliptic operator $Q(y, D_{y'})$ and $K(y_n)$ has the order -1 for each $y_n \in [0,1]$. Denote

$$Q^{-1}(y,\xi') = \begin{pmatrix} Q^{11}(y,\xi') & Q^{12}(y,\xi') \\ Q^{21}(y,\xi') & Q^{22}(y,\xi') \end{pmatrix}.$$

If $|\xi| \ge 1$, then the i-th column of the matrix $Q^{-1}(y,\xi')$ equals the i-th vector from our basic $\mathcal{B}$. In case (2.3) of the Dirichlet boundary condition, we have $V_1|_{y_n=0} = Q^{12}(y, D_{y'})U_2|_{y_n=0}$.

Note that

$$|\det Q^{12}(0,\xi')| \ge \beta > 0 \quad \text{for all } |\xi'| \ge 1.$$

Henceforth we set $\|\cdot\|_k = \|\cdot\|_{H^k(\mathbb{R}^{n-1})}$ for $k \ge 0$.

Therefore there exists a constant $\kappa > 0$ such that

$$\|V_1(\cdot,0)\|_{\frac{1}{2}} \ge \kappa \|U_2(\cdot,0)\|_{\frac{1}{2}} - c(\delta)\|U_2(\cdot,0)\|_{-\frac{1}{2}},$$

where $c(\delta) \to 0$ as $\delta \to 0$. Taking the parameter δ sufficiently small, we obtain $\|V_1(\cdot,0)\|_{\frac{1}{2}} \ge \frac{\kappa}{2}\|U_2(\cdot,0)\|_{\frac{1}{2}}$. Hence there exists a constant $C > 0$ such that

$$\|V_2(\cdot,0)\|_{\frac{1}{2}} \le C(\|V_1(\cdot,0)\|_{\frac{1}{2}} + \|g\|_X). \tag{2.21}$$

In the case of stress boundary condition (2.4), the operator of the boundary condition $\mathcal{B}(y, D_{y'})$ has a form $\mathcal{B}(y, D_{y'}) = (\mathcal{B}_-(y, D_{y'}), \mathcal{B}_+(y, D_{y'}))$. The principal symbol of the operator $\mathcal{B}_+(y, D_{y'})$ is given by the formula

$$\mathcal{B}_+(0,\xi') = (B_1(0,\xi')|\xi'|^{-1}, A(0,\xi'))(Q^{11}(0,\xi'), Q^{12}(0,\xi')).$$

In [24] (Lemma 1.4), it is proved that the determinant of the matrix $\mathcal{B}_+(0,\xi')$ does not vanish for $|\xi'| = 1$. Therefore, as in the case of the Dirichlet boundary condition, we have inequality (2.21).

Note that the principal symbol of the operator S is $Q(y,\xi')$. Consequently we have

$$U = Q(y, D_{y'})V + K(y_n)U. \tag{2.22}$$

Applying the operator $Q^{-1}(y, D_{y'})$ to equation (2.11), we have

$$Q^{-1}(y, D_{y'})(D_{y_n} - M(y, D_{y'}))U = Q(y, D_{y'})^{-1}\widetilde{f} \quad \text{in } \mathcal{G}. \tag{2.23}$$

Substituting (2.22) into (2.23), we obtain

$$Q^{-1}(D_{y_n}(QV + K(y_n)U) - M(y, D_{y'})(QV + K(y_n)U)) = Q^{-1}\widetilde{f} \quad \text{in } \mathcal{G}.$$

Hence

$$D_{y_n}V - J(y,D_{y'})V = T_0U + T_1\widetilde{f} \quad \text{in } \mathcal{G}, \quad V(\cdot,1)=0. \tag{2.24}$$

Here T_0 and T_1 are pseudodifferential operators of the order 0. Denote $z = Ve^{s\varphi}$. We recall that in the new coordinate system, we have $\varphi(y) = e^{\tau y_n}$. By (2.24), the function z satisfies

$$D_{y_n}z + is\tau\varphi z - J(y,D_{y'})z = (T_0U + T_1\widetilde{f})e^{s\varphi} \quad \text{in } \mathcal{G}, \quad z(\cdot,1)=0. \tag{2.25}$$

In the Jordan matrix J, the first n diagonal elements are $is\tau\varphi - \alpha_-(y,\xi')$. By (2.12), we have

$$\mathrm{Re}\,(s\tau\varphi - i\alpha_-(y,\xi')) \ge \kappa(s\tau\varphi + |\xi'|), \quad \forall|\xi'|\ge 1. \tag{2.26}$$

Now we will formulate two lemmata and the proof of the second one will be given in Appendix I. Let us consider the initial value problem

$$\frac{\partial z}{\partial y_n} + s\tau\varphi z + r(y,D_{y'})z = \mathbf{f} \quad \text{in}\,\mathcal{G}, \quad z(\cdot,0)=0. \tag{2.27}$$

Assume that there exists $\kappa > 0$ such that

$$\mathrm{Re}\,r(y,\xi') \ge \kappa|\xi'|, \quad r(y,\xi')\in C^2S^1_{c\ell}. \tag{2.28}$$

By $C^2S^1_{c\ell}$, we mean the class of symbols introduced in [21, p.36]. Then

Lemma 2 (see [12, Lemma 3.2]). *Let $\mathbf{f}\in L^2(\mathcal{G}), z\in H^1(\mathcal{G})$ be a solution to equation (2.27) and let condition (2.28) hold true. Then there exists a constant C independent of s,τ such that*

$$\|\nabla z\|_{L^2(\mathcal{G})} + s\tau\|\varphi z\|_{L^2(\mathcal{G})} \le C\|\mathbf{f}\|_{L^2(\mathcal{G})}. \tag{2.29}$$

Second let us consider the equation

$$\widetilde{L}z = \frac{\partial z}{\partial y_n} - s\tau\varphi z + r(y,D_{y'})z = \mathbf{f}, \quad \text{in } \mathcal{G}, \quad z(\cdot,1)=0. \tag{2.30}$$

Lemma 3. *Let $z \in H^1(\mathcal{G})$ be a solution to problem (2.30), $\mathbf{f} \in L^2(\mathcal{G})$ and let condition (2.28) hold true. Then there exists $\widehat{\tau} > 1$ such that for all $\tau > \widehat{\tau}$ there exists $s_0(\tau) > 0$ such that the following estimate holds true*

$$\int_{\mathcal{G}} s\tau^2\varphi|z|^2 dy + \tau\|z\|^2_{L^2(0,1;H^{\frac{1}{2}}(\mathbb{R}^{n-1}))} + \int_G \frac{1}{s\varphi}|\nabla z|^2 dy$$
$$\leq C(\|\mathbf{f}\|^2_{L^2(\mathcal{G})} + \|z(\cdot,0)\|^2_{\frac{1}{2}}), \quad \forall s \geq s_0(\tau). \tag{2.31}$$

The constant C is independent of s, τ.

Thus by Lemma 2, the first n components of the function z satisfy the estimate

$$\sum_{k=1}^n \int_0^1 \left(\left\| \frac{\partial z_k}{\partial y_n} \right\|_0^2 + s^2\tau^2\varphi^2\|z_k\|_0^2 + \|z_k\|_1^2 \right) dy_n + \sum_{k=1}^n (s\tau\|z_k(\cdot,0)\|_0^2 + \|z_k(\cdot,0)\|^2_{\frac{1}{2}})$$
$$\leq C\int_0^1 (\|U\|_0^2 + \|f\|_0^2)e^{2s\varphi}dy_n, \quad \tau > 1, \ s > 1, \tag{2.32}$$

where the constant C is independent of s, τ. By (2.21) and Lemma 3, we have

$$\sum_{k=n+2}^{2n} \int_0^1 \left(\frac{1}{s\varphi} \left\| \frac{\partial z_k}{\partial y_n} \right\|_0^2 + s\tau^2\varphi\|z_k\|_0^2 + \frac{1}{s\varphi}\|z_k\|_1^2 \right) dy_n$$
$$\leq C\left(\int_0^1 (\|U\|_0^2 + \|f\|_0^2)e^{2s\varphi}dy_n + \|g\|_X^2 \right), \quad \tau > \widehat{\tau}, \ s > s_0(\tau), \tag{2.33}$$

where the constant C is independent of s, τ. Let us estimate z_{n+1}. Note that

$$D_{y_n}\left(\frac{z_{n+1}}{\sqrt{\varphi}}\right) + is\tau\sqrt{\varphi}z_{n+1} + \alpha_+(y, D_{y'})\frac{z_{n+1}}{\sqrt{\varphi}} + r(y, D_{y'})\frac{z_{n+2}}{\sqrt{\varphi}} + \frac{\tau}{2i}\frac{z_{n+1}}{\varphi^{\frac{3}{2}}}$$
$$= (T_0U + T_1 f)_{n+1}\frac{e^{s\varphi}}{\sqrt{\varphi}} \quad \text{in } \mathcal{G} \tag{2.34}$$

and

$$z_{n+1}(\cdot, 1) = 0. \tag{2.35}$$

By Lemma 3 we have

$$
\begin{aligned}
&\int_0^1 (\tau^2\|z_{n+1}\|_0^2 + \frac{1}{s^2\varphi^2}\|z_{n+1}\|_1^2) dy_n \\
\le & C\left(\int_0^1 (\|f\|_0^2 + \|U\|_0^2)e^{2s\varphi}dy_n + \int_0^1 \frac{\|z_{n+2}\|_1^2}{s\varphi}dy_n + \|z_{n+1}(\cdot,0)\|^2_{\frac12}\right) \\
\le & C\left(\int_0^1 (\|f\|_0^2 + \|U\|_0^2)e^{2s\varphi}dy_n + \|g\|_X^2\right).
\end{aligned} \tag{2.36}
$$

Combining (2.35) and (2.36), we obtain

$$
\int_0^1 \tau^2\|V\|_0^2 e^{2s\varphi}dy_n \le C\int_0^1 (\|f\|_0^2 + \|U\|_0^2)e^{2s\varphi}dy_n, \quad \forall \tau > \widehat{\tau},\ s \ge s_0(\tau). \tag{2.37}
$$

Noting that

$$
\|V\|_0^2 \ge C\|U\|_0^2 - \widetilde{C}\|U\|_{-1}^2
$$

and taking the parameter τ sufficiently large, we have

$$
\int_0^1 \tau^2\|U\|_0^2 e^{2s\varphi}dy_n \le C\int_0^1 (\|f\|_0^2 + \tau^2\|U\|_{-1}^2)e^{2s\varphi}dy_n. \tag{2.38}
$$

Next we note that

$$
\begin{aligned}
\|U\|_{-1}^2 = & \left\|\frac{\partial u}{\partial y_n}\right\|_{-1}^2 + \|\Lambda u\|_{-1}^2 \le C(\delta)\left\|\frac{\partial u}{\partial y_n}\right\|_0^2 + \sup_{\psi\in H^1(\mathbb{R}^{n-1})} < \Lambda u, \psi > \\
\le & C(\delta)\left\|\frac{\partial u}{\partial y_n}\right\|_0^2 + C(\delta)\|u\|_0^2,
\end{aligned}
$$

where $C(\delta)\to 0$ as $\delta\to 0$. Thus taking the parameter δ sufficiently small, we obtain

$$
\tau^2\int_0^1 (\|\nabla u\|_0^2 + \|u\|_0^2)e^{2s\varphi}dy_n \le C\left(\int_0^1 \|f\|_0^2 e^{2s\varphi}dy_n + \|g\|_X^2\right). \tag{2.39}
$$

Denote $w = ue^{s\varphi}$. Then

$$
\partial_{y_n} w - s\tau\varphi w = \frac{\partial u}{\partial y_n}e^{s\varphi}.
$$

Taking the scalar product in $L^2(\mathcal{G})$ of this equation and the function $s\tau\varphi w$, we obtain

$$s\tau\|w(\cdot,0)\|_0^2 + \frac{1}{2}\int_0^1 \tau^2\varphi s\|w\|_0^2 dy_n + \int_0^1 s^2\tau^2\varphi^2\|w\|_0^2 dy_n$$
$$= -\int_0^1 \left(\frac{\partial u}{\partial y_n}e^{s\varphi}, s\tau\varphi w\right)_{L^2(\mathbb{R}^{n-1})} dy_n.$$

Therefore for $s \geq 2$ we have

$$\frac{1}{2}\int_0^1 s^2\tau^2\varphi^2\|w\|_0^2 dy_n \leq \left|\int_0^1 \left(\frac{\partial u}{\partial y_n}e^{s\varphi}, s\tau\varphi w\right)_0 dy_n\right|$$
$$\leq 4\int_0^1 \left\|\frac{\partial u}{\partial y_n}\right\|_0^2 e^{2s\varphi} dy_n + \frac{1}{4}s^2\tau^2\int_0^1 \varphi^2\|w\|_0^2 dy_n.$$

Combining this estimate with (2.39), we obtain (2.8). Thus the proof of the theorem is complete. ■

§3. Conditional stability in the Cauchy problem for a stationary Lamé equation.

In this section, we consider

$$Pu = f \qquad \text{in } \Omega. \tag{3.1}$$

Our goal is to estimate a solution u to (3.1) by $u|_\omega$ or $\{u|_\Gamma, \sigma(u)\nu|_\Gamma\}$, where $\omega \subset \Omega$ is an arbitrarily fixed subdomain and $\Gamma \subset \partial\Omega$ is an arbitrary relatively open subset of $\partial\Omega$.

Let us recall that ψ is defined in Lemma 1 in Section 2. We set

$$\Omega(\delta) = \{x \in \Omega; \psi(x) > \delta\} \tag{3.2}$$

for small $\delta > 0$. Then, by (2.6), we have $\Omega(0) = \Omega$. Without the loss of generality, by reducing ω, one can assume that $\omega \subset \Omega(\delta)$ for all sufficiently small positive δ.

Now we can state conditional stability by the interior observation $u|_\omega$:

Theorem 2. *Let $u \in (H^2(\Omega))^n$ satisfy (3.1) with $f \in (L^2(\Omega))^n$ and*

$$\|u\|_{H^1(\Omega)} \le M \tag{3.3}$$

with a constant $M > 0$. Then there exist $\theta \in (0,1)$ and $C > 0$ depending on Ω, ω, δ, λ and μ such that

$$\|u\|_{H^2(\Omega(\delta))} \le CM^{1-\theta}(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\omega)})^\theta, \tag{3.4}$$

where $\lim_{\delta\downarrow 0}\theta = 0$.

Estimate (3.4) makes a sense as an estimate of u, under the condition that $\|u\|_{H^1(\Omega)}$ is a priori bounded. Thus (3.4) is called conditional stability. Our conditional stability is of Hölder type, but when we will estimate the H^2-norm of u over Ω, our estimate (3.4) becomes trivial by $\lim_{\delta\downarrow 0}\theta = 0$.

Next we will show conditional stability by boundary measurements u and $\sigma(u)\nu$ on $\Gamma \subset \partial\Omega$.

Theorem 3. *Let $u \in (H^2(\Omega))^n$ satisfy (3.1) and (3.3) with $f \in (L^2(\Omega))^n$ and a constant $M > 0$. Then there exist $\theta \in (0,1)$ and $C > 0$ depending on Ω, Γ, δ, λ and μ such that*

$$\|u\|_{H^2(\Omega(\delta))} \le CM^{1-\theta}(\|f\|_{L^2(\Omega)} + \|u\|_{H^{\frac{3}{2}}(\Gamma)} + \|\sigma(u)\nu\|_{H^{\frac{1}{2}}(\Gamma)})^\theta. \tag{3.5}$$

The proof is based on a usual argument (e.g., Isakov [16]) by means of a cutoff function.

Proof of Theorem 2. We take a cutoff function $\chi \in C_0^\infty(\Omega)$, $0 \le \chi \le 1$ on $\overline{\Omega}$ such that

$$\chi(x) = \begin{cases} 1, & x \in \Omega(\frac{2\delta}{3}), \\ 0, & x \in \Omega \setminus \overline{\Omega(\frac{\delta}{3})}. \end{cases} \tag{3.6}$$

By (3.6), we see that $\chi = 0$ in a neighbourhood of $\partial\Omega$. We set

$$v = \chi u.$$

Then

$$v|_{\partial\Omega} = 0. \tag{3.7}$$

Moreover, by the Leibniz formula and (3.6), we have

$$Pv = \chi P u + \eta_\delta Q_1 u, \tag{3.8}$$

where η_δ is the characteristic function of $\Omega(\frac{\delta}{3}) \setminus \overline{\Omega(\frac{2\delta}{3})}$ and Q_1 is a differential operator of the first order. By (3.7), we can apply Theorem 1 to (3.8), and, noting (3.6), we obtain

$$\begin{aligned}
&\int_{\Omega(2\delta/3)} \left(\frac{1}{s^2\varphi^2} \sum_{j,k=1}^n \left| \frac{\partial^2 u}{\partial x_j \partial x_k} \right|^2 + \tau^2 |\nabla u|^2 + s^2\tau^4\varphi^2 |u|^2 \right) e^{2s\varphi} dx \\
\le & C_1 \int_\Omega |\chi f|^2 e^{2s\varphi} dx + C_1 \int_{\Omega(\delta/3)\setminus\overline{\Omega(2\delta/3)}} |Q_1 u|^2 e^{2s\varphi} dx \\
+ & C_1 \int_\omega (\tau^2 |\nabla(\chi u)|^2 + s^2\tau^4\varphi^2 |\chi u|^2) e^{2s\varphi} dx
\end{aligned}$$

for sufficiently large $\tau > 0$ and $s > 0$. Therefore, recalling definition (3.2) and that $\Omega(\delta) \subset \Omega(2\delta/3)$, we obtain

$$\begin{aligned}
& e^{2se^{\tau\delta}} \int_{\Omega(\delta)} \left(\frac{1}{s^2\varphi^2} \sum_{j,k=1}^n \left| \frac{\partial^2 u}{\partial x_j \partial x_k} \right|^2 + \tau^2 |\nabla u|^2 + s^2\tau^4\varphi^2 |u|^2 \right) dx \\
\le & C_1 \int_\Omega |f|^2 e^{2s\varphi} dx + C_1 \exp(2se^{\frac{2\delta\tau}{3}}) \|u\|^2_{H^1(\Omega)} \\
+ & C_1 \int_\omega (\tau^2 |\nabla u|^2 + s^2\tau^4\varphi^2 |u|^2) e^{2s\varphi} dx.
\end{aligned}$$

Fixing a sufficiently large $\tau > 0$, taking a constant $C_2 > 0$ which is independent of $s > 0$ and noting $s^2\varphi^2 \le C_2 e^{2s\varphi}$ in Ω, we see that there exists a constant $\varepsilon > 0$

independent of $s > 0$ such that

$$\int_{\Omega(\delta)} \left(\sum_{j,k=1}^{n} \left| \frac{\partial^2 u}{\partial x_j \partial x_k} \right|^2 + s^2 |\nabla u|^2 + s^4 |u|^2 \right) dx$$
$$\le C_2 s^2 e^{-s\varepsilon} M^2 + C_2 s^2 e^{C_2 s} F^2 \tag{3.9}$$

for all large $s > 0$. Here and henceforth we set $F = \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\omega)}$. Without loss of generality, we may assume that F is sufficiently small, so that

$$\frac{1}{C_2 + \varepsilon} \log \frac{M^2}{F^2} > s_0(\tau).$$

In fact, if $F > C_0$ with some constant $C_0 > 0$, then conclusion (3.4) is trivial if the constant $C > 0$ in (3.4) is chosen sufficiently large for $C_0 > 0$ (but independently of a special choice of u). Consequently we can choose $s = \frac{1}{C_2+\varepsilon} \log \frac{M^2}{F^2}$ in (3.9) to obtain

$$\|u\|_{H^2(\Omega(\delta))} \le C_3 \left(\log \frac{M}{F} \right) M^{\frac{C_2}{C_2+\varepsilon}} F^{\frac{\varepsilon}{C_2+\varepsilon}}.$$

Since for any sufficiently small $\varepsilon_0 > 0$, there exists $C_{\varepsilon_0} > 0$ such that $\log t \le C_{\varepsilon_0} t^{\varepsilon_0}$ for all $t \ge \varepsilon_0$, we see that

$$\|u\|_{H^2(\Omega(\delta))} \le C_3 C_{\varepsilon_0} M^{\frac{C_2}{C_2+\varepsilon} + \varepsilon_0} F^{\frac{\varepsilon}{C_2+\varepsilon} - \varepsilon_0}.$$

Thus the proof of Theorem 2 is complete. ■

Proof of Theorem 3. First we show

Lemma 4. *Under assumption (1.2), there exists a constant $C_4 > 0$ such that*

$$\left\| \frac{\partial y}{\partial \nu} \right\|_{H^{1/2}(\partial\Omega)} + \|y\|_{H^{3/2}(\partial\Omega)} \le C_4 \|\sigma(y)\nu\|_{H^{1/2}(\partial\Omega)} + C_4 \|y\|_{H^{3/2}(\partial\Omega)}$$

for all $y \in (H^2(\Omega))^n$.

For completeness, we will prove the lemma in Appendix II.

Now we will proceed to the proof of Theorem 3. By the extension theorem, we can take $u_0 \in H^2(\Omega)$ such that

$$\|u_0\|_{H^2(\Omega)} \le C\left(\|u\|_{H^{3/2}(\Gamma)} + \left\|\frac{\partial u}{\partial \nu}\right\|_{H^{1/2}(\Gamma)}\right) \tag{3.10}$$

and

$$u_0|_\Gamma = u|_\Gamma, \quad \frac{\partial u_0}{\partial \nu}|_\Gamma = \frac{\partial u}{\partial \nu}|_\Gamma. \tag{3.11}$$

We set $v = u - u_0$ and choose a bounded domain $\widetilde{\Omega}$ such that $\partial\widetilde{\Omega}$ is of class C^3, $\widetilde{\Omega} \supset \Omega$ and $(\widetilde{\Omega} \setminus \Omega) \cap \partial\Omega = \Gamma$. We set

$$\widetilde{v} = \begin{cases} v & \text{in } \Omega \\ 0 & \text{in } \widetilde{\Omega} \setminus \overline{\Omega}. \end{cases}$$

Then, by (3.11), we have $\widetilde{v} \in H^2(\Omega)$ and

$$P\widetilde{v} = \begin{cases} f - Pu_0 & \text{in } \Omega \\ 0 & \text{in } \widetilde{\Omega} \setminus \overline{\Omega}. \end{cases}$$

Moreover, by (3.10), we see

$$\|P\widetilde{v}\|_{L^2(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|u\|_{H^{3/2}(\Gamma)} + \left\|\frac{\partial u}{\partial \nu}\right\|_{H^{1/2}(\Gamma)}\right).$$

Thus we can apply Theorem 2 to $\widetilde{v}$ in $\widetilde{\Omega}$ with $\omega = \widetilde{\Omega} \setminus \overline{\Omega}$, so that the proof of Theorem 3 is complete in view of Lemma 4. ■

§4. Determination of source terms by interior measurements.

In this section, we mainly consider a stationary Lamé system in the whole space:

$$(P\widetilde{y})(x) + \widetilde{\rho}\kappa^2\widetilde{y}(x) = F(x), \quad x \in \mathbb{R}^n \tag{4.1}$$

where $\widetilde{\rho}(x) > 0$ is a density and $\kappa > 0$ is a frequency. This is the equation of motion in the frequency domain corresponding to

$$\widetilde{\rho}\frac{\partial^2 z}{\partial t^2}(x,t) = Pz(x,t) - e^{i\kappa t}F(x), \quad x \in \mathbb{R}^n,\ t > 0,$$

where F is an external force.

In the case of $F(x) = \delta(x - x_0)q(x)$ where $\delta(\cdot - x_0)$ is the Dirac delta function centred at $x_0 \in \mathbb{R}^n$ and $q : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a smooth vector field, stationary Lamé system (4.1) is a typical model equation of seismic sources (e.g., Chapter 4 in Ben-Menahem and Singh [3]). Henceforth we set $x = (x', x_n)$ and $x' = (x_1, ..., x_{n-1})$. Let us approximate $\delta(x - x_0)q(x)$ by a function whose support is restricted in a neighbourhood of x^0 and let us assume that x'-depending factors of q are unknown and the source term is modelled to be separated as follows:

$$\delta(x - x_0)q(x) \approx R(x)(f_1(x'), ..., f_n(x'))^T. \tag{4.2}$$

One interesting inverse problem is determination of f by observations of u in some part of $\mathbb{R}^n$, provided that a real-valued function R is given. In particular, related with model (4.2) of sources, the inverse problem is determination of x'-depending factors of centre of point-like source.

Taking into consideration the above motivation from the seismology, we formulate our inverse problem in a general form. We consider

$$(Py)(x) + \rho(x)y(x) = R(x)f(x'), \qquad x \in \mathbb{R}^n. \tag{4.3}$$

Here we assume that

(4.4) the Lamé coefficients λ and μ are independent of x_n,

and condition (1.2) is satisfied, $\rho \in W^{1,\infty}_{loc}(\mathbb{R}^n)$, $R(x)$ is real-valued and $f(x') = (f_1(x'), ..., f_n(x'))^T$. Now we discuss

Inverse source problem. Let $r > 0$ be given, let $E \subset \mathbb{R}^n$ be an arbitrary bounded domain such that $E \subset \{(x', x_n); |x'| < r, x_n \in \mathbb{R}\}$ and $E \cap \{x_n = 0\}$ is a non-empty open set in $\mathbb{R}^{n-1}$, and let function $R(x)$ be known. We wish to determine $f(x')$, $|x'| < r$ from the observations of $y(x', 0)$, $|x'| < r$ and $y(x)$, $x \in E$.

Needless to say, in the case where $E \cap \{x_n = 0\} \supset \{(x', 0); |x'| < r\}$, our inverse problem is trivial. The application of Carleman estimate to inverse problems has been done firstly by Bukhgeim and Klibanov [6]. We refer to Bukhgeim [4], Bukhgeim, Cheng, Isakov and Yamamoto [5], Imanuvilov and Yamamoto [13], Isakov [15], Isakov and Yamamoto [17], Khaĭdarov [18], Klibanov [19] concerning such applications. As for inverse problems for the non-stationary Lamé system, we can refer to Isakov [14], Imanuvilov, Isakov and Yamamoto [11]. However there are not many applications to inverse elliptic problems and we have to assume that unknown coefficients should be independent of one fixed component (say, x_n) of x. As for the application to a Carleman estimate to such inverse problems for a single elliptic equation, see Isakov [15] and Klibanov [19]. In particular, in Klibanov [19], it is not necessary to assume that the coefficients of a (single) elliptic equation under consideration are independent of x_n, thanks to a special weight function in the Carleman estimate, which was proved in Lavrent'ev, Romanov and Shishat·skiĭ[20]. The papers concerning inverse problems for the stationary Lamé system by Carleman estimates, are rare.

In contrast to the inverse problems for a single elliptic equation as in [19], with our Carleman estimate, we do not know whether assumption (4.4) can be omitted.

We set

$$B_0 = \{x'; |x'| < r\} \tag{4.5}$$

and

$$B_0(\delta) = \{x'; |x'| < r - \delta\} \tag{4.6}$$

for small $\delta > 0$.

Now we are ready to state our main result for the inverse source problem.

Theorem 4. *We assume that*

$$R, \frac{\partial R}{\partial x_n} \in W^{2,\infty}_{loc}(\mathbb{R}^n), \quad |R(x', 0)| > 0, \quad |x'| \le r. \tag{4.7}$$

Then there exists constants $C = C(\Omega, R, \lambda, \mu, \delta) > 0$ *and* $\theta = \theta(\Omega, R, \lambda, \mu, \delta) \in (0,1)$ *such that*

$$\|f\|_{L^2(B_0(\delta))} \le CM^{1-\theta}\left(\|y(\cdot, 0)\|_{H^2(B_0)} + \left\|\frac{\partial y}{\partial x_n}\right\|_{H^1(E)} + \|y\|_{H^1(E)}\right)^{\theta} \tag{4.8}$$

provided that $\|y\|_{H^2_{loc}(\mathbb{R}^n)} \le M$ *where* $M > 0$ *is an arbitrarily fixed constant.*

Remark. We note that $\lim_{\delta \downarrow 0} \theta = 0$. Therefore we cannot replace the left hand side by $\|f\|_{L^2(B_0)}$.

Proof. Let us choose a domain $\omega_0 \subset \mathbb{R}^{n-1}$ such that $\omega_0 \subset E \cap \{x_n = 0\}$ and $\omega_0 \times (-\delta', \delta') \subset E$ with some $\delta' > 0$. Since $\omega_0 \ne \emptyset$ is a subdomain in $\mathbb{R}^{n-1}$ by the assumption, by Lemma 1, we can choose $\psi_0 = \psi_0(x') \in C^3(\overline{B_0})$ such that

$$\begin{cases} \psi_0 = 0 \quad \text{on } \partial B_0, \\ \psi_0 > 0 \quad \text{in } B_0, \\ \left(\dfrac{\partial \psi_0}{\partial x_1}(x'), ..., \dfrac{\partial \psi_0}{\partial x_{n-1}}(x')\right) \ne 0, \quad x' \in \overline{B_0 \setminus \omega_0}. \end{cases} \tag{4.9}$$

We set

$$\psi(x', x_n) = \varepsilon_0 \psi_0(x') - x_n^2, \tag{4.10}$$

where $\varepsilon_0 > 0$ is a small parameter. We define a domain $\Omega \subset \mathbb{R}^n$ by

$$\Omega = \{(x', x_n); |x_n| < \sqrt{\varepsilon_0 \psi_0(x')},\ x' \in B_0\}. \tag{4.11}$$

Then $\partial\Omega$ is of class C^3. In fact, by the definition of Ω, it is easy to verify that $\partial\Omega \cap \{x_n \neq 0\}$ is of class C^3. Next we have to verify that $\partial\Omega$ is given by a C^3-function near $\{(x', 0); x' \in B_0,\ \psi_0(x') = 0\}$. Since $\{x' \in B_0; \psi_0(x') = 0\} \subset \partial B_0 \subset \overline{B_0 \setminus \omega_0}$, we have that either of $\frac{\partial \psi_0}{\partial x_j}(x')$, $1 \le j \le n-1$, does not vanish by (4.9). Therefore the implicit function theorem implies that $\partial\Omega$ is represented by a C^3-function $\widetilde{g}$; there exists $j_0 \in \{1, ..., n-1\}$ such that $x_{j_0} = \widetilde{g}(x_1, ..., x_{j_0-1}, x_{j_0+1}, ..., x_n)$ near $(x', 0)$. Thus the proof that $\partial\Omega$ is of C^3, is complete.

We define a subdomain $\omega \subset \Omega$ by

$$\omega = \{(x', x_n); x' \in \omega_0,\ |x_n| < \sqrt{\varepsilon_0 \min_{x' \in \overline{\omega_0}} \psi_0(x')}\}. \tag{4.12}$$

For sufficiently small $\varepsilon_0 > 0$, we see that $\omega \subset E$. Then $\nabla\psi \neq 0$ on $\overline{\Omega \setminus \omega}$. In fact, we have

$$\nabla\psi(x', x_n) = \left(\varepsilon_0 \frac{\partial \psi_0}{\partial x_1}(x'), ..., \varepsilon_0 \frac{\partial \psi_0}{\partial x_{n-1}}(x'), -2x_n\right).$$

Therefore $\nabla\psi \neq 0$ if $x_n \neq 0$. Let $(x', 0) \in \overline{\Omega \setminus \omega}$. Then, by the definition of ω, we see that $x' \in \overline{B_0 \setminus \omega_0}$. Consequently (4.9) implies that $\nabla\psi(x', 0) \neq 0$. Thus we have seen that ψ given by (4.10), satisfies the conditions in Lemma 1 for Ω and ω defined by (4.11) and (4.12).

We recall the definition (3.2) of $\Omega(\delta)$ for $\delta > 0$. For given $\delta > 0$, we can choose constants $\delta_0, \delta_1, \delta_2$ such that $0 < \delta_2 < \delta_1 < \delta_0$ and

$$B_0(\delta) \subset \Omega(\delta_0) \cap \{x_n = 0\}. \tag{4.13}$$

We introduce a cutoff function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \chi \le 1$ and

$$\chi(x) = \begin{cases} 1 & \text{in } \Omega(\delta_1) \\ 0, & \text{in } \Omega \setminus \overline{\Omega(\delta_2)}. \end{cases} \tag{4.14}$$

We set

$$z = \chi y.$$

Then, by (4.13) and (4.14),

$$z \text{ vanishes in a neighbourhood of } \partial\Omega. \tag{4.15}$$

On the other hand, since $R(x',0)^{-1}$ exists for $x' \in \overline{B_0}$ by (4.7), we see that $R(x)^{-1}$ exists for $x \in \overline{\Omega}$ if we choose $\varepsilon_0 > 0$ sufficiently small. Henceforth, without fear of confusion, we denote $Py(x) + \rho(x)y(x)$ by $\widetilde{P}y(x)$. We note that Theorem 1 is true for this $\widetilde{P}$ if we take $s_0(\tau) > 0$ large. Therefore, since $\widetilde{P}y(x) = R(x)f(x')$, $x \in \Omega$, so that $\frac{\partial}{\partial x_n}(R^{-1}\widetilde{P}y) = 0$ in Ω, namely,

$$\begin{aligned} &P\left(\frac{\partial y}{\partial x_n} - \left(\frac{\partial R}{\partial x_n}R^{-1}\right)y\right) + \left(\frac{\partial \rho}{\partial x_n}\right)y \\ &+ \rho\frac{\partial y}{\partial x_n} - \frac{\partial R}{\partial x_n}R^{-1}\rho y - \left[\frac{\partial R}{\partial x_n}R^{-1}, P\right]y = 0 \quad \text{in } \Omega. \end{aligned} \tag{4.16}$$

Here we have used assumption (4.4). We note that

$$Q_1 y \equiv -\frac{\partial \rho}{\partial x_n}y - \rho\frac{\partial y}{\partial x_n} + \frac{\partial R}{\partial x_n}R^{-1}\rho y + \left[\frac{\partial R}{\partial x_n}R^{-1}, P\right]y$$

is a differential operator of at most first order whose coefficients are in $L^\infty_{loc}(\mathbb{R}^n)$ by (4.7).

We set

$$Nw = \left(\frac{\partial}{\partial x_n} - \frac{\partial R}{\partial x_n}R^{-1}\right)w.$$

Then, by (4.16), we have $PNy = Q_1 y$. Consequently we have

$$PNz = PN(\chi y) = \chi PNy + [PN, \chi]y = \chi Q_1 y + [PN, \chi]y = Q_1 z + [\chi, Q_1]y + [PN, \chi]y.$$

Noting the definitions of Q_1 and N, in terms of (4.7) and (4.14), we obtain

$$PNz = Q_1 z + Q(\nabla\chi)y \tag{4.17}$$

where $Q(\nabla\chi)$ is a partial differential operator of at most second order whose coefficients are in $L^\infty_{loc}(\mathbb{R}^n)$ and are linear combinations of the derivatives of χ. In particular, by (4.14), we have

$$Q(\nabla\chi)y \ne 0 \quad \text{only if in } \Omega(\delta_2) \setminus \overline{\Omega(\delta_1)}. \tag{4.18}$$

By (4.15), noting that $\omega \subset E$, we apply Theorem 1 to Nz in Ω:

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{s^2\varphi^2} \sum_{j,k=1}^n \left| \frac{\partial^2 (Nz)}{\partial x_j \partial x_k} \right|^2 + \tau^2 |\nabla(Nz)|^2 + s^2\tau^4\varphi^2 |Nz|^2 \right) e^{2s\varphi} dx \\
\le& C \int_\Omega |Q_1 z|^2 e^{2s\varphi} dx + C \int_\Omega |Q(\nabla\chi)y|^2 e^{2s\varphi} dx \\
&+ C \int_\omega (\tau^2 |\nabla(Nz)|^2 + s^2\tau^4\varphi^2 |Nz|^2) e^{2s\varphi} dx \\
\le& C \int_\Omega (|\nabla z|^2 + |z|^2) e^{2s\varphi} dx \\
&+ C \exp(2se^{\tau\delta_1}) \|y\|^2_{H^2(\Omega)} + Ce^{sC} \left(\left\| \frac{\partial y}{\partial x_n} \right\|^2_{H^1(E)} + \|y\|^2_{H^1(E)} \right) \quad \forall s \ge s_0(\tau).
\end{aligned} \tag{4.19}$$

by (4.14), (4.18) and $z = \chi y$.

On the other hand, we can prove

$$
\left\{
\begin{aligned}
&\int_\Omega \left\vert \frac{\partial^2 z}{\partial x_j\partial x_k}\right\vert^2 e^{2s\varphi}dx\\
\le& C\int_\Omega \left(\sum_{j,k=1}^n \left\vert \frac{\partial^2 (Nz)}{\partial x_j\partial x_k}\right\vert^2 + \vert\nabla(Nz)\vert^2 + \vert Nz\vert^2\right) e^{2s\varphi}dx\\
+& Ce^{Cs}\Vert y(\cdot,0)\Vert^2_{H^2(B_0)},\\
&\int_\Omega \vert\nabla z\vert^2 e^{2s\varphi}dx \le C\int_\Omega (\vert\nabla(Nz)\vert^2 + \vert Nz\vert^2)e^{2s\varphi}dx + Ce^{Cs}\Vert y(\cdot,0)\Vert^2_{H^1(B_0)},\\
&\int_\Omega \vert z\vert^2 e^{2s\varphi}dx \le C\int_\Omega \vert Nz\vert^2 e^{2s\varphi}dx + Ce^{Cs}\Vert y(\cdot,0)\Vert^2_{L^2(B_0)}
\end{aligned}
\right. \tag{4.20}
$$

for all large $s > 0$ and $\tau > 0$. In fact, it suffices to verify the first inequality for $1 \le j,k \le n-1$, because the rest inequalities are proved similarly. The proof is done along the line of Klibanov [18], and, for completeness, we will give it. Noting that $Nz = \left(\frac{\partial}{\partial x_n} - \frac{\partial R}{\partial x_n}R^{-1}\right) z$ and $z(x',0) = \chi(x',0)y(x',0)$ for $x \in B_0$, we can represent z by means of the fundamental solution $L(x,\xi)$:

$$
z(x) = \int_0^{x_n} L(x,\xi)(Nz)(x',\xi)d\xi + \chi(x',0)y(x',0), \quad x = (x',x_n) \in \Omega. \tag{4.21}
$$

Moreover, by (4.7), we see that L is in $W^{2,\infty}$ with respect to x,ξ.

Noting (4.11) and the Schwarz inequality, we have

$$\begin{aligned}
&\int_\Omega \left|\frac{\partial^2 z}{\partial x_j \partial x_k}\right|^2 e^{2s\varphi} dx \\
\le &\int_{B_0}\int_{-\sqrt{\varepsilon_0\psi_0(x')}}^{\sqrt{\varepsilon_0\psi_0(x')}} \Bigg|\int_0^{x_n}\left(\frac{\partial^2 L}{\partial x_j\partial x_k}(x,\xi)(Nz)(x',\xi) + \frac{\partial L}{\partial x_j}(x,\xi)\frac{\partial (Nz)}{\partial x_k}(x',\xi)\right.\\
&\left.+\frac{\partial L}{\partial x_k}(x,\xi)\frac{\partial(Nz)}{\partial x_j}(x',\xi) + L(x,\xi)\frac{\partial^2 (Nz)}{\partial x_j \partial x_k}(x',\xi)\right)d\xi \\
&+\frac{\partial^2}{\partial x_j \partial x_k}(\chi(x',0)y(x',0))\Bigg|^2 e^{2s\varphi(x',x_n)}dx_n dx' \\
\le &C\int_{B_0}\int_{-\sqrt{\varepsilon_0\psi_0(x')}}^{\sqrt{\varepsilon_0\psi_0(x')}} \left|\int_0^{x_n} |A(x',\xi)|^2 d\xi\right| e^{2s\varphi(x',x_n)}dx_n dx' + Ce^{Cs}\|y(\cdot,0)\|^2_{H^2(B_0)} \\
= &C\int_{B_0}\int_0^{\sqrt{\varepsilon_0\psi_0(x')}} \left|\int_0^{x_n} |A(x',\xi)|^2 d\xi\right| e^{2s\varphi}dx \\
+ &C\int_{B_0}\int_{-\sqrt{\varepsilon_0\psi_0(x')}}^{0} \left|\int_0^{x_n} |A(x',\xi)|^2 d\xi\right| e^{2s\varphi}dx + Ce^{Cs}\|y(\cdot,0)\|^2_{H^2(B_0)} \\
\equiv &I_1 + I_2 + Ce^{Cs}\|y(\cdot,0)\|^2_{H^2(B_0)}.
\end{aligned}$$

Here and henceforth we set

$$A(x',\xi) = |(Nz)(x',\xi)| + |\nabla(Nz)(x',\xi)| + \sum_{j,k=1}^n \left|\frac{\partial^2(Nz)}{\partial x_j\partial x_k}(x',\xi)\right|.$$

We can estimate I_1 and I_2 similarly. Changing the orders of integrals and using $\varphi(x',x_n) \le \varphi(x',\xi)$ for $\xi \le x_n \le \sqrt{\varepsilon_0\psi_0(x')}$ by (4.10), we see that

$$\begin{aligned}
&I_1 = C\int_{B_0}\int_0^{\sqrt{\varepsilon_0\psi_0(x')}}\left(\int_\xi^{\sqrt{\varepsilon_0\psi_0(x')}} e^{2s\varphi(x',x_n)}dx_n\right)|A(x',\xi)|^2 d\xi dx' \\
\le &C\int_{B_0}\int_0^{\sqrt{\varepsilon_0\psi_0(x')}} e^{2s\varphi(x',\xi)}|A(x',\xi)|^2 d\xi dx',
\end{aligned}$$

which completes the verification of the first inequality of (4.20).

Therefore, applying (4.20) to (4.19) and absorbing the first integral at the right hand side of (4.19) into the left hand side by fixing $\tau > 0$ sufficiently large, we obtain

$$\int_\Omega \left(\left| \frac{1}{s^2} \sum_{j,k=1}^n \frac{\partial^2 (Nz)}{\partial x_j \partial x_k} \right|^2 + |\nabla (Nz)|^2 + |Nz|^2 \right) e^{2s\varphi} dx$$

$$\le C e^{2se^{\tau\delta_1}} M^2 + C e^{Cs} F^2 \tag{4.22}$$

for all large $s > 0$. Here and henceforth we set

$$F = \left(\|y(\cdot,0)\|^2_{H^2(B_0)} + \left\| \frac{\partial y}{\partial x_n} \right\|^2_{H^1(E)} + \|y\|^2_{H^1(E)} \right).$$

Therefore, by (4.20), we have

$$\int_\Omega \left(\left| \frac{1}{s^2} \sum_{j,k=1}^n \frac{\partial^2 z}{\partial x_j \partial x_k} \right|^2 + |\nabla z|^2 + |z|^2 \right) e^{2s\varphi} dx$$

$$\le C e^{2se^{\tau\delta_1}} M^2 + C e^{Cs} F^2. \tag{4.23}$$

In (4.22) and (4.23), we replace the integral over Ω by the one over $\Omega(\delta_0)$, so that by (4.13), (4.14) and $0 < \delta_2 < \delta_1 < \delta_0$, we obtain

$$\int_{\Omega(\delta_0)} \left(\sum_{j,k=1}^n \left| \frac{\partial^2 y}{\partial x_j \partial x_k} \right|^2 + |\nabla y|^2 + |y|^2 \right) dx$$

$$+ \int_{\Omega(\delta_0)} \left(\sum_{j,k=1}^n \left| \frac{\partial^2 (Ny)}{\partial x_j \partial x_k} \right|^2 + |\nabla (Ny)|^2 + |Ny|^2 \right) dx$$

$$\le C s^2 \exp(2s(e^{\tau\delta_1} - e^{\tau\delta_0})) M^2 + C s^2 e^{Cs} F^2 \tag{4.24}$$

for all large $s > 0$.

Similarly to the argument for (3.9), we can assume that F is sufficiently small and so in (4.22) we can set

$$s = \frac{2}{\kappa} \log \frac{M}{F}$$

where $\kappa = C + 2(e^{\tau\delta_0} - e^{\tau\delta_1})$. Then, by the definition of the operator N, we can obtain

$$\|y\|_{H^2(\Omega(\delta_0))} + \left\| \frac{\partial y}{\partial x_n} \right\|_{H^2(\Omega(\delta_0))} \le \frac{4C}{\kappa} \left(\log \frac{M}{F} \right) M^{\frac{C}{\kappa}} F^{1-\frac{C}{\kappa}} \le C_1 M^{1-\theta} F^\theta$$

with some constant $\theta \in (0,1)$. By (4.13), the trace theorem yields

$$\left\| \frac{\partial^2 y}{\partial x_j \partial x_n}(\cdot,0) \right\|_{L^2(B_0(\delta))} + \left\| \frac{\partial y}{\partial x_n}(\cdot,0) \right\|_{L^2(B_0(\delta))} \le C_1 M^{1-\theta} F^\theta, \quad 1 \le j \le n.$$

Therefore

$$\begin{aligned} &\left\| \frac{\partial^2 y}{\partial x_j \partial x_k}(\cdot,0) \right\|_{L^2(B_0(\delta))} + \left\| \frac{\partial y}{\partial x_j}(\cdot,0) \right\|_{L^2(B_0(\delta))} + \|y(\cdot,0)\|_{L^2(B_0(\delta))} \\ \le & C_1 M^{1-\theta} F^\theta + C_1 \|y(\cdot,0)\|_{H^2(B_0)} \le C_2 M^{1-\theta} F^\theta, \qquad 1 \le j,k \le n, \end{aligned}$$

because $0 < \theta < 1$. Since $R(x',0)f(x') = (Py)(x',0)$, $x' \in B_0$ and $R(x',0)^{-1}$ exists for $x' \in \overline{B_0}$, we see that

$$\|f\|_{L^2(B_0(\delta))} \le C_3 \|R(\cdot,0)f\|_{L^2(B_0(\delta))} \le C_2 C_3 M^{1-\theta} F^\theta.$$

Thus the proof of Theorem 4 is complete. ■

Appendix I. Proof of Lemma 3.

Proof of Lemma 3. We introduce the operators L_1, L_2:

$$L_1(y,D) = -s\tau\varphi + \frac{1}{2}(r(y,D_{y'}) + r^*(y,D_{y'}))$$

and

$$L_2(y,D) = \frac{\partial}{\partial y_n} + \frac{1}{2}(r(y,D_{y'}) - r^*(y,D_{y'})).$$

Here r^* denotes the formal adjoint operator of r. Obviously $\widetilde{L}z = L_1 z + L_2 z$. Taking the L^2-norm of the both sides of (2.30), we obtain:

$$\begin{aligned} &\|\mathbf{f}\|^2_{L^2(\mathcal{G})} = \|L_1 z\|^2_{L^2(\mathcal{G})} + \|L_2 z\|^2_{L^2(G)} \\ &+([L_1,L_2]z,z)_{L^2(\mathcal{G})} - (L_1(0,D_{y'})z(0),z(0))_{L^2(\mathbb{R}^{n-1})}. \end{aligned} \tag{1}$$

Let us compute the commutator $[L_1, L_2]$. We note that

$$\left[L_1, \frac{\partial}{\partial y_n}\right] = s\tau^2\varphi + \widetilde{c}(y, D_{y'}), \tag{2}$$

where $\widetilde{c}(y, D_{y'})$ is the pseudo differential operator with the symbol obtained by the differentiation of the symbol of the operator $\frac{1}{2}(r(y, D_{y'}) + r^*(y, D_{y'}))$ with respect to the variable y_n. On the other hand,

$$\begin{aligned}
&\left[L_1, \frac{1}{2}(r(y, D_{y'}) - r^*(y, D_{y'}))\right] \\
=&\frac{1}{4}\left[(r(y, D_{y'}) + r^*(y, D_{y'})), (r(y, D_{y'}) - r^*(y, D_{y'}))\right] \\
\in&\mathcal{L}(H^{\frac{1}{2}}(\mathbb{R}^{n-1}), H^{-\frac{1}{2}}(\mathbb{R}^{n-1}))
\end{aligned} \tag{3}$$

for each $y_n \in [0, 1]$. By (2) and (3),

$$[L_1, L_2] = s\tau^2\varphi + T(y_n), \tag{4}$$

where the operator $T(y_n) \in C([0, 1]; \mathcal{L}(H^1(\mathbb{R}^{n-1}), L^2(\mathbb{R}^{n-1})))$ is independent of s and τ. Note that by Proposition 2.1.D ([21, p.47]), there exists $C_1 > 0$ such that

$$-(L_1(0, D_{y'})z(\cdot, 0), z(\cdot, 0))_{L^2(\mathbb{R}^{n-1})} \ge -C_1\|z(\cdot, 0)\|^2_{\frac{1}{2}}. \tag{5}$$

Using (4) and (5) in (1), we obtain

$$\begin{aligned}
&\|L_1 z\|^2_{L^2(\mathcal{G})} + \|L_2 z\|^2_{L^2(\mathcal{G})} + \int_{\mathcal{G}} s\tau^2\varphi|z|^2 dy \\
\le&C_2(\|\mathbf{f}\|^2_{L^2(\mathcal{G})} + \|z(\cdot, 0)\|^2_{\frac{1}{2}} + \|z\|^2_{L^2(0,1;H^{\frac{1}{2}}(\mathbb{R}^{n-1}))}).
\end{aligned} \tag{6}$$

On the other hand, there exist constants $C_3 > 0$ and $C_4 > 0$, independent of s and τ, such that

$$\begin{aligned}
&\tau\mathrm{Re}(L_1 z, z)_{L^2(\mathcal{G})} = \frac{\tau}{2}\mathrm{Re}((r(y, D_{y'}) + r^*(y, D_{y'}))z, z)_0 - \int_G s\tau^2\varphi|z|^2 dy \\
\ge&C_3\tau\|z\|^2_{L^2(0,1;H^{\frac{1}{2}}(\mathbb{R}^{n-1}))} - C_4 s\tau^2\int_{\mathcal{G}}\varphi|z|^2 dy.
\end{aligned} \tag{7}$$

From (6) and (7), taking the parameter τ sufficiently large, we obtain

$$\|L_1 z\|^2_{L^2(\mathcal{G})} + \int_{\mathcal{G}} s\tau^2\varphi|z|^2 dy + \tau\|z\|^2_{L^2(0,1;H^{\frac{1}{2}}(\mathbb{R}^{n-1}))}$$

$$\leq C_5(\|\mathbf{f}\|^2_{L^2(\mathcal{G})} + \|z(0,\cdot)\|^2_{\frac{1}{2}}). \tag{8}$$

By Corollary 4.3.C ([21, p.108]), we have

$$\int_{\mathcal{G}} \frac{1}{s\varphi}|\nabla z|^2 dy \leq C\int_0^1 \left(\frac{1}{s\varphi}\|r(y,D_{y'}) + r^*(y,D_{y'})\|_0^2 + s\tau^2\varphi|z|^2\right) dy$$

$$+C\|L_1 z\|^2_{L^2(\mathcal{G})}. \tag{9}$$

The inequalities (8) and (9) imply (2.31). ■

Appendix II. Proof of Lemma 4.

We set $a\otimes b = (a_j b_k)_{1\leq j,k\leq n}$ for $a = (a_1,...,a_n)^T$ and $b = (b_1,...,b_n)^T$, and SymA $= \frac{1}{2}(A + A^T)$, $A\cdot B = \sum_{j,k=1}^n a_{jk}b_{jk}$ and $|A|^2 = \sum_{j,k=1}^n a_{jk}^2$ for square matrices $A = (a_{jk})_{1\leq j,k\leq n}$ and $B = (b_{jk})_{1\leq j,k\leq n}$. We define an $n\times n$ matrix $B = B(x)$ by

$$B(x)a = \lambda(x)(a\cdot\nu(x))\nu(x) + 2\mu(x)\{\mathrm{Sym}\,(a\otimes\nu(x))\}\nu(x)$$

for $a\in\mathbb{R}^n$. Then

$$B = B(x) \text{ is invertible for all } x\in\partial\Omega. \tag{1}$$

In fact, since

$$(\mathrm{Sym}\,(a\otimes\nu)\nu\cdot a = (\mathrm{Sym}\,(a\otimes\nu)\cdot(a\otimes\nu) = |\mathrm{Sym}\,(a\otimes\nu)|^2$$

by direct calculations, we see that

$$(Ba\cdot a) = \lambda|a\cdot\nu|^2 + 2\mu|\mathrm{Sym}\,(a\otimes\nu(x))|^2 = \lambda|\mathrm{tr}\,A|^2 + 2\mu|A|^2. \tag{2}$$

Here we set $A = \mathrm{Sym}\,(a \otimes \nu)$ and

$$D = A - \frac{\mathrm{tr}\, A}{n} I_n. \tag{3}$$

Then $\mathrm{tr}\, D = 0$, so that

$$D \cdot I_n = 0 \tag{4}$$

by the identity $D \cdot I_n = \mathrm{tr}\, D$. Therefore (2) - (4) imply

$$\begin{aligned} Ba \cdot a = & \lambda |\mathrm{tr}\, A|^2 + 2\mu \left| \frac{\mathrm{tr}\, A}{n} I_n + D \right|^2 \\ = & \lambda |\mathrm{tr}\, A|^2 + 2\mu \left(\left| \frac{\mathrm{tr}\, A}{n} I_n \right|^2 + |D|^2 + 2 \frac{\mathrm{tr}\, A}{n} I_n \cdot D \right) \\ = & \frac{n\lambda + 2\mu}{n} |\mathrm{tr}\, A|^2 + 2\mu |D|^2 \ge \frac{\delta_0}{n} |\mathrm{tr}\, A|^2 + \delta_0 |D|^2. \end{aligned}$$

At the last inequality, we have used (1.2). By (3), we have $A = D + \frac{\mathrm{tr}\, A}{n} I_n$, so that $\delta_0 |A|^2 = \frac{\delta_0}{n} |\mathrm{tr}\, A|^2 + \delta_0 |D|^2$ by (4). Therefore $Ba \cdot a \ge \delta_0 |\mathrm{Sym}\,(a \otimes \nu)|^2$. Moreover we have

$$\begin{aligned} & |\mathrm{Sym}\,(a \otimes \nu)|^2 = \frac{1}{4} (|a \otimes \nu|^2 + 2(a \otimes \nu) \cdot (\nu \otimes a) + |\nu \otimes a|^2) \\ = & \frac{1}{4} (|a|^2 + 2|a \cdot \nu| + |a|^2) \ge \frac{1}{2} |a|^2, \end{aligned}$$

so that

$$Ba \cdot a \ge \frac{\delta_0}{2} |a|^2 \quad \text{on } \partial\Omega.$$

Moreover direct calculations verify that $Ba \cdot b = Bb \cdot a$ for every $a, b \in \mathbb{R}^n$, which means that B is a symmetric matrix. Therefore the proof of (1) is complete.

Henceforth $\frac{\partial y}{\partial \tau} \tau$ denotes the tangential component of ∇y on $\partial\Omega$. We note that $\nabla y = \frac{\partial y}{\partial \nu} \nu + \frac{\partial y}{\partial \tau} \tau$ on $\partial\Omega$. Next we will prove

$$\nabla y = \{(\nabla y)\nu\} \otimes \nu + \frac{\partial y}{\partial \tau} \tau^T \quad \text{on } \partial\Omega \tag{5}$$

and

$$\operatorname{div} y = (\nabla y)\nu \cdot \nu \quad \text{on } \partial\Omega. \tag{6}$$

In fact, setting $y = (y_1, ..., y_n)^T$ and $\nu = (\nu_1, ..., \nu_n)^T$, we have

$$\nabla y_j = (\nabla y_j \cdot \nu)\nu + \frac{\partial y_j}{\partial \tau}\tau = \left(\frac{\partial y_j}{\partial \nu}\right)\nu + \frac{\partial y_j}{\partial \tau}\tau, \quad 1 \le j \le n.$$

Therefore we have

$$\begin{aligned}\nabla y = &\begin{pmatrix} (\nabla y_1)^T \\ \vdots \\ (\nabla y_n)^T \end{pmatrix} = \begin{pmatrix} (\nabla y_1 \cdot \nu)\nu^T + \frac{\partial y_1}{\partial \tau}\tau^T \\ \vdots \\ (\nabla y_n \cdot \nu)\nu^T + \frac{\partial y_n}{\partial \tau}\tau^T \end{pmatrix} \\ =&((\nabla y_j \cdot \nu)\nu_k)_{1 \le j,k \le n} + \begin{pmatrix} \frac{\partial y_1}{\partial \tau}\tau^T \\ \vdots \\ \frac{\partial y_n}{\partial \tau}\tau^T \end{pmatrix}, \end{aligned} \tag{7}$$

which means (5). Moreover by $\nu\nu^T = 1$ and $(\tau^T)\nu = 0$, we have

$$(\nabla y)\nu = \begin{pmatrix} (\nabla y_1 \cdot \nu)\nu^T \\ \vdots \\ (\nabla y_n \cdot \nu)\nu^T \end{pmatrix} \nu = \begin{pmatrix} (\nabla y_1 \cdot \nu) \\ \vdots \\ (\nabla y_n \cdot \nu) \end{pmatrix} = \frac{\partial y}{\partial \nu}$$

and so

$$(\nabla y)\nu \cdot \nu = (\nabla y_1 \cdot \nu)\nu_1 + \cdots + (\nabla y_n \cdot \nu)\nu_n = \operatorname{tr} \nabla y = \operatorname{div} y$$

by (7). Thus the proof of (5) and (6) is complete. ■

Now we will complete the proof of Lemma 4. By the definition of B, (5) and (6), we have

$$\begin{aligned} &B((\nabla y)\nu) = B\left(\frac{\partial y}{\partial \nu}\right) = \lambda((\nabla y)\nu \cdot \nu)\nu + 2\mu\{\mathrm{Sym}\,((\nabla y)\nu \otimes \nu)\}\nu \\ =&\lambda(\operatorname{div} y)\nu + 2\mu(\mathrm{Sym}\,\nabla y)\nu - \mathrm{Sym}\left(\frac{\partial y}{\partial \tau}\tau^T\right) = \sigma(y)\nu - \mathrm{Sym}\left(\frac{\partial y}{\partial \tau}\tau^T\right) \quad \text{on } \partial\Omega. \end{aligned}$$

Therefore, by (1), we have

$$\begin{aligned} &\left\|\frac{\partial y}{\partial \nu}\right\|_{H^{1/2}(\partial\Omega)} \le C\|\sigma(y)\nu\|_{H^{1/2}(\partial\Omega)} + C\left\|\mathrm{Sym}\left(\frac{\partial y}{\partial \tau}\tau^T\right)\right\|_{H^{1/2}(\partial\Omega)} \\ \le&C(\|\sigma(y)\nu\|_{H^{1/2}(\partial\Omega)} + \|y\|_{H^{3/2}(\partial\Omega)}). \end{aligned}$$

Thus the proof of Lemma 4 is complete. ■

Acknowledgements. Oleg Imanuvilov was supported in part by NSF Grant DMS 02-05148.

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