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NEW REALIZATION OF THE PSEUDOCONVEXITY AND ITS APPLICATION TO AN INVERSE PROBLEM

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Abstract. We consider a hyperbolic differential operator $P = a_0(x)^2\partial_t^2 - \Delta$ with variable principal term. We first give a condition for the pseudoconvexity which yields a Carleman estimate. Our condition implies that level sets generated by the weight function in the Carleman estimate, is convex with respect to the set of rays given by $a_0(x)$, and gives a more general explicit condition of a_0 for the pseudoconvexity. Second we apply the Carleman estimate to an inverse problem of determining a_0 by Cauchy data on a lateral boundary with relaxed constraints on a_0.

§1. Introduction.

We consider a hyperbolic differential operator

$$(Pu)(x,t) = a_0(x)^2\partial_t^2 u(x,t) - \Delta u(x,t), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{1.1}$$

where $a_0 > 0$ is a real-valued smooth function, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $1 \le j \le n$, $\Delta = \sum_{j=1}^n \partial_j^2$.

One of the fundamental problems is the uniqueness in the initial value problem for the equation $Pu = 0$ or the unique continuation. For these purposes, a basic tool is a Carleman estimate, and for general theories, we refer to Hörmander [5],

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Imanuvilov [6], Isakov [13] - [15], for example. According to Isakov [13] - [15], we will state a sufficient condition for a relevant Carleman estimate. Let us define the principal symbol $P_m(x,t,\zeta)$ by

$$P_m(x,t,\zeta) = -a_0(x)^2\zeta_{n+1}^2 + \sum_{j=1}^{n}\zeta_j^2 \tag{1.2}$$
$$,x \in \mathbb{R}^n,\ t \in \mathbb{R},\ \zeta = (\zeta_1, ..., \zeta_{n+1}) \in \mathbb{C}^{n+1}.$$

We set

$$t = x_{n+1}, \quad \partial_{n+1} = \partial_t,\ \nabla' = (\partial_1, ..., \partial_n),\ \nabla = (\partial_1, ..., \partial_n, \partial_t).$$

Let $Q \subset \mathbb{R}^n \times \mathbb{R}$ be a domain and let $\varphi \in C^2(\overline{Q})$ satisfy $\nabla\varphi \ne 0$ on $\overline{Q}$. Then

Theorem A. *(Isakov [13]). Let $K \subset Q$ be an arbitrarily fixed bounded domain. We assume*

$$\sum_{1\le j,k\le n+1}(\partial_j\partial_k\varphi)\frac{\partial P_m}{\partial\zeta_j}\overline{\frac{\partial P_m}{\partial\zeta_k}} + \sum_{k=1}^{n+1} s^{-1} Im\left(\partial_k P_m \overline{\frac{\partial P_m}{\partial\zeta_k}}\right) > 0 \tag{1.3}$$

if $(x, x_{n+1}) \in \overline{Q}$ and

$$\zeta = \xi + \sqrt{-1}s\nabla\varphi, \quad P_m(x, x_{n+1}, \zeta) = 0, \quad s \ne 0 \quad or \tag{1.4}$$
$$\sum_{k=1}^{n+1}\frac{\partial P_m(x,\xi)}{\partial\xi_k}\partial_k\varphi = 0, \quad P_m(x, x_{n+1}, \xi) = 0, \quad \xi = (\xi_1, ..., \xi_{n+1}) \in \mathbb{R}^{n+1}\setminus\{0\}.$$

Then there exist constants $s_0 > 0$ and $C > 0$ such that

$$s\int_Q |\nabla u|^2 e^{2s\varphi}dxdx_{n+1} + s^3\int_Q u^2 e^{2s\varphi}dxdx_{n+1} \le C\int_Q |Pu|^2 e^{2s\varphi}dxdx_{n+1} \tag{1.5}$$

for $s > s_0$ and $u \in H_0^2(K)$.

Here and henceforth, for $\alpha \in \mathbb{C}$, Im α and Re α denote the imaginary part and the real part respectively, and $\overline{\alpha}$ is the complex conjugate.

An estimate of form (1.5) is called a Carleman estimate with the weight function φ, by which we can establish the unique continuation or stability in the Cauchy problem (e.g., Isakov [13] - [15]), observability inequalites (e.g., Cheng, Isakov, Yamamoto and Zhou [4], Kazemi and Klibanov [17], Lasiecka and Triggiani [20], Tataru [24]) and inverse problems (e.g., Bukhgeim [2], Bukhgeim and Klibanov [3], Imanuvilov, Isakov and Yamamoto [7], Imanuvilov and Yamamoto [8] - [11], Isakov [12], [13], Isakov and Yamamoto [16], Khaĭdarov [18], Klibanov [19], Yamamoto [25]). By the inverse problem, we mean the determination of $a_0(x)$ by overdetermining data of u on ∂Q. Thus it is seriously important to find a weight function φ satisfying (1.3) under condition (1.4). However the existing searches for φ are restricted and as φ, one mainly takes $|x-x_0|^2 - \beta t^2$ or $e^{\lambda(|x-x_0|^2-\beta t^2)}$ where $x^0 \in \mathbb{R}^n$, $\beta > 0$ and $\lambda > 0$ are parameters, and after such a fixed choice of φ, we have to assume conditions on a_0 in order that condition (1.3) is satisfied. That is, the following is known:

Proposition B. *Let us set $\mathcal{D} = \{x; (x,t) \in Q \quad \text{for some } t \in \mathbb{R}\}$ and let $a_0 \in C^2(\overline{\mathcal{D}})$ satisfy $a_0 > 0$ on $\overline{\mathcal{D}}$ and there exists $x^0 \in \mathbb{R}^n \setminus \overline{\mathcal{D}}$ such that*

$$(\nabla' \log a_0(x) \cdot (x - x^0)) > -1 \tag{1.6}$$

for any $x \in \overline{\mathcal{D}}$. If we set

$$\varphi(x,t) = e^{\lambda(|x-x_0|^2-\beta t^2)} \tag{1.7}$$

with sufficiently large $\lambda > 0$ and small $\beta > 0$, then (1.4) implies (1.3). In particular, Carleman estimate (1.5) holds.

Here and henceforth $(\zeta \cdot \zeta')$ denotes the scalar product in $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. For the proof, it suffices to verity that (1.7) satisfies (1.3) under assumption (1.6), and see

Imanuvilov and Yamamoto [10] for example. Condition (1.6) is restrictive, and we have to limit unknown coefficients to a class meeting (1.6) when we consider the inverse problem of determining a_0. We note that condition (1.6) is merely one sufficient condition for (1.5). In other words, even if a_0 does not satisfy (1.6), other choice of φ may be able to satisfy (1.3).

The main purpose of this paper is to propose more flexible choices of φ in Theorem A to relax constraint (1.6) for the principal term. Next we will apply such a Carleman estimate to the inverse problem of determining a principal term within a more general class.

As related papers, for a more general hyperbolic operator $\partial_t^2 - \sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_k)$, Lasiecka, Triggiani and Yao [21] and Yao [26] introduce the weight function of the form

$$\varphi(x,t) = d(x) - \beta t^2 \tag{1.8}$$

where d is strictly convex with respect to the Riemann metric derived by the elliptic part, and establish an inequality of Carleman's type. In our case of $a_{jk}(x) = \delta_{jk}a_0(x)^{-2}$ where $\delta_{jk} = 1$ if $j = k$ and $= 0$ if $j \ne k$, we can verify that d is strictly convex with respect to the Riemann metric if and only if the following $n \times n$ matrix $(m_{jk})_{1\le j,k\le n}$ is positive definite in the domain under consideration:

$$\begin{aligned} m_{jk} &= \partial_j\partial_k d - 2a_0^{-1}(\partial_j a_0)\partial_k d \\ &+ a_0^{-1}\sum_{\ell=1}^n (\partial_\ell d)\{\partial_j(\delta_{\ell k}a_0) + \partial_\ell(\delta_{jk}a_0) - \partial_k(\delta_{j\ell}a_0)\}. \end{aligned}$$

In [21], the second large parameter $\lambda > 0$ is not considered unlike (1.7) and such a parameter is generally useful for guaranteeing the relevant convexity (e.g., Isakov

[15]). In particular, we below require some convexity (1.9) of d with respect to the rays in choosing $e^{\lambda(d(x)-\beta t^2)}$, as a weight function, not $d(x)-\beta t^2$. In [21] and [26], the inequality of Carleman's type yields observability inequalites with a generous condition on the principal term, but their inequality includes some extra lower order term, so that it is not directly applicable to our inverse problem. As for weight functions with factor $d(x)-\beta t^2$, see further Isakov and Yamamoto [16], Lasiecka, Triggiani and Zhang [22].

Now we state our new realization for the pseudoconvexity. Let $Q \subset \mathbb{R}^{n+1}$ and $K \subset Q$ be an arbitrarily fixed bounded domain, and let us set $\Omega = \{x; (x,t) \in K \text{ for some } t \in \mathbb{R}\}$.

Theorem 1. *We assume that $a_0 > 0$ on $\overline{Q}$ and $a_0 \in C^2(\overline{\Omega})$. For some costant $\varepsilon_0 > 0$, we suppose that $d \in C^2(\overline{\Omega})$ satisfies*

$$\sum_{j,k=1}^{n} (\partial_j \partial_k d)\xi_j \xi_k + (\nabla' d \cdot \nabla' \log a_0) \geq \varepsilon_0 \tag{1.9}$$

for $x \in \overline{\Omega}$, $\xi_1, ..., \xi_n \in \mathbb{R}$ with $|\xi_1|^2 + \cdots |\xi_n|^2 = 1$, and

$$|\nabla' d| \neq 0 \qquad \text{on } \overline{\Omega}. \tag{1.10}$$

We set

$$\psi(x,t) = d(x) - \beta t^2, \qquad \varphi = e^{\lambda \psi(x,t)}. \tag{1.11}$$

Then there exist constants $\beta_0 > 0$ and $\lambda_0 > 0$ such that if $0 < \beta < \beta_0$ and $\lambda > \lambda_0$, then for φ, condition (1.4) implies (1.3). In particular, Carleman estimate (1.5) holds.

From Theorem 1, we directly derive

Corollary. *Let an $n \times n$ matrix $(\partial_j \partial_k a_0(x))_{1\le j,k\le n}$ be non-negative definite for $x \in \overline{\Omega}$ and let $|\nabla' a_0| \ne 0$ on $\overline{\Omega}$. be true. Then Carleman estimate (1.5) holds true with the weight function $\varphi = e^{\lambda(a_0(x)-\beta t^2)}$.*

The proof of Theorem 1 is given in Section 2, which is straightforward on the basis of Theorem A.

Our condition (1.9) for the Carleman estimate can be interpreted in terms of the ray, so that condition (1.9) is natural although technical apparently. For the interpretation of (1.9), we define the ray (e.g., Chapter 3 in Romanov [23]). Let us consider the three dimensional case and let $L(x, x^0)$ denote an arbitrary smooth curve connecting x, $x^0 \in \mathbb{R}^3$ and ds be an element of the arc length of $L(x, x^0)$. Then a ray $\Gamma(x, x^0)$ is defined as L attaining an extremal of the functional of L:

$$\int_{L(x,x^0)} a_0 dx.$$

Note that a_0^{-1} corresponds to the wave speed and that the ray is not necessarily determined uniquely for given x and x^0. Then (1.9) is interpreted that each surface $d(x) = C$ for any constant C is convex with respect to the set of rays, and, under some conditions on a smooth real-valued function d, we know the following fact:

Let us assume that (1.9) holds for any $\xi' = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ satisfying $|\xi'| = 1$ and $\xi' \cdot \nabla' d = 0$. Then any ray touching the surface $\{x; d(x) = C\}$, belongs to the domain $\{x; d(x) > C\}$ at any other point.

As for the details, see Chapter 3 in Romanov [23]. Intuitively we can understand that rays remaining on a surface prevent us from detecting interior information of solutions inside the domain, so that if such remaining rays exist, then the property of unique continuation may be very complicated. Since the Carleman estimate implies the unique continuation (e.g., Hörmander [5], Isakov [15]), the above fact suggests that our condition (1.9) is reasonable for proving the Carleman estimate. However we do not know whether (1.9) is a necessary condition for Carleman estimates.

We note that if in (1.9), we set $d(x) = |x - x^0|^2$, then (1.9) is rewritten as

$$(\nabla' \log a_0(x) \cdot (x - x^0)) > -1 + \frac{\varepsilon_0}{2} > -1$$

which implies (1.6). Therefore condition (1.6) is a special case of (1.9) with a fixed choice $d(x) = |x - x^0|^2$.

Remark. Our theorem really generalizes condition (1.6) in Proposition B. Let us set

$$\Omega = \left\{ x \in \mathbb{R}^n; \sqrt{\frac{9}{10}} < |x| < 1 \right\}, \qquad a_0(x) = 1 - \frac{2}{3}|x|^2, \quad x \in \overline{\Omega}.$$

Then (1.6) can not be satisfied for any $x^0 \in \mathbb{R}^n$. In fact, (1.6) is equivalent to

$$\frac{4|x|^2 - 4(x \cdot x^0)}{3 - 2|x|^2} < 1 \quad \text{if} \ \sqrt{\frac{9}{10}} \le |x| \le 1. \tag{1.12}$$

For any $x^0 \in \mathbb{R}^n$, we can choose $x^1 \in \Omega$ such that $(x^1 \cdot x^0) = 0$ and $|x^1| = \sqrt{\frac{10}{11}}$ for example, which breaks condition (1.12). However if we take $d(x) = -|x|^2$ for $x \in \overline{\Omega}$, then (1.9) holds trues: [the left hand side of (1.9)] $= -2 + \frac{8|x|^2}{3-2|x|^2} \ge 4$ if $\sqrt{\frac{9}{10}} \le |x| \le 1$.

§2. Proof of Theorem 1.

It suffices to verify the assumptions in Theorem A. We denote the left hand side of (1.3) by $H(x, x_{n+1}, \zeta)$. Henceforth we set $\zeta' = (\zeta_1, ..., \zeta_n)$ and $\xi' = (\xi_1, ..., \xi_n)$. Since $\frac{\partial P_m}{\partial \zeta_j} = 2\zeta_j$ for $1 \le j \le n$, $\frac{\partial P_m}{\partial \zeta_{n+1}} = -2a_0^2\zeta_{n+1}$, $\partial_k P_m = -2a_0(\partial_k a_0)\zeta_{n+1}^2$, $1 \le k \le n$ and $\partial_{n+1} P_m = 0$, dividing $\sum_{j,k=1}^{n+1}$ into $\sum_{j,k=1}^{n}$, $\sum_{j=1}^{n}$ [the terms with $k = n+1$],

$\sum_{k=1}^n$ [the terms with $j = n+1$] and the term with $j = k = n+1$, we can directly calculate to obtain

$$H(x, x_{n+1}, \zeta) = \sum_{j,k=1}^{n} 4\zeta_j\overline{\zeta_k}\partial_j\partial_k\varphi - 4a_0^2\sum_{k=1}^{n}(\zeta_{n+1}\overline{\zeta_k} + \overline{\zeta_{n+1}}\zeta_k)(\partial_{n+1}\partial_k\varphi)$$
$$+4a_0^4(\partial_{n+1}^2\varphi)|\zeta_{n+1}|^2 - \frac{4a_0}{s}\mathrm{Im}\left[\left(\sum_{k=1}^{n}(\partial_k a_0)\overline{\zeta_k}\right)\zeta_{n+1}^2\right]$$
$$\equiv H_1 + H_2 + H_3 + H_4. \tag{2.1}$$

On the other hand, by $\varphi = e^{\lambda\psi}$, $\partial_j\varphi = \lambda(\partial_j\psi)\varphi$, etc. and $\zeta_j = \xi_j + \sqrt{-1}s\partial_j\varphi$, we see that (1.4) holds if and only if

$$(\xi' \cdot \nabla'\psi) = -2\beta t a_0^2\xi_{n+1} \tag{2.2}$$

and

$$|\xi'|^2 - s^2\lambda^2|\nabla'\psi|^2\varphi^2 = a_0^2\xi_{n+1}^2 - 4s^2\lambda^2a_0^2\beta^2t^2\varphi^2. \tag{2.3}$$

Moreover we have

$$H_1(x, x_{n+1}, \zeta) = 4\lambda^2\varphi(\xi' \cdot \nabla'\psi)^2 + 4s^2\lambda^4\varphi^3|\nabla'\psi|^4$$
$$+4\lambda\varphi\sum_{j,k=1}^{n}(\partial_j\partial_k\psi)\xi_k\xi_j + 4s^2\lambda^3\varphi^3\sum_{j,k=1}^{n}(\partial_j\partial_k\psi)(\partial_j\psi)(\partial_k\psi),$$

$$H_2(x, x_{n+1}, \zeta) = -8a_0^2\lambda^2\varphi(\partial_{n+1}\psi)\xi_{n+1}(\nabla'\psi \cdot \xi') - 8a_0^2\lambda^4\varphi^3s^2(\partial_{n+1}\psi)^2|\nabla'\psi|^2,$$

$$H_3(x, x_{n+1}, \zeta) = 4a_0^4\lambda^2\varphi\xi_{n+1}^2\left\{(\partial_{n+1}\psi)^2 + \frac{1}{\lambda}\partial_{n+1}^2\psi\right\}$$
$$+4a_0^4s^2\lambda^4\varphi^3\left\{(\partial_{n+1}\psi)^4 + \frac{1}{\lambda}(\partial_{n+1}\psi)^2(\partial_{n+1}^2\psi)\right\}$$

and

$$H_4(x,x_{n+1},\zeta) = -\frac{4a_0}{s}\mathrm{Im}\left[\sum_{k=1}^{n}(\partial_k a_0)(\xi_k - \sqrt{-1}s\lambda(\partial_k\psi)\varphi)\right.$$
$$\left.\times\{\xi_{n+1}^2 - s^2\lambda^2(\partial_{n+1}\psi)^2\varphi^2 + 2\sqrt{-1}s\lambda\varphi(\partial_{n+1}\psi)\xi_{n+1}\}\right]$$
$$=4a_0\lambda\varphi(\nabla' a_0\cdot\nabla'\psi)\mathrm{Re}\,(\zeta_{n+1}^2) - 8a_0\lambda\varphi(\partial_{n+1}\psi)\sum_{k=1}^{n}(\partial_k a_0)(\mathrm{Re}\,\zeta_k)(\mathrm{Re}\,\zeta_{n+1})$$
$$=4a_0\lambda\varphi(\nabla' a_0\cdot\nabla'\psi)(\xi_{n+1}^2 - s^2\lambda^2(\partial_{n+1}\psi)^2\varphi^2) - 8a_0\lambda\varphi(\nabla' a_0\cdot\xi')(\partial_{n+1}\psi)\xi_{n+1}.$$

Therefore, by $\partial_{n+1}\psi = -2\beta t$ and (2.2), we have

$$H(x,x_{n+1},\zeta) = \left\{4\lambda\varphi\sum_{j,k=1}^{n}(\partial_j\partial_k\psi)\xi_j\xi_k + 16a_0^4\lambda^2\varphi\beta^2t^2\xi_{n+1}^2\right.$$
$$\left.+4s^2\lambda^4\varphi^3|\nabla'\psi|^4 + 4s^2\varphi^3\lambda^3\sum_{j,k=1}^{n}(\partial_j\partial_k\psi)(\partial_j\psi)(\partial_k\psi)\right\}$$
$$-\{32a_0^4\lambda^2\varphi\beta^2t^2\xi_{n+1}^2 + 32a_0^2\lambda^4\varphi^3s^2\beta^2t^2|\nabla'\psi|^2\}$$
$$+\left\{4a_0^4\lambda^2\varphi\left(4\beta^2t^2 - \frac{2\beta}{\lambda}\right)\xi_{n+1}^2 + 4a_0^4s^2\lambda^4\varphi^3\left(16\beta^4t^4 - \frac{8}{\lambda}\beta^3t^2\right)\right\}$$
$$+\{4a_0\lambda\varphi(\nabla' a_0\cdot\nabla'\psi)\xi_{n+1}^2 - 16a_0\lambda^3s^2\varphi^3\beta^2t^2(\nabla' a_0\cdot\nabla'\psi)$$
$$+16a_0\lambda\varphi\beta t(\nabla' a_0\cdot\xi')\xi_{n+1}\}. \tag{2.4}$$

Noting the forms of H_1, H_2, H_3, H_4, we see that H is homogeneous in ζ. Therefore we can assume that $|\zeta| = 1$, that is,

$$|\xi'|^2 + \xi_{n+1}^2 + s^2\lambda^2\varphi^2|\nabla\psi|^2 = 1. \tag{2.5}$$

In particular,

$$|\xi'|^2 + \xi_{n+1}^2 \le 1. \tag{2.6}$$

Moreover (2.2), (2.3) and (2.5) yield

$$
\begin{cases}
|\xi'|^2 = \dfrac{a_0^2}{1+a_0^2} - \dfrac{s^2\lambda^2\varphi^2}{1+a_0^2}(4a_0^2\beta^2t^2 + a_0^2|\nabla\psi|^2 - |\nabla'\psi|^2) \\
\equiv \dfrac{a_0^2}{1+a_0^2} + s^2\lambda^2\varphi^2 e_1(x,t), \\
|\xi_{n+1}|^2 = \dfrac{1}{1+a_0^2} + \dfrac{s^2\lambda^2\varphi^2}{1+a_0^2}(4a_0^2\beta^2t^2 - |\nabla\psi|^2 - |\nabla'\psi|^2) \\
\equiv \dfrac{1}{1+a_0^2} + s^2\lambda^2\varphi^2 e_2(x,t).
\end{cases}
\tag{2.7}
$$

Henceforth we set $\varepsilon = \inf_{x\in\Omega} 4|\nabla'\psi|^4$, $T = \sup_{(x,t)\in K}|t|$. Then, by (1.10), we see that $\varepsilon > 0$. Henceforth $C > 0$ denotes generic constants depending only on Q, K, $\|a_0\|_{C^1(\overline{\Omega})}$, ψ, but independent of s, λ. Let us assume

$$
0 < \beta < 1, \quad \lambda > 1, \quad \sqrt{\beta}T < 1, \quad \lambda\beta^2T^2 < 1. \tag{2.8}
$$

Applying (2.6) and (2.8) in (2.4) and noting that $|(\nabla'a_0\cdot\xi')\xi_{n+1}| \le \frac{|\nabla'a_0|}{2}(|\xi'|^2 + \xi_{n+1}^2)$, we have

$$
\begin{aligned}
H(x,x_{n+1},\zeta) \ge &\Bigg\{4\lambda\varphi\sum_{j,k=1}^n(\partial_j\partial_k\psi)\xi_j\xi_k + 4a_0\lambda\varphi(\nabla'a_0\cdot\nabla'\psi)\xi_{n+1}^2 \\
&-16a_0^4\lambda^2\varphi\beta^2t^2 - 4a_0^4\lambda^2\varphi|4\beta^2t^2 - 2\lambda^{-1}\beta| - 8a_0\lambda\varphi\beta|t||\nabla'a_0|\Bigg\} \\
&+\Bigg\{4s^2\lambda^4\varphi^3|\nabla'\psi|^4 - 4s^2\varphi^3\lambda^3\left|\sum_{j,k=1}^n(\partial_j\partial_k\psi)(\partial_j\psi)\partial_k\psi\right| \\
&-32a_0^2\lambda^4\varphi^3s^2\beta^2t^2|\nabla'\psi|^2 - 4a_0^4s^2\lambda^4\varphi^3(16\beta^4t^2 + 8\lambda^{-1}\beta^3t^2) \\
&-16a_0\lambda^3s^2\varphi^3\beta^2t^2|\nabla'a_0||\nabla'\psi|\Bigg\} \\
\ge &4\lambda\varphi\left\{\sum_{j,k=1}^n(\partial_j\partial_k\psi)\xi_j\xi_k + a_0(\nabla'a_0\cdot\nabla'\psi)\xi_{n+1}^2 - C\lambda\beta - C\sqrt{\beta}\right\} \\
&+\{s^2\lambda^4\varphi^3\varepsilon - Cs^2\lambda^3\varphi^3 - Cs^2\lambda^4\varphi^3\beta^2T^2\}.
\end{aligned}
\tag{2.9}
$$

Moreover substitution of (2.7) into (2.9), yields

$$
\begin{aligned}
&H(x,x_{n+1},\zeta)\\
\geq&\frac{4\lambda\varphi a_0^2}{1+a_0^2}\left[(\nabla'\log a_0\cdot\nabla'\psi)+\sum_{j,k=1}^{n}(\partial_j\partial_k\psi)\xi_j\xi_k\frac{1+a_0^2}{a_0^2}-C\lambda\beta-C\sqrt{\beta}\right]\\
&+4s^2\lambda^3\varphi^3a_0e_2(\nabla'a_0\cdot\nabla'\psi)+s^2\lambda^4\varphi^3(\varepsilon-C\lambda^{-1}-C\beta).
\end{aligned}
\tag{2.10}
$$

First let us assume $|\xi'|=0$. Then, by (2.2), we obtain

$$
t=0\quad\text{or}\quad\xi_{n+1}=0. \tag{2.11}
$$

Let $t=0$. Then (2.3) yields $-s^2\lambda^2|\nabla'\psi|\varphi^2=a_0^2\xi_{n+1}^2$. Since $|\nabla'\psi|\neq 0$, we obtain $s=0$ and $\xi_{n+1}=0$. This contradicts (2.5). Therefore we have $\xi_{n+1}=0$ and $t\neq 0$. Then (2.5) implies $s^2\lambda^2\varphi^2|\nabla\psi|^2=1$, and so

$$
s^2\lambda^2\varphi^2=\frac{1}{|\nabla\psi|^2} \tag{2.12}
$$

by $|\nabla\psi|\neq 0$ on $\overline{Q}$. Consequently, substituting $|\xi'|=\xi_{n+1}=0$ and (2.12) into (2.9), we obtain

$$
\begin{aligned}
&H(x,x_{n+1},\zeta)\geq\varepsilon\lambda^2\varphi\frac{1}{|\nabla\psi|^2}-C\lambda\varphi\frac{1}{|\nabla\psi|^2}-C\lambda^2\varphi\beta\frac{1}{|\nabla\psi|^2}\\
&-C\lambda^2\beta\varphi-C\sqrt{\beta}\lambda\varphi\\
&\geq C\lambda^2\varphi(1-C\lambda^{-1}-C\beta-C\sqrt{\beta}\lambda^{-1}).
\end{aligned}
$$

We choose sufficiently large $\lambda>0$ and sufficiently small $\beta\in(0,1)$ such that

$$
1-C\lambda^{-1}-C\beta-C\sqrt{\beta}\lambda^{-1}>0. \tag{2.13}
$$

After choosing (2.13), we see that $H(x,x_{n+1},\zeta)>0$ if $|\xi'|=0$.

Second let us assume $|\xi'| \ne 0$. Then (2.7) and (2.10) yield

$$H(x, x_{n+1}, \zeta) \ge \frac{4\lambda\varphi a_0^2}{1+a_0^2}\Bigg[(\nabla' \log a_0 \cdot \nabla'\psi) + \sum_{j,k=1}^n (\partial_j\partial_k\psi)\frac{\xi_j}{|\xi'|}\frac{\xi_k}{|\xi'|}$$
$$+ \sum_{j,k=1}^n (\partial_j\partial_k\psi)\frac{\xi_j}{|\xi'|}\frac{\xi_k}{|\xi'|} s^2\lambda^2\varphi^2 \frac{e_1(1+a_0^2)}{a_0^2} - C\lambda\beta - C\sqrt{\beta}\Bigg]$$
$$-Cs^2\lambda^3\varphi^3 + s^2\lambda^4\varphi^3(\varepsilon - C\lambda^{-1} - C\beta).$$

Consequently (1.9) implies

$$H(x, x_{n+1}, \zeta) \ge \frac{4\lambda\varphi a_0^2}{1+a_0^2}(\varepsilon_0 - C\lambda\beta - C\sqrt{\beta}) + s^2\lambda^4\varphi^3(\varepsilon - C\lambda^{-1} - C\beta).$$

Here we have used

$$\left|\sum_{j,k=1}^n (\partial_j\partial_k\psi)\frac{\xi_j}{|\xi'|}\frac{\xi_k}{|\xi'|} s^2\lambda^2\varphi^2 \frac{e_1(1+a_0^2)}{a_0^2}\right| \le Cs^2\lambda^2\varphi^2 \sum_{j,k=1}^n |\partial_j\partial_k\psi|.$$

We further take sufficiently large $\lambda > 0$ and sufficiently small $\beta > 0$ such that

$$\varepsilon_0 - C\lambda\beta - C\sqrt{\beta} > 0, \quad \varepsilon - C\lambda^{-1} - C\beta > 0. \tag{2.14}$$

Hence if $\lambda > 0$ and $\beta > 0$ satisfy (2.8), (2.13) and (2.14), then we see that $H(x, x_{n+1}, \zeta) > 0$ if $(x, x_{n+1}) \in \overline{Q}$ and (x, x_{n+1}, ζ) satisfies (1.4). Thus by Theorem A, the proof of Theorem 1 is complete.

§3. Application to an inverse problem of determining principal terms.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2- boundary $\partial\Omega$ and let us consider

$$(P_k u)(x,t) = (a_k(x)^2\partial_t^2 - \Delta)u(x,t), \quad k = 0, 1,\ x \in \Omega,\ t \in \mathbb{R}, \tag{3.1}$$

where $a_k > 0$ on $\overline{\Omega}$ and $a_k \in C^2(\overline{\Omega})$. We discuss

Uniqueness in Inverse Problem. Let $S \subset \partial\Omega$ be given and let u_k satisfy $P_k u_k = 0$ in $\Omega \times (-T, T)$, $k = 0, 1$. Then can we conclude that $a_0 = a_1$ in some subdomain Ω_0 by

$$\begin{cases} u_0(x,0) = u_1(x,0), \quad \partial_t u_0(x,0) = \partial_t u_1(x,0), \qquad x \in \Omega_0, \\ u_0 = u_1, \quad \dfrac{\partial u_0}{\partial \nu} = \dfrac{\partial u_1}{\partial \nu} \quad \text{on } S \times (-T,T)? \end{cases} \tag{3.2}$$

Here and henceforth, $\nu = \nu(x)$ is the unit outward normal vector to $\partial\Omega$ at x and $\frac{\partial}{\partial\nu}$ denotes the normal derivative: $\frac{\partial u}{\partial \nu} = \nabla' u \cdot \nu$.

In this kind of inverse problems, unknown coefficients appear in principal terms and for the Carleman estimate which is the key, we have to assume conditions of type (1.6) in Imanuvilov, Isakov and Yamamoto [7], Imanuvilov and Yamamoto [10], [11], Isakov [13]. Condition (1.6) definitely restricts an admissible set of unknown coefficients and the relaxation of the condition for the Carleman estimate is very desirable. As generalization of the condition, see also Bellassoued [1].

In this section, for simplicity, we mainly discuss the uniqueness in determining $a_1(x)$ around a given $a_0(x)$. For known a_0, we assume that there exists $d \in C^2(\overline{\Omega})$ satisfying (1.9) and (1.10) for $x \in \overline{\Omega}$, $\xi' = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ with $|\xi'| = 1$. We set $\psi(x,t) = d(x) - \beta t^2$ and $\varphi(x,t) = e^{\lambda\psi(x,t)}$, where $\beta > 0$ and $\lambda > 0$ are defined in Theorem 1.

We set

$$\begin{aligned} &Q(\delta) = \{(x,t) \in \Omega \times (-T,T); \varphi(x,t) > \delta\}, \\ &\Omega(\delta) = \{x \in \Omega; \varphi(x,0) > \delta\}. \end{aligned} \tag{3.3}$$

Now we are ready to state the main result on the uniqueness.

Theorem 2. *Let $S \subset \partial\Omega$ satisfy*

$$\overline{Q(0)} \subset (\Omega \cup S) \times (-T, T), \tag{3.4}$$

and let $u_k \in C^2(\overline{\Omega} \times [-T, T])$, $k = 0, 1$, satisfy $\partial_t u_k \in C^2(\overline{\Omega} \times [-T, T])$,

$$P_k u_k = 0 \qquad \text{in } \Omega \times (-T, T), \tag{3.5}$$

$$u_0(x, 0) = u_1(x, 0), \quad \partial_t u_0(x, 0) = \partial_t u_1(x, 0), \quad x \in \Omega(0), \tag{3.6}$$

and

$$u_0 = u_1, \quad \frac{\partial u_0}{\partial \nu} = \frac{\partial u_1}{\partial \nu} \quad \text{on } S \times (-T, T). \tag{3.7}$$

Moreover let

$$\Delta u_0(x, 0) > 0, \qquad x \in \overline{\Omega(0)}. \tag{3.8}$$

Then

$$a_0(x) = a_1(x), \qquad x \in \overline{\Omega(0)}. \tag{3.9}$$

This theorem asserts the uniqueness which holds in some subdomain under condition (3.8) assumed in the same subdomain. If we take $\partial\Omega$ as S and assume $\Delta u_0(x, 0) > 0$ for $x \in \overline{\Omega}$, then we will be able to conclude the global uniqueness: $a_0(x) = a_1(x)$ for $x \in \overline{\Omega}$ under an extra condition $\overline{\Omega(0)} = \overline{\Omega}$. Moreover within a suitable admissible set of a_k's, we can prove the conditional stability which estimates $a_0 - a_1$ by means of $u_0 - u_1$ and $\frac{\partial u_0}{\partial \nu} - \frac{\partial u_1}{\partial \nu}$ on $S \times (-T, T)$.

Proof. Now that we have established a Carleman estimate in Theorem 1, the proof is done along the line of Imanuvilov and Yamamoto [8]. The difference $y = u_1 - u_0$ satisfies

$$P_0 y = R(x,t)f(x) \qquad \text{in } \Omega \times (-T,T), \tag{3.10}$$

$$y(x,0) = \partial_t y(x,0) = 0, \qquad x \in \Omega(0), \tag{3.11}$$

and

$$y = \frac{\partial y}{\partial \nu} = 0 \qquad \text{on } S \times (-T,T). \tag{3.12}$$

Here we set

$$f(x) = a_0^2(x) - a_1^2(x), \quad R(x,t) = \partial_t^2 u_1(x,t) = \frac{1}{a_1^2(x)} \Delta u_1(x,t), \tag{3.13}$$

for $x \in \Omega$ and $t \in (-T,T)$. We arbitrarily fix a sufficiently small $\delta > 0$.

For application of the Carleman estimate, we have to introduce a cutoff function $\chi \in C_0^\infty(Q(0))$ such that $0 \le \chi \le 1$ and

$$\chi(x,t) = \begin{cases} 1, & (x,t) \in Q(3\delta), \\ 0, & (x,t) \in Q(\delta) \setminus \overline{Q(2\delta)}. \end{cases} \tag{3.14}$$

We set

$$z = (\partial_t y) e^{s\varphi} \chi \in C^2(\overline{\Omega} \times [-T,T]). \tag{3.15}$$

Then, by (3.10), we have

$$\begin{aligned} P_0 z = {} & f(\partial_t R) e^{s\varphi}\chi + s(-2(\nabla'\varphi \cdot \nabla' z) + 2a_0^2(\partial_t\varphi)\partial_t z + (P_0\varphi)z) \\ & - s^2(a_0^2|\partial_t\varphi|^2 - |\nabla'\varphi|^2)z \\ & + 2e^{s\varphi}(a_0^2(\partial_t^2 y)\partial_t\chi - (\nabla'(\partial_t y)\cdot\nabla'\chi)) + (\partial_t y)e^{s\varphi}P_0\chi \quad \text{in } Q(0). \end{aligned} \tag{3.16}$$

In fact,

$$\partial_j z = (\partial_j\partial_t y)e^{s\varphi}\chi + s(\partial_j\varphi)z + (\partial_t y)e^{s\varphi}\partial_j\chi,$$

and

$$(\partial_j\partial_t y)e^{s\varphi}\chi = \partial_j z - s(\partial_j\varphi)z - (\partial_t y)e^{s\varphi}\partial_j\chi. \tag{3.17}$$

Hence, by (3.17), we see

$$\partial_j^2 z = (\partial_j^2\partial_t y)e^{s\varphi}\chi + (\partial_j\partial_t y)s(\partial_j\varphi)e^{s\varphi}\chi + 2(\partial_j\partial_t y)e^{s\varphi}(\partial_j\chi) + s(\partial_j^2\varphi)z + s(\partial_j\varphi)\partial_j z$$

$$+(\partial_t y)s(\partial_j\varphi)e^{s\varphi}\partial_j\chi + (\partial_t y)e^{s\varphi}\partial_j^2\chi$$

$$=(\partial_j^2\partial_t y)e^{s\varphi}\chi + s(\partial_j\varphi)\{\partial_j z - s(\partial_j\varphi)z - (\partial_t y)e^{s\varphi}\partial_j\chi\}$$

$$+s(\partial_j^2\varphi)z + s(\partial_j\varphi)\partial_j z + 2(\partial_j\partial_t y)e^{s\varphi}(\partial_j\chi) + (\partial_t y)s(\partial_j\varphi)e^{s\varphi}\partial_j\chi + (\partial_t y)e^{s\varphi}\partial_j^2\chi$$

$$=(\partial_j^2\partial_t y)e^{s\varphi}\chi + 2s(\partial_j\varphi)\partial_j z + s(\partial_j^2\varphi)z - s^2(\partial_j\varphi)^2 z$$

$$+2(\partial_j\partial_t y)e^{s\varphi}\partial_j\chi + (\partial_t y)e^{s\varphi}\partial_j^2\chi.$$

Substitution into $(a_0^2\partial_{n+1}^2 - \Delta)z$ yields (3.16).

Moreover, setting $w = (\partial_t y)\chi$, we obtain

$$P_0 w = f(\partial_t R)\chi + 2a_0^2(\partial_t^2 y)\partial_t\chi - 2(\nabla'(\partial_t y)\cdot\nabla'\chi)$$

$$+(\partial_t y)P_0\chi \quad \text{in } Q(0). \tag{3.18}$$

From (3.3) and (3.4), it follows that $Q(0)$ is a bounded domain and $\partial Q(\delta) \subset (S\times(-T,T))\cup Q(0)$. Therefore $w \in H_0^2(Q(\delta))$ by (3.12) and (3.14), so that we can apply Theorem 1 to w in $Q(\delta)$:

$$\int_{Q(\delta)}(s^3w^2 + s|\nabla w|^2)e^{2s\varphi}dxdt \le C\int_{Q(\delta)} f^2|\partial_t R|^2\chi^2 e^{2s\varphi}dxdt$$

$$+C\int_{Q(\delta)} |2a_0^2(\partial_t^2 y)\partial_t\chi - 2(\nabla'(\partial_t y)\cdot\nabla'\chi) + (\partial_t y)P_0\chi|^2 e^{2s\varphi}dxdt$$

$$\le C\int_{Q(\delta)} f^2\chi^2 e^{2s\varphi}dxdt + Ce^{6s\delta} \tag{3.19}$$

for all sufficiently large $s > 0$. At the last inequality, we have used

$$\partial_t \chi = |\nabla' \chi| = P_0 \chi = 0 \quad \text{in } \overline{Q(3\delta)} \cup \overline{(Q(\delta) \setminus Q(2\delta))}, \tag{3.20}$$

which follows from (3.14), and $e^{2s\varphi} \le e^{6s\delta}$ in $Q(2\delta) \setminus Q(3\delta)$. Noting that $z = we^{s\varphi}$, we can rewrite (3.19) in terms of z:

$$\int_{Q(\delta)} (s^3|z|^2 + s|\nabla z|^2) dxdt \le C \int_{Q(\delta)} f^2 \chi^2 e^{2s\varphi} dxdt + Ce^{6s\delta} \tag{3.21}$$

for sufficiently large $s > 0$.

We set $Q_- = \{(x,t) \in Q(\delta); t < 0\}$. We multiply (3.16) by $\partial_t z$ and integrate over Q_-:

$$\begin{aligned}
I_1 \equiv \int_{Q_-} (P_0 z)\partial_t z dxdt = \int_{Q_-} f(\partial_t R)e^{s\varphi}\chi \partial_t z dxdt \\
+ \int_{Q_-} s(-2(\nabla'\varphi \cdot \nabla' z) + 2a_0^2(\partial_t\varphi)\partial_t z + (P_0\varphi)z)\partial_t z dxdt \\
-s^2 \int_{Q_-} (a_0^2|\partial_t\varphi|^2 - |\nabla'\varphi|^2) z\partial_t z dxdt \\
+ \int_{Q_-} \{2a_0^2(\partial_t^2 y)\partial_t\chi - 2(\nabla'(\partial_t y) \cdot \nabla'\chi) + (\partial_t y)P_0\chi\} e^{s\varphi}\partial_t z dxdt \equiv I_2.
\end{aligned} \tag{3.22}$$

By (3.12) and (3.14), we integrate I_1 by parts:

$$\begin{aligned}
I_1 = \int_{Q_-} (a_0^2(\partial_t^2 z)\partial_t z - (\Delta z)\partial_t z) dxdt = \int_{Q_-} \left\{ \frac{1}{2}\partial_t(|\partial_t z|^2 a_0^2) + \frac{1}{2}\partial_t(|\nabla' z|^2) \right\} dxdt \\
= \int_{Q(\delta)\cap\{t=0\}} \frac{1}{2}(|\partial_t z|^2 a_0^2 + |\nabla' z|^2)\nu_{n+1} dx
\end{aligned}$$

where ν_{n+1} is the $(n+1)$-component of the unit outward normal vector to ∂Q_-.

Hence (3.8), (3.10), (3.11) and (3.15) imply

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{\Omega(\delta)} |(\partial_t z)(x,0)|^2 a_0^2(x)dx \\ &= \frac{1}{2}\int_{\Omega(\delta)} a_0^2(x)|(\partial_t^2 y)(x,0)|^2 \chi^2(x,0)e^{2s\varphi(x,0)}dx \\ &= \frac{1}{2}\int_{\Omega(\delta)} f^2(x)\frac{|\Delta u_1(x,0)|^2}{a_0^2(x)a_1^4(x)}\chi^2(x,0)e^{2s\varphi(x,0)}dx \\ &\ge C_1\int_{\Omega(\delta)} f^2(x)\chi^2(x,0)e^{2s\varphi(x,0)}dx. \end{aligned} \tag{3.23}$$

For I_2, we use Schwarz's inequality and (3.20), (3.21) to obtain

$$I_2 \le C\int_{Q(\delta)} f^2\chi^2 e^{2s\varphi}dxdt + Ce^{6s\delta} \tag{3.24}$$

for all large $s>0$. Consequently (3.23) and (3.24) yield

$$\int_{\Omega(\delta)} f^2(x)\chi^2(x,0)e^{2s\varphi(x,0)}dx \le C\int_{Q(\delta)} f^2\chi^2 e^{2s\varphi}dxdt + Ce^{6s\delta}$$

for all large $s>0$. Noting (3.14) and $e^{2s\varphi}\le e^{6s\delta}$ in $\overline{Q(\delta)\setminus Q(3\delta)}$, we obtain

$$\begin{aligned} &\int_{\Omega(3\delta)} f^2(x)\chi^2(x,0)e^{2s\varphi(x,0)}dx \le C\left(\int_{Q(\delta)\setminus Q(3\delta)} + \int_{Q(3\delta)}\right) f^2\chi^2 e^{2s\varphi}dxdt \\ &+Ce^{6s\delta}, \end{aligned}$$

so that

$$\int_{\Omega(3\delta)} f^2(x)e^{2s\varphi(x,0)}dx \le C\int_{Q(3\delta)} f^2 e^{2s\varphi}dxdt + Ce^{6s\delta} \tag{3.25}$$

for all large $s>0$.

On the other hand, by (3.3) and $\varphi(x,0)\ge\varphi(x,t)$, we see that $Q(3\delta)\subset\Omega(3\delta)\times(-T,T)$. Hence,

$$\begin{aligned} \int_{Q(3\delta)} f^2 e^{2s\varphi}dxdt &\le \int_{\Omega(3\delta)}\left(\int_{-T}^{T} e^{2s\varphi(x,t)}dt\right) f^2(x)dx \\ &= \int_{\Omega(3\delta)} f^2(x)e^{2s\varphi(x,0)}\left(\int_{-T}^{T} e^{2s(\varphi(x,t)-\varphi(x,0))}dt\right)dx. \end{aligned}$$

Recalling form (1.11) of φ and applying the Lebesgue theorem, we have

$$\sup_{x\in\Omega}\left|\int_{-T}^{T} e^{2s(\varphi(x,t)-\varphi(x,0))}dt\right| = \sup_{x\in\Omega}\left|\int_{-T}^{T}\exp(2se^{\lambda d(x)}(e^{-\lambda\beta t^2}-1))dt\right|$$
$$\le \int_{-T}^{T}\exp(2se^{\lambda d_0}(e^{-\lambda\beta t^2}-1))dt = o(1),$$

where $d_0 = \inf_{x\in\Omega} d(x)$, as $s \longrightarrow \infty$. Therefore

$$\int_{Q(3\delta)} f^2 e^{2s\varphi}dxdt = o(1)\int_{\Omega(3\delta)} f^2(x)e^{2s\varphi(x,0)}dx,$$

with which inequality (3.25) yields

$$(1-o(1))\int_{\Omega(3\delta)} f^2(x)e^{2s\varphi(x,0)}dx \le Ce^{6s\delta}$$

as $s \longrightarrow \infty$. Hence

$$(1-o(1))e^{8s\delta}\int_{\Omega(4\delta)} f^2(x)dx \le Ce^{6s\delta},$$

so that

$$\int_{\Omega(4\delta)} f^2(x)dx \le Ce^{-2s\delta}$$

as $s \longrightarrow \infty$. Consequently, by letting $s \longrightarrow \infty$, we see that $f(x) = 0$ in $\Omega(4\delta)$. Since $\delta > 0$ is arbitrary, we have $f(x) = a_0^2(x) - a_1^2(x) = 0$, $x \in \Omega(0)$. Thus the proof of Theorem 2 is complete.

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