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Abstract

We study the scattering theory for the Zakharov equation in three space dimensions. We show the unique existence of the solution for this equation which tends to the given free profile with no restriction on the size of the scattered states and on the support of the Fourier transform of them. This yields the existence of the pseudo wave operators.

1 Introduction

We study the scattering theory for the Zakharov equation in three space dimensions:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = uv, \\ \partial_t^2 v - \Delta v = \Delta |u|^2. \end{cases} \tag{Z}$$

Here u and v are $\mathbb{C}^n$-valued and real valued unknown functions of $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, respectively. In the present paper, we prove the unique existence of the solution for the equation (Z) which tends to the given free profile with no restriction on the size of the scattered states and on the support of the Fourier transform of them.

A large amount of works has been devoted to the asymptotic behavior of solutions for the nonlinear Schrödinger equation (see [3, 4, 6, 7, 8, 12, 17, 20, 21, 22, 23, 24, 32, 34, 35, 37]) and for the nonlinear wave equation (see [11, 14, 16, 18, 19, 27, 28, 32, 33]). We consider the scattering theory for the coupled systems of the Schrödinger equation and the second order hyperbolic equation, in particular, the Klein-Gordon-Schrödinger, Wave-Schrödinger, Maxwell-Schrödinger and Zakharov equations. In the scattering theory for the linear Schrödinger equation, (ordinary) wave operators are defined as follows. Assume that for a solution of the free Schrödinger equation with given initial data ϕ, there exists a unique time global solution u for the perturbed Schrödinger equation such that u behaves like the given free solution as $t \to \infty$. (This case is called the short range case, and otherwise we call the long range case). Then we define a wave operator W_+ by the mapping from ϕ to $u|_{t=0}$. In the long range case, ordinary wave operators do not exist and we have to construct modified wave operators including a suitable phase correction in their definition. For the nonlinear Schrödinger equation, the nonlinear wave equation and systems centering on the Schrödinger equation, we can define the wave operators and introduce the modified wave operators in the same way (for the nonlinear Schrödinger and wave equation, see the references mentioned above, and for systems, see [9, 25, 29, 36]).

There exist some results of the scattering theory for nonlinear equations and systems. Ozawa [23] and Ginibre and Ozawa [6] proved the existence of modified wave operators in the borderline case for the nonlinear Schrödinger equation in one space dimension and in two and three space dimensions, respectively. Those results have been extended to the Klein-Gordon-Schrödinger equation in two space dimensions by Ozawa and Tsutsumi [25] and the author [29], to the Wave-Schrödinger equation in three space dimensions by Ginibre and Velo [9] and the author [30], to the Maxwell-Schrödinger equation in three space dimensions by Ginibre and Velo [10], Tsutsumi [36] and the author [31] and to the Zakharov equation in three space dimensions by Ozawa and Tsutsumi [26].

The quadratic nonlinearities in the equation (Z) cause the difficulty of constructing global solution for (Z) and investigating asymptotic behavior of it. Klainerman [15] introduced the null condition technique to construct the global existence of small amplitude solution for the wave equation with quadratic nonlinearity in three space dimensions. We note that the null condition technique is mainly besed on the Lorentz invariance of the equations. However, since the Schrödinger equation does not have that invariance, we do not apply the null condition technique to the equation (Z). In this sense, the Schödinger equation and the wave equation are not compatible.

To overcome this difficulty, in the result for the equation (Z) by Ozawa and Tsutsumi [26], they assumed either the restriction on the size of the scattered states or that on the support of the Fourier transform of the scattered state ϕ of the Schrödinger part. More precisely, the restriction on the support of the Fourier transform of ϕ is as follows: $\operatorname{supp}\hat{\phi} \subset \{\xi \in \mathbb{R}^3 \colon |\xi| \geq 1+\varepsilon\} \cup \{\xi \in \mathbb{R}^3 \colon |\xi| \leq 1-\varepsilon\}$ for some $\varepsilon > 0$. Roughly speaking, the reason why they assumed this condition is as follows. Let u_0 and v_0 be the solutions for the free Schrödinger and wave equations, respectively. It is well-known that $\|u_0(t)v_0(t)\|_{L^2(\mathbb{R}^3)} = O(t^{-3/2})$ if no restriction on the support of the Fourier transform of data is supposed. When the smallness of the scattered states is not assumed, we have to introduce the function space such that $\|u(t)-u_0(t)\|_{L^2}$ decay faster than $\|\nabla(u(t)-u_0(t))\|_{L^2}$ in order to apply the Cook-Kuroda method. Hence we need good time decay rate of $\|u(t) - u_0(t)\|_{L^2}$. Therefore above time decay estimate is not sufficient to prove the existence of the solution of (Z) which tends to the free profile, and the improved time decay estimate of the interaction term is needed. Their proof is based on the improved decay estimates of the interaction term which take account of the difference between the propagation property of the solution to the Schrödinger and wave equation. The property of finite propagation speed and the Huygens principle for the three dimensional wave equation imply the following time decay estimate $\|v_0(t)\|_{L^\infty(|x|\geq(1+\varepsilon)t)} + \|v_0(t)\|_{L^\infty(|x|\leq(1-\varepsilon)t)} = O_{\varepsilon,N}(t^{-N})$, for any $\varepsilon, N > 0$. This yields an improved time decay estimate of the L^2-norm of the cross term u_0v_0, where u_0 is the solution of the free Schrödinger equation, $\|u_0(t)v_0(t)\|_{L^2(\mathbb{R}^3)} \sim t^{-3/2}\|\hat{\phi}(\cdot/t)v_0(t)\|_{L^2(\mathbb{R}^3)} = t^{-3/2}(\|\hat{\phi}(\cdot/t)v_0(t)\|_{L^2(|x|\geq(1+\varepsilon)t)} + \|\hat{\phi}(\cdot/t)v_0(t)\|_{L^2(|x|\leq(1-\varepsilon)t)}) = O_{\varepsilon,N}(t^{-N})$ as $t \to \infty$ for any $N > 0$. On the other hand, under the restriction on the size of the scattered states, they could obtain the same conclusion, because the second equation (the wave part) of the system (Z) had the second derivative at the interaction term, which implied the improved time decay rate of that term.

Recently, in [29] and [30], the author has proved the existence of wave operators for the two dimensional Klein-Gordon-Schrödinger equation with the Yukawa type interaction and of the modified wave operators for the three dimensional Wave-Schrödinger equation with same interaction, respectively, for small scattered states without any restrictions on the support of the Fourier transform of them. (Since these equations do not have second derivatives at the interaction terms as the Zakharov equation (Z), the scattering problems of them are more difficult than that of the Zakharov equation). The proof for the Klein-Gordon-Schrödinger equation is mainly based on the construction of suitable second approximations $[u_2, v_2]$ of the solution to the equation (Z) so that $(i\partial_t + \frac{1}{2}\Delta)u_2 - u_0v_0$ and $(\partial_t^2 - \Delta + 1)v_2 + |u_0|^2$ decay faster than u_0v_0 and $-|u_0|^2$ as $t \to \infty$, respectively. This enables us to apply the Cook-Kuroda method. Here u_0 and v_0 are the solutions of the free Schrödinger and Klein-Gordon equations, respectively.

In this paper, we prove the unique existence of the solution for the equation (Z) which tends to the given free profile with no restriction on the size of the scattered states and on the support of the Fourier transform of them. Our main idea of proof is as follows. Let u_0 and v_0 be the solutions of the free Schrödinger and wave equations, respectively. The principal term of our asymptotic profile is the free profile $[u_0, v_0]$. In order to improve time decay estimate of the interaction term of the Schrödinger part under no restriction on the size of the scattered states and on the support of the Fourier transform of them, we construct a suitable second correction term u_2 of the asymptotic profile for the Schrödinger part such that $(i\partial_t + \frac{1}{2}\Delta)u_2 - u_0 v_0$ decays faster than $u_0 v_0$, as in [29, 30, 31] (Section 2.2). Since the time decay rate of the interaction term for the wave part of the equation (Z) is sufficient for our problem, the second correction term of the asymptotic profile of the wave part, which appears in [29, 30, 31], is not needed. Our proof for the existence argument is based on the energy estimates and the compactness argument.

Before stating our main result, we introduce some notations.

Notations. We use the following symbols:

$$\partial_0 = \partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j} \quad \text{for } j = 1, 2, 3,$$
$$\partial^\alpha = \partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} \quad \text{for a multi-index } \alpha = (\alpha_1, \alpha_2, \alpha_3),$$
$$\nabla = (\partial_1, \partial_2, \partial_3), \quad \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2,$$

for $t \in \mathbb{R}$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Let

$$L^q \equiv L^q(\mathbb{R}^3) = \left\{ \psi\colon \|\psi\|_{L^q} = \left(\int_{\mathbb{R}^3} |\psi(x)|^q \, dx \right)^{1/q} < \infty \right\} \text{ for } 1 \le q < \infty,$$
$$L^\infty \equiv L^\infty(\mathbb{R}^3) = \left\{ \psi\colon \|\psi\|_{L^\infty} = \operatorname{ess.sup}_{x\in\mathbb{R}^3} |\psi(x)| < \infty \right\}.$$

We denote the L^2-scalar product by

$$(\varphi, \psi) \equiv \int_{\mathbb{R}^3} \varphi(x)\overline{\psi(x)} \, dx.$$

We denote the set of rapidly decreasing functions on $\mathbb{R}^3$ by $\mathcal{S}$. Let $\mathcal{S}'$ be the set of tempered distributions on $\mathbb{R}^3$. For $w \in \mathcal{S}'$, we denote the Fourier transform of w by $\hat{w}$. For $w \in L^1(\mathbb{R}^n)$, $\hat{w}$ is represented as

$$\hat{w}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x) e^{-ix\cdot\xi} \, dx.$$

For $s, m \in \mathbb{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space L^2 as follows:

$$H^{s,m} \equiv \{\psi \in \mathcal{S}' \colon \|\psi\|_{H^{s,m}} \equiv \|(1+|x|^2)^{m/2}(1-\Delta)^{s/2}\psi\|_{L^2} < \infty\}.$$

We also denote $H^{s,0}$ by H^s. For $1 \le p \le \infty$ and a positive integer k, we define the Sobolev space W_p^k corresponding to the Lebesgue space L^p by

$$W_p^k \equiv \left\{ \psi \in L^p \colon \|\psi\|_{W_p^k} \equiv \sum_{|\alpha| \le k} \|\partial^\alpha \psi\|_{L^p} < \infty \right\}.$$

Note that for a positive integer k, $H^k = W_2^k$ and the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W_2^k}$ are equivalent.

For $s > 0$, we define the homogeneous Sobolev spaces $\dot{H}^s$ by the completion of $\mathcal{S}$ with respect to the norm

$$\|w\|_{\dot{H}^s} \equiv \|(-\Delta)^{s/2} w\|_{L^2}. \tag{1.1}$$

If $s < 0$, we set

$$\dot{H}^s \equiv \{w \in \mathcal{S}' \colon (-\Delta)^{s/2} w \in L^2\}.$$

Then $\dot{H}^s$ is a Banach space with the norm (1.1) for $s > 0$. On the other hand, $\dot{H}^s$ is a semi-normed space with the semi-norm (1.1) for $s < 0$.

Let Y and Z be two Banach spaces with the norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. We denote

$$\|w\|_{Y \cap Z} \equiv \|w\|_Y + \|w\|_Z,$$

for $w \in Y \cap Z$. Then $Y \cap Z$ is a Banach space with the norm $\|\cdot\|_{Y \cap Z}$. We use the following notation:

$$[z; Y, k](t) \equiv \sup_{\tau \ge t} (\tau^k \|z(\tau)\|_Y),$$

for a Y-valued function z of $t \in \mathbb{R}$.

We set for $t \in \mathbb{R}$,

$$U(t) \equiv e^{\frac{it}{2}\Delta}, \quad \omega \equiv (-\Delta)^{1/2},$$

$$\mathcal{L} \equiv i\partial_t + \frac{1}{2}\Delta, \quad \Box \equiv \partial_t^2 - \Delta.$$

We denote various constants by C and so forth. They may differ from line to line, when it does not cause any confusion.

Let (ϕ, ψ_0, ψ_1) be given scattered states, where $\phi = (\phi^{(1)}, \phi^{(2)}, \phi^{(3)})$ is $\mathbb{C}^n$-valued and ψ_0 and ψ_1 are real valued, and let

$$u_0(t,x) \equiv (U(t)\phi)(x), \tag{1.2}$$

$$v_0(t,x) \equiv ((\cos \omega t)\psi_0)(x) + ((\omega^{-1} \sin \omega t)\psi_1)(x). \tag{1.3}$$

The functions u_0 and v_0 are unique solutions of the Cauchy problems for the free Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = 0, \\ u(0,x) = \phi(x), \end{cases}$$

and for the free wave equation

$$\begin{cases} \partial_t^2 v - \Delta v = 0, \\ v(0,x) = \psi_0(x), \quad \partial_t v(0,x) = \psi_1(x), \end{cases}$$

respectively.

Our main result is as follows.

Theorem. *Assume that $\phi \in H^{6,9}$, $\psi_0 \in H^3 \cap \dot{H}^{-2}$, $x\omega^{-1}\psi_0 \in L^2$, $\omega^{-2}\psi_0 \in W_1^7$, $\psi_1 \in H^2 \cap \dot{H}^{-3}$, $x\omega^{-2}\psi_1 \in L^2$ and $\omega^{-2}\psi_0 \in W_1^6$. Then there exists a constant $T > 0$ such that the equation* (Z) *has a unique solution $[u, v]$ satisfying*

$$u \in C([T,\infty); H^3), \tag{1.4}$$

$$v \in C([T,\infty); H^2), \tag{1.5}$$

$$\partial_t v \in C([T,\infty); H^1 \cap \dot{H}^{-1}), \tag{1.6}$$

$$\sup_{t \geq T}(t^{5/4}\|u(t) - u_0(t)\|_{L^2} + t\|u(s) - u_0(t)\|_{\dot{H}^1 \cap \dot{H}^3}) < \infty, \tag{1.7}$$

$$\sup_{t \geq T}[t\{\|v(t) - v_0(t)\|_{H^2} + \|\partial_t v(t) - \partial_t v_0(t)\|_{H^1 \cap \dot{H}^{-1}}\}] < \infty. \tag{1.8}$$

A similar result holds for negative time.

Remark 1.1. The assumptions $\psi_0 \in \dot{H}^{-2}$ and $\psi_1 \in \dot{H}^{-3}$ in Theorem implies that the their Fourier transforms $\hat{\psi}_0$ and $\hat{\psi}_1$ vanish at the origin.

The constant T which appears in Theorem depends only on

$$\begin{aligned} \eta \equiv &\|\phi\|_{H^{6,9}} + \|\psi_0\|_{H^3} + \|\psi_0\|_{\dot{H}^{-2}} + \|x\omega^{-1}\psi_0\|_{L^2} + \|\omega^{-2}\psi_0\|_{W_1^7} \\ &+ \|\psi_1\|_{H^2} + \|\psi_1\|_{\dot{H}^{-3}} + \|x\omega^{-2}\psi_1\|_{L^2} + \|\omega^{-2}\psi_1\|_{W_1^6}. \end{aligned} \tag{1.9}$$

In Theorem, we do not restrict the size of η.

Let $\mathcal{V}$ be the set of all scattered states (ϕ, ψ_0, ψ_1) satisfying the assumptions of Theorem.

The following corollary is an immediate consequence of Theorem.

Corollary. *For the equation* (Z)*, the pseudo wave operator* $W_+ \colon (\phi, \psi_0, \psi_1) \mapsto (u(T), v(T), \partial_t v(T))$ *is well-defined on* $\mathcal{V}$*, where* $[u, v]$ *is the solution to the equation* (Z) *obtained in Theorem and* T *is a constant which appears in Theorem. Similarly the modified wave operator* W_- *for negative time is also well-defined on* $\mathcal{V}$*.*

Remark 1.2. The Zakharov equation (Z) is invariant under the translation in the time variable t. Translating the solution $[u, v]$ obtained in Theorem in t by T, we see that for any initial data $(\tilde{\phi}, \tilde{\psi_0}, \tilde{\psi_1})$ belonging to the range of W_+, there exists a unique global solution $[u, v]$ such that

$$\begin{aligned} u &\in C([0, \infty); H^3), \\ v &\in C([0, \infty); H^2), \\ \partial_t v &\in C([0, \infty); H^1 \cap \dot{H}^{-1}), \end{aligned}$$

where W_+ is defined in Corollary. We note that it is not clear what initial data belong to the range of W_+.

Outline of this paper is as follows. We prove the statement for positive time in Theorem. The statement for negative time is proved in the same way. In Sections 2, we construct a suitable asymptotic profile and derive the estimate of each term of it. In Section 3, we prove Theorem by the energy estimates. Hereafter we always assume that the space dimension is three.

2 Preliminaries

2.1 The Principal Term of the Asymptotic profiles

In this section, we study time decay estimates of the solutions for the free Schrödinger and Klein-Gordon equations, which are the principal part of the asymptotic profile. We introduce the asymptotics of the solution for the free Schrödinger equation and time decay estimates of it.

The time decay estimates of the free solutions u_0 and v_0, which is defined in (1.2) and (1.3), respectively, are well-known (see, e.g., Section 2 in Ozawa and Tsutsumi [26]):

Lemma 2.1. *Let k be a non-negative integer. There exists a constant $C > 0$ such that for $t \geq 1$,*

$$\sum_{|\alpha|+2j\leq k} \|\partial_x^\alpha \partial_t^j u_0(t)\|_{L^2} \leq C\|\phi\|_{H^k},$$
$$\sum_{|\alpha|+2j\leq k} \|\partial_x^\alpha \partial_t^j u_0(t)\|_{L^\infty} \leq C\|\phi\|_{W_1^k} t^{-3/2},$$
$$\sum_{|\alpha|+2j\leq k} \|\partial_x^\alpha \partial_t^j u_0(t)\|_{L^\infty} \leq C\|\phi\|_{H^{k,2}} t^{-3/2},$$
$$\sum_{|\alpha|+j\leq k} \|\partial_x^\alpha \partial_t^j v_0(t)\|_{L^2} \leq C(\|\psi_0\|_{H^k} + \|\psi_1\|_{H^{k-1}} + \|\psi_1\|_{\dot{H}^{-1}}),$$
$$\sum_{|\alpha|+j\leq k} \|\partial_x^\alpha \partial_t^j v_0(t)\|_{L^\infty} \leq C(\|\psi_0\|_{W_1^{k+2}} + \|\psi_1\|_{W_1^{k+1}}) t^{-1}.$$

2.2 The Second Correction Term of the Asymptotic Profile for the Schrödinger Part

According to Lemma 2.1, $\|u_0(t)v_0(t)\|_{L^2} = O(t^{-3/2})$. This time decay estimate is not sufficient to prove Theorem directly with no restriction on the size of the scattered states. To overcome this difficulty, we construct the second correction term u_2 of the asymptotic profile of the Schrödinger part such that $(\partial_t + \frac{1}{2}\Delta)u_2 - u_0 v_0$ decays faster than $u_0 v_0$ as $t \to \infty$.

We construct the second correction u_2 of the form

$$u_2(t,x) = u_0(t,x)V(t,x), \tag{2.1}$$

where

$$V(t,x) = ((\cos \omega t)Q_0)(x) + ((\omega^{-1} \sin \omega t)Q_1)(x). \tag{2.2}$$

We determine functions Q_0 and Q_1 of $x \in \mathbb{R}^3$. We first note the following identity:

$$\mathcal{L}(wz) = w\frac{1}{2}\Delta z + z\mathcal{L}w + \frac{1}{t}\left(-i\sum_{k=1}^{3}(J_k w)(\partial_k z) + iwPz\right) \tag{2.3}$$

for a $\mathbb{C}^3$-valued function w and a real valued function z, where

$$J_k \equiv x_k + it\partial_k \ (k = 1,2,3), \quad J \equiv (J_1, J_2, J_3),$$
$$P \equiv t\partial_t + x \cdot \nabla.$$

It is well-known that if w and z solve the free Schrödinger and wave equations, then so do $J_k w$ Pz because $J\mathcal{L} - \mathcal{L}J = 0$ and $\Box P = (P+2)\Box$. Noting this fact and putting $w = u_0$ and $z = V$, we expect that the most slowly decaying part of $\mathcal{L}u_2$ is $(1/2)u_0 \Delta V$. Now we set

$$Q_0(x) \equiv -2(-\Delta)^{-1}\psi_0(x) = -2\omega^{-2}\psi_0(x), \tag{2.4}$$

$$Q_1(x) \equiv -2(-\Delta)^{-1}\psi_1(x) = -2\omega^{-2}\psi_1(x), \tag{2.5}$$

so that the equality

$$\frac{1}{2}u_0 \Delta V = u_0 v_0$$

holds. Then it is expected that $\mathcal{L}u_2 - u_0 v_0$ decays faster than $u_0 v_0$ as $t \to \infty$.

From the equality (2.3), we have

$$\mathcal{L}u_2 - u_0 v_0 = \frac{1}{t}\left(-i\sum_{k=1}^{3}(J_k u_0)(\partial_k V) + iu_0 PV\right). \tag{2.6}$$

Remark 2.1. It is well known that

$$J_k u_0(t,\cdot) = J_k(t)U(t)\phi = U(t)(\mathcal{M}_{x_k}\phi) \quad (k=1,2,3),$$
$$PV(t,\cdot) = (\cos\omega t)(\mathcal{M}_x \cdot \nabla Q_0) + (\omega^{-1}\sin\omega t)((1+\mathcal{M}_x\cdot\nabla)Q_1),$$

where $\mathcal{M}_{x_k}$ and $\mathcal{M}_x$ are the maltiplication operators by the function x_k and x, respectively.

The time decay estimates of u_2 and $\mathcal{L}u_2 - u_0 v_0$ are as follows.

Lemma 2.2. *There exists a constant* $C > 0$ *such that for* $t \geq 1$*,*

$$\sum_{j=0}^{2}\|\partial_t^j u_2(t)\|_{H^{4-j}} \leq C\eta^2 t^{-3/2},$$

$$\sum_{j=0}^{2}\|\partial_t^j u_2(t)\|_{W^{4-j}} \leq C\eta^2 t^{-5/2},$$

$$\|\mathcal{L}u_2(t) - u_0(t)v_0(t)\|_{H^3} + \|\partial_t(\mathcal{L}u_2(t) - u_0(t)v_0(t))\|_{H^1} \leq C\eta^2 t^{-5/2},$$

where $\eta > 0$ *is defined in* (1.9)*.*

Noting Lemmas 2.1, Remark 2.1 and the equality (2.6), we can prove this lemma exactly in the same way as in the proof of Lemma 3.3 in [30].

3 Proof of Theorem

In this section, we prove Theorem for positive time. The statement for negative time in Theorem is proved in the same way. Throughout this section, we always assume that the assumptions of Theorem are satisfied.

Let u_0 and v_0 be the functions defined in (1.2) and (1.3), respectively, and let u_2 be the function defined by (2.1), (2.2), (2.4) and (2.5). We consider the following final value problem:

$$\begin{cases} i\partial_t F + \dfrac{1}{2}\Delta F = FG + Fv_0 + hG + f, \\ \partial_t^2 G - \Delta G = \Delta |F|^2 + 2\mathrm{Re}\Delta(F\overline{h}) + g \end{cases} \tag{3.1}$$

with the condition

$$\begin{cases} \|F(t)\|_{H^3} \to 0, \quad \text{as } t \to \infty, \\ \|G(t)\|_{H^2} + \|\partial_t G(t)\|_{H^1} + \|\partial_t G(t)\|_{\dot{H}^{-1}} \to 0, \quad \text{as } t \to \infty, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} h &\equiv u_0 + u_2, \\ f &\equiv h v_0 - \mathcal{L}u_2 \\ &= u_2 v_0 - (\mathcal{L}u_2 - u_0 v_0), \\ g &\equiv \Delta |h|^2 \\ &= \Delta |u_0 + u_2|^2. \end{aligned}$$

Remark 3.1. If we put $F = u - h = u - u_0 - u_2$ and $G = v - v_0$, then the system (Z) is equivalent to the system (3.1). Hence we solve the equation (3.1) instead of the equation (Z)

From Lemmas 2.1, 2.2, Hölder's inequality and the Sobolev embedding theorem, we have the time decay estimates for the interaction terms.

Lemma 3.1. *There exists a constant $C > 0$ such that for $t \geq 1$,*

$$\begin{aligned} \|f(t)\|_{H^3} + \|\partial_t f(t)\|_{H^1} &\leq C(\eta^2 + \eta^3) t^{-5/2}, \\ \sum_{j=0}^{2} \|\partial_t^j g(t)\|_{H^{2-j}} &\leq C(\eta^2 + \eta^4) t^{-7/2}, \\ \|\omega^{-1} g(t)\|_{L^2} &\leq C(\eta^2 + \eta^4) t^{-5/2}, \end{aligned}$$

where $\eta > 0$ is defined in (1.9).

Now we prove Theorem. The proof of the existence argument in Theorem is based on the energy estimates for the equation (3.1) and the compactness argument. Since that of the uniqueness argument is easy (see Ozawa [23] and Ozawa and Tsutsumi [25]), we omit the detailed proof of it.

Proof of Theorem. To solve the final value problem (3.1)–(3.2), we consider the final value problem of the following regularized equation:

$$\begin{cases} i\partial_t F_{a,b} + \dfrac{1}{2}\Delta F_{a,b} = (1+bt)^{-5}\rho_a * [(\rho_a * F_{a,b})(\rho_a * G_{a,b}) \\ \qquad + (\rho_a * F_{a,b})(\rho_a * v_0) + (\rho_a * h)(\rho_a * G_{a,b}) \\ \qquad + \rho_a * f], \\ \partial_t^2 G_{a,b} - \Delta G_{a,b} = (1+bt)^{-5}\rho_a * [\Delta|\rho_a * F_{a,b}|^2 \\ \qquad + 2\mathrm{Re}\Delta((\rho_a * F)(\overline{\rho_a * h})) + \rho_a * g] \end{cases} \tag{3.3}$$

with the condition

$$\begin{cases} \|F_{a,b}(t)\|_{H^3} \to 0, \quad \text{as } t \to \infty, \\ \|G_{a,b}(t)\|_{H^2} + \|\partial_t G_{a,b}(t)\|_{H^1} + \|\partial_t G_{a,b}(t)\|_{\dot{H}^{-1}} \to 0, \quad \text{as } t \to \infty \end{cases} \tag{3.4}$$

for $0 < a < 1$ and $0 < b < 1$. Here $\rho_a(x) = a^{-3}\rho(x/a)$ for $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\|\rho\|_{L^1} = 1$ and $\rho(x) = \rho(-x)$.

Using the contraction mapping principle, we easily see that for any $0 < a, b < 1$, there exists a constant $\tilde{T}_{a,b} > 0$ such that the equation (3.3) has a unique solution $[F_{a,b}, G_{a,b}]$ satisfying

$$F_{a,b} \in \bigcap_{j=1}^{\infty} C^2([\tilde{T}_{a,b}, \infty); H^j), \tag{3.5}$$

$$G_{a,b} \in \bigcap_{j=1}^{\infty} C^2([\tilde{T}_{a,b}, \infty); H^j), \tag{3.6}$$

$$\partial_t G_{a,b} \in C([\tilde{T}_{a,b}, \infty); \dot{H}^j), \tag{3.7}$$

$$\sup_{t \geq \tilde{T}_{a,b}} \left[(1+bt)^4 \sum_{|\alpha|+2j \leq 3} \|\partial_x^\alpha \partial_t^j F_{a,b}(t)\|_{L^2} \right] < \infty, \tag{3.8}$$

$$\sup_{t \geq \tilde{T}_{a,b}} \left[(1+bt)^4 \left(\sum_{|\alpha|+j \leq 2} \|\partial_x^\alpha \partial_t^j G_{a,b}(t)\|_{L^2} + \|\partial_t G_{a,b}(t)\|_{\dot{H}^{-1}} \right) \right] < \infty. \tag{3.9}$$

Since the initial value problem of the equation (3.3) is time globally solvable, we can extend the above solution $[F_{a,b}, G_{a,b}]$ to the time interval $[0, \infty)$. We note that we do not assume the smallness of η.

We set

$$\begin{aligned}X_{a,b}(t) \equiv&[F_{a,b}; L^2, 5/4](t) + [\nabla \cdot F_{a,b}; L^2, 1](t) \\&+ [\Delta F_{a,b}; L^2, 1](t) + [\nabla \cdot \Delta F_{a,b}; L^2, 1](t) \\&+ [G_{a,b}; L^2, 1](t) + [\nabla G_{a,b}; L^2, 1](t) \\&+ [\Delta G_{a,b}; L^2, 1](t) + [\partial_t G_{a,b}; \dot{H}^{-1}, 1](t) \\&+ [\partial_t G_{a,b}; L^2, 1](t) + [\nabla \partial_t G_{a,b}; L^2, 1](t).\end{aligned} \tag{3.10}$$

In order to to estimate $X_{a,b}(t)$ independent of a and b, we have to derive the various a priori estimates of $F_{a,b}$ and $G_{a,b}$ independent of a and b. Since the detail proof for the equation (3.3) is rather complicated and the regularizing factors $\rho_a *$ and $(1+bt)^{-5}$ cause no trouble, we discribe only the formal caluculations for the equation (3.1).

Let $T > 0$ be a constant determined later, and let $[F, G]$ be the solution for the equation (3.1) on $[T, \infty)$, which are smooth and decay rapidly enough as $t \to \infty$. For $t \geq T$, we put

$$\begin{aligned}X(t) \equiv&[F; L^2, 5/4](t) + [\nabla \cdot F; L^2, 1](t) \\&+ [\Delta F; L^2, 1](t) + [\nabla \cdot \Delta F; L^2, 1](t) \\&+ [G; L^2, 1](t) + [\nabla G; L^2, 1](t) + [\Delta G; L^2, 1](t) \\&+ [\partial_t G; \dot{H}^{-1}, 1](t) + [\partial_t G; L^2, 1](t) + [\nabla \partial_t G; L^2, 1](t).\end{aligned}$$

To estimate $X(t)$, we derive the various a priori estimates for F and G. Hereafter we assume $t \geq T$.

We begin with the L^2-norm of F. Recalling the equation

$$\frac{1}{2}\frac{d}{dt}\|F(t)\|_{L^2}^2 = \mathrm{Im}(h(t)G(t) - f(t), F(t))$$

(see, e.g., the equation (3.23) in Ozawa and Tsutsumi [26]), integrating this equality and using Hölder's inequality, Lemmas 2.1, 2.2 and 3.1, we obtain

$$\begin{aligned}\|F(t)\|_{L^2}^2 \leq& C\int_t^\infty (\|h(s)\|_{L^\infty}\|G(s)\|_{L^2}\|F(s)\|_{L^2} + \|f(s)\|_{L^2}\|F(s)\|_{L^2})\, ds \\ \leq& C\int_t^\infty [(\eta+\eta^2)s^{-15/4}[G; L^2, 1](t)[F; L^2, 5/4](t) \\&+ (\eta+\eta^3)s^{-15/4}[F; L^2, 5/4](t)]\, ds \\ \leq& C(\eta+\eta^3)t^{-11/4}(1 + [G; L^2, 1](t))[F; L^2, 5/4](t).\end{aligned}$$

Therefore there exists a constant $M_1(\eta) > 0$ such that for $t \geq T$,

$$\begin{aligned}[F; L^2, 5/4](t) \leq& M_1(\eta)T^{-1/4}(1 + [G; L^2, 1](t)) \\ \leq& M_1(\eta)T^{-1/4}(1 + X(t)).\end{aligned} \tag{3.11}$$

We next estimate the L^2-norm of $\nabla \cdot F$, G and $\omega^{-1}\partial_t G$. Noting the equation

$$\begin{aligned}
&\|\nabla F(t)\|_{L^2}^2 + \|G(t)\|_{L^2}^2 + \|\partial_t G(t)\|_{\dot{H}^{-1}}^2 \\
&= -\,(F(t)G(t), F(t)) - (F(t)v_0(t), F(t)) \\
&\quad + 2\mathrm{Re}[-(h(t)G(t), F(t)) + (f(t), F(t))] \\
&\quad + \int_t^\infty [-(F(t)v_0(t), F(t)) + 2\mathrm{Re}\{-(h(s)G(s), F(s)) \\
&\qquad + (\partial_s f(s), F(s)) - (F(s)v_0(s), \partial_s G(s))\} \\
&\qquad - (\omega^{-1} g(s), \omega^{-1}\partial_s G(s))]\, ds,
\end{aligned}$$

(see the equation (3.29) in Ozawa and Tsutsumi [26]), and using the Hölder and Gagliardo-Nirenberg inequalities, Lemmas 2.1, 2.2 and 3.1, we obtain

$$\begin{aligned}
&\|\nabla F(t)\|_{L^2}^2 + \|G(t)\|_{L^2}^2 + \|\partial_t G(t)\|_{\dot{H}^{-1}}^2 \\
&\le C(\|F(t)\|_{L^2}^{1/2} \|\nabla \cdot F(t)\|_{L^2}^{3/2} \|G(t)\|_{L^2} + \|F(t)\|_{L^2}^2 \|v_0(t)\|_{L^\infty} \\
&\quad + \|h(t)\|_{L^\infty} \|F(t)\|_{L^2} \|G(t)\|_{L^2} + \|f(t)\|_{L^2} \|F(t)\|_{L^2} \|G(t)\|_{L^2}) \\
&\quad + C\int_t^\infty [\|F(s)\|_{L^2}^2 \|v_0(s)\|_{L^\infty} + \|h(s)\|_{L^\infty} \|F(s)\|_{L^2} \|G(s)\|_{L^2} \\
&\qquad + \|\partial_s f(s)\|_{L^2} \|F(s)\|_{L^2} + \|h(t)\|_{L^\infty} \|F(t)\|_{L^2} \|\partial_s G(s)\|_{L^2}\} \\
&\qquad + \|g(s)\|_{\dot{H}^{-1}} \|\partial_s G(s)\|_{\dot{H}^{-1}}]\, ds \\
&\le C(t^{-25/8}[F; L^2, 5/4](t)^{1/2} [\nabla \cdot F; L^2, 1](t)^{3/2} [G; L^2, 1](t) \\
&\quad + \eta t^{-5/2} [F; L^2, 5/4](t)^2 + (\eta + \eta^2) t^{-11/4} [F; L^2, 5/4](t) [G; L^2, 1](t) \\
&\quad + (\eta^2 + \eta^3) t^{-11/4} [F; L^2, 5/4](t) \\
&\quad + (\eta + \eta^2) t^{-11/4} [F; L^2, 5/4](t) [\partial_t G; L^2, 1](t) \\
&\quad + (\eta + \eta^3) t^{-5/2} [\partial_t G; \dot{H}^{-1}, 1](t)).
\end{aligned}$$

Therefore there exists a constant $M_2(\eta) > 0$ such that for $t \ge T$,

$$\begin{aligned}
&[\nabla \cdot F; L^2, 1](t)^2 + [G; L^2, 1](t)^2 + [\partial_t G; \dot{H}^{-1}, 1](t)^2 \\
&\qquad \le M_2(\eta) T^{-1/2} (X(t)^3 + X(t)^2 + X(t)).
\end{aligned} \tag{3.12}$$

We evaluate the L^2-norm of ΔF, ∇G and $\partial_t G$. We note the equality

$$\begin{aligned}
&\|\Delta F(t)\|_{L^2}^2+\|\nabla G(t)\|_{L^2}^2+\|\partial_t G(t)\|_{L^2}^2\\
&=4\mathrm{Re}[(F(t)G(t),F(t))+(F(t)v_0(t),\Delta F(t))\\
&\quad+(h(t)G(t),\Delta F(t))-(f(t),\Delta F(t))]\\
&\quad+4\int_t^\infty\Big[\mathrm{Im}\{(F(s)G(s)^2,\Delta F(s))+(F(s)v_0(s)G(s),\Delta F(s))\\
&\qquad+(h(s)G(s)^2,\Delta F(s))-(G(s)f(s),\Delta F(s))\\
&\qquad+(F(s)v_0(s)G(s),\Delta F(s))+(F(s)v_0(s)^2,\Delta F(s))\\
&\qquad-(G(s)f(s),\Delta F(s))\}\\
&\qquad+\mathrm{Re}\Big\{(F(s)\partial_s v_0(s),\Delta F(s))+(h(s)\partial_s G(s),\Delta F(s))\\
&\qquad+(\partial_s h(s)G(s),\Delta F(s))-(\partial_s f(s),\Delta F(s))\\
&\qquad+(\Delta(F(s)\cdot h(s)),\partial_s G(s))\\
&\qquad+\sum_{j=1}^{3}(\partial_s G(s)\nabla F^{(j)}(s),\nabla F^{(j)}(s))+\frac{1}{2}(g(s),\partial_s G(s))\Big\}\Big]\,ds,
\end{aligned}\tag{3.13}$$

where $F^{(j)}$ is the j-th component of F for $j=1,2,3$ (see the equation (3.37) in Ozawa and Tsutsumi [26]). We show only the estimates of several typical terms in the right hand side of the equation (3.13). By the Hölder and Gagliardo-Nirenberg inequalities, Lemmas 2.1, 2.2 and 3.1, we have the following estimates:

$$\begin{aligned}
&\int_t^\infty |(F(s)G(s)^2,\Delta F(s))|\,ds\\
&\le\int_t^\infty \|G(s)\|_{L^6}^2\|F(s)\|_{L^6}\|\Delta F(s)\|_{L^2}\,ds\\
&\le C\int_t^\infty \|\nabla G(s)\|_{L^2}^2\|F(s)\|_{L^2}^{1/2}\|\Delta F(s)\|_{L^2}^{3/2}\,ds\\
&\le C\int_t^\infty s^{-25/8}\,ds[F;L^2,5/4](t)^{1/2}[\Delta F;L^2,1](t)^{3/2}[\nabla G;L^2,1](t)^2\\
&\le Ct^{-17/8}[F;L^2,5/4](t)^{1/2}[\Delta F;L^2,1](t)^{3/2}[\nabla G;L^2,1](t)^2,
\end{aligned}$$

$$
\begin{aligned}
&\sum_{j=1}^{3}\int_t^\infty |(\partial_s G(s)\nabla F^{(j)}(s), \nabla F^{(j)}(s))|\,ds \\
&\le \sum_{j=1}^{3}\int_t^\infty \|\partial_s G(s)\|_{L^2}\|\nabla F^{(j)}(s)\|_{L^4}^2\,ds \\
&\le C\int_t^\infty \|\partial_s G(s)\|_{L^2}\|F(s)\|_{L^2}^{1/4}\|\Delta F(s)\|_{L^2}^{7/4}\,ds \\
&\le C\int_t^\infty s^{-49/16}\,ds[F;L^2,5/4](t)^{1/4}[\Delta F;L^2,1](t)^{7/4}[\partial_t G;L^2,1](t) \\
&\le Ct^{-33/16}[F;L^2,5/4](t)^{1/4}[\Delta F;L^2,1](t)^{7/4}[\partial_t G;L^2,1](t),
\end{aligned}
$$

$$
\begin{aligned}
\int_t^\infty |(g(s),\partial_s G(s))|\,ds \le &\int_t^\infty \|g(s)\|_{L^2}\|\partial_s G(s)\|_{L^2}\,ds \\
\le &C(\eta^2+\eta^4)\int_t^\infty s^{-9/2}\,ds[\partial_t G;L^2,1](t) \\
\le &C(\eta^2+\eta^4)t^{-7/2}[\partial_t G;L^2,1](t).
\end{aligned}
$$

Since the rest terms in the right hand side of the equality (3.13) can be evaluated in the same way as above, there exists a constant $M_3(\eta) > 0$ such that for $t \ge T$,

$$
\begin{aligned}
&\|\Delta F(t)\|_{L^2}^2 + \|\nabla G(t)\|_{L^2}^2 + \|\partial_t G(t)\|_{L^2}^2 \\
&\le M_3(\eta)t^{-33/16}(X(t) + X(t)^2 + X(t)^3 + X(t)^4).
\end{aligned}
$$

This implies that for $t \ge T$,

$$
\begin{aligned}
&[\Delta F;L^2,1](t)^2 + [\nabla G;L^2,1](t)^2 + [\partial_t G;L^2,1](t)^2 \\
&\le M_3(\eta)T^{-1/16}(X(t) + X(t)^2 + X(t)^3 + X(t)^4).
\end{aligned}
\tag{3.14}
$$

Finally, we evaluate the L^2-norm of $\nabla\cdot\Delta F$, ΔG and $\nabla\partial_t G$. The following equality holds:

$$
\begin{aligned}
&-\frac{1}{4}(\|\nabla\cdot\Delta F(t)\|_{L^2}^2 + \|\Delta G(t)\|_{L^2}^2 + \|\nabla\partial_t G(t)\|_{L^2}^2) \\
&= \sum_{j=1}^{3}\Big[\mathrm{Re}\{-(F^{(j)}(t)G(t), \nabla\Delta F^{(j)}(t)) \\
&\quad - (\nabla F^{(j)}(t)G(t), \nabla\Delta F^{(j)}(t)) - (\nabla F^{(j)}(t)v_0(t), \nabla\Delta F^{(j)}(t)) \\
&\quad - (h^{(j)}(t)\nabla G(t), \nabla\Delta F^{(j)}(t)) - (\nabla h^{(j)}(t)G(t), \nabla\Delta F^{(j)}(t)) \\
&\quad + (\nabla f^{(j)}(t), \nabla\Delta F^{(j)}(t))\}
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^\infty [\mathrm{Re}\{(F^{(j)}(s)\nabla\partial_t G(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (\nabla F^{(j)}(s)\partial_t G(s), \nabla\Delta F^{(j)}(s)) + (F^{(j)}(s)\nabla\partial_s v_0(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (\nabla F^{(j)}(s)\partial_s v_0(s), \nabla\Delta F^{(j)}(s)) + (h^{(j)}(s)\nabla\partial_t G(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (\partial_s h^{(j)}(s)\nabla G(s), \nabla\Delta F^{(j)}(s)) + (\nabla h^{(j)}(s)\partial_t G(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (\nabla\partial_s h^{(j)}(s)G(s), \nabla\Delta F^{(j)}(s)) - (\nabla\partial_s f^{(j)}(s), \nabla\Delta F^{(j)}(s)) \\
&\quad - (F^{(j)}(s)\nabla\partial_s G(s), \nabla\Delta F^{(j)}(s)) - (\nabla F^{(j)}(s)\Delta\overline{F^{(j)}(s)}, \nabla\partial_s G(s)) \\
&\quad - 2(\Delta F^{(j)}(s)\nabla\overline{F^{(j)}(s)}, \nabla\partial_s G(s)) + (\nabla\Delta(F^{(j)}(s)\overline{h^{(j)}(s)}), \nabla\partial_s G(s))\} \\
&\quad + \mathrm{Im}\{-(\Delta F^{(j)}(s)\nabla G(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (F^{(j)}(s)G(s)\nabla G(s), \nabla\Delta F^{(j)}(s)) + (F^{(j)}(s)\nabla G(s)v_0(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (h^{(j)}(s)G(s)\nabla G(s), \nabla\Delta F^{(j)}(s)) - (f^{(j)}(s)\nabla G(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (G(s)\nabla(F^{(j)}(s)G(s)), \nabla\Delta F^{(j)}(s)) + (G(s)\nabla(F^{(j)}(s)v_0(s)), \nabla\Delta F^{(j)}(s)) \\
&\quad + (G(s)\nabla(h^{(j)}(s)G(s)), \nabla\Delta F^{(j)}(s)) - (\nabla f^{(j)}(s)G(s), \nabla\Delta F^{(j)}(s)) \\
&\quad - (\Delta F^{(j)}(s)\nabla v_0(s), \nabla\Delta F^{(j)}(s)) + (F^{(j)}(s)G(s)\nabla v_0(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (F^{(j)}(s)v_0(s)\nabla v_0(s), \nabla\Delta F^{(j)}(s)) + (h^{(j)}(s)G(s)\nabla v_0(s), \nabla\Delta F^{(j)}(s)) \\
&\quad - (h^{(j)}(s)\nabla G(s), \nabla\Delta F^{(j)}(s)) + (\nabla(F^{(j)}(s)G(s))v_0(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (\nabla(F^{(j)}(s)v_0(s))v_0(s), \nabla\Delta F^{(j)}(s)) + (\nabla(h^{(j)}(s)G(s))v_0(s), \nabla\Delta F^{(j)}(s)) \\
&\quad + (\nabla f^{(j)}(s)v_0(s), \nabla\Delta F^{(j)}(s))\}]\, ds\Bigg] \\
&+\frac{1}{2}\int_t^\infty (\nabla g(s), \nabla\partial_s G(s))\, ds,
\end{aligned}
\tag{3.15}
$$

where $f^{(j)}$ and $h^{(j)}$ are the j-th components of f and h,respectively, (see the equations (3.42) and (3.43) in Ozawa and Tsutsumi [26]). We show only the estimates of several typical terms in the right hand side of the equation (3.15). By the Hölder and Gagliardo-Nirenberg inequalities, Lemmas 2.1, 2.2 and 3.1, we have the following estimates:

$$
\begin{aligned}
&\sum_{j=1}^{3}\int_t^\infty |(F^{(j)}(s)\nabla\partial_s G(s), \nabla\Delta F^{(j)}(s))|\, ds \\
&\le \int_t^\infty \|F(s)\|_{L^\infty}\|\nabla\cdot\Delta F(s)\|_{L^2}\|\nabla\partial_s G(s)\|_{L^2}\, ds \\
&\le C\int_t^\infty \|F(s)\|_{H^2}\|\nabla\cdot\Delta F(s)\|_{L^2}\|\nabla\partial_s G(s)\|_{L^2}\, ds
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_t^\infty (\|F(s)\|_{L^2}+\|F(s)\|_{L^2}^{1/3}\|\nabla\cdot\Delta F(s)\|_{L^2}^{2/3})\|\nabla\cdot\Delta F(s)\|_{L^2}\|\nabla\partial_s G(s)\|_{L^2}\,ds\\
&\le C\int_t^\infty s^{-37/12}\,ds[F;L^2,5/4](t)^{1/3}[\nabla\cdot\Delta F;L^2,1](t)^{5/3}[\nabla\partial_s G;L^2,1](t)\\
&\le Ct^{-25/12}[F;L^2,5/4](t)^{1/3}[\nabla\cdot\Delta F;L^2,1](t)^{5/3}[\nabla\partial_s G;L^2,1](t),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^3\int_t^\infty |(\Delta F^{(j)}(s)\nabla G(s),\nabla\Delta F^{(j)}(s))|\,ds\\
&\quad\le \int_t^\infty \|\Delta F(s)\|_{L^3}\|\nabla\cdot\Delta F(s)\|_{L^2}\|\nabla G(s)\|_{L^6}\,ds\\
&\quad\le C\int_t^\infty \|\omega^{5/2}F(s)\|_{L^2}\|\nabla\cdot\Delta F(s)\|_{L^2}\|\Delta G(s)\|_{L^2}\,ds\\
&\quad\le C\int_t^\infty \|F(s)\|_{L^2}^{1/6}\|\nabla\cdot\Delta F(s)\|_{L^2}^{11/6}\|\Delta G(s)\|_{L^2}\,ds\\
&\quad\le C\int_t^\infty s^{-73/24}\,ds[F;L^2,5/4](t)^{1/6}[\nabla\cdot\Delta F;L^2,1](t)^{11/6}[\Delta G;L^2,1](t)\\
&\quad\le Ct^{-49/24}[F;L^2,5/4](t)^{1/6}[\nabla\cdot\Delta F;L^2,1](t)^{11/6}[\Delta G;L^2,1](t),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^3\int_t^\infty |(\Delta F^{(j)}(s)\nabla v_0(s),\nabla\Delta F^{(j)}(s))|\,ds\\
&\quad\le \int_t^\infty \|\nabla v_0(s)\|_{L^\infty}\|\Delta F(s)\|_{L^2}\|\nabla\cdot\Delta F(s)\|_{L^2}\,ds\\
&\quad\le C\int_t^\infty \|\nabla v_0(s)\|_{L^\infty}\|F(s)\|_{L^2}^{1/3}\|\nabla\cdot\Delta F(s)\|_{L^2}^{5/3}\,ds\\
&\quad\le C\eta\int_t^\infty s^{-49/12}\,ds[F;L^2,5/4](t)^{1/3}[\nabla\cdot\Delta F;L^2,1](t)^{5/3}[\Delta G;L^2,1](t)\\
&\quad\le C\eta t^{-49/24}[F;L^2,5/4](t)^{1/3}[\nabla\cdot\Delta F;L^2,1](t)^{5/3}[\Delta G;L^2,1](t).
\end{aligned}
$$

Since the rest terms in the right hand side of the equality (3.15) can be evaluated in the same way as above, there exists a constant $M_4(\eta)>0$ such that for $t\ge T$,

$$
\begin{aligned}
&\|\nabla\cdot\Delta F(t)\|_{L^2}^2+\|\Delta G(t)\|_{L^2}^2+\|\nabla\partial_t G(t)\|_{L^2}^2\\
&\quad\le M_4(\eta)t^{-49/24}(X(t)+X(t)^2+X(t)^3+X(t)^4).
\end{aligned}
$$

This implies that for $t\ge T$,

$$
\begin{aligned}
&[\nabla\cdot\Delta F;L^2,1](t)^2+[\Delta G;L^2,1](t)^2+[\nabla\partial_t G;L^2,1](t)^2\\
&\quad\le M_4(\eta)T^{-1/24}(X(t)+X(t)^2+X(t)^3+X(t)^4).
\end{aligned}
\tag{3.16}
$$

Combining with the estimates (3.11), (3.12), (3.14) and (3.16), we see that there exists a constant $M_0(\eta) > 0$ such that for $t \geq T$,

$$X(t) \leq M_0(\eta)T^{-1/24}(1 + X(t) + X(t)^2 + X(t)^3). \tag{3.17}$$

The above proof of (3.17) is rather formal. But exactly in the same way as above, we can show that there exists a constant $M(\eta) > 0$ independent of a and b such that for $t \geq T$,

$$X_{a,b}(t) \leq M(\eta)T^{-1/24}(1 + X_{a,b}(t) + X_{a,b}(t)^2 + X_{a,b}(t)^3), \tag{3.18}$$

where $X_{a,b}$ is defined in (3.10). According to (3.5)–(3.9), $X_{a,b}(t) \to 0$ as $t \to \infty$. Therefore it follows from the estimate (3.18) that if $T_\eta > 0$ is sufficiently large, there exists a constant $L_\eta > 0$ independent of a and b such that for any $t \geq T_\eta$,

$$X_{a,b}(t) \leq L_\eta. \tag{3.19}$$

Here we note that the constants T_η and L_η depend only on η, and that the estimate (3.19) is independent of a and b. The estimate (3.19) and the standerd compactness argument show that there exists a solution $[F, G]$ of the equation (3.1) such that

$$\begin{gathered}
F \in C([T_\eta, \infty); H^3), \\
G \in C([T_\eta, \infty); H^2), \\
\partial_t G \in C([T_\eta, \infty); H^1 \cap \dot{H}^{-1}), \\
\sup_{t \geq T_\eta}(t^{5/4}\|F(t)\|_{L^2} + t\|F(t)\|_{\dot{H}^1 \cap \dot{H}^3}) \leq L_\eta, \\
\sup_{t \geq T_\eta}[t\{\|G(t)\|_{H^2} + \|\partial_t G(t)\|_{H^1 \cap \dot{H}^{-1}}\}] \leq L_\eta.
\end{gathered}$$

According to Remark 3.1 and Lemma 2.2, this implies the existence of a solution $[u, v]$ for the equation (Z) satisfying the conditions (1.4)–(1.8).

It remains only to prove the uniqueness. If $T_\eta > 0$ is sufficiently large, we can prove the uniqueness of the solution $[u, v]$ for the equation (Z) satisfying the conditions (1.4)–(1.8). (For detailed proof of this, see Ozawa [23] and Ozawa and Tsutsumi [25]). This completes the proof of Theorem. □

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