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Euler systems on Drinfeld modular curves and zeta values

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1 Introduction

A conjecture of Beilinson relates elements in the K-group of schemes to special values of L-functions. Beilinson gave such elements in the K-group of modular curves and showed that they were related to the special values of L-functions attached to modular forms ([1],[13]). We follow the analogy between Drinfeld modules of rank two and elliptic curves, and show that the elements in K-theory constructed in [12] are related to, under a regulator map, special values of L-functions attached to automorphic forms in positive characteristic.

We work over the global field $K = \mathbb{F}_q(T)$ of positive characteristic. For an ideal $I \subset A = \mathbb{F}_q[T]$, the moduli space M_I of level I Drinfeld modules of rank two is a curve over K. The localization at infinity of the curve $M_I \otimes K_\infty (K_\infty = \mathbb{F}_q((1/T)))$ is known to be analytically uniformized by the Drinfeld upper half plane $\Omega = \mathbb{P}^1(C) \setminus \mathbb{P}^1(K_\infty)$ where C is the completion of the algebraic closure of K_∞. The reduction (or the model) of a uniformized variety is described in a combinatorial manner, as the quotient of the Bruhat-Tits tree $\mathscr{T}$ of $\mathrm{PGL}_2(K_\infty)$ by some congruence subgroup of $\mathrm{GL}_2(A)$. In [12], a series of elements $\kappa_I \in K_2(M_I)$ in the K_2 of the moduli space was constructed, and was shown to form an Euler system.

We construct a homomorphism *reg*, which we call regulator, from the K-group of Drinfeld modular curves to the space of harmonic cochains. The harmonic cochains are automorphic forms on $\mathrm{GL}_2(K)$ with prescribed property at infinity (automorphic forms of Drinfeld type in [14]), and we regard them as the analogue of modular forms. A pairing $\langle , \rangle$ is defined ([7]). Our main result is

Theorem 1.1. *Let $I \subset A$ be an ideal. There exists an element $\kappa_{I,0}$ in the K-group $K_2(M_{I,0})$ of level $I, 0$ Drinfeld modular curve $M_{I,0}$ such that for any normalized Hecke eigen cusp form f with rational eigenvalues,*

$$\langle \mathrm{reg}(\kappa_{I,0}), f\rangle = (1-q^2)(\log_e q)^{-1}\frac{1-W(f)}{2}\,\frac{\partial}{\partial s}L(f,s)\bigg|_{s=0} L(f,1)$$

where $W(f) = \pm 1$ is a constant which is equal to 1 only if $L(f,1)$ is zero, and the logarithm is taken with respect to the natural base $e = \sum_{n\geqslant 0}\frac{1}{n!}$.

See Theorem 5.6 for precise statement.

We give an outline of this article. We construct regulator map for unifomized curves in Chapter 2. We collect some facts on Bruhat-Tits tree and Drinfeld upper half plane in sections 2.2 and 2.3. The source and the target of the regulator are defined in sections 2.5 and 2.6, as K-cohomology of the uniformized curves and harmonic cochains, respectively. In section 2.7, we define regulator on the affinoid domain, which corresponds to an edge of the Bruhat-Tits tree. We then glue the translates of it and obtain the whole regulator in section 2.8. The regulator on global units has a nice analytic description in terms of logarithm and logarithmic derivative, which resembles the classical regulator for curves over the complex numbers (section 2.9).

In [10] and [11], an analogue of real analytic Eisenstein series for the group SL_2 was defined and the Fourier coefficients were computed. In Chapter 3, we define an analogue of real analytic Eisenstein series for arbitrary level. We collect some facts on Fourier analysis following [6] in section 3.1; the Fourier coefficients are computed in section 3.3.

We prove an analogue of Kronecker limit formula in Chapter 4 (Theorem 4.1). In the case of elliptic curves, Kronecker limit formula says the logarithm of the absolute value of the delta function (the weight 12 modular form) is equal to real analytic Eisenstein series. We consider, instead of the delta function, Siegel units. Siegel units are global units on the modular variety. They were studied for ranks 1 and 2 by Goss [8] and by Gekeler [4]. For the proof of the formula, we compute the Fourier coefficients of the logarithm of Siegel units (section 4.2), and compare with those of real analytic Eisenstein series.

We prove our main result in Chapter 5. In section 5.1, we collect some facts on automorphic forms. The notions of Hecke operators, cusp forms, and inner product are recalled, and the definition of the L-function attached to cusp form is given. We define special elements in K-group in section 5.2. For the function field analogue of Rankin-Selberg integral, we follow [14]. The pairing of a cusp form and the image under regulator of the special element is then expressed as a Dirichlet series (section 5.3). The main result is stated and proved in the last section.

2 Regulator

We use the following notation throughout this paper.

2.1 Notation

p : a prime number
$q = p^i$: a power of p, i is a positive integer
$\mathbb{F}_q$: the finite field with q elements
$A = \mathbb{F}_q[T]$: the polynomial ring in one variable T
$K = \mathbb{F}_q(T)$: the field of fractions of A
∞ : prime at infinity, the prime of K defined by $1/T$
$K_\infty = \mathbb{F}_q((1/T))$: the completion at ∞
$O_\infty = \mathbb{F}_q[[1/T]]$: the ring of integers
$\pi = 1/T$: (fixed) uniformizer of K_∞
$C = \hat{\overline{K_\infty}}$: the completion of the algebraic closure of K_∞
$\deg a$: the degree of the polynomial $a \in A$
$\deg \mathfrak{m}$: the degree of the divisor $\mathfrak{m}$ of K
$|\cdot|$: the valuation of K_∞ extended to C, normalized so that $|\pi| = q^{-1}$
$N\mathfrak{m} = q^{\deg \mathfrak{m}}$: the norm of the divisor $\mathfrak{m}$
$\mathrm{ord}_\infty c$: the order at infinity, for $a \in C, \mathrm{ord}_\infty a = -\log_q |a|$

2.2 The Bruhat-Tits tree of $\mathrm{PGL}_2(K_\infty)$

We collect some facts on Bruhat-Tits tree. We copy from [7] and [5]. We let $\mathscr{T}$ be the Bruhat-Tits tree of $\mathrm{PGL}_2(K_\infty)$. Its set of vertices $X(\mathscr{T})$ consists of the classes $[L]$ of O_∞-lattices L in the K_∞-vector space $V = K_\infty^2$, where lattices L, L' define the same class if $L' = c \cdot L$ for some $c \in K_\infty^*$. Two vertices $[L]$ and $[L']$ are adjacent or neighbors if they are represented by lattices $L' \subset L$ such that $\dim_{\mathbb{F}_q} L/L' = 1$. We let $Y(\mathscr{T})$ denote the set of the oriented edges of $\mathscr{T}$. Two vertices are connected by an edge if and only if they are adjacent. There is a $\mathrm{GL}_2(K_\infty)$-action on $X(\mathscr{T})$ and $Y(\mathscr{T})$ (see [7](1.3.2)) which induces an action on $\mathscr{T}$. Let $v_0 = [O_\infty \oplus O_\infty]$ and $v_1 = [\pi^{-1}O_\infty \oplus O_\infty]$ be two vertices, and $e_0 = \overrightarrow{v_0 v_1}$ be the oriented edge with origin v_0 and terminus v_1. We have (see [5](1.3))

$$\begin{array}{rcl} \mathrm{GL}_2(K_\infty)/\mathscr{K} \cdot \mathrm{Z}(K_\infty) & \xrightarrow{\cong} & X(\mathscr{T}) \\ g & \mapsto & g \cdot v_0 \\ \mathrm{GL}_2(K_\infty)/\mathscr{I} \cdot \mathrm{Z}(K_\infty) & \xrightarrow{\cong} & Y(\mathscr{T}) \\ g & \mapsto & g \cdot e_0 \end{array}$$

where $\mathscr{K} = \mathrm{GL}_2(O_\infty)$, $\mathscr{I} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{K} \,\middle|\, c \equiv 0 \bmod (\pi) \right\}$ is the Iwahori subgroup, and $\mathrm{Z} \subset \mathrm{GL}_2$ is the center. We fix a set of representatives of $X(\mathscr{T})$ and $Y(\mathscr{T})$ (see [5](1.5)). Let

$$\begin{aligned} S_X &= \left\{ \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \,\middle|\, k \in \mathbb{Z}, u \in K_\infty \text{ ranges over a fixed set of } K_\infty/\pi^k O_\infty \right\} \\ S_U &= \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \,\middle|\, c \in \mathbb{F}_q \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \\ S_Y &= \{ gh |\, g \in S_X, h \in S_U \}. \end{aligned}$$

then S_X (resp. S_Y) is a set of representatives of $X(\mathscr{T})$ (resp. $Y(\mathscr{T})$).

2.3 The building map

We follow and copy [7](1.4).

We let $\Omega = \mathbb{P}^1(C) \setminus \mathbb{P}^1(K_\infty) = C \setminus K_\infty$ denote the Drinfeld upper half plane. It has a natural structure of rigid analytic space over K_∞(see [7](1.2)). We write $\mathscr{T}(\mathbb{R})$ for the realization (see [7](1.4)) of $\mathscr{T}$. It is a topological space that consists of a real unit interval for every non-oriented edge of $\mathscr{T}$, glued together at their extremities according to the incidence relations of $\mathscr{T}$. For an oriented edge e, we write $e(\mathbb{R})$ (resp. $e^0(\mathbb{R})$) for the closed (resp. open) interval to which it corresponds. There is the $G(K_\infty)$-equivariant building map $\lambda : \Omega \to \mathscr{T}(\mathbb{R})$ (see [7] (1.5), Proposition 1.5.3). If we let D to be the affinoid subspace of Ω defined by

$$\begin{aligned} &1 \leqslant |\tau| \leqslant |\pi|^{-1}, \text{and} \\ &|\tau - c\pi^{-1}| \geqslant |\pi|^{-1}, |\tau - c| \geqslant 1 \text{ for all } c \in \mathbb{F}_q^* \hookrightarrow K_\infty^*, \end{aligned}$$

then $\lambda^{-1}(e(\mathbb{R})) = D$ (see [7](1.5.4)).

2.4 Uniformized curves

Let Γ be an arithmetic subgroup (see [7](2.1) for the definition). We will take Γ to be $\Gamma_0(I)$ (see section 5.1.1 for definition) for our application. Let $X = \Gamma \setminus \Omega$ be a uniformized curve. By this, we mean that X is an algebraic curve, defined over some finite extension of K_∞, whose set of C-valued points is $\Gamma \setminus \Omega$ (see [7](2.2)).

We define a homomorphism, which we call regulator, from the K_2 of a uniformized curve to a certain subgroup of functions on the set of oriented edges $Y(\mathscr{T})$. We give the precise definition of the target and the source of the regulator in sections 2.5, 2.6. In section 2.7, we define a regulator for a certain analytic subspace of Ω. Its translates will be glued together to give the whole regulator map in section 2.8.

2.5 The source of regulator

Let

$$H^0(X, \mathscr{K}_2) := \mathrm{Ker}[K_2(K(X)) \stackrel{\oplus \partial_x}{\to} \bigoplus_{x \in X^1} K_1(\kappa(x))]$$

where X^1 is the set of codimension 1 points of X, K_i is the i-th K-group, $\kappa(x)$ is the residue field at x, and ∂_x is the tame symbol at x.

2.6 The target of regulator

The definition in this section is taken from [16]. We define $C_{har}(Y(\mathscr{T}), \mathbb{C})^\Gamma$ to be the group of $\mathbb{C}$-valued functions $f \in \mathrm{Map}(Y(\mathscr{T}), \mathbb{C})$ satisfying the following conditions.

1. (alternating) $f(\overrightarrow{v_1 v_2}) = -f(\overrightarrow{v_2 v_1})$
2. (harmonicity) $\sum_i f(\overrightarrow{v v_i}) = 0$
3. (Γ-invariance) $f(\gamma v) = f(v)$ $(\gamma \in \Gamma)$

Later, we define cusp forms and Hecke operators acting on this group.

2.7 Regulator on D

We define a map from $K_2(K(D))$ to $\mathbb{Z}$ where $K(D)$ is the function field of D. First, we write explicitly the coordinate ring of D. Let $\{c_0 = 0, c_1, \ldots, c_{q-1}\} = \mathbb{F}_q$. Further let

$$\begin{aligned}
&T = \varprojlim_n O_\infty[X_0, \ldots, X_{q-1}, Y_0, \ldots, Y_{q-1}]/(\pi^n) \\
&T' = T \otimes_{O_\infty} K_\infty \\
&\mathfrak{a} = (X_0 Y_0 - \pi, (X_0 - c_i) X_i - 1, (X_0 - \pi c_i) Y_i - \pi \mid 1 \leqslant i \leqslant q-1) \subset T \\
&\mathfrak{a}' = \text{the ideal of } T' \text{ generated by } \mathfrak{a} \\
&B = T/\mathfrak{a} \\
&B' = T'/\mathfrak{a}.
\end{aligned}$$

Then T' is the algebra of strict convergent power series (see [2] Ch. 5), B' is the coordinate ring of D, and B is the ring of integers of B'. The embedding of D into Ω is given by $\tau \mapsto X_0/\pi$ where τ is the coordinate of $\Omega = C \setminus K_\infty$. We let $B_{(X_0,Y_0)}$ denote the localization at (X_0, Y_0) of B. We call the following map regulator on D:

$$\begin{aligned} &\mathrm{reg}_D : K_2(K(D)) \cong K_2(K(B_{(X_0,Y_0)})) \xrightarrow{\partial_1} K_1(K(B_{(X_0,Y_0)}/(X_0))) \\ &\xrightarrow{\partial_2} K_0(K(B_{(X_0,Y_0)})/(X_0,Y_0)) \cong \mathbb{Z}. \end{aligned}$$

where ∂_1 (resp. ∂_2) is the tame symbol map at (X_0) (resp. (X_0, Y_0)), and $K(R)$ denotes the field of fractions of the ring R.

2.8 Regulator on X

We define a map $\mathrm{reg} : K_2(K(X)) \to \mathrm{Map}(Y(\mathscr{T}), \mathbb{C})$. Then we show that the image of $H^0(X, \mathscr{K}_2)$ is contained in $C_{har}(Y(\mathscr{T}), \mathbb{C})^\Gamma$. Let $\gamma \in S_Y$, and $e = \gamma e_0$ be the corresponding oriented edge. By the equivariance of the building map λ, we have $\lambda^{-1}(e(\mathbb{R})) = \gamma D \hookrightarrow X$. We define reg to be the composition

$$\begin{aligned} &\mathrm{reg} : K_2(K(X)) \xrightarrow{\mathrm{res}} K_2(K(\gamma D)) \xrightarrow{\gamma} K_2(K(D)) \xrightarrow{\mathrm{reg}_D} \mathbb{Z} \\ &\mathrm{reg}(\{f,g\})(e) = \mathrm{reg}_D \circ \gamma \circ \mathrm{res}(\{f,g\}), \end{aligned}$$

where $\gamma : K_2(K(\gamma D)) \tilde{\to} K_2(K(D))$ is the isomorphism induced by the action of $\gamma \in \mathrm{GL}_2(K_\infty)$, and $\mathrm{res} : K_2(K(X)) \to K_2(K(\gamma D))$ is the restriction map. One may check that this does not depend on the choice of the representatives.

Proposition 2.1 (alternating). *Let e be an oriented edge, and $\bar{e}$ be the opposite edge (i.e., the origin and the terminus are interchanged). Then*

$$\mathrm{reg}(\{f,g\})(e) + \mathrm{reg}(\{f,g\})(\bar{e}) = 0$$

for $\{f,g\} \in H^0(X, \mathscr{K}_2)$.

Proof. It suffices to prove the statement for $e = e_0 = \overrightarrow{v_0 v_1}, \bar{e} = \overrightarrow{v_1 v_0}$. We apply Lemma 2.2 below with $R = B_{(X_0,Y_0)}$ and $r_0 = (X_0, Y_0)$. Then the composition

$$\begin{aligned} &K_2(K(B_{(X_0,Y_0)})) \xrightarrow{\oplus \partial_x} \textstyle\bigoplus_{x \in (\mathrm{Spec}\, B_{(X_0,Y_0)})^1} K_1(\kappa(x)) \\ &\xrightarrow{\sum \partial_{(X_0,Y_0)}} K_0(\kappa(B_{(X_0,Y_0)}/(X_0,Y_0))) \end{aligned}$$

is zero. Here $(\mathrm{Spec}\, B_{(X_0,Y_0)})^1$ denotes the codimension one points of $\mathrm{Spec}\, B_{(X_0,Y_0)}$. Since $\{f,g\} \in H^0(X, \mathscr{K}_2)$, we have $\partial_x(\{f,g\}) = 0$ for those x whose support is not contained in the closed subset of $\mathrm{Spec}\, B_{(X_0,Y_0)}$ defined by (π). Those points whose support is contained in (π) are exactly (X_0) and (Y_0). The proposition follows. ☐

Lemma 2.2. *Let R be a 2-dimensional local ring, and $\mathfrak{m}$ be the maximal ideal. Then the composition*

$$K_2(K(R)) \xrightarrow{\oplus \partial_r} \bigoplus_{r \in (\mathrm{Spec}\, R)^1} K_1(\kappa(r)) \xrightarrow{\sum \partial_{\mathfrak{m}}} K_0(\kappa(\mathfrak{m})),$$

where $\kappa(x)$ denotes the residue field at x, and ∂_x denotes the tame symbol at x.

Proof. It follows from the spectral sequence in Theorem 5.20 of [15]. $\square$

Proposition 2.3 (harmonicity). *Let $v \in X(\mathscr{T})$ be a vertex, and $\{\overrightarrow{vv_i}\}(1 \leqslant i \leqslant q+1)$ be the set of all edges with origin v. Then*

$$\sum_{i=1}^{q+1} \mathrm{reg}(\{f,g\})(\overrightarrow{vv_i}) = 0$$

for $\{f,g\} \in H^0(X, \mathscr{K}_2)$.

Proof. Again it suffices to show it for $v = v_0$. We let $e_{c_i} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_i & 1 \end{pmatrix} \in Y(\mathscr{T})$ for $1 \leqslant i \leqslant q-1$, $e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $e_\infty = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then these are all the edges whose origin is v_0. Note that $K(B_{(X_0,Y_0)}/(X_0)) \to K(\mathbb{F}_q(\bar{Y_0})); Y_0 \mapsto \bar{Y_0}$ is an isomorphism. Let $h(\bar{Y_0}) = \partial_1(\{f,g\}) \in \mathbb{F}_q(\bar{Y_0})^*$. Then $\mathrm{reg}(\{f,g\})(e_0) = \mathrm{ord}_{\infty \bar{Y_0}=0} h(\bar{Y_0})$. In a similar manner, $\mathrm{reg}(\{f,g\})(e_{c_i}) = \mathrm{ord}_{\bar{Y_0}=c_i} h(\bar{Y_0})$ for $1 \leqslant i \leqslant q-1$, and $\mathrm{reg}(\{f,g\})(e_\infty) = \mathrm{ord}_{\bar{Y_0}=\infty} h(\bar{Y_0})$. Since $\{f,g\} \in H^0(X, \mathscr{K}_2)$, the rational function $h(\bar{Y_0})$ does not have any pole or zero outside $\{0, c_1, \cdots, c_{q-1}, \infty\}$. Hence $\sum_{a \in \{0, c_1, \ldots, c_{q-1}, \infty\}} \mathrm{ord}_{\bar{Y_0}=a} h(\bar{Y_0}) = 0$. $\square$

Since the Γ-invariance is clear from the definition, the image of the regulator is contained in $C_{har}(Y(\mathscr{T}), \mathbb{C})^\Gamma$.

2.9 On global units

Let $O(X)^*$ be the group of global units on X.

2.9.1 Logarithm

Given $v \in X(\mathscr{T})$, choose $\omega_v \in \Omega$ such that $\lambda(\omega_v) = v$. We define logarithm to be the following map.

$$\log|\cdot| : O(X)^* \to \mathrm{Map}(X(\mathscr{T}), \mathbb{Z})$$
$$\log|f|(v) = \log|f(\omega_v)|$$

where log is taken with base q. This does not depend on the choice of ω_v (see (1.12) [6]).

2.9.2 Logarithmic derivative

We define logarithmic derivative to be the following map:

$$\mathrm{dlog} : O(X)^* \to \mathrm{Map}(Y(\mathscr{T}), \mathbb{Z})$$
$$\mathrm{dlog}\, f(e) = \log|f|(v_2) - \log|f|(v_1),$$

for an oriented edge $e = \overrightarrow{v_1 v_2} \in Y(\mathscr{T})$.

2.9.3 Regulator

On the subgroup of K-group $O(X)^* \otimes O(X)^* \subset K_2(K(X))$ generated by the global units, the regulator is expressed as follows:

$$\mathrm{reg} : O(X)^* \otimes O(X)^* \to C_{har}(Y(\mathscr{T}), \mathbb{C})^\Gamma$$
$$\mathrm{reg}(\{f,g\})(e) = \log|f|(v_1)\mathrm{dlog}\, g(e) - \log|g|(v_2)\mathrm{dlog}\, f(e).$$

3 Eisenstein series

The function field analogue of real analytic Eisenstein series was treated in [10] and [11]. In this section, we give the definition of real analytic Eisenstein series and compute their Fourier coefficients.

3.1 Fourier analysis

We collect some facts on Fourier analysis. For proofs, see section 2 of [5].

We fix an additive character $\eta : K_\infty \to \mathbb{C}^*$ as the composition:

$$\begin{array}{llll} K_\infty & \to \mathbb{F}_q & \xrightarrow{tr} & \mathbb{F}_p \hookrightarrow \mathbb{C}^* \\ \sum_i a_i \pi^i & \mapsto a_1 & & 1 \mapsto \exp(2\pi\sqrt{-1}/p) \end{array}$$

where tr is the trace map. Let $\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \middle| c = 0 \right\}$. A function f on $Y(\mathscr{T})$ which is invariant under Γ_∞ has Fourier expansion of the form

$$f \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} = c_0(f, \pi^k) + \sum_{\substack{0 \neq m \in A \\ \deg m \leqslant k-2}} c(F, \mathrm{div}\,(m) \cdot \infty^{k-2}) \eta(-mu),$$

where the coefficients are determined uniquely. The constant coefficient is a function on $k \in \mathbb{Z}$:

$$c_0(f, \pi^k) = \begin{cases} f \begin{pmatrix} \pi^k & 0 \\ 0 & 1 \end{pmatrix} & (k \leqslant 0) \\ q^{1-k} \displaystyle\sum_{u \in (\pi)/(\pi^k)} f \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} & (k \geqslant 1), \end{cases}$$

and the non-constant coefficients are defined for divisors $\mathfrak{m}$:

$$c(f, \mathfrak{m}) = q^{1-k} \sum_{u \in (\pi)/(\pi^k)} f \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \eta(mu).$$

Here m is an element of A such that $\mathfrak{m}_{\mathrm{fin}} = \mathrm{div}(m)_{\mathrm{fin}}$, where $()_{\mathrm{fin}}$ denotes the finite part of a divisor, and $k = 2 + \deg \mathfrak{m}$. They are zero for non-positive $\mathfrak{m}$. We may consider the Fourier expansion of the functions on $X(\mathscr{T})$ by pulling back to $Y(\mathscr{T})$ by the map $S_X \hookrightarrow S_Y$. The following lemma by Gekeler will be useful.

Lemma 3.1. *Let $c \in A \setminus \{0\}$, and let the two functions f, g on $Y(\mathscr{T})$, which are Γ_∞-invariant, be related by*

$$f(e) = g\left(\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} e \right).$$

Then for divisors $\mathfrak{m}$, and for $k \in \mathbb{Z}$,

$$c(f, \mathfrak{m}) = c(g, \mathfrak{m} \cdot \mathrm{div}(c)_{\mathrm{fin}}^{-1}) \text{ and } c_0(f, \pi^k) = c_0(g, \pi^{k - \deg c}).$$

Proof. See [5] Proposition 2.10. □

3.1.1 Variant

Fix an ideal I and its monic generator i. Let

$$\Gamma_{\infty,I} = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(A) \middle| a \equiv 0 \bmod (I) \right\}.$$

We may consider Fourier expansion of a $\Gamma_{\infty,I}$-invariant function f on $Y(\mathscr{T})$ or on $X(\mathscr{T})$:

$$f\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} = c_0^i(f,\pi^k) + \sum_{\substack{m\in A\setminus\{0\} \\ \deg m \leqslant k-2}} c^i(f, \mathrm{div}(m)\cdot \infty^{k-2})\eta(\frac{mu}{i}).$$

The Fourier coefficients are determined uniquely. For $k \in \mathbb{Z}$, the constant coefficient is given by

$$c_0^i(f,\pi^k) = \begin{cases} f\begin{pmatrix} \pi^k & 0 \\ 0 & 1 \end{pmatrix} & (k \leqslant \mathrm{ord}_\infty i) \\ q^{1+\mathrm{ord}_\infty i - k} \displaystyle\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} f\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} & (k \geqslant \mathrm{ord}_\infty i + 1). \end{cases}$$

For a positive divisor $\mathfrak{m}$, the non-constant coefficients are given by

$$c^i(f,\mathfrak{m}) = q^{1+\mathrm{ord}_\infty i - k} \sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} f\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}\eta(-\frac{mu}{i}),$$

and $c^i(f,\mathfrak{m}) = 0$ for non-positive $\mathfrak{m}$. Here m is an element of A such that $\mathfrak{m}_{\mathrm{fin}} = \mathrm{div}(m)_{\mathrm{fin}}$, and $k = 2 + \deg \mathfrak{m}$. One can check that for a Γ_∞-invariant function f,

1. $c_0^i(f,\pi^k) = c_0(f,\pi^k)$,
2. $c^i(f,\mathfrak{m}) = c(f, \mathfrak{m}\cdot \mathrm{div}(i))$.

We omit the superscript i when there is no confusion. The following lemmas are due to Gekeler [6].

Lemma 3.2. *([6] Proposition 2.10) Let $c \in A \setminus \{0\}$, and let the two functions f, g on $Y(\mathscr{T})$, which are $\Gamma_{\infty,I}$-invariant be related by*

$$f(e) = g\left(\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} e\right).$$

Then, for divisors $\mathfrak{m}$, and for $k \in \mathbb{Z}$, we have

$$c(f,\mathfrak{m}) = c(g, \mathfrak{m}\cdot(\mathrm{div}(c))_{\mathrm{fin}}^{-1}),$$
$$c_0(f,\pi^k) = c_0(g,\pi^{k-\deg c}).$$

Lemma 3.3. *([6] Lemma 2.13) Let f be a harmonic, alternating function on $Y(\mathscr{T})$. Then the constant Fourier coefficient for $k \in \mathbb{Z}$ is given by*

$$c_0(f,\pi^k) = C \cdot q^{-k}$$

for some constant C independent of k.

3.2 Definitions

We define a function on $X(\mathscr{T})$. For $c,d \in A$, $(c,d) \neq (0,0)$, and $s \in \mathbb{C}$, let

$$\phi^s_{c,d}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} = \begin{cases} q^{(k-\deg c)s} & (\omega \geqslant k - \deg c)\\ q^{\omega s} & (\omega < k - \deg c)\end{cases}$$

where $\omega = \mathrm{ord}_\infty(cu+d)$. Let $I = (i) \subset A$ be an ideal, $\alpha, \beta \in A, (\alpha,\beta) \neq (0,0), \deg \alpha < \deg i$, and $\deg \beta < \deg i$. We let

$$F^s_{c,\beta/i} = \sum_{d \equiv \beta \,(\mathrm{mod}\, I)} \phi^s_{c,d},\ E^s_{\alpha/i,\beta/i} = \sum_{c \equiv \alpha \,(\mathrm{mod}\, I)} F^s_{c,\beta}.$$

The sums are absolutely convergent for $\mathrm{Re}\, s \gg 0$. We call $E^s_{\alpha/i,\beta/i}$ real analytic Eisenstein series.

Remark 3.4. Let $e = \begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} \in Y(\mathscr{T})$, and $\gamma \in \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(A)$. If we let $I_s(e) = q^{-ks}$, we have $I_s(\gamma e) = q^{-ks}\phi^{2s}_{c,d}(e)$.

3.3 The Fourier coefficients of Eisenstein series

We compute the Fourier coefficients of Eisenstein series. Each coefficient can be analytically continued to the whole complex plane, hence so can the Eisenstein series itself (Corollary 3.11).

3.3.1 The constant coefficients

Let $\alpha, \beta, i \in A$, $(\alpha,\beta) \neq (0,0)$, $\deg \alpha < \deg i$, and $\deg \beta < \deg i$.

Proposition 3.5. *We have*

$$c_0(E^s_{\alpha/i,\beta/i}, \pi^k) = \begin{cases} q^{-k(1-s)}\left[\dfrac{q-1}{1-q^{2-s}}q^{\deg i(1-s)} + q^{\deg \alpha(1-s)}\right] q^{-\deg i}\dfrac{q^s-1}{q^{s-1}-1} & (\alpha \neq 0)\\ q^{-k(1-s)}q^{-\deg i}\dfrac{q^s-1}{q^{s-1}-1} + q^{(k-\deg i)s}\dfrac{q-1}{1-q^{1-s}} + q^{(-\deg\beta-k)s} & (\alpha = 0).\end{cases}$$

Lemma 3.6. *With notation as above, we have*

$$c_0(F^s_{1,\beta/i}, \pi^k) = q^{k(s-1)}q^{\mathrm{ord}_\infty i}\frac{q^s-1}{q^{s-1}-1}.$$

Proof. (i) The case $k \leqslant \mathrm{ord}_\infty i$.

Start from the left hand side,

$$\begin{aligned} F^s_{1,\beta/i}\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix} &= \sum_{d\equiv\beta(i)} \phi^s_{1,d}\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}\\ &= \sum_{\substack{d\equiv\beta(i)\\ \deg d < -k}} q^{ks} + \sum_{\substack{d\equiv\beta(i)\\ \deg d \geqslant -k}} q^{(-\deg d)s}.\end{aligned}$$

The first term is

$$\begin{aligned}
\sum_{\substack{d'\in A\\ \deg(d'i+\beta)<-k}} q^{ks} &= \sum_{\substack{d'\in A\\ \deg d'i<-k}} q^{ks} = \sum_{\substack{d'\in A\\ \deg d'<-k-\deg i}} q^{ks}\\
&= \sum_{j=0}^{-k-\deg i-1} \sum_{\deg d'=j} q^{ks} + q^{ks}\\
&= \sum_{j=0}^{-k-\deg i-1} (q-1)q^j q^{ks} + q^{ks}\\
&= q^{ks}(q-1)\frac{q^{-k-\deg i}-1}{q-1} + q^{ks}\\
&= q^{-k(1-s)}q^{-\deg i}.
\end{aligned}$$

The second term is

$$\begin{aligned}
&\sum_{\substack{d'\in A\\ \deg(d'i+\beta)\geqslant -k}} q^{-\deg(d'i+\beta)s}\\
&= \sum_{\substack{0\neq d'\in A\\ \deg d'i\geqslant -k}} q^{(-\deg d'i)s}\\
&= \sum_{\substack{0\neq d'\in A\\ \deg d'\geqslant -k-\deg i}} q^{-\deg is}q^{-\deg d's}\\
&= q^{(-\deg i)s}\sum_{j=-k-\deg i}^{\infty}\sum_{\deg d'=j} q^{-\deg d's}\\
&= q^{-\deg is}\sum_{j=-k-\deg i}^{\infty}(q-1)q^{j(1-s)}\\
&= q^{-2\deg is}(q-1)\frac{1}{1-q^{1-s}}q^{(-k-\deg i)(1-s)}\\
&= q^{-k(1-s)}q^{-\deg i}(q-1)\frac{1}{1-q^{1-s}}.
\end{aligned}$$

Taking the sum,

$$\begin{aligned}
c_0(F_{1,\beta/i},\pi^k) &= q^{-k(1-s)}q^{-\deg i}[1+(q-1)\frac{1}{1-q^{1-s}}]\\
&= q^{k(s-1)}q^{-\deg i}\frac{q^s-1}{q^{s-1}-1}.
\end{aligned}$$

(ii) When $k\geqslant \mathrm{ord}_\infty i+1$,

$$\begin{aligned}
&c_0(F^s_{1,\beta/i},\pi^k)\\
&= q^{1+\mathrm{ord}_\infty i-k}\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} F^s_{1,\beta/i}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\\
&= q^{1+\mathrm{ord}_\infty i-k}\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} \phi^s_{1,\beta}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\\
&+q^{1+\mathrm{ord}_\infty i-k}\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)}\sum_{d\equiv\beta(i),d\neq\beta} \phi^s_{1,d}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}.
\end{aligned}$$

The first term is

$$
\begin{aligned}
&q^{1+\mathrm{ord}_\infty i-k}\sum_{\substack{u+\beta\not\equiv 0(\pi^k)\\ u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)}} q^{\mathrm{ord}_\infty(u+\beta)s}+q^{1+\mathrm{ord}_\infty i-k}q^{ks}\\
&=q^{1+\mathrm{ord}_\infty i-k}\sum_{\substack{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)\\ u\neq 0}} q^{(\mathrm{ord}_\infty u)s}+q^{1+\mathrm{ord}_\infty i-k}q^{ks}\\
&=q^{1+\mathrm{ord}_\infty i-k}\sum_{j=1+\mathrm{ord}_\infty i}^{k-1}\sum_{\substack{\mathrm{ord}_\infty u=j\\ u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)}} q^{(\mathrm{ord}_\infty u)s}+q^{1+\mathrm{ord}_\infty i-k}q^{ks}\\
&=q^{1+\mathrm{ord}_\infty i-k}\sum_{j=1+\mathrm{ord}_\infty i}^{k-1}(q-1)q^{k-j-1}q^{js}+q^{1+\mathrm{ord}_\infty i-k}q^{ks}\\
&=q^{1+\mathrm{ord}_\infty i-k}\sum_{j=1+\mathrm{ord}_\infty i}^{k-1}(q-1)q^{k-1}q^{j(s-1)}+q^{1+\mathrm{ord}_\infty i-k}q^{ks}\\
&=q^{1+\mathrm{ord}_\infty i-k}(q-1)q^{k-1}\frac{1-q^{(k-\mathrm{ord}_\infty i-1)(s-1)}}{1-q^{s-1}}q^{(1+\mathrm{ord}_\infty i)(s-1)}+q^{1+\mathrm{ord}_\infty i-k}q^{ks}\\
&=(q-1)q^{(\mathrm{ord}_\infty i)s}\frac{1-q^{(k-\mathrm{ord}_\infty i-1)(s-1)}}{q^{1-s}-1}+q^{k(s-1)}q^{1+\mathrm{ord}_\infty i}\\
&=(q-1)q^{(\mathrm{ord}_\infty i)s}\frac{1}{q^{1-s}-1}+\frac{1-q}{q^{1-s}-1}q^{(\mathrm{ord}_\infty i)s}q^{k(s-1)}q^{(1-s)}q^{\mathrm{ord}_\infty i(1-s)}\\
&+q^{k(s-1)}q^{1+\mathrm{ord}_\infty i}\\
&=(q-1)q^{(\mathrm{ord}_\infty i)s}\frac{1}{q^{1-s}-1}+q^{k(s-1)}q^{\mathrm{ord}_\infty i}\left[\frac{1-q}{1-q^{s-1}}+\frac{q-q^s}{1-q^{s-1}}\right]\\
&=(q-1)q^{(\mathrm{ord}_\infty i)s}\frac{1}{q^{1-s}-1}+q^{k(s-1)}q^{\mathrm{ord}_\infty i}\frac{1-q^s}{1-q^{s-1}}.
\end{aligned}
$$

Note that for $u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)$ and for $d\in A$ such that $\deg d\geqslant \deg i$, we have $\omega=\mathrm{ord}_\infty(u+d)=\mathrm{ord}_\infty d$. Hence $\phi^s_{1,d}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}=q^{(\mathrm{ord}_\infty d)s}$. Now the second term is

$$
\begin{aligned}
&q^{1+\mathrm{ord}_\infty i-k}\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)}\ \sum_{d\equiv\beta(i),d\neq\beta}\phi^s_{1,d}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\\
&=q^{1+\mathrm{ord}_\infty i-k}\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)}\ \sum_{d\equiv\beta(i),d\neq\beta}q^{(\mathrm{ord}_\infty d)s}\\
&=\sum_{d\equiv\beta(i),d\neq\beta}q^{(\mathrm{ord}_\infty d)s}\\
&=\sum_{d'\in A\setminus\{0\}}q^{(\mathrm{ord}_\infty d')s}q^{(\mathrm{ord}_\infty i)s}\\
&=q^{(\mathrm{ord}_\infty i)s}\frac{q-1}{1-q^{1-s}}.
\end{aligned}
$$

Taking the sum,

$$
c_0(F^s_{1,\beta/i},\pi^k)=q^{k(s-1)}q^{\mathrm{ord}_\infty i}\frac{1-q^s}{1-q^{s-1}}.
$$

$\square$

Lemma 3.7. *Let $c \in A \setminus \{0\}$. Then*

$$c_0(F^s_{c,\beta/i}, \pi^k) = c_0(F^s_{1,\beta/i}, \pi^{k+\mathrm{ord}_\infty c}).$$

Proof. We note that $\phi^s_{c,\beta/i}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} = \phi^s_{1,\beta/i}\begin{pmatrix}c\pi^k & cu\\ 0 & 1\end{pmatrix}$ for $\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} \in X(\mathscr{T})$. By taking the sum, the same relation holds for $F^s_{c,\beta/i}$ and $F^s_{1,\beta/i}$. Now apply Lemma 3.2. ☐

Lemma 3.8. *With notation as above, we have*

$$c_0(F^s_{0,\beta/i}, \pi^k) = \zeta_{\equiv\beta(i)}(s),$$

where $\zeta_{\equiv\beta(i)}(s) = q^{(\mathrm{ord}_\infty \beta)s} + q^{(\mathrm{ord}_\infty i)s}\dfrac{q-1}{1-q^{1-s}}$ *is the partial zeta function.*

Proof. Note that $\phi^s_{0,d}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} = q^{(\mathrm{ord}_\infty d)s}$ for all $d \in A \setminus \{0\}$. Hence,

$$\begin{aligned}
&F^s_{0,\beta/i}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} = \sum_{d\equiv\beta(i)} \phi^s_{0,d}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\\
&= \phi^s_{0,\beta}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} = \sum_{d\equiv\beta(i), d\neq\beta} \phi^s_{0,d}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\\
&= q^{(\mathrm{ord}_\infty b)s} + \sum_{d\equiv\beta(i), d\neq\beta} q^{(\mathrm{ord}_\infty d)s}\\
&= q^{(\mathrm{ord}_\infty b)s} + q^{(\mathrm{ord}_\infty i)s}\frac{q-1}{1-q^{1-s}} = \zeta_{\equiv\beta(i)}(s).
\end{aligned}$$

Hence, $c_0(F^s_{0,\beta/i}, \pi^k) = \zeta_{\equiv\beta(i)}(s)$ for $k \leqslant \mathrm{ord}_\infty i$, and

$$\begin{aligned}
c_0(F^s_{0,\beta/i}, \pi^k) &= q^{1+\mathrm{ord}_\infty i - k} \sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} \zeta_{\equiv\beta(i)}(s)\\
&= \zeta_{\equiv\beta(i)}(s),
\end{aligned}$$

for $k \geqslant \mathrm{ord}_\infty i + 1$. ☐

Proof of Proposition 3.5.

(i) When $\alpha = 0$. By definition,

$$\begin{aligned}
&c_0(E_{\alpha/i,\beta/i}, \pi^k) = \sum_{c\equiv0(i)} c_0(F^s_{c,\beta/i}, \pi^k)\\
&= c_0(F_{0,\beta/i}, \pi^k) + \sum_{c\equiv0(i), c\neq0} c_0(F^s_{c,\beta/i}, \pi^k).
\end{aligned}$$

The second term is

$$
\begin{aligned}
&\sum_{c\equiv 0(i),c\neq 0} c_0(F^s_{0,\beta/i},\pi^k)\\
&= \sum_{c\equiv 0(i),c\neq 0} q^{(k-\deg c)(s-1)} q^{-\deg i}\frac{q^s-1}{q^{s-1}-1}\\
&= q^{-\deg i}\frac{q^s-1}{q^{s-1}-1}q^{k(s-1)}\sum_{c\equiv 0(i),c\neq 0} q^{(-\deg c)(s-1)}\\
&= q^{-\deg i}\frac{q^s-1}{q^{s-1}-1}q^{k(s-1)}\sum_{c'=ci,i\in A\setminus\{0\}} q^{(-\deg c')(s-1)}\\
&= q^{-\deg i}\frac{q^s-1}{q^{s-1}-1}q^{k(s-1)}\sum_{c'=ci,i\in A\setminus\{0\}} q^{(-\deg c)(s-1)}q^{(-\deg i)(s-1)}\\
&= q^{k(s-1)}q^{-\deg i s}\frac{q^s-1}{q^{s-1}-1}\sum_{j=0}^{\infty}\sum_{\deg c=j} q^{(-\deg c)(s-1)}\\
&= q^{k(s-1)}q^{(-\deg i)s}\frac{q^s-1}{q^{s-1}-1}\sum_{j=0}^{\infty}(q-1)q^{j(2-s)}\\
&= q^{k(s-1)}q^{(-\deg i)s}\frac{q^s-1}{q^{s-1}-1}(q-1)\frac{1}{1-q^{2-s}}.
\end{aligned}
$$

Now add $c_0(F^s_{0,\beta/i},\pi^k)=\zeta_{\equiv\beta(i)}(s)$.

(ii) When $\alpha\neq 0$.

$$
\begin{aligned}
&\sum_{c\equiv\alpha(i)} c_0(F_{c,\beta/i},\pi^k)\\
&= q^{k(s-1)}q^{-\deg i}\frac{q^s-1}{q^{s-1}-1}\sum_{c\equiv\alpha(i)} q^{(\deg c)(1-s)}\\
&= q^{k(s-1)}q^{-\deg i}\frac{q^s-1}{q^{s-1}-1}\left[\sum_{c'\in A\setminus\{0\}} q^{(\deg c')(1-s)}q^{(\deg i)(1-s)} + q^{(\deg\alpha)(1-s)}\right]\\
&= q^{k(s-1)}q^{-\deg i}\frac{q^s-1}{q^{s-1}-1}\left[(q-1)\frac{1}{1-q^{2-s}}q^{(\deg i)(2-s)} + q^{(\deg\alpha)(1-s)}\right].
\end{aligned}
$$

□

3.3.2 The non-constant coefficients

We express the non-constant Fourier coefficients of real analytic Eisenstein series in terms of those of $F^s_{1,0/1}$.

Lemma 3.9. *Let $m\in A$, $\mathfrak{m}=\operatorname{div}(m)\cdot\infty^k$ be a positive divisor. Given $\beta,i,c\in A$, we have*

1. $c^i(F^s_{1,0/i},\mathfrak{m}) = q^{(\operatorname{ord}_\infty i)s}c^i(F^s_{1,0/i},\mathfrak{m})$
2. $c^i(F^s_{1,\beta/i},\mathfrak{m}) = c^i(F^s_{1,0/i},\mathfrak{m})\eta(m\beta/i)$
3. $c^i(F^s_{c,\alpha/i},\mathfrak{m}) = \begin{cases} c^i(F_{1,\beta/i},\mathfrak{m}\cdot(\operatorname{div}c)^{-1}_{\text{fin}}) & \operatorname{div}c\,|\,\mathfrak{m}\\ 0 & \operatorname{div}c\nmid\mathfrak{m},\end{cases}$

where $()_{\text{fin}}$ denotes the finite part of a divisor.

Proof. Use Lemma 3.2. $\square$

Proposition 3.10. *Let* $\alpha, \beta, i \in A, \deg\alpha < \deg i$*, and* $\deg\beta < \deg i$*. We have*

$$c(E^s_{\alpha/i,\beta/i}, \mathfrak{m}) = \sum_{\substack{c|\mathfrak{m} \\ c\equiv\alpha(i)}} c(F^s_{c,\beta/i}, \mathfrak{m}).$$

Proof. By definition, $c(E^s_{\alpha/i,\beta/i}, \mathfrak{m}) = \sum_{c\equiv\alpha(i)} c(F^s_{c,\beta/i}, \mathfrak{m})$. We know from the previous lemma that $c(F^s_{c,\beta/i}, \mathfrak{m}) = 0$ if $\operatorname{div} c \nmid \mathfrak{m}$. The proposition follows. $\square$

Corollary 3.11. Let notation be as in the proposition. The real analytic Eisenstein series $E^s_{\alpha/i,\beta/i}$ has analytic continuation as a meromorphic function to the whole complex plane.

Proof. We know that the constant coefficient of real analytic Eisenstein series is a rational function in q^s. We may check that each $c(F^s_{1,\beta/i}, \mathfrak{m})$ is a rational function in q^s. By the previous lemma, so are $c(F^s_{c,\beta/i}, \mathfrak{m})$. Then by the above proposition, the non-constant Fourier coefficient of the real analytic Eisenstein series is a finite sum of rational functions in q^s. $\square$

Definition 3.12. For $\alpha, \beta, i \in A$, $\deg\alpha < \deg i$, and $\deg\beta < \deg i$, we let

$$E^{(0)}_{\alpha/i,\beta/i} = \lim_{s\to 0} \frac{1}{s} E^s_{\alpha/i,\beta/i}.$$

We know that the non-constant Fourier coefficients of $F^s_{1,\beta/i}$ have a zero at $s = 0$. It follows that the coefficients of $F^s_{c,\beta/i}$ for $c \in A$ have a zero at $s = 0$. Since each coefficient of $E^s_{\alpha/i,\beta/i}$ is a finite sum of the coefficients of $F^s_{c,\beta/i}$, it also has a zero at $s = 0$, and the expression above is well defined.

4 Siegel units

In this section, we give some results on Siegel units. Siegel units in the function field case have been studied by Goss [8] and by Gekeler [4]. We assume from now on that the ideal I has at least two distinct prime factors. We give the analytic description of Siegel units defined in [12]. We then compute the Fourier coefficients of the logarithm and the logarithmic derivative of Siegel units.

4.1 Siegel units on Drinfeld upper half plane

We follow [7](2.3). We will use τ as the coordinate of Ω. We define $\Lambda_\tau = \langle 1, \tau\rangle$ to be the rank 2 free A-module in C generated by 1 and τ. Put

$$e_{\Lambda_\tau}(z) \quad = z \prod_{\lambda\in\Lambda_\tau\setminus\{0\}} \left(1 - \frac{z}{\lambda}\right) = z \prod_{a,b\in A}{}' \left(1 - \frac{z}{a\tau + b}\right).$$

Here $\prod'$ means that we omit the term $(a, b) = (0, 0)$. The product converges and defines an entire, $\mathbb{F}_q$-linear, surjective, and Λ_τ-periodic function $e_{\Lambda_\tau} : C \to C$. It provides the additive group scheme $\mathbb{G}_{a,C} = C \xleftarrow{\sim} C/\mathrm{Ker}(e_{\Lambda_\tau}) = C/\Lambda_\tau$ with a structure of Drinfeld module (see [7](2.3.2)). We define another analytic function $\Delta : \Omega \to C$ (see [6]) by

$$\Delta(\tau) = \prod_{\substack{\alpha,\beta \in T^{-1}A/A \\ (\alpha,\beta)\neq(0,0)}} e_{\Lambda_\tau}(\alpha\tau + \beta).$$

Let $I \subset A$ be an ideal, and $\mathscr{K}_{\mathrm{fin}} = \mathrm{Ker}[\mathrm{GL}_2(\hat{A}) \to \mathrm{GL}_2(\hat{A}/\hat{A}I)]$ where $\hat{A}$ denotes the profinite completion of A. As discussed in Chapter 2.4, 2.5 of [3], there corresponds a Drinfeld modular curve, which we denote M_I. Its analytic points are given by

$$\begin{aligned} M_I^2(C) &= \coprod_{(A/I)^*} \Gamma(I) \setminus \Omega \\ &= \mathrm{GL}_2(K) \setminus \mathrm{GL}_2(\hat{A} \otimes K) \times \Omega / \mathscr{K}_{\mathrm{fin}} \end{aligned}$$

where $\Gamma(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \,\middle|\, b \equiv c \equiv 0, a \equiv d \equiv 1 \bmod I \right\}$ (see [3](5.7)). We denote by ν the map $\nu : \Omega \to \Gamma(I) \setminus \Omega \hookrightarrow M_I(C)$ which maps to the component which contains $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tau \right) \in \mathrm{GL}_2(\hat{A} \otimes K) \times \Omega$. Let $\alpha, \beta, i \in A$. Siegel units $g_{\alpha/i,\beta/i} \in O(M_I)^*$ were defined in §2 of [12] as invertible functions on M_I. The pull-back by ν to Ω has the form

$$g_{\alpha/i,\beta/i}(\tau) = \pi\Delta(\tau) e_{\Lambda_\tau}(\alpha\tau/i + \beta/i)^{1-q^2}.$$

4.2 Logarithm

We prove an analogue of Kronecker limit formula. We compute the Fourier coefficients of the logarithm of Siegel units, and then compare them with those of $E^{(0)}_{\alpha/i,\beta/i}$.

4.2.1 Limit Formula

Let $\alpha, \beta, i \in A, \deg \alpha < \deg i$, and $\deg \beta < \deg i$. We prove the following theorem in this section.

Theorem 4.1. *We have*

$$\log |g_{\alpha/i,\beta/i}| = E^{(0)}_{\alpha/i,\beta/i} (\log_e q)^{-1} (1 - q^2).$$

Here the logarithm on the left hand side was defined in section 2.9.1 , and the one on the right hand side is taken with respect to the natural base $e = \sum_{n \geqslant 0} \frac{1}{n!}$.

Proof. We use the propositions in the following two sections. By Propositions 4.2 and 4.6, we see that the Fourier coefficients are equal. □

4.2.2 The constant coefficient

We compute the Fourier coefficients of $\log |g_{\alpha/i,\beta/i}|$. Let $\alpha, \beta, i \in A, \deg \alpha < \deg i$, and $\deg \beta < \deg i$.

Proposition 4.2. *We have*

$$c_0(\log|g_{\alpha/i,\beta/i}|,\pi^k)=\begin{cases} -q^{1-k}(1-(q+1)q^{-\mathrm{ord}_\infty\frac{\alpha}{i}}) & (\alpha\neq 0)\\ q^2+q-q^{1-k}+(1-q^2)\mathrm{ord}_\infty(\frac{\beta}{i}) & (\alpha=0).\end{cases}$$

Lemma 4.3. *For $k\leqslant \mathrm{ord}_\infty i$,*

$$\log\left|e_{\Lambda_\tau}(\frac{\alpha}{i}\tau+\frac{\beta}{i})\right|\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}=\begin{cases}\mathrm{ord}_\infty\left(\frac{\beta}{i}\right) & (\alpha=0)\\ q\cdot\dfrac{1-q^{-k-\mathrm{ord}_\infty\frac{\alpha}{i}}}{q-1} & (\alpha\neq 0).\end{cases}$$

Proof. Here is a tool for computing the logarithm of a function $f(\tau)\in O(\Omega)^*$. We fix, once and for all, a unit $\mu\in C$ which is transcendental over K_∞ (exists for cardinality reason). If a vertex $v\in X(\mathscr{T})$ is represented by the matrix $\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}$, using the equivariance of the building map λ, we have

$$\begin{aligned}\log|f(\tau)|(v) &= \log|f(\pi^k\mu+u)|\\ &= \mathrm{ord}_\infty f(\pi^k\mu+u),\end{aligned}$$

not depending on the choice of μ (See [6](1.12)(1.13)). The left hand side is

$$\begin{aligned}&\log|e_{\Lambda_\tau}(\tfrac{\alpha}{i}\tau+\tfrac{\beta}{i})|\\ &=\log|\tfrac{\alpha}{i}\tau+\tfrac{\beta}{i}|+\sum_{a,b\in A}{}'\log\left|\frac{(a-\frac{\alpha}{i})\tau+(b-\frac{\beta}{i})}{a\tau+b}\right|\\ &=\log|\tfrac{\alpha}{i}\tau+\tfrac{\beta}{i}|+\sum_{a,b\in A}{}'\left[\log\left|(a-\frac{\alpha}{i})\tau+(b-\frac{\beta}{i})\right|-\log|a\tau+b|\right].\end{aligned}$$

Let e be the oriented edge which corresponds to $\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}$ $(k\leqslant\mathrm{ord}_\infty i)$. Then

$$\begin{aligned}&\left[\log\left|(a-\tfrac{\alpha}{i})\tau+(b-\tfrac{\beta}{i})\right|-\log|a\tau+b|\right](e)\\ &=\mathrm{ord}_\infty((a-\tfrac{\alpha}{i})\pi^k\mu+(b-\tfrac{\beta}{i}))-\mathrm{ord}_\infty(a\pi^k\mu+b).\end{aligned}$$

(i) When $a\neq 0$ and $b\neq 0$,

$$\begin{aligned}&\mathrm{ord}_\infty((a-\tfrac{\alpha}{i})\pi^k\mu+(b-\tfrac{\beta}{i}))=\begin{cases}-\deg a+k & (-\deg a+k\leqslant-\deg b)\\ -\deg b & (-\deg a+k\geqslant -\deg b),\end{cases}\\ &\mathrm{ord}_\infty(a\pi^k\mu+b)=\begin{cases}-\deg a+k & (-\deg a+k\leqslant-\deg b)\\ -\deg b & (-\deg a+k\geqslant-\deg b).\end{cases}\end{aligned}$$

We used that $\mathrm{ord}_\infty(a-\frac{\alpha}{i})=\mathrm{ord}_\infty a=-\deg a$ if $a\neq 0$. Hence in this case the difference is zero.
(ii) When $a\neq 0$ and $b=0$,

$$\begin{aligned}&\mathrm{ord}_\infty((a-\tfrac{\alpha}{i})\pi^k\mu+\tfrac{\beta}{i})=\begin{cases}-\deg a+k & (-\deg a+k\leqslant-\mathrm{ord}_\infty(\frac{\beta}{i}))\\ -\mathrm{ord}_\infty\frac{\beta}{i} & (-\deg a+k\geqslant-\mathrm{ord}_\infty(\frac{\beta}{i})),\end{cases}\\ &\mathrm{ord}_\infty(a\pi^k\mu+b)=-\deg a+k.\end{aligned}$$

The difference is equal to

$$\begin{cases}0 & (-\deg a+k\leqslant\mathrm{ord}_\infty(\frac{\beta}{i}))\\ \deg a-k+\mathrm{ord}_\infty\frac{\beta}{i} & (-\deg a+k\geqslant\mathrm{ord}_\infty(\frac{\beta}{i})).\end{cases}$$

But the set of a satisfying the second condition is empty if $k \leqslant -\mathrm{ord}_\infty i$.
(iii) When $a = 0$ and $b \neq 0$,

$$\begin{aligned}
&\mathrm{ord}_\infty(-\tfrac{\alpha}{i}\pi^k\mu + (b - \tfrac{\beta}{i})) \\
&= \begin{cases} \mathrm{ord}_\infty(\frac{\alpha}{i} + k) & (\mathrm{ord}_\infty(\frac{\alpha}{i}) + k \leqslant \mathrm{ord}_\infty(b - \frac{\beta}{i})) \\ \mathrm{ord}_\infty(b - \frac{\beta}{i}) & (\mathrm{ord}_\infty(\frac{\alpha}{i}) + k \geqslant \mathrm{ord}_\infty(b - \frac{\beta}{i})) \end{cases} \\
&= \begin{cases} \mathrm{ord}_\infty(\frac{\alpha}{i} + k) & (\mathrm{ord}_\infty(\frac{\alpha}{i}) + k \leqslant -\deg b) \\ \mathrm{ord}_\infty(b - \frac{\beta}{i}) & (\mathrm{ord}_\infty(\frac{\alpha}{i}) + k \geqslant -\deg b). \end{cases}
\end{aligned}$$

Hence the difference is equal to

$$\begin{cases} \mathrm{ord}_\infty(\frac{\alpha}{i}) + k + \deg b & (\mathrm{ord}_\infty(\frac{\alpha}{i}) + k \leqslant -\deg b) \\ 0 & (\mathrm{ord}_\infty(\frac{\alpha}{i}) + k \geqslant -\deg b). \end{cases}$$

The remaining term is

$$\mathrm{ord}_\infty(\frac{\alpha}{i}\pi^k\mu + \frac{\beta}{i}) = \begin{cases} \mathrm{ord}_\infty \frac{\alpha}{i} + k & (k + \mathrm{ord}_\infty \frac{\alpha}{i} \leqslant \mathrm{ord}_\infty \frac{\beta}{i}) \\ \mathrm{ord}_\infty \frac{\beta}{i} & (k + \mathrm{ord}_\infty \frac{\alpha}{i} \geqslant \mathrm{ord}_\infty \frac{\beta}{i}). \end{cases}$$

We now take the sum. When $\alpha = 0$, the difference is 0 and the remaining term is equal to $\mathrm{ord}_\infty \frac{\beta}{i}$. When $\alpha \neq 0$, the difference is equal to

$$\sum_{\mathrm{ord}_\infty \frac{\alpha}{i} + k \leqslant -\deg b} \Big(\mathrm{ord}_\infty \frac{\alpha}{i} + k + \deg b\Big) = -(k + \mathrm{ord}_\infty \frac{\alpha}{i}) + q \cdot \frac{1 - q^{-k - \mathrm{ord}_\infty \frac{\alpha}{i}}}{q - 1}.$$

The remaining term is $\mathrm{ord}_\infty \frac{\alpha}{i} + k$. Hence the lemma. $\square$

Lemma 4.4. *Let notation be as above. We have*

$$c_0(\log |e_{\Lambda_\tau}(\alpha\tau/i + \beta/i)|, \pi^k) = \begin{cases} \mathrm{ord}_\infty(\beta/i) & (\alpha = 0) \\ q \cdot \dfrac{1 - q^{-k - \mathrm{ord}_\infty(\alpha/i)}}{q - 1} & (\alpha \neq 0) \end{cases}$$

Proof. By definition,

$$\begin{aligned}
&c_0(\log |e_{\Lambda_\tau}(\alpha\tau/i + \beta/i)|, \pi^k) \\
&= \begin{cases} \log |e_{\Lambda_\tau}(\frac{\alpha}{i}\tau + \frac{\beta}{i})| \begin{pmatrix} \pi^k & 0 \\ 0 & 1 \end{pmatrix} & (k \leqslant \mathrm{ord}_\infty i) \\ q^{1 + \mathrm{ord}_\infty i - k} \displaystyle\sum_{u \in (\pi^{1 + \mathrm{ord}_\infty i})/(\pi^k)} \log |e_{\Lambda_\tau}(\frac{\alpha}{i}\tau + \frac{\beta}{i})| \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} & \\ & (k \geqslant \mathrm{ord}_\infty i + 1). \end{cases}
\end{aligned}$$

We need to compute, for fixed a and b,

$$\begin{aligned}
\sum_{u \in (\pi^{1 + \mathrm{ord}_\infty i})/(\pi^k)} &[\mathrm{ord}_\infty((a - \alpha/i)\pi^k\mu + (a - \alpha/i)u + (b - \beta/i)) \\
&-\mathrm{ord}_\infty(a\pi^k\mu + (au + b))].
\end{aligned}$$

We first treat the case $\alpha \neq 0$.
(i) The case $a \neq 0$ and $b \neq 0$.

In this case, we note that

$$\begin{aligned}
&\operatorname{ord}_\infty((a-\alpha/i)\pi^k\mu+(a-\alpha/i)u+(b-\beta/i))\\
&=\begin{cases} k+\operatorname{ord}_\infty a & (k+\operatorname{ord}_\infty a \leqslant \operatorname{ord}_\infty((a-\frac{\alpha}{i})u+(b-\frac{\beta}{i})))\\ \operatorname{ord}_\infty((a-\frac{\alpha}{i})u+(b-\frac{\beta}{i})) & \\ & (k+\operatorname{ord}_\infty a \geqslant \operatorname{ord}_\infty((a-\frac{\alpha}{i})u+(b-\frac{\beta}{i}))),\end{cases}
\end{aligned}$$

$$\begin{aligned}
&\operatorname{ord}_\infty(a\pi^k\mu+(au+b))\\
&=\begin{cases} k+\operatorname{ord}_\infty a & (k+\operatorname{ord}_\infty a \leqslant \operatorname{ord}_\infty(au+b))\\ \operatorname{ord}_\infty(au+b) & (k+\operatorname{ord}_\infty a \geqslant \operatorname{ord}_\infty(au+b)).\end{cases}
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{u\in(\pi^{1+\operatorname{ord}_\infty i})/(\pi^k)} \operatorname{ord}_\infty((a-\alpha/i)\pi^k\mu+(a-\alpha/i)u+(b-\beta/i))\\
&=\begin{cases} \displaystyle\sum_{u\in(\pi^{1+\operatorname{ord}_\infty i})/(\pi^k)} \operatorname{ord}_\infty((a-\alpha/i)\pi^k\mu+au) & \\ \qquad\qquad (\operatorname{ord}_\infty a+1+\operatorname{ord}_\infty i\leqslant \operatorname{ord}_\infty b) & \\ \displaystyle\sum_{u\in(\pi^{1+\operatorname{ord}_\infty i})/(\pi^k)} \operatorname{ord}_\infty((a-\alpha/i)\pi^k\mu+(b-\beta/i)) & \\ \qquad\qquad (\operatorname{ord}_\infty a+1+\operatorname{ord}_\infty i> \operatorname{ord}_\infty b) & \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\sum_{u\in(\pi^{1+\operatorname{ord}_\infty i})/(\pi^k)} \operatorname{ord}_\infty((a-\alpha/i)\pi^k\mu+(a-\alpha/i)u+(b-\beta/i))\\
&=\begin{cases} \displaystyle\sum_{u\in(\pi^{1+\operatorname{ord}_\infty i})/(\pi^k)} \operatorname{ord}_\infty(a\pi^k\mu+au) & (\operatorname{ord}_\infty a+1+\operatorname{ord}_\infty i\leqslant \operatorname{ord}_\infty b)\\ \displaystyle\sum_{u\in(\pi^{1+\operatorname{ord}_\infty i})/(\pi^k)} \operatorname{ord}_\infty(a\pi^k\mu+b) & (\operatorname{ord}_\infty a+1+\operatorname{ord}_\infty i> \operatorname{ord}_\infty b)\end{cases}
\end{aligned}$$

Hence the difference is zero.
(ii) The case $a\neq 0$ and $b=0$. We need to compute the difference $\sum_u \operatorname{ord}_\infty((a-\alpha/i)\pi^k\mu+(a-\alpha/i)u+(-\beta/i))-\sum_u \operatorname{ord}_\infty(a\pi^k\mu+au)$. For the first term, we have

$$\begin{aligned}
&\sum_{u\in(\pi^{1+\operatorname{ord}_\infty i})/(\pi^k)} \operatorname{ord}_\infty((a-\alpha/i)\pi^k\mu+(a-\alpha/i)u+(-\beta/i))\\
&=\sum_{u\in(\pi^{1+\operatorname{ord}_\infty i})/(\pi^k)} \operatorname{ord}_\infty((a-\alpha/i)\pi^k\mu+(a-\alpha/i)u)\\
&=\sum_{u\in(\pi^{1+\operatorname{ord}_\infty i})/(\pi^k)} \operatorname{ord}_\infty(a\pi^k\mu+au).
\end{aligned}$$

The second equality follows from the fact that $\operatorname{ord}_\infty(\beta/i)>\operatorname{ord}_\infty a+\operatorname{ord}_\infty i+1$. Hence the difference is zero.
(iii) The case $a=0$ and $b\neq 0$. We need to compute the difference $\sum_u \operatorname{ord}_\infty((-\alpha/i)\pi^k\mu+$

$((-\alpha/i)u + (b-\beta/i)) - \sum_u \mathrm{ord}_\infty b$. Since

$$\begin{aligned}
&\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} \mathrm{ord}_\infty((-\alpha/i)\pi^k\mu + ((-\alpha/i)u + (b-\beta/i)) \\
&= \begin{cases} \displaystyle\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} \mathrm{ord}_\infty((-\alpha/i)\pi^k\mu + b)) \\ \qquad\qquad\qquad (\mathrm{ord}_\infty b < \mathrm{ord}_\infty(\alpha/i) + 1 + \mathrm{ord}_\infty i) \\ \displaystyle\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} \mathrm{ord}_\infty((-\alpha/i)\pi^k\mu + (-\alpha/i)u) \\ \qquad\qquad\qquad (\mathrm{ord}_\infty b \geqslant \mathrm{ord}_\infty(\alpha/i) + 1 + \mathrm{ord}_\infty i), \end{cases}
\end{aligned}$$

the difference is zero if $\deg b \geqslant -\mathrm{ord}_\infty \alpha$. We compute the sum over b of the difference:

$$\begin{aligned}
&q^{1+\mathrm{ord}_\infty i - k} \sum_{\substack{\deg b < -\mathrm{ord}_\infty \alpha \\ b\neq 0}} [\sum_{\substack{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k) \\ u\neq 0}} (\mathrm{ord}_\infty(\alpha/i) + \mathrm{ord}_\infty u - \mathrm{ord}_\infty b) \\
&+ \mathrm{ord}_\infty(\alpha/i) + k - \mathrm{ord}_\infty b].
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{\deg b < \deg \alpha} \deg b &= \sum_{j=0}^{\deg\alpha - 1} \sum_{j=\deg b} \deg b \\
&= \sum_{j=0}^{\deg\alpha-1} (q-1)q^j j \\
&= \sum_{j=0}^{\deg\alpha-1} (jq^{j+1} - jq^j) \\
&= -q^1 - q^2 - \cdots - q^{\deg\alpha} + \deg\alpha\, q^{\deg\alpha} \\
&= (-q)\frac{1-q^{\deg\alpha}}{1-q} + \deg\alpha\, q^{\deg\alpha},
\end{aligned}$$

and that

$$\begin{aligned}
&\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k), u\neq 0} \mathrm{ord}_\infty u \\
&= \sum_{j=1+\mathrm{ord}_\infty i}^{k-1} \sum_{\mathrm{ord}_\infty u = j} j = \sum_{j=1+\mathrm{ord}_\infty i}^{k-1} (q-1)jq^{k-1-j} \\
&= q^{k-1} \sum_{j=1+\mathrm{ord}_\infty i}^{k-1} [jq^{-j+1} - jq^{-j}] \\
&= q^{k-1}[(1+\mathrm{ord}_\infty i)q^{-(1+\mathrm{ord}_\infty i)+1} - (1+\mathrm{ord}_\infty i)q^{-(1+\mathrm{ord}_\infty i)} \\
&+ (1+\mathrm{ord}_\infty i + 1)q^{-(1+\mathrm{ord}_\infty i+1)+1} - (1+\mathrm{ord}_\infty i+1)q^{-(1+\mathrm{ord}_\infty i+1)} \\
&+ \cdots + (k-1)q^{-(k-1)+1} - (k-1)q^{-(k-1)}] \\
&= q^{k-1}[(1+\mathrm{ord}_\infty i)q^{-(1+\mathrm{ord}_\infty i)+1} + q^{-(1+\mathrm{ord}_\infty i)} \\
&+ \cdots + q^{-(k-1)+1} + q^{-(k-1)} - kq^{-(k-1)}] \\
&= \mathrm{ord}_\infty i\, q^{-\mathrm{ord}_\infty i} q^{k-1} - k + [q^{k-\mathrm{ord}_\infty i - 1} + \cdots + 1] \\
&= \mathrm{ord}_\infty i\, q^{-\mathrm{ord}_\infty i} q^{k-1} - k + \frac{q^{k-\mathrm{ord}_\infty i} - 1}{q-1}.
\end{aligned}$$

Hence the above expression is equal to

$$\begin{aligned}&(-q)\frac{1-q^{\deg\alpha}}{1-q}+\deg\alpha\, q^{\deg\alpha}+(-\deg\alpha+\deg i)(q^{\deg\alpha}-1)\\&+\sum_b\left(\mathrm{ord}_\infty i+\frac{q-q^{1+\mathrm{ord}_\infty i-k}}{q-1}\right).\end{aligned}$$

(iv) The remaining term.

We need to compute

$$q^{1+\mathrm{ord}_\infty i-k}\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)}\mathrm{ord}_\infty((-(\alpha/i)\pi^k\mu+((\alpha/i)u+\beta/i)).$$

When $\alpha\neq 0$, we have

$$\begin{aligned}&q^{1+\mathrm{ord}_\infty i-k}\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)}\mathrm{ord}_\infty((-(\alpha/i)\pi^k\mu+((\alpha/i)u+\beta/i))\\&=q^{1+\mathrm{ord}_\infty i-k}[\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k),u\neq0}(\mathrm{ord}_\infty(\alpha/i)+\mathrm{ord}_\infty u)+(\mathrm{ord}_\infty(\alpha/i)+k)]\\&=q^{1+\mathrm{ord}_\infty i-k}[\mathrm{ord}_\infty(\alpha/i)+\mathrm{ord}_\infty i\, q^{-\mathrm{ord}_\infty i}q^{k-1}+\frac{q^{k-\mathrm{ord}_\infty i}-1}{q-1}]\\&=\mathrm{ord}_\infty(\alpha/i)+\mathrm{ord}_\infty i+\frac{q-q^{1+\mathrm{ord}_\infty i-k}}{q-1}.\end{aligned}$$

The sum of the four parts (i)(ii)(iii)(iv) is then

$$\begin{aligned}&(-q)\frac{1}{1-q}+\frac{q\cdot q^{\deg\alpha}}{1-q}+\deg\alpha\, q^{\deg\alpha}+(-\deg\alpha+\deg i)(q^{\deg\alpha-1})\\&+(-\deg i)(q^{\deg\alpha}-1)+\frac{q-q^{1+\mathrm{ord}_\infty i-k}}{q-1}(q^{\deg\alpha}-1)\\&-\deg\alpha+\deg i-\deg i+\frac{q-q^{1+\mathrm{ord}_\infty i-k}}{q-1}\\&=\frac{-q}{1-q}+\frac{q^{\deg\alpha+1}}{1-q}+\deg\alpha\, q^{\deg\alpha}\\&+(-\deg\alpha+\deg i)q^{\deg\alpha}-\deg i\, q^{\deg\alpha}+\frac{q-q^{1+\mathrm{ord}_\infty i-k}}{q-1}q^{\deg\alpha}\\&=\frac{q}{q-1}-\frac{q^{1-\mathrm{ord}_\infty(\alpha/i)-k}}{q-1}.\end{aligned}$$

We now assume $\alpha=0$. The cases (i) and (ii) are same as above.

(iii) The case $a=0,b\neq0$.

We have

$$\begin{aligned}&\mathrm{ord}_\infty((-\alpha/i)\pi^k\mu+(-\alpha/i)u+(b-\beta/i))-\mathrm{ord}_\infty b\\&=\mathrm{ord}_\infty(b-\beta/i)-\mathrm{ord}_\infty b\\&=0.\end{aligned}$$

Hence the difference is zero.

(iv) The remaining term.

We have

$$\begin{aligned}&q^{1+\mathrm{ord}_\infty i-k}\sum_{u\in(\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)}\mathrm{ord}_\infty(\beta/i)\\&=\mathrm{ord}_\infty(\beta/i).\end{aligned}$$

The sum of the four parts is $\mathrm{ord}_\infty(\beta/i)$. $\square$

Corollary 4.5. ([6]Theorem 2.13) For $k \in \mathbb{Z}$, we have

$$c_0(\log|\Delta(\tau)|, \pi^k) = q^2 + q - 1 - q^{1-k}.$$

Proof. We apply the previous proposition.

$$\begin{aligned} c_0(\log|\Delta(\tau)|, \pi^k) &= \sum_{\substack{u,v \in T^{-1}A/A \\ (u,v)\neq(0,0)}} c_0(\log|e_{\Lambda_\tau}(u\tau+v)|, \pi^k) \\ &= q - 1 + \frac{q - q^{-k}}{q-1}(q^2 - q) \\ &= q^2 + q - 1 - q^{1-k}. \end{aligned}$$

The first term is for $\alpha = 0$, and $q^2 - q = \#\{(u,v) | u \neq 0\}$. $\square$

Proof of Proposition 4.2. We combine the previous lemmas. When $\alpha \neq 0$,

$$\begin{aligned} & c_0(\log|g_{\alpha/i,\beta/i}|, \pi^k) \\ &= (1-q^2)c_0(\log|e_{\Lambda_\tau}(\alpha\tau/i + \beta/i)|, \pi^k) + c_0(\Delta, \pi^k) + 1 \\ &= (1-q^2)q \cdot \frac{1 - q^{-k}q^{-\mathrm{ord}_\infty(\alpha/i)}}{q-1} + q^2 + q - 1 - q^{1-k} + 1 \\ &= -q^{1-k}[1 - (q+1)q^{-\mathrm{ord}_\infty(\alpha/i)}]. \end{aligned}$$

The case $\alpha = 0$ is simpler. $\square$

4.2.3 The non-constant Fourier coefficients

We compare the non-constant Fourier coefficients of the logarithm of Siegel units with those of real analytic Eisenstein series.

Proposition 4.6. *Let $\alpha, \beta, i \in A$, $\deg \alpha < \deg i$, and $\deg \beta < \deg i$. Suppose $\mathfrak{m} = \mathfrak{m}_{\mathrm{fin}} \cdot \infty^k$ is a positive divisor. Then*

$$c(\log|g_{\alpha/i,\beta/i}|, \mathfrak{m}) = c(E^{(0)}_{\alpha/i,\beta/i}, \mathfrak{m})(\log_e q)^{-1}(1-q^2).$$

We first compute the Fourier coefficients of $\log|e_{\Lambda_\tau}(\frac{\alpha\tau}{i} + \frac{\beta}{i})|$. We then obtain as a corollary the Fourier coefficients of Δ.

Lemma 4.7. *Let $m \in A$, and $\mathfrak{m} = \mathrm{div}(m)_{\mathrm{fin}} \cdot \infty^k$ be a positive divisor. Then, for any integer $N \geqslant \deg a$, we have*

$$\begin{aligned} & \frac{\partial}{\partial s} c(F^s_{a,0/1}, \mathfrak{m})|_{s=0} \\ &= q^{-1-\deg\mathfrak{m}} \sum_{u \in (\pi)/(\pi^{2+\deg\mathfrak{m}})} \sum_{\substack{b \in A \\ \deg b \leqslant N}} \log|a\tau + b| \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \eta(-mu)(\log_e q). \end{aligned}$$

Proof.

$$
\begin{aligned}
&\frac{\partial}{\partial s} c(F^s_{a,0/1}, \mathfrak{m})|_{s=0} \\
&= \frac{\partial}{\partial s}\left[q^{-1-\deg \mathfrak{m}} \sum_{u \in (\pi)/(\pi^{2+\deg \mathfrak{m}})} \sum_{d \in A} \phi^s_{a,d}\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \eta(-mu) \right]\Bigg|_{s=0} \\
&\overset{(*)}{=} \frac{\partial}{\partial s}\left[q^{-1-\deg \mathfrak{m}} \sum_{u \in (\pi)/(\pi^{2+\deg \mathfrak{m}})} \sum_{\substack{d \in A \\ \deg d \leqslant N}} \phi^s_{a,d}\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \eta(-mu) \right]\Bigg|_{s=0} \\
&= q^{-1-\deg \mathfrak{m}} \sum_{u \in (\pi)/(\pi^{2+\deg \mathfrak{m}})} \sum_{\substack{d \in A \\ \deg d \leqslant N}} \frac{\partial}{\partial s}\left(\phi^s_{a,d}\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}\right)\Bigg|_{s=0} \eta(-mu) \\
&\overset{(**)}{=} q^{-1-\deg \mathfrak{m}} \sum_{u \in (\pi)/(\pi^{2+\deg \mathfrak{m}})} \sum_{\substack{d \in A \\ \deg d \leqslant N}} \log|a\tau + d| \eta(-mu)(\log_e q).
\end{aligned}
$$

When $\deg d \geqslant \deg a$, $\phi^s_{a,d}$ is independent of u, hence has no contribution to the Fourier coefficient. The equality $(*)$ follows. For $(**)$, we use the following.

$$
\begin{aligned}
\log|a\tau + b| \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} &= \mathrm{ord}_\infty(a\pi^k\mu + (au + b)) \\
&= \begin{cases} k - \deg a & (k - \deg a \leqslant \mathrm{ord}_\infty(au+b)) \\ \mathrm{ord}_\infty(au+b) & (k - \deg a \geqslant \mathrm{ord}_\infty(au+b)) \end{cases}
\end{aligned}
$$

$$
\phi^s_{a,d}\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} = \begin{cases} q^{(k-\deg a)s} & (k - \deg a \leqslant \mathrm{ord}_\infty(au+d)) \\ q^{\mathrm{ord}_\infty(au+d)s} & (k - \deg a \geqslant \mathrm{ord}_\infty(au+d)). \end{cases}
$$

□

Slight modification of the proof yields the following lemma.

Lemma 4.8. *Let $m \in A$, and $\mathfrak{m} = \mathrm{div}(m)_{\mathrm{fin}} \cdot \infty^k$ be a positive divisor. Then, for any integer $N \geqslant \deg a$, we have*

$$
\begin{aligned}
&\frac{\partial}{\partial s} c(F^s_{a,\beta/i}, \mathfrak{m})|_{s=0} = q^{-1-\deg \mathfrak{m}} \\
&\times \sum_{u \in (\pi^{1+\mathrm{ord}_\infty i})/(\pi^{2+\deg \mathfrak{m}})} \sum_{\substack{b \in A, b \equiv \beta(i) \\ \deg b \leqslant N}} \log|a\tau + b| \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \eta(\frac{-mu}{i})(\log_e q)^{-1}.
\end{aligned}
$$

Proof of Proposition 4.6.
By definition,

$$
c(\log|g_{\alpha/i,\beta/i}|, \mathfrak{m}) = q^{-1-\deg \mathfrak{m}} \sum_{u \in (\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} \log|g_{\alpha/i,\beta/i}| \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \eta(-mu).
$$

Choose an integer N_τ large enough so that

1. $N_\tau > \deg \mathfrak{m}$

2. For all k and u as above,
$$\log|g_{\alpha/i,\beta/i}|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} = \sum_{\substack{\deg a\leqslant N_\tau\\ \deg b\leqslant N_\tau}}{}' \log\left|1-\frac{(\alpha/i)\tau+(\beta/i)}{a\tau+b}\right|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}.$$

Then

$$\begin{aligned}
&\log|g_{\alpha/i,\beta/i}|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\\
&= \sum_{\deg a,\deg b<N_\tau+1}{}' \log|a\tau+b|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} - \sum_{\deg a,\deg b<N_\tau}{}' \log|a\tau+b|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\\
&+(1-q^2)\sum_{\deg a,\deg b<N_\tau}{}' \log|(a+\alpha/i)\tau+(b+\beta/i)|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
&c(\log|g_{\alpha/i,\beta/i}|,\mathfrak{m})\\
&= q^{-1-\deg\mathfrak{m}}\sum_u\sum_{a,b}{}' \log|a\tau+b|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\eta(\frac{-mu}{i})\\
&-q^{-1-\deg\mathfrak{m}}\sum_u\sum_{a,b}{}' \log|a\tau+b|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\eta(\frac{-mu}{i})\\
&+(1-q^2)q^{-1-\deg\mathfrak{m}}\sum_u\sum_{a,b}{}' \log|(a+\alpha/i)\tau+(b+\beta/i)|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\eta(\frac{-mu}{i})\\
&= c(\Delta,\mathfrak{m})-c(\Delta,\mathfrak{m})+(1-q^2)c(E^{(0)}_{\alpha/i,\beta/i},\mathfrak{m}).
\end{aligned}$$

We used that if $a\nmid\mathfrak{m}$,

$$q^{-1-\deg\mathfrak{m}}\sum_u\sum_{b\in A}\log|a\tau+b|\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\eta(-mu) = \frac{\partial}{\partial s}c(F^s_{a,0/1},\mathfrak{m})\Big|_{s=0} = 0.$$

□

4.3 Logarithmic derivative

We calculate the Fourier coefficients of the logarithmic derivative of Siegel units. The result of this section will be used to prove Lemma 5.3.

4.3.1 The constant Fourier coefficient

Let $\alpha,\beta,i\in A$, with i monic, $\deg\alpha<\deg i$, and $\deg\beta<\deg i$.

Proposition 4.9. *For all k, we have*

$$c_0(\operatorname{dlog} g_{\alpha/i,\beta/i},\pi^k) = \begin{cases} -q^{1-k}(1-q)(1-(q+1)q^{-\operatorname{ord}_\infty(\frac{\alpha}{i})}) & (\alpha\neq 0)\\ -(1-q)q^{1-k} & (\alpha=0).\end{cases}$$

Lemma 4.10. *For $k<-\deg i$, we have*

$$\operatorname{dlog} g_{\alpha/i,\beta/i}\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix} = \begin{cases} -q^{1-k}(1-q)(1-(q+1)q^{-\operatorname{ord}_\infty(\frac{\alpha}{i})}) & (\alpha\neq 0)\\ -(1-q)q^{1-k} & (\alpha=0).\end{cases}$$

Proof. By definition,

$$\operatorname{dlog} g_{\alpha/i,\beta/i}\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix} = \log|g_{\alpha/i,\beta/i}|\begin{pmatrix}\pi^{k-1} & 0\\ 0 & 1\end{pmatrix} - \log|g_{\alpha/i,\beta/i}|\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}.$$

Now use Proposition 4.2. $\square$

Proof of Proposition 4.9. We know from Lemma 3.3 that the constant Fourier coefficient of a harmonic function is a constant multiple of q^{-k}. The constant can be determined at $k \ll 0$ where the coefficient is equal to the value given in the proposition. $\square$

4.3.2 Partial sums

Let $a, \beta, i \in A$, and $\deg\beta < \deg i$. We define a function $G_{a,\beta/i}$ on $Y(\mathscr{T})$, which is invariant under $\Gamma_{\infty,I}$. For an oriented edge $e \in Y(\mathscr{T})$, we let

$$G_{a,\beta/i}(e) = \sum_{b\in A, b\equiv\beta(i)} \operatorname{dlog}(a\tau + b)(e).$$

To compute their Fourier coefficients, we need the values at $e \in Y(\mathscr{T})$.

Lemma 4.11.

$$G_{1,0/1}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} = \begin{cases} -q^{-k+1} & (k \leqslant 0, u = 0)\\ -1 & (k \geqslant 1, u = 0)\\ 0 & (k \geqslant 1, u \neq 0, u \in (\pi)).\end{cases}$$

Proof. For $\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} \in Y(\mathscr{T})$, we are interested in

$$\operatorname{dlog}(\tau + b)\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} = \operatorname{ord}_\infty(\pi^{k-1}\mu + (u+b)) - \operatorname{ord}_\infty(\pi^k\mu + (u+b)).$$

We have

$$\operatorname{ord}_\infty(\pi^k\mu + (u+b)) = \begin{cases} k & (k \leqslant \operatorname{ord}_\infty(u+b))\\ \operatorname{ord}_\infty(u+b) & (k \geqslant \operatorname{ord}_\infty(u+b)).\end{cases}$$

(i) When $k \geqslant 1$, $b \neq 0$, and $u \in (\pi)$,

$$\operatorname{ord}_\infty(\pi^k\mu + (u+b)) = \begin{cases} k & (k \leqslant \operatorname{ord}_\infty b)\\ \operatorname{ord}_\infty b & (k \geqslant \operatorname{ord}_\infty b).\end{cases}$$

(ii) When $k \geqslant 1$, $b = 0$, and $u \in (\pi)$,

$$\begin{aligned}\operatorname{ord}_\infty(\pi^k\mu + (u+b)) &= \begin{cases} k & (k \leqslant \operatorname{ord}_\infty u)\\ \operatorname{ord}_\infty u & (k \geqslant \operatorname{ord}_\infty u).\end{cases}\\ &= \begin{cases} k & (u = 0)\\ \operatorname{ord}_\infty u & (u \neq 0, u \in (\pi)).\end{cases}\end{aligned}$$

(iii) When $k \geqslant 1$ and $u = 0$,

$$\operatorname{ord}_\infty(\pi^k\mu + (u+b)) = \begin{cases} k & (k \leqslant \operatorname{ord}_\infty b)\\ \operatorname{ord}_\infty b & (k \geqslant \operatorname{ord}_\infty b).\end{cases}$$

Hence,

$$\begin{aligned}&\mathrm{ord}_\infty(\pi^{k-1}\mu+(u+b))-\mathrm{ord}_\infty(\pi^k\mu+(u+b))\\ =&\begin{cases}0 & (k\geqslant 1, b\neq 0)\\ -1 & (k\geqslant 1, b=0, u=0)\\ 0 & (k\geqslant 1, b=0, u\neq 0, u\in(\pi))\\ -1 & (k\leqslant 1, k\leqslant \mathrm{ord}_\infty b, u=0)\\ 0 & (1\geqslant k\geqslant \mathrm{ord}_\infty b, u=0).\end{cases}\end{aligned}$$

Taking the sum, when $k\geqslant 0$,

$$\begin{aligned}G_{1,0/1}\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix} &= -\sum_{\substack{b\neq 0\\ \deg b\leqslant -k}} 1-1\\ &= -q^{-k+1}.\end{aligned}$$

When $k\geqslant 1$,

$$G_{1,0/1}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix} = \begin{cases}-1 & (u=0)\\ 0 & (u\neq 0, u\in(\pi)).\end{cases}$$

□

Lemma 4.12. *Let notation be as above. Suppose* $\mathfrak{m}=(\mathrm{div} m)\cdot\infty^{k-2}$*, for* $m\in A$ *and* $k\in\mathbb{Z}$*, is a positive divisor. Then*

$$c(G_{a,\beta/i},\mathfrak{m}) = \begin{cases}-q^{-1-\deg\,\mathfrak{m}+\deg a}\eta(\frac{m\beta}{i}) & (a|\mathfrak{m})\\ 0 & (a\nmid\mathfrak{m})\end{cases}$$

Proof. By definition,

$$\begin{aligned}c(G_{1,0/1},\mathfrak{m}) &= q^{1-k}\sum_{u\in(\pi)/(\pi^k)} G_{1,0/1}\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\eta(mu)\\ &= q^{1-k}G_{1,0/1}\begin{pmatrix}\pi^k & 0\\ 0 & 1\end{pmatrix}\\ &= -q^{-1-\deg\mathfrak{m}}.\end{aligned}$$

We apply Lemma 3.2 and obtain

$$c(G_{a,0/i},\mathfrak{m}) = \begin{cases}-q^{-1-\deg\mathfrak{m}+\deg a} & (a|\mathfrak{m})\\ 0 & (a\nmid\mathfrak{m}).\end{cases}$$

for $a\in A\setminus\{0\}$. It then follows by a change of variables that

$$c(G_{a,\beta/i},\mathfrak{m}) = \begin{cases}-q^{-1-\deg\,\mathfrak{m}+\deg a}\eta(\frac{m\beta}{i}) & (a|\mathfrak{m})\\ 0 & (a\nmid\mathfrak{m}).\end{cases}$$

□

4.3.3 The non-constant coefficients

Proposition 4.13. *Let notation be as above. Suppose* $\mathfrak{m}$ *is a positive divisor. Then*

$$c(\mathrm{dlog}\, g_{\alpha/i,\beta/i},\mathfrak{m}) = (q^2-1)\sum_{\substack{a|\mathfrak{m}\\ a\equiv\alpha(i)}} q^{-1-\deg\mathfrak{m}+\deg a}\eta(\frac{m\beta}{i}).$$

Proof. The right hand side is $(1-q^2)\sum_{\substack{a|\mathfrak{m}\\ a\equiv\alpha(i)}} c(G_{a,\beta/i},\mathfrak{m})$. The left hand side is

$$
\begin{aligned}
&c(\mathrm{dlog}\, g_{\alpha/i,\beta/i},\mathfrak{m})\\
&= q^{-1-\deg\mathfrak{m}} \sum_{u\in(\pi)/(\pi^{2+\deg\mathfrak{m}})} \sum_{a,b}{}' \mathrm{dlog}\,(a\tau+b)\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\eta(\frac{mu}{i})\\
&= q^{-1-\deg\mathfrak{m}} \sum_{u\in(\pi)/(\pi^{2+\deg\mathfrak{m}})} [\sum_{\deg a,\deg b<N+1}{}' \quad \mathrm{dlog}\,(a\tau+b)\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\\
&-\sum_{\deg a,\deg b<N} \mathrm{dlog}\,(a\tau+b)\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\\
&+(1-q^2)\sum_{\substack{\deg a,\deg b<N+1\\ a\equiv\alpha(i),b\equiv\beta(i)}} \mathrm{dlog}\,(a\tau+b)\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}]\eta(\frac{mu}{i})\\
&= \sum_{\deg a,\deg b<N+1} c(G_{a,b/i},\mathfrak{m}) - \sum_{\deg a,\deg b<N} c(G_{a,b/i},\mathfrak{m})\\
&+(1-q^2)\sum_a c(G_{a,b/i},\mathfrak{m})\\
&= (1-q^2)\sum_a c(G_{a,b/i},\mathfrak{m}).
\end{aligned}
$$

□

5 Values of L-functions

5.1 Automorphic forms

5.1.1 Cusp forms

For an ideal $I\subset A$, we let

$$\Gamma_0(I)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(A)\,\middle|\, c\equiv 0 \bmod I\right\}.$$

We call an element f of $C_{har}(Y(\mathscr{T}),\mathbb{C})^{\Gamma_0(I)}$ level $I, 0$ automorphic form of Drinfeld type. We say that f is a cusp form if f has finite support.

5.1.2 Hecke operators

Fix an ideal $I\subset A$. We have a Hecke operator $T_\wp$ on $C_{har}(Y(\mathscr{T}),\mathbb{C})^{\Gamma_0(I)}$ for each prime ideal $\wp=(p)$. For $f\in C_{har}(Y(\mathscr{T}),\mathbb{C})^{\Gamma_0(I)}$,

1. $(T_\wp f)(e)=f\left(\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}e\right)+\sum_{q \bmod \wp} f\left(\begin{pmatrix}1 & q\\ 0 & p\end{pmatrix}e\right)$ when $\wp\nmid I$.

2. $(T_\wp f)(e)=\sum_{q \bmod \wp} f\left(\begin{pmatrix}1 & q\\ 0 & p\end{pmatrix}e\right)$ when $\wp|I$.

Here p is the monic generator of $\wp$, $e\in Y(\mathscr{T})$ is an oriented edge, and q runs through the set $A/\wp$.

5.1.3 Fourier coefficients

The Fourier coefficients of a normalized Hecke eigen cusp form satisfy the following conditions.

1. $c_0(\pi^k) = 0 (k \in \mathbb{Z})$.
2. $c((1)) = 1$.
3. $c(\mathfrak{m})c(\mathfrak{n}) = c(\mathfrak{m}\mathfrak{n})$ whenever $\mathfrak{m}$ and $\mathfrak{n}$ are relatively prime.
4. $c(\wp^{n-1}) - \lambda_\wp c(\wp^n) + |\wp| c(\wp^{n+1}) = 0$ when $\wp \nmid I\infty$.
5. $c(\wp^n) - \lambda_\wp c(\wp^{n+1}) = 0$ when $\wp | I$.
6. $c(\infty^n) = q^{-n}$ $(n \geqslant 0)$.

where $\lambda_\wp$ is a complex number.

5.1.4 L-functions

The definition is taken from [5](3.4) but we shift by 1 and discard the component at infinity. Given an automorphic form $f \in C_{har}(Y(\mathscr{T}), \mathbb{C})^{\Gamma_0(I)}$, we define its L-function to be

$$L(f, s) = \sum_{\substack{\mathfrak{m}:\text{positive divisor} \\ (\mathfrak{m},\infty)=1}} \frac{c(f, \mathfrak{m})}{|\mathfrak{m}|^{s-1}}$$

where s is a complex variable.

5.1.5 Petersson inner product

Let $f, g \in C_{har}(Y(\mathscr{T}), \mathbb{C})^{\Gamma_0(I)}$ be automorphic forms. Suppose f is a cusp form. We define Petersson inner product to be

$$\langle f, g \rangle \quad := \sum_{e \in \Gamma_0(I) \backslash Y(\mathscr{T})} f(e) \cdot \overline{g(e)}$$

where the bar denotes complex conjugation. We also extend the definition to any function on $Y(\mathscr{T})$ not necessarily harmonic.

5.1.6 Involution

Let I be an ideal, and i be its monic generator. Let f be a $\Gamma_0(I)$-invariant function on $Y(\mathscr{T})$ or on $X(\mathscr{T})$. We write $f|_\tau(e) := f(\tau e)$ for $\tau = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}$.

Lemma 5.1. *Let f be a Hecke eigen cusp form with rational Fourier coefficients. Then*

$$f|_\tau = W(f) \cdot f$$

for a constant $W(f) = \pm 1$. In particular,

$$L(f|_\tau, s) = W(f) L(f, s).$$

The proof is similar to the elliptic modular case, and hence is omitted. See [9](6.3).

5.2 Special elements

5.2.1 Elements in K-theory

Fix an ideal $I \subset A$. We let

$$\mathscr{K}_{I,0} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{A}) \middle| c \equiv 0 \bmod I \right\}.$$

This compact subgroup of $\mathrm{GL}_2(\hat{A})$ defines a Drinfeld modular curve, which we denote $M_{I,0}$. Its C-valued points are given by the quotient

$$M_{I,0}(C) = \Gamma_0(I) \setminus \Omega.$$

We define a level $I, 0$ K-theory element to be

$$\kappa_{I,0} = \left\{ \prod_{\substack{a \in (A/I)^* \\ b \in A/I}} g_{a/i,b/i}, \prod_{d \in A/I} g_{0,d/i} \right\} \in K_2(M_{I,0}).$$

5.2.2 Automorphic forms

We label some special automorphic forms. Let

$$\tilde{E}^2_{I,0} = \sum_{\substack{a \in (A/I)^* \\ b \in A/I}} \mathrm{dlog}\, g_{\frac{a}{i},\frac{b}{i}}, \quad E^s_{I,0} = \sum_{\substack{a \in (A/I)^* \\ b \in A/I}} E^s_{\frac{a}{i},\frac{b}{i}}, \quad E^{(0)}_{I,0} = \sum_{\substack{a \in (A/I)^* \\ b \in A/I}} E^{(0)}_{\frac{a}{i},\frac{b}{i}},$$

$$\tilde{E}^{2'}_{I,0} = \sum_{d \in A/I} \mathrm{dlog}\, g_{0,\frac{d}{i}}, \quad E^{s'}_{I,0} = \sum_{d \in A/I} E^s_{0,\frac{d}{i}}, \quad E^{(0)'}_{I,0} = \sum_{d \in A/I} E^{(0)}_{0,\frac{d}{i}}.$$

It is immediate from the definition that

$$\mathrm{reg}(\kappa_{I,0}) = \tilde{E}^2_{I,0} E^{(0)'}_{I,0} - \tilde{E}^{2'}_{I,0} E^{(0)}_{I,0}.$$

Lemma 5.2. *We have*

$$\tilde{E}^2_{I,0}|_\tau = \tilde{E}^{2'}_{I,0},\ E^s_{I,0}|_\tau = E^{s'}_{I,0},\ E^{(0)}_{I,0}|_\tau = E^{(0)'}_{I,0}.$$

Proof. The first and the third claim follows from

$$\prod_{b \in A/I} g_{\frac{a}{i},\frac{b}{i}} \left(-\frac{1}{i\tau} \right) = g_{0,\frac{a}{i}}(\tau),$$

for $a \in (A/I)^*$. The second from

$$\phi^s_{c,d} \left(\begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix} e \right) = \phi^s_{di,-c}(e)$$

for $c, d \in A, (c,d) \neq (0,0)$. □

Lemma 5.3. *For a positive divisor* $\mathfrak{m}$*, let* $c(\mathfrak{m})$ *denote the* $\mathfrak{m}$*-th Fourier coefficient of the function* $\frac{q}{1-q^2} E^2_{I,0}$*. Then the following conditions are satisfied:*

1. $c((1)) = 1$
2. $c(\mathfrak{m}\mathfrak{n}) = c(\mathfrak{m})c(\mathfrak{n})$ *whenever* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are relatively prime.*
3. $c(\mathfrak{p}^n) + (1 + |\mathfrak{p}|^{-1})c(\mathfrak{p}^{n+1}) + |\mathfrak{p}|^{-1}c(\mathfrak{p}^{n+2}) = 0 (n \geqslant 0)$ *when* $\mathfrak{p} \nmid I\infty, \mathfrak{p}$*:prime.*
4. $c(\mathfrak{p}^n) = |\mathfrak{p}|^{-n} (n \geqslant 0)$ *when* $\mathfrak{p}|I$, $\mathfrak{p}$*:prime*
5. $c(\infty^n) = q^{-n} (n \geqslant 0)$.

Proof. We use Proposition 4.13. Let $m \in A \setminus \{0\}$, and $\mathfrak{m} = \mathrm{div}(m) \cdot \infty^k$ be a positive divisor.

$$
\begin{aligned}
&c(\tilde{E}^2_{I,0}, \mathfrak{m}) \\
&= \sum_{u \in (\pi)/(\pi^k)} \tilde{E}^2_{I,0} \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \eta(-mu) \\
&= q^{-\deg i} \sum_{u \in (\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} \tilde{E}^2_{I,0} \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \eta(-\frac{mu}{i}) \\
&= q^{-\deg i} \sum_{u \in (\pi^{1+\mathrm{ord}_\infty i})/(\pi^k)} \sum_{\substack{\alpha \in (A/I)^* \\ \beta \in A/I}} \mathrm{dlog}\, g_{\alpha/i, \beta/i}\, \eta(-\frac{mu}{i}) \\
&= q^{-\deg i} \sum_{\substack{\alpha \in (A/I)^* \\ \beta \in A/I}} c(\mathrm{dlog}\, g_{\alpha/i,\beta/i}, \mathfrak{m}) \\
&= q^{-\deg i} \frac{1-q^2}{q} \sum_{\alpha \in (A/I)^*} \sum_{\substack{a \mid \mathfrak{m} \\ a \equiv \alpha (i)}} q^{-\deg \mathfrak{m} + \deg a} \sum_{\beta \in A/I} \eta(-\frac{m\beta}{i}) \\
&= \frac{1-q^2}{q} \sum_{\substack{a \mid \mathfrak{m} \\ (a,i)=1}} q^{-\deg \mathfrak{m} + \deg a} \sum_{\beta} \eta(-\frac{m\beta}{i}) \\
&= \frac{1-q^2}{q} \sum_{\substack{a \mid \mathfrak{m} \\ (a,i)=1}} q^{-\deg \mathfrak{m} + \deg a} \\
&= \frac{1-q^2}{q} q^{-\deg \mathfrak{m}} \sum_{\substack{a \mid \mathfrak{m} \\ (a,i)=1}} q^{\deg a}.
\end{aligned}
$$

This shows the conditions 1 and 2 hold for the Fourier coefficients of the function $\frac{q}{1-q^2}\tilde{E}^2_{I,0}$. If $\mathfrak{p}$ is a prime ideal, and n is a non-negative integer, one can check that

$$
\frac{q}{1-q^2} c(\tilde{E}^2_{I,0}, \mathfrak{p}^n) = q^{-\deg \mathfrak{p}^n} \sum_{j=0}^{n} q^{-\deg \mathfrak{p}^j}
$$

satisfies the conditions 3, 4 and 5. □

5.3 Rankin-Selberg integral

The function field analogue of Rankin-Selberg integral is treated in [14]. The next proposition is essentially proved in §3 of [14].

Proposition 5.4. *Let $f, g \in C_{har}(Y(\mathscr{T}), \mathbb{C})^{\Gamma_0(I)}$ be automorphic forms. Suppose f is a cusp form. Then*

$$
\langle g \cdot E^{2s}_{I,0} q^{-ks}, f \rangle = \frac{1}{q} \cdot D(f, g, s)
$$

where $D(f, g, s)$ is the Dirichlet series

$$
\sum_{\mathfrak{m}:\text{positive divisor}} \frac{\overline{c(f,\mathfrak{m})} c(G, \mathfrak{m})}{(\mathrm{N}\mathfrak{m})^{s-1}}.
$$

Proof.

$$\begin{aligned}
\langle g \cdot E_{I,0}^{2s} q^{-ks}, f\rangle &= \sum_{e \in \Gamma_0(I)\backslash Y(\mathscr{T})} \overline{f(e)} \cdot g(e) \cdot E_{I,0}^{2s}(e) q^{-ks} \\
&= \sum_{e \in \Gamma_\infty \backslash Y(\mathscr{T})} \overline{f(e)} \cdot g(e) q^{-ks} \\
&= \sum_{k=2}^{\infty} \sum_{u \in (\pi)/(\pi^k)} \overline{f\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}} \cdot g\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} q^{-ks} \\
&= \sum_{k=2}^{\infty} \sum_{u \in (\pi)/(\pi^k)} \sum_{\mathfrak{m},\mathfrak{n}} \overline{c(f,\mathfrak{m})} c(g,\mathfrak{n}) \eta(\mathrm{mu}) \eta(-\mathrm{nu}) q^{-ks} \\
&= \sum_{k=2}^{\infty} \sum_{\deg \mathfrak{m} \leqslant k-2} \overline{c(f,\mathfrak{m})} c(g,\mathfrak{m}) q^{k-1} q^{-ks} \\
&= \frac{1}{q} D(f,g,s).
\end{aligned}$$

The second equality follows from Remark 3.4. $\square$

Lemma 5.5. *Let $f \in C_{har}(Y(\mathscr{T}), \mathbb{C})^{\Gamma_0(I)}$ be a Hecke eigen cusp form. Then*

$$D(\tilde{E}_{I,0}^2, f, s) = \frac{q^2 - 1}{q} \frac{1}{1 - q^{-2+s}} \zeta_I(2s+2)^{-1} L_I(f,s) L(f, s+1).$$

where $L_I(f,s) = \prod_{\wp \nmid I} (1 - \lambda_\wp (\mathrm{N}\wp)^{-s} + (\mathrm{N}\wp)^{1-2s})$ (resp. $\zeta_I(s)$) is the L-function (resp. zeta function) with bad factors removed.

Proof. The computation of the Dirichlet series is similar to the elliptic modular case, and hence is omitted (see [9] p.155). $\square$

5.4 Special values

We come to our main result.

Theorem 5.6. *Let $I \subset A$ be an ideal. Suppose $f \in C_{har}(Y(\mathscr{T}), \mathbb{C})^{\Gamma_0(I)}$ is a normalized Hecke eigen cusp form with rational Fourier coefficients. Then we have*

$$\langle \mathrm{reg}(\kappa_{I,0}), f \rangle = (1 - q^2)(\log_e q)^{-1} \frac{1 - W(f)}{2} \zeta_I(2)^{-1} \left. \frac{\partial}{\partial s} L_I(f,s) \right|_{s=0} \cdot L(f,1).$$

See section 5.1 for the definition of $W(f)$.

Proof. The left hand side is equal to

$$(1 - q^2)(\log_e q)^{-1} [-\langle \tilde{E}_{I,0}^2 E_{I,0}^{(0)'}, f\rangle + \langle \tilde{E}_{I,0}^{2'} E_{I,0}^{(0)}, f\rangle].$$

The first term is

$$
\begin{aligned}
&\langle \tilde{E}^2_{I,0} E^{(0)'}_{I,0}, f\rangle \\
&= \sum_{e \in \Gamma_0(I)\backslash Y(\mathscr{T})} \overline{f(e)} \tilde{E}^2_{I,0}(e) E^{(0)'}_{I,0}(e) \\
&= \lim_{s\to 0} \frac{1}{s} \sum_{e \in \Gamma_0(I)\backslash Y(\mathscr{T})} \overline{f(e)} \tilde{E}^2_{I,0}(e) E^{s'}_{I,0}(e) q^{-ks/2} \\
&= \lim_{s\to 0} \frac{1}{s}\frac{1}{q} D(f, \tilde{E}^2_{I,0}, \frac{s}{2}) \\
&= \lim_{s\to 0} \frac{1}{s}\frac{1}{q}\frac{q^2-1}{q} \zeta_I(\frac{2s}{2}+2)^{-1} L_I(f, \frac{s}{2}) L(f, \frac{s}{2}+1) \\
&= \frac{q^2-1}{q^2}\frac{1}{1-q^{-2}} \zeta_I(2)^{-1} \frac{1}{2} \frac{\partial}{\partial s} L(f,s)\Big|_{s=0} \cdot L(f,1).
\end{aligned}
$$

where we define $k \in \mathbb{Z}$ by $e = \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}$ for $e \in Y(\mathscr{T})$. Here we used Proposition 5.4 and Lemma 5.3. The second term is

$$
\begin{aligned}
&\langle \tilde{E}^{2'}_{I,0} E^{(0)}_{I,0}, f\rangle \\
&= \sum_{e \in \Gamma_0(I)\backslash Y(\mathscr{T})} \overline{f(e)} \tilde{E}^{2'}_{I,0}(e) E^{(0)}_{I,0}(e) \\
&= \sum_{e \in \Gamma_0(I)\backslash Y(\mathscr{T})} \overline{f(\tau e)} \tilde{E}^{2}_{I,0}(e) E^{(0)'}_{I,0}(e) \\
&= \lim_{s\to 0} \frac{1}{s}\frac{1}{q} D(f|_\tau, \tilde{E}^2_{I,0}, \frac{s}{2}) \\
&= W(f) \lim_{s\to 0} \frac{1}{s}\frac{1}{q} D(f, \tilde{E}^2_{I,0}, \frac{s}{2}) \\
&= W(f) \frac{q^2-1}{q^2}\frac{1}{1-q^{-2}} \zeta_I(2)^{-1} \frac{1}{2} \frac{\partial}{\partial s} L(f,s)\Big|_{s=0} \cdot L(f,1).
\end{aligned}
$$

The second equality follows from Lemma 5.2. We take the difference and obtain

$$
\langle \mathrm{reg}(\kappa_{I,0}), f\rangle = (1-q^2)(\log_e q)^{-1} \cdot \frac{1-W(f)}{2} \zeta_I(2)^{-1} \frac{\partial}{\partial s} L_I(f,s)\Big|_{s=0} \cdot L(f,1).
$$

□

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