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On the representations of $\mathbf{Sp}(p,q)$ and $\mathbf{SO}^*(2n)$ unitarily induced from derived functor modules

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Abstract

We obtain a decomposition formula of a representation of Sp(p, q) and SO$^*(2n)$ unitarily induced from a derived functor module, which enables us to reduce the problem of irreducible decompositions to the study of derived functor modules. In particular, we show such an induced representation is decomposed into a direct sum of irreducible unitarily induced modules from derived functor modules under some regularity condition on the parameters. In particular, representations of $SO^*(2n)$ and Sp(p, q) induced from one-dimensional unitary representations of their parabolic subgroups are irreducible.

§ 0. Introduction

Our object of study is the decomposition of unitarily induced modules of a real reductive Lie groups from derived functor modules. In [Matumoto 1996], the case of $U(m, n)$ is treated. In this article, we study the case of Sp(p, q) and SO$^*(2n)$. Reducibilities of the the representations of $U(m, n)$ unitarily induced from derived functor modules is coming from the reducibility of particular degenerate principal series of U(n, n) found by Kashiwara-Vergne [Kashiwara-Vergne 1979]. In the case of Sp(p, q) and SO$^*(2n)$ the situation is quite similar, at least in regular region of the parameter, the reducibilities also reduce to the Kashiwara-Vergne decomposition.

We are going into more details.

Let $G = \mathrm{Sp}(p, q) \quad (p \geqslant q)$ or $G = \mathrm{SO}^*(2n)$. We fix a Cartan involution θ as usual. Let $\kappa = (k_1, ..., k_s)$ be a finite sequence of positive integers such that

$$k_1 + \cdots + k_s \leqslant \begin{cases} q & \text{if } G = \mathrm{Sp}(p, q), \\ \frac{n}{2} & \text{if } G = \mathrm{SO}^*(2n). \end{cases} .$$

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If $G = \mathrm{Sp}(p,q)$, put $p' = p - k_1 - \cdots - k_s$ and $q' = q - k_1 - \cdots - k_s$. If $G = \mathrm{SO}^*(2n)$, put $r = n - 2(k_1 - \cdots - k_s)$. Then, there is a parabolic subgroup P_κ of G, whose Levi subgroup M_κ is written as

$$M_\kappa \cong \begin{cases} \mathrm{GL}(k_1,\mathbb{H}) \times \cdots \times \mathrm{GL}(k_s,\mathbb{H}) \times \mathrm{Sp}(p',q') & \text{if } G = \mathrm{Sp}(p,q) \\ \mathrm{GL}(k_1,\mathbb{H}) \times \cdots \times \mathrm{GL}(k_s,\mathbb{H}) \times \mathrm{SO}^*(2r) & \text{if } G = \mathrm{SO}^*(2n) \end{cases}.$$

Here, formally, we denote by $\mathrm{Sp}(0,0)$ and $\mathrm{SO}^*(0)$ the trivial group $\{1\}$. Any parabolic subgroup of G is G-conjugate to some P_κ. $\mathrm{GL}(k,\mathbb{H})$ has some particular irreducible unitary representation so-called quarternionic Speh representations defined as follows. We consider $\mathrm{GL}(k,\mathbb{C})$ as a subgroup of $\mathrm{GL}(k,\mathbb{H})$. For $\ell \in \mathbb{Z}$ and $t \in \sqrt{-1}\mathbb{R}$, we define a one-dimensional unitary representation $\xi_{\ell,t}$ of $\mathrm{GL}(k,\mathbb{C})$ as follows.

$$\xi_{\ell,t}(g) = \left(\frac{\det(g)}{|\det(g)|}\right)^{\ell} |\det(g)|^t.$$

$\mathrm{GL}(k,\mathbb{C})$ is the centerizer in $\mathrm{GL}(k,\mathbb{H})$ of the group consisting of scalar matrices with the eigenvalue of the absolute value one. So, there is a θ-stable parabolic subalgebra $\mathfrak{q}(k)$ with a Levi subgroup $\mathrm{GL}(k,\mathbb{C})$. We choose the nilradical $\mathfrak{n}(k)$ so that $\xi_{\ell,t}$ is good with respect to $\mathfrak{q}(k)$ for sufficient large ℓ. Derived functor modules with respect to $\mathfrak{q}(k)$ is called quarternionic Speh representations.

For $t \in \sqrt{-1}\mathbb{R}$, there is a one-dimensional unitary representation $\tilde{\xi}_t$ of $\mathrm{GL}(k,\mathbb{H})$ whose restriction to $\mathrm{GL}(k,\mathbb{C})$ is $\xi_{0,t}$. We put

$$A_k(\ell,t) = ({}^u\mathcal{R}^{\mathfrak{gl}(k,\mathbb{H})\otimes_{\mathbb{R}}\mathbb{C},\mathrm{Sp}(k)}_{\mathfrak{q}(k),\mathrm{O}(k)})^{k(k+1)}(\xi_{\ell+2k,t}) \quad (\ell \in \mathbb{Z}).$$

We also put

$$A_k(-\infty,t) = \tilde{\xi}_t.$$

For $\ell \in \mathbb{Z}$, $A_k(\ell,t)$ is derived functor module in the good (resp. weakly fair) range in the sense of [Vogan 1988] if and only if $\ell \geqslant 0$ (resp. $\ell \geqslant -k$). It is more or less known by [Vogan 1986] that any derived functor module of $\mathrm{GL}(k,\mathbb{H})$ is a unitarity parabolic induction from one-dimensional representations or quarternionic Speh representations. So, it suffices to consider the following induced representation.

$$(\circledast) \qquad \mathsf{Ind}^G_{P_\kappa}(A_{k_1}(\ell_1,t_1) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s,t_s) \boxtimes Z).$$

Here, Z is a derived functor module of $\mathrm{Sp}(p', q')$ or $\mathrm{SO}^*(2r)$ in the weakly fair range. Moreover, $\ell_i \in \{\ell \in \mathbb{Z} \mid \ell \geqslant -k_i\} \cup \{-\infty\}$, and $t_i \in \sqrt{-1}\mathbb{R}$ for $1 \leqslant i \leqslant s$. If we apply well-known Harish-Chandra's result on unitary induction, we may freely permute $A_{k_i}(\ell_i, t_i)$'s. We assume that $\ell_i + 1 \in 2\mathbb{Z}$ and $t_i = 0$ for some $1 \leqslant i \leqslant s$. Then, we may assume $i = s$. Let $\kappa' = (k_1, ..., k_{s-1})$. Then from the induction-by-stage, we have

$$\begin{aligned}&\mathsf{Ind}_{P_\kappa}^{G}(A_{k_1}(\ell_1, t_1) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s, t_s) \boxtimes Z)\\ &\qquad \cong \mathsf{Ind}_{P'_\kappa}^{G}(A_{k_1}(\ell_1, t_1) \boxtimes \cdots \boxtimes A_{k_{s-1}}(\ell_{s-1}, t_{s-1}) \boxtimes \mathsf{Ind}_{P_{(k_s)}}^{M^\circ_{\kappa'}}(A_{k_s}(\ell_s, 0) \boxtimes Z)).\end{aligned}$$

Here, $M_{\kappa'}$ is $\mathrm{Sp}(p' + k_s, q' + k_s)$ or $\mathrm{SO}^*(2(r + 2k_s))$.

Our reducibility result is:

Theorem A (Theorem3.6.5)

$\mathsf{Ind}_{P_{(k_s)}}^{M_{\kappa'}}(A_{k_s}(\ell_s, 0) \boxtimes Z)$ *is decomposed into a direct sum of derived functor modules of* $M^\circ_{\kappa'}$ *in weakly fair range. We obtain an explicit decomposition formula.*

Whenever there is $1 \leqslant i \leqslant s$ such that $\ell_i + 1 \in 2\mathbb{Z}$ and $t_i = 0$, we can apply the above procedure. Assuming that we understand the reducibility of derived functor modules, we can reduce the irreducible decomposition of the above induced module to the following.

$$(\Diamond) \qquad \mathsf{Ind}_{P_\kappa}^{G}(A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0) \boxtimes A_{k_{h+1}}(\ell_{h+1}, t_{h+1}) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s, t_s) \boxtimes Z).$$

Here, ℓ_i is not odd integer if $1 \leqslant i \leqslant h$, $\sqrt{-1}t_i > 0$ if $h < i \leqslant s$, and Z is an irreducible representation of M°_κ whose infinitesimal character plus the half sum of positive roots is appearing as some weights of finite dimensional representation of G. Put $\tau = (k_1, ..., k_h)$ and $\tau' = (k_{h+1}, ..., k_s)$. Also put $a = k_1 + \cdots + k_h$ and $b = k_{h+1} + \cdots + k_s$. In this setting we have:

Theorem B (Theorem 4.1.2) *The following is equivalent.*

(1) *The above* $\Diamond$ *is irreducible.*

(2) *The following induced module is irreducible.*

$$\mathsf{Ind}_{P_\tau}^{SO^*(4a)}(A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0)).$$

Under an adequate regularity condition on $\ell_1, ..., \ell_h$, we may show the irreducibility of the induced module in the above (2).

On the above (2), we have a partial answer:

Lemma C (Theorem 5.1.1)

If $\ell_1, ..., \ell_h$ are all $-\infty$ (namely, if $A_{k_1}(\ell_1, 0)$,..., $A_{k_h}(\ell_h, 0)$ are trivial representations,) $\mathsf{Ind}_{P_\tau}^{SO^*(4a)}(A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0))$ *is irreducible.*

As a corollary of this result, we have:

Corollary D (Corollary 5.1.2) *Representations of $SO^*(2n)$ and $Sp(p, q)$ induced from one-dimensional unitary representations of their parabolic subgroups are irreducible.*

For some of special parabolic subgroups, the irreducibility of the above kind of induced representation has been known. If the parabolic subgroup is minimal, the irreducibility of the induced representation is a special case of a general result in [Kostant 1969] (also see [Helgason 1970]). Studies of Johnson, Sahi, and Howe-Tan ([Johnson 1990], [Sahi 1993], [Howe-Tan 1993]) also include the irreducibility of the induced modules from a unitary one-dimensional representations of some maximal parabolic subgroups.

The remaining problems on the reducibility of the representations of $\mathrm{Sp}(p, q)$ and $\mathrm{SO}^*(2n)$ unitarily induced from derived functor modules in the weakly fair region are:

(1) Vanishing and irreducibilities of derived functor modules of $\mathrm{Sp}(p, q)$ and $\mathrm{SO}^*(2n)$ in the weakly fair range.

(2) Irreducibilities of the induced representation of the form:

$$\mathsf{Ind}_{P_\tau}^{SO^*(4a)}(A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0)).$$

(Here, $\ell_i \quad (1 \leqslant i \leqslant h)$) are even integers or $-\infty$.)

Regrettably, I do not have a complete answer on the above problem. For a type A group $U(m, n)$, general theories on translation principle are applicable to the above problem on irreduciblilty. Together with Trapa's result [Trapa 2001], we have a complete answer. Unfortunately, neither $\mathrm{Sp}(p, q)$ nor $\mathrm{SO}^*(2n)$ are of type A. So, situation is more difficult than the case of $U(m, n)$. In fact, irreducibility of a derived functor module of $\mathrm{Sp}(p, q)$ fails in some bad parameter (Vogan). If the degeneration of the parameter is not so bad, Vogan ([Vogan 1988]) found an idea to control irreducibilities. Using the idea, he proved irreducibility of discrete series of semisimple symmetric spaces. This idea works in this case. In fact, using Vogan's idea Kobayashi studied irreducibilities of derived functor modules of $\mathrm{Sp}(p, q)$ in [Kobayashi 1992]. In subsequent article, I would like to take up this problem.

One of the main ingredient of this article is the change-of-polarization formula (Theorem 2.2.3). It means we may exchange, under some positivity condition, the order of cohomological induction and parabolic induction in the Grothendieck group. The change of polarization for standard module was originated by Vogan ([Vogan 1983]) and completed by Hecht, Miličić, Schmid, and Wolf (cf. [Schmid 1988]). Also see [Knapp-Vogan 1995] Theorem 11.87. For the degenerate setting, some case is observed for $\mathrm{GL}(n)$ in [Vogan 1986]. We apply this idea in [Matumoto 1996]. In Theorem 2.2.3, we gave a formulation of the change-of-polarization in the general setting.

The other ingredient of this article is comparison of the Hecke algebra module structures. Using this idea, we show the above Theorem B.

§ 1. Preliminaries

1.1 General notations

In this article, we use the following notations.

As usual we denote the Hamilton quarternionic field, the complex number field, the real number field, the rational number field, the ring of integers, and the set of non-negative integers by $\mathbb{H}$, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively.

For a complex vector space V, we denote by V^* the dual vector space. For a real vector space V_0, we denote by V_0' the real dual vector space of V_0. We denote by $\emptyset$ the empty set and denote by $A - B$ the set theoretical difference of A from B. For each set A, we denote by $\mathrm{card} A$ the cardinality of A. For a complex number a (resp. a matrix X over $\mathbb{C}$), we denote by $\bar{a}$ (resp. $\overline{X}$) the complex conjugation. If $p > q$, we put $\sum_{i=p}^{q} = 0$.

Let R be a ring and let M be a left R-module. We denote by $\mathrm{Ann}_R(M)$ the annihilator of M in R.

In this article, a character of a Lie group G means a (not necessarily unitary) continuous homomorphism of G to $\mathbb{C}^\times$.

For a matrix $X = (a_{ij})$, we denote by ${}^tX$, $\mathrm{tr}X$, and $\det X$ the transpose (a_{ji}) of X, the trace of X, and the determinant of X respectively.

For a positive integer k, we denote by I_k (resp. 0_k) the $k \times k$-identity (resp. zero) matrix.

Let $n, n_1, ..., n_\ell$ be positive integers such that $n = n_1 + \cdots + n_\ell$. For $n_i \times n_i$-matrices X_i $(1 \leqslant i \leqslant \ell)$, we put

$$\mathrm{diag}(X_1, ..., X_\ell) = \begin{pmatrix} X_1 & 0 & \cdots\cdots & & 0 \\ 0 & X_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots\cdots & & 0 & X_\ell \end{pmatrix}$$

We denote by $\mathfrak{S}_\ell$ the ℓ-th symmetric group.

For a complex Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g})$ its universal enveloping algebra. We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$.

For a Harish-Chandra module V, we denote by $[V]$ the corresponding distribution character. In this article, an irreducible Harish-Chandra module should be non-zero.

1.2 Notations for root systems

Let G be a connected real reductive linear group, and let $G_{\mathbb{C}}$ be its complexification.

We fix a maximal compact subgroup K of G and denote by θ the corresponding Cartan involution. We denote by $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$) the Lie algebras of G (resp. K).

Let H be a θ-stable Cartan subgroup of G and let $\mathfrak{h}_0$ be its Lie algebra.

We denote by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{h}$ the complexification of $\mathfrak{g}_0$, $\mathfrak{k}_0$, and $\mathfrak{h}_0$, respectively.

We denote by $\mathfrak{h}^*$ the complex dual of $\mathfrak{h}$. We denote the induced involution from θ on $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{h}^*$ by the same letter θ. We denote by σ he complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

We denote by $W(\mathfrak{g},\mathfrak{h})$ (resp. $\Delta(\mathfrak{g},\mathfrak{h})$) the Weyl group (resp. the root system) with respect to the pair $(\mathfrak{g}, \mathfrak{h})$. Let $\langle\ .\ \rangle$ be the nondegenerate $W(\mathfrak{g},\mathfrak{h})$-invariant bilinear form on $\mathfrak{h}^*$ induced from the Killing form of $\mathfrak{g}$.

A root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ is called imaginary (resp. real) if $\theta(\alpha) = \alpha$ (resp. $\theta(\alpha) = -\alpha$). A root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ is called complex if α is neither real nor imaginary. A imaginary root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ is called compact (resp. noncompact) if the root space for α is contained (resp. not contained) in $\mathfrak{k}$.

We denote by $\mathcal{P}(\mathfrak{h})$ the integral weight lattice in $\mathfrak{h}^*$. Namely, we put

$$\mathcal{P}(\mathfrak{h}) = \left\{ \lambda \in \mathfrak{h}^* \,\middle|\, 2\frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}\ \ (\alpha \in \Delta(\mathfrak{g},\mathfrak{h})) \right\}.$$

We also put

$$\mathcal{P}_G(\mathfrak{h}) = \{\lambda \in \mathfrak{h}^* | \lambda \text{ appear as a weight of some finite dimensional representation of } G.\}.$$

We denote by $\mathcal{Q}(\mathfrak{h})$ the root lattice, namely the set of integral linear combination of elements of $\Delta(\mathfrak{g},\mathfrak{h})$. We have $\mathcal{Q}(\mathfrak{h}) \subseteq \mathcal{P}_G(\mathfrak{h}) \subseteq \mathcal{P}(\mathfrak{h}) \subseteq \mathfrak{h}^*$.

For $\lambda \in \mathfrak{h}^*$, we denote by χ_λ the corresponding Harish-Chandra homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$.

We fix a θ-stable maximally split Cartan subgroup ${}^sH$ of G and denote by ${}^s\mathfrak{h}$ its complexified Lie algebra. For simplicity, we write Δ, W, $\mathcal{P}$, $\mathcal{P}_G$, $\mathcal{Q}$ for $\Delta(\mathfrak{g},{}^s\mathfrak{h})$, $W(\mathfrak{g},{}^s\mathfrak{h})$, $\mathcal{P}({}^s\mathfrak{h})$, $\mathcal{P}_G({}^s\mathfrak{h})$, $\mathcal{Q}({}^s\mathfrak{h})$, respectively.

We choose regular weights $\lambda \in \mathfrak{h}^*$ and ${}^s\lambda \in {}^s\mathfrak{h}^*$ such that $\chi_\lambda = \chi_{{}^s\lambda}$. Then, there is a unique isomorphism $\mathbf{i}_{{}^s\lambda,\lambda} : {}^s\mathfrak{h}^* \to \mathfrak{h}^*$ induced from an inner automorphism of G such that $\mathbf{i}_{{}^s\lambda,\lambda}({}^s\lambda) = \lambda$. We denote by the same letter $\mathbf{i}_{{}^s\lambda,\lambda}$ the corresponding isomorphism of W onto $W(\mathfrak{g},\mathfrak{h})$.

1.3 Cohomological inductions

Next, we fix the notations on the Vogan-Zuckerman cohomological inductions of Harish-Chandra modules. Here, we adapt the definition found in [Knapp-Vogan 1995]. Let G be a real reductive linear Lie group which is contained in the complexification $G_{\mathbb{C}}$ which is a connected complex reductive linear group.

Definition 1.3.1. *Assume that a parabolic subalgebra* $\mathfrak{q}$ *has a Levi decomposition* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *such that* $\mathfrak{l}$ *is stable under* θ *and* σ*. Such a Levi decomposition is called an orderly Levi decomposition.*

A θ-stable or σ-stable parabolic subalgebra has a unique orderly Levi decomposition. In fact, if $\mathfrak{q}$ is θ (resp. σ)-stable, then $\mathfrak{l} = \mathfrak{q} \cap \sigma(\mathfrak{q})$ (resp. $\mathfrak{l} = \mathfrak{q} \cap \theta(\mathfrak{q})$).

Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with an orderly Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. We fix a θ and σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ and a Weyl group invariant non-degenerate bilinear form $\langle\ ,\ \rangle$. Let L be the corresponding Levi subgroup in G to $\mathfrak{l}$.

We denote by ${}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K}$ the right adjoint functor of the forgetful functor of the category of $(\mathfrak{g}, K)$-modules to the category of $(\mathfrak{q}, L \cap K)$-modules. Introducing trivial $\mathfrak{u}$-action, we regard an $(\mathfrak{l}, L \cap K)$-module as a $(\mathfrak{q}, L \cap K)$-module. So, we also regard ${}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K}$ as a functor of the category of $(\mathfrak{l}, L \cap K)$-modules to the category of $(\mathfrak{g}, K)$-modules. We denote by $({}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}$ the i-th right derived functor. (See [Knapp-Vogan 1995] p671)

Next, we review a normalized version. We denote by $\delta(\mathfrak{u})$ a one-dimensional representation of $\mathfrak{l}$ defined by $\delta(\mathfrak{u})(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{u}})$. Following [Knapp-Vogan 1995] p720, we define a one-dimensional representation $\mathbb{C}_{2\delta(\mathfrak{u})'}$ of L as follows. Here, we consider slightly more general setting. Let V be a finite dimensional semisimple $\mathfrak{l}$-module. We denote by $\delta(V)$ a one-dimensional representation of $\mathfrak{l}$ defined by $\delta(V)(X) = \frac{1}{2}\mathrm{tr}(X|_V)$. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be the decomposition of V into irreducible $\mathfrak{l}$-modules. We distinguish between those V_i that are self-conjugate with respect to σ and those are not. We define a one dimensional representation $\xi_{2\delta(V)'}$ of L on a space $\mathbb{C}_{2\delta(V)'}$ by

$$\xi_{2\delta(V)'}(\ell) = \left(\prod_{i \text{ with } V_i \text{ self-conjugate}} |\det(\ell|_{V=i})| \right) \left(\prod_{i \text{ with } V_i \text{ not self-conjugate}} \det(\ell|_{V=i}) \right).$$

Let $L^{\sim}$ be the metaplectic double cover of L with respective to $\mathbb{C}_{2\delta(V)'}$. Namely,

$$L^{\sim} = \{(\ell, z) \in L \times \mathbb{C}^{\times} \mid \xi_{2\delta(V)'}(\ell) = z^2\}.$$

We denote by $\mathbb{C}_{\delta(V)'}$ the one-dimensional $L^{\sim}$-module defined by the projection to the second factor of $L^{\sim} \subseteq L \times \mathbb{C}^{\times}$. Of course, the definition of $L^{\sim}$ depends on V. Hereafter, we consider the case of $V = \mathfrak{u}$ (the adjoint action of L on $\mathfrak{u}$). Let $(K \cap L)^{\sim}$ be the maximal compact subgroup of L corresponding to $L \cap K$.

Let Z be a Harish-Chandra $(\mathfrak{l},(K\cap L)^{\sim})$-module such that $Z\otimes\mathbb{C}_{\delta(\mathfrak{u})}$ is a Harish-Chandra $(\mathfrak{l},K\cap L)$-module. Introducing the trivial action of $\mathfrak{u}$, we also regard $Z\otimes\mathbb{C}_{\delta(\mathfrak{u})'}$ as a $\mathfrak{q}$-module. We put

$$({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}(Z)=({}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}(Z\otimes\mathbb{C}_{\delta(\mathfrak{u})'}).$$

Let λ be the infinitesimal character of Z with respect to $\mathfrak{h}$. (It is well-defined up to the Weyl group action of $\mathfrak{l}$.) Then $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}(Z)$ is a Harish-Chandra $(\mathfrak{g},K)$-module of an infiniteimal character λ.

We consider three particular cases.

(1) (*Hyperbolic case*) If $\mathfrak{q}$ is stable under the complex conjugation of $\mathfrak{g}$ with respect to G, there is a parabolic subgroup $Q=LU$ whose complexified Lie algebra is $\mathfrak{q}$ and whose nilradical is U. In this case, we have $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}(Z)=0$ for all $i>0$. In fact, $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{0}(Z)$ is nothing but the parabolic induction $\mathsf{Ind}_{Q}^{G}(Z)$.

We clarify the definition of the parabolic induction. First, we remark that $L^{\sim}$ is just a direct product $L\times\{\pm1\}$ in this case and $\mathbb{C}_{\delta(\mathfrak{u})}$ can be reduced to a representation of L (say $(\xi_{\delta(\mathfrak{u})},\mathbb{C}_{\delta(\mathfrak{u})})$) .

$\mathsf{Ind}_{Q}^{G}(Z)$ (or we also write $\mathsf{Ind}(Q\uparrow G;Z)$) is the K-finite part of

$$\{f\in C^{\infty}(G)\otimes H\mid f(g\ell n)=\pi(\ell^{-1})f(g)\quad(g\in G,\ell\in L,n\in U)\}.$$

Here, (π,H) is any Hilbert globalization of $Z\otimes\mathbb{C}_{\delta(\mathfrak{u})}$. If Z is unitarizable, so is $\mathsf{Ind}(Q\uparrow G;Z)$ (unitary induction). We also consider the unnormalized parabolic induction as follows.

$${}^{u}\mathsf{Ind}(Q\uparrow G;Z)=\mathsf{Ind}(Q\uparrow G;Z\otimes\mathbb{C}_{\delta(\bar{\mathfrak{u}})}).$$

We have the following additive property.

Let Z_0, Z_1,...,Z_k be Harish-Chandra $(\mathfrak{l},(L\cap K)^{\sim})$-modules such that $Z_0\otimes\mathbb{C}_{\rho(\mathfrak{u})}$,...,$Z_k\otimes\mathbb{C}_{\rho(\mathfrak{u})'}$ are reduced to Harish-Chandra $(\mathfrak{l},L\cap K)$-modules. Let n_1,..., n_k be integers. If we have a character identity $[Z]=\sum_{i=1}^{k}n_i[Z_i]$, we have $[\mathsf{Ind}(Q\uparrow G;Z)]=\sum_{i=1}^{k}n_k[\mathsf{Ind}(Q\uparrow G;Z_i)]$.

(2) (*Elliptic case*) Assume $\mathfrak{q}$ is θ-stable and put $S=\dim(\mathfrak{u}\cap\mathfrak{k})$. We call Z weakly good (or λ is in the weakly good range), if $\mathrm{Re}\langle\lambda,\alpha\rangle\geqslant 0$ holds for each root α of $\mathfrak{h}$ in $\mathfrak{u}$. We call Z integrally good (resp. weakly integrally good), if $\langle\lambda,\alpha\rangle>0$ (resp. $\langle\lambda,\alpha\rangle\geqslant 0$) holds for each root α of $\mathfrak{h}$ in $\mathfrak{u}$ such that $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$.

Theorem 1.3.2. *([Vogan 1984] Theorem2.6)*

(1) If Z is weakly integrally good, then $({}^n\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^i = 0$ for $i \neq S$.

(2) If Z is irreducible and weakly integrally good, $({}^n\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^S(Z)$ is irreducible or zero.

(3) If Z is irreducible and integrally good, $({}^n\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^S(Z)$ is irreducible.

(4) If Z is unitarizable and weakly good, $({}^n\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^S(Z)$ is unitarizable.

The additivity property of $({}^n\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^S$ in this case is described as follows. We fix an infinitesimal character λ in the integrally weakly good range with respect to $\mathfrak{q}$. Let Z_0, Z_1,...,Z_k be Harish-Chandra $(\mathfrak{l}, (L\cap K)^{\sim})$-modules with infinitesimal character λ and let n_1,..., n_k be integers. We also assume $Z_i \otimes \mathbb{C}_{\delta(\mathfrak{u})'}$ is reduced to a Harish-Chandra $(\mathfrak{l}, L\cap K)$-module for each $0 \leq i \leq k$. Moreover we assume a character identity $[Z] = \sum_{i=1}^k n_i[Z_i]$ holds. Then, we have $[({}^n\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^S(Z)] = \sum_{i=1}^k n_k[({}^n\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^S(Z_i)]$.

1.4 Standard modules

We retain notations in 1.2. Since we assume all the Cartan subgroups are connected, we can simplify description of some fundamental material on this subject.

A regular character (H, Γ, λ) is a pair satisfying the following conditions (R1)-(R6).

(R1) H is a θ-stable Cartan subgroup of G.

(R2) Γ is a (non-unitary) character of H.

(R3) λ is in $\mathfrak{h}^*$. (Here, $\mathfrak{h}$ is the complexified Lie algebra of H.)

(R4) λ is regular (with respect to $\Delta(\mathfrak{g},\mathfrak{h})$).

(R5) $\langle\lambda,\alpha\rangle$ is real for any imaginary root α in $\Delta(\mathfrak{g},\mathfrak{h})$.

In order to write down the last condition (R6), we introduce some notations. Let $\mathfrak{t}$ (resp. $\mathfrak{a}$) the $+1$ (resp. -1) eigenspace in $\mathfrak{h}$ with respect to θ. We denote by $\mathfrak{m}$ the centerizer of $\mathfrak{a}$ in $\mathfrak{g}$. Then $\Delta(\mathfrak{m},\mathfrak{h})$ is nothing but the set of imaginary roots in $\Delta(\mathfrak{g},\mathfrak{h})$. Under the above condition (R4), there is a unique positive system $\Delta^+_\lambda(\mathfrak{m},\mathfrak{h})$ of $\Delta(\mathfrak{m},\mathfrak{h})$ such that $\langle\alpha,\lambda\rangle > 0$. We denote by $\rho_\lambda(\mathfrak{m},\mathfrak{h})$ (resp. $\rho^c_\lambda(\mathfrak{m},\mathfrak{h})$) the half sum of positive imaginary roots (resp. compact imaginary roots) with respect to $\Delta^+_\lambda(\mathfrak{m},\mathfrak{h})$. We put $\mu_\lambda = \lambda + \rho_\lambda(\mathfrak{m},\mathfrak{h}) - 2\rho^c_\lambda(\mathfrak{m},\mathfrak{h})$.

(R6) μ_λ is the differential of Γ.

We fix a regular character $\gamma = (H,\Gamma,\lambda)$. We denote by M the centerizer of $\mathfrak{a}$ in G. Since H is connected, M is the analytic subgroup of G with respect to $\mathfrak{m}$. The above conditions (R1)-(R5) assure that there is a unique relative discrete series representation σ with infinitesimal character

λ such that the highest weight of the minimal $K \cap M$-type of σ is μ_λ. Here, a relative discrete series means a representation whose restriction to semisimple part is in discrete series. We do not require the unitarizability of σ itself. We fix a parabolic subgroup P of G such that M is a Levi part of P. We define the standard module $\pi_G(\gamma)$ (We often simply denote by $\pi(\gamma)$, if there is no confusion.) for a regular character $\gamma = (H, \Gamma, \lambda)$ by $\pi_G(\gamma) = \mathsf{Ind}_P^G(\sigma)$. The distribution character $[\pi_G(\gamma)]$ is independent of the choice of P.

We may describe $\pi_G(\gamma)$ in terms of the cohomological induction as follows. First, let $\mathfrak{b}_0$ be the Borel subalgebra of $\mathfrak{m}$ correspoinding to $(\mathfrak{h}, \Delta_\lambda^+(\mathfrak{m}, \mathfrak{h}))$ and let $\mathfrak{u}_1$ be its nilradical. Then $\mathfrak{b}_1$ is θ-stable and $\sigma \cong ({}^u\mathcal{R}_{\mathfrak{b}_1, H}^{\mathfrak{m}, M\cap K})^{\dim \mathfrak{u}_1 \cap \mathfrak{k}}(\Gamma \otimes \mathbb{C}_{2\delta(\mathfrak{u}_1 \cap \mathfrak{k})'})$. Let $\mathfrak{n}$ be the nilradical of the complexified Lie algebra of P. We put $\mathfrak{b} = \mathfrak{b}_1 + \mathfrak{n}$ and $\mathfrak{u} = \mathfrak{u}_1 + \mathfrak{n}$. $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{u}$ is the nilradical of $\mathfrak{b}$. Using the induction-by-stage formula ([Knapp-Vogan 1995] Corollary 11.86), we have

$$\pi_G(\gamma) \cong ({}^u\mathcal{R}_{\mathfrak{b}, H}^{\mathfrak{g}, K})^{\dim \mathfrak{u} \cap \mathfrak{k}}(\Gamma \otimes \mathbb{C}_{2\delta(\mathfrak{u}_1 \cap \mathfrak{k})'} \otimes \mathbb{C}_{\delta(\mathfrak{n})})$$

Ther are various presentations of the standard representation as a cohomological induction from a character on a Borel subalgebra (cf. [Schmid 1988], [Kanpp-Vogan 1995] XI).

For a regular character $\gamma = (H, \Gamma, \lambda)$ and $k \in K$, we put $k \cdot \gamma = (\mathrm{Ad}(k)H, \lambda \circ \mathrm{Ad}(k^{-1}))$. Then, $k \cdot \gamma$ is also a regular character. For two regular characters γ_1 and γ_2, $[\pi_G(\gamma_1)] = [\pi_G(\gamma_2)]$ if and only if $k \cdot \gamma_1 = \gamma_2$ for some $k \in K$.

A standard module $\pi_G(\gamma)$ has a unique irreducible subquotient (Langlands subquotient) $\bar{\pi}_G(\gamma)$ such that all the minimal K-types of $\pi_G(\gamma)$ is contained in $\bar{\pi}_G(\gamma)$. $\bar{\pi}_G(\gamma)$ is independent of the choice of P. Any irreducible Harish-Chandra $(\mathfrak{g}, K)$-module with a regular infinitesimal character is isomorphic to some $\bar{\pi}_G(\gamma)$, and for two regular characters γ_1 and γ_2, $\bar{\pi}_G(\gamma_1) \cong \bar{\pi}_G(\gamma_2)$ if and only if $k \cdot \gamma_1 = \gamma_2$ for some $k \in K$ (Langlands classification).

For a θ-stable Cartan subgroup H of G and a regular weight $\eta \in {}^s\mathfrak{h}^*$, we denote by $R_G(H, \eta)$ the set of the regular characters (H, Γ, λ) such that $\chi_\lambda = \chi_\eta$. For a regular weight $\eta \in {}^s\mathfrak{h}^*$, we denote by $R_G(\eta)$, the set of all the regular character γ such that $\pi(\gamma)$ has an infinitesimal character η. Namely, $R_G(\eta)$ is the union of $R_G(H, \eta)$'s.

We call a θ-stable cartan subgroup H of G η-integral if $R_G(H, \eta) \neq \emptyset$.

A root $\alpha \in \Delta$ is called real, complex, compact imaginary, noncompact imaginary with respect to $\gamma = (H, \Gamma . \lambda) \in R_G(\eta)$, if $\mathbf{i}_{\eta,\lambda}(\alpha)$ is real, complex, compact imaginary, noncompact imaginary, respectively. For $\gamma = (H, \Gamma . \lambda) \in R_G(\eta)$, we put $\theta_\gamma = \mathbf{i}_{\eta,\lambda}^{-1} \circ \theta \circ \mathbf{i}_{\eta,\lambda}$. θ_γ acts on Δ. Obviously, θ_γ

only depends on the K-conjugacy class of γ.

1.5 Coherent families

We retain notations in 1.2., and 1.4. In this section, we assume that all the Cartan subgroup of G is connected, for simplicity. ($SO^*(2n)$, $Sp(m,n)$, $GL(m,\mathbb{H})$, and their Levi subgroups satisfy this condition.) Under this assumption, we may write the regular character $(H,\Gamma,\lambda))$ as $(H,\lambda))$, since Γ is uniquely determined by λ We fix a regular weight ${}^s\lambda \in {}^s\mathfrak{h}^*$. Put $\Lambda = {}^s\lambda + \mathcal{P}_G$.

We denote by $W_{{}^s\lambda}$ (resp. $\Delta_{{}^s\lambda}$) the integral Weyl group (resp. the integral root system) for λ. Namely, we put

$$W_{{}^s\lambda} = \{w \in W \mid w^s\lambda - {}^s\lambda \in \mathcal{Q}\},$$

$$\Delta_{{}^s\lambda} = \left\{\alpha \in \Delta \mid \frac{\langle\alpha, {}^s\lambda\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}\right\}.$$

We put

$$\Delta^+_{{}^s\lambda} = \{\alpha \in \Delta_{{}^s\lambda} \mid \langle\alpha, {}^s\lambda\rangle > 0\}.$$

Then, $\Delta^+_{{}^s\lambda}$ is a positive system for $\Delta_{{}^s\lambda}$. We denote by $\Pi_{{}^s\lambda}$ the set of simple roots in $\Delta^+_{{}^s\lambda}$.

A map Θ of Λ to the space of invariant eigendistributions on G is called a coherent family on Λ if it satisfies the following conditions.

(C1) For all $\eta \in \Lambda$, $\Theta(\eta)$ is a complex linear combination of the distribution characters of Harish-Chandra modules with infinitesimal character η.

(C2) For any finite dimensional representation E, we have

$$[E]\Theta(\eta) = \sum_{\mu\in\mathcal{P}_G} [\mu : E]\Theta(\eta+\mu) \qquad (\eta \in \Lambda).$$

Here, $[\mu : E]$ means the multiplicity of the weight μ in E.

We denote by $\mathcal{C}(\Lambda)$ the set of coherent families on Λ. For $w \in W_{{}^s\lambda}$ and $\Theta \in \mathcal{C}(\Lambda)$, we define $w\cdot\Theta$ by $(w\cdot\Theta)(\eta) = \Theta(w^{-1}\eta)$. We see $\mathcal{C}(\Lambda)$ is a $W_{{}^s\lambda}$-representation. This representation is called the coherent continuation representation for Λ.

For any Harish-Chandra $(\mathfrak{g},K)$-module V with infinitesimal character ${}^s\lambda$, there is a unique coherent family Θ_V such that $\Theta_V({}^s\lambda) = [V]$. For a regular character $\gamma = (H,\Gamma.\lambda)$ such that $\chi_\lambda = \chi_{{}^s\lambda}$, we put $\Theta^G_\gamma = \Theta_{\pi_G(\gamma)}$ and $\bar{\Theta}^G_\gamma = \Theta_{\bar{\pi}_G(\gamma)}$. If $\eta \in \Lambda$ is regular and dominant (with respect to $\Delta^+_{^s\lambda}$), then $(H, \mathbf{i}_{^s\lambda,\lambda}))$ is a regular character and we have $\Theta^G_\gamma(\eta) = [\pi_G(H, \mathbf{i}_{^s\lambda,\lambda}(\eta))]$ and $\bar{\Theta}^G_\gamma(\eta) = [\bar{\pi}_G(H, \mathbf{i}_{^s\lambda,\lambda}(\eta))]$. Put $\mathbf{St}_G(^s\lambda) = \{\Theta^G_\gamma \mid \gamma \in R_G(^s\lambda)\}$ and $\mathbf{Irr}_G(^s\lambda) = \{\bar{\Theta}^G_\gamma \mid \gamma \in R_G(^s\lambda)\}$. We define a bijection $\Theta \rightsquigarrow \bar{\Theta}$ of $\mathbf{St}_G(^s\lambda)$ onto $\mathbf{Irr}_G(^s\lambda)$ by $\bar{\Theta^G_\gamma} = \bar{\Theta}^G_\gamma$ for $\gamma \in R_G(^s\lambda)$. $\mathbf{St}_G(^s\lambda)$ forms a basis of $\mathcal{C}(\Lambda)$ and so does $\mathbf{Irr}_G(^s\lambda)$.

We write $\bar{\Theta}^G_\gamma = \sum_{\Theta \in \mathbf{St}_G(^s\lambda)} M_G(\gamma, \Theta)\Theta$ and $M_G(\gamma, \delta) = M_G(\gamma, \Theta_\delta) \in \mathbb{C}$.

We also denote θ_γ by $\theta_{\Theta^G_\gamma}$.

For $\gamma = (H, \Gamma.\lambda) \in R_G(^s\lambda)$ and $w \in W_{^s\lambda}$, the cross product is defined as follows.

$$w \times \gamma = (H, \mathbf{i}_{^s\lambda,\lambda}(w)^{-1}\lambda).$$

Then, we have $w \times \gamma \in R_G(^s\lambda)$. Moreover, for any $\gamma, \gamma' \in R_G(^s\lambda)$ such that $\Theta^G_\gamma = \Theta^G_{\gamma'}$, we have $\Theta^G_{w\times\gamma} = \Theta^G_{w\times\gamma'}$ for all $w \in W_{^s\lambda}$. So, we put $w \times \Theta^G_\gamma = \Theta^G_{w\times\gamma}$.

1.6 Cayley transform

We fix the notation for Cayley transforms. We retain the notations in 1,2, etc.

Let H be a θ-stable Cartan subgroup of G and let $\mathfrak{h}$ be its complexified Lie algebra. We choose a non-compact imaginary root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$. We denote by $\check{H}_\alpha$ the element of $\mathfrak{h}$ such that $\langle \check{H}_\alpha, X\rangle = \alpha(X)$ for all $X \in \mathfrak{h}$. We may choose $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ such that $\sigma(X_\alpha) = X_{-\alpha}$ and $[X_\alpha, X_{-\alpha}] = \check{H}_\alpha$, where σ is the complex conjugation with respect to the real form of $\mathfrak{g}$ corresponding to G. Hence $\check{H}_\alpha$, X_α, and $X_{-\alpha}$ form an $\mathfrak{s}l(2)$-triple.

We introduce a standard complexified Cartan involution θ_0 on $\mathfrak{s}l(2,\mathbb{C})$ by $\theta_0(X) = -{}^tX$. We denote by σ_0 the complex conjugation with respect to $\mathfrak{s}l(2.\mathbb{R})$.

There is a Lie algebra homomorphism $\phi_\alpha : \mathfrak{s}l(2,\mathbb{C}) \to \mathfrak{g}$ which satisfies the following properties.

$$\phi_\alpha\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right) = \check{H}_\alpha, \quad \phi_\alpha\left(\frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \right) = X_\alpha, \quad \phi_\alpha\left(\frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}\right) = X_{-\alpha}.$$

ϕ_α satisfies $\phi_\alpha(\theta_0(X)) = \theta(\phi_\alpha(X))$ for all $X \in \mathfrak{s}l(2,\mathbb{C})$ and $\phi_\alpha(\sigma_0(X)) = \sigma(\phi_\alpha(X))$ for all $X \in \mathfrak{s}l(2,\mathbb{C})$. So, it induces a Lie group homomorphism $\Phi_\alpha : SL(2,\mathbb{C}) \to G_{\mathbb{C}}$, which maps $SL(2,\mathbb{R})$ into G. We put $c_\alpha = \Phi_\alpha\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right) \in G_{\mathbb{C}}$. $\mathrm{Ad}(c_\alpha)$ is called a Cayley transform. The image $\mathrm{Ad}(c_\alpha)(\mathfrak{h})$ of the complexified Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ under the Cayley transform is also a θ-stable Cartan subgroup of $\mathfrak{g}$, which is invariant under the complex conjugation with respect to the real form of $\mathfrak{g}$ corresponding to G. Formally, we denote by $\mathrm{Ad}(c_\alpha)(H)$ the corresponding θ-stable Cartan subgroup of G to $\mathrm{Ad}(c_\alpha)(\mathfrak{h})$.

Conversely, if $\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ is real, we can define $C^{\alpha} \in G_{\mathbb{C}}$ as follows. We choose $\check{H}_{\beta} \in \mathfrak{h}$, $X_{\beta} \in \mathfrak{g}_{\beta}$, and $X_{-\beta} \in \mathfrak{g}_{-\beta}$ similarly to the case of α. Then, there is a Lie algebra homomorphism $\phi^{\beta} : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ satisfying:

$$\phi^{\beta}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \check{H}_{\beta}, \quad \phi^{\beta}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = X_{\beta}, \quad \phi^{\beta}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = X_{-\beta}.$$

ϕ^{β} also satisfies $\phi^{\beta}(\theta_0(X)) = \theta(\phi^{\beta}(X))$ for all $X \in \mathfrak{sl}(2, \mathbb{C})$ and $\phi^{\beta}(\sigma_0(X)) = \sigma(\phi^{\beta}(X))$ for all $X \in \mathfrak{sl}(2, \mathbb{C})$. So, it induces a Lie group homomorphism $\Phi^{\beta} : SL(2, \mathbb{C}) \to G_{\mathbb{C}}$, which maps $SL(2, \mathbb{R})$ into G.

We put $c^{\beta} = \Phi^{\beta}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\right) \in G_{\mathbb{C}}$. $\mathrm{Ad}(c^{\beta})$ is also called a Cayley transform. Similarly, we define $\mathrm{Ad}(c^{\beta})(H)$ as the case of α. In this case, $\alpha = \beta \circ \mathrm{Ad}(c^{\beta})^{-1}$ is a noncompact imaginary root for $\mathrm{Ad}(c^{\beta})(\mathfrak{h})$ and $c^{\beta} = c_{\alpha}^{-1}$.

Next, we consider the Cayley transform of regular characters. Again, we assume that all the Cartan subgroup of G is connected. Fix $\gamma = (H, \lambda) \in R_G(H, {}^s\lambda)$, and choose $\alpha \in \Delta_{{}^s\lambda}$ such that α is noncompact imaginary with respect to γ. we put $c_{\alpha}(\gamma) = (Ad(c_{\mathbf{i}_{{}^s\lambda,\lambda}(\alpha)})(H), \lambda\cdot\mathrm{Ad}(c_{\mathbf{i}_{{}^s\lambda,\lambda}(\alpha)})^{-1})$. Then, we have $c_{\alpha}(\gamma) \in R_G({}^s\lambda)$ and α is real with respect to $c_{\alpha}(\gamma)$. It is easy to see $c_{\alpha}(\Theta_{\gamma}^G) = \Theta_{c_{\alpha}(\gamma)}^G$ is well-defined.

Conversely, consider $\gamma \in R_G({}^s\lambda)$ and $\alpha \in \Delta_{{}^s\lambda}$ which is real with respect to γ. We call α satisfies the parity condition with respect to γ, if there is some $\gamma' \in R_G({}^s\lambda)$ such that α is noncompact imaginary with respect to γ' and $\gamma = c_{\alpha}(\gamma')$. If α satisfies the parity condition with respect to γ, there are just two regular characters in $R_G(\mathrm{Ad}(c^{\mathbf{i}_{{}^s\lambda,\lambda}(\alpha)})(H), {}^s\lambda)$, say $c_{+}^{\alpha}(\gamma)$ and $c_{-}^{\alpha}(\gamma)$, in the preimage of γ with respect to c_{α}. Since we assume that all the Cartan subgroups of G are connected, $c_{\pm}^{\alpha}(\gamma)$ are not K-conjugate to each other. It is easy to see $c_{\pm}^{\alpha}(\Theta_{\gamma}^G) = \Theta_{c_{\pm}^{\alpha}(\gamma)}^G$ is well-defined.

1.7 Hecke algebra module structure

We retain the notations of 1.2, etc. and fix a regular weight ${}^s\lambda \in {}^s\mathfrak{h}^*$. Put $\Lambda = {}^s\lambda + \mathcal{P}_G$ as before.

First, we recall the definition of the Hecke algebra $H(W_{{}^s\lambda})$ for $W_{{}^s\lambda}$. For $w \in W_{{}^s\lambda}$, we denote by $\ell(w)$ the length of w with respect to the simple system $\Pi_{{}^s\lambda}$. Let q be an indeterminant. The Hecke algebra $H(W_{{}^s\lambda})$ is a $\mathbb{C}[q]$-algebra with a basis $\{T_w \mid w \in W_{{}^s\lambda}\}$, subject to the relations:

$$T_{w_1}T_{w_2} = T_{w_1 w_2} \quad (w_i \in W_{{}^s\lambda}, \ell(w_1) + \ell(w_2) = \ell(w_1 w_2)),$$

$$(T_{s_\alpha}+1)(T_{s_\alpha}-q)=0 \quad (\alpha \in \Pi_{^s\lambda}).$$

Put $\mathcal{C}(\Lambda)_q = \mathcal{C}(\Lambda)\otimes_{\mathbb{C}} \mathbb{C}[q]$. We introduce $H(W_{^s\lambda})$-module structure on $\mathcal{C}(\Lambda)_q$ as follows. For each $\gamma \in R_G(^s\lambda)$ and $\alpha \in \Pi_{^s\lambda}$, we put

$$T_{s_\alpha}\Theta^G_\gamma = q\Theta^G_\gamma \quad \text{if } \alpha \text{ is compact imaginary with respect to } \gamma,$$

$$T_{s_\alpha}\Theta^G_\gamma = s_\alpha \times \Theta^G_\gamma + c_\alpha(\Theta^G_\gamma) \quad \text{if } \alpha \text{ is noncompact imaginary with respect to } \gamma,$$

$$T_{s_\alpha}\Theta^G_\gamma = s_\alpha \times \Theta^G_\gamma \quad \text{if } \alpha \text{ is complex with respect to } \gamma \text{ and } \theta_\gamma(\alpha) \in \Delta^+_{^s\lambda},$$

$$T_{s_\alpha}\Theta^G_\gamma = q(s_\alpha \times \Theta^G_\gamma) + (q-1)\Theta^G_\gamma \quad \text{if } \alpha \text{ is complex with respect to } \gamma \text{ and } \theta_\gamma(\alpha) \notin \Delta^+_{^s\lambda},$$

$$T_{s_\alpha}\Theta^G_\gamma = (q-2)\Theta^G_\gamma + (q-1)(c^\alpha_+(\Theta^G_\gamma) + c^\alpha_-(\Theta^G_\gamma)) \quad \text{if } \alpha \text{ is real and satisfies the parity condition,}$$

$$T_{s_\alpha}\Theta^G_\gamma = -\Theta^G_\gamma \quad \text{if } \alpha \text{ is real and does not satisfy the parity condition.}$$

The important thing is that the Hecke algebra module structure is completely determined by the action of cross product and Cayley transforms on the K-conjugacy classes of regular characters in $R_G(^s\lambda)$.

If we consider the specialization at $q=1$ of this Hecke algebra module $\mathcal{C}(\Lambda)_q$, then we have a $W_{^s\lambda}$-representation on $\mathcal{C}(\Lambda)$. The relation to the coherent continuation representation is given as follows.

Theorem 1.7.1. *([Vogan (green), 1982])*

We have an isomorphism

(Specialization of $\mathcal{C}(\Lambda)_q$ at $q=1$) $\cong$ (Coherent continuation representation) $\otimes$ sgn,

where sgn means the signature representation of $W_{^s\lambda}$. This isomorphism preserves the basis $\mathbf{St}_G(^s\lambda)$.

The following result is crucial in our proof.

Theorem 1.7.2. *(see [Vogan 1983], [Adams-Barbasch-Vogan 1992] Chapter 16)*

For $\gamma, \delta \in R_G(^s\lambda)$, the complex number $M(\gamma,\delta)$ defined in 1.5 is computed from a certain algorithm (the Kazhdan-Lusztig type algorithm) which depends only on the Hecke algebra structure on $\mathcal{C}(\Lambda)_q$.

1.8 Cell structure

We retain the notations of 1.2, etc. and fix a regular weight ${}^s\lambda \in {}^s\mathfrak{h}^*$. Put $\Lambda = {}^s\lambda + \mathcal{P}_G$ as before.

A subrepresentation of $\mathcal{C}(\Lambda)$ is called basal, if it is generated by a subset of $\mathbf{Irr}_G({}^s\lambda)$ as a $\mathbb{C}$-vector space. For $\gamma \in R_G({}^s\lambda)$, we denote by $\mathrm{Cone}(\gamma)$ the smallest basal subrepresentation of $\mathcal{C}(\Lambda)$ which contains $\bar{\Theta}^G_\gamma$. For $\gamma, \eta \in R_G({}^s\lambda)$, we write $\gamma \sim \eta$ (resp. $\gamma \leq \eta$) if $\mathrm{Cone}(\gamma) = \mathrm{Cone}(\eta)$ (resp. $\mathrm{Cone}(\gamma) \supseteq \mathrm{Cone}(\eta)$). Obviously $\sim$ is an equivalence relation on $R_G({}^s\lambda)$. For $\gamma \in R_G({}^s\lambda)$ let $s(\gamma)$ be the set of regular characters $\eta \in R_G({}^s\lambda)$ such that $\lambda \leq \eta$ and $\lambda \not\sim \eta$. We define $\mathrm{Cell}(\gamma) = \mathrm{Cone}(\gamma)/\sum_{\eta \in s(\gamma)} \mathrm{Cone}(\eta)$. A cell (resp. cone) for $\mathcal{C}(\Lambda)$ is a subquotient (resp. subrepresentation) of $\mathcal{C}(\Lambda)$ of the form $\mathrm{Cell}(\gamma)$ (resp. $\mathrm{Cone}(\gamma)$) for some $\gamma \in R_G({}^s\lambda)$.

For each cell, we can associate a nilpotent orbit in $\mathfrak{g}$ as follows. For $\mathrm{Cell}(\gamma)$, we consider an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module $\bar{\pi}(\gamma)$. The annihilator (say I) of $\bar{\pi}(\gamma)$ in $U(\mathfrak{g})$ is a primitive ideal of $U(\mathfrak{g})$ and its associated variety is the closure of a single nilpotent orbit in $\mathfrak{g}$. The nilpotent orbit constructed above is independent of the choice of γ and we say it the associated nilpotent orbit for the cell $\mathrm{Cell}(\gamma)$.

For each cone $\mathrm{Cone}(\gamma)$, some canonical (up to scalar factor) construction of W-homomorphism (say ϕ_γ) of $\mathrm{Cone}(\gamma)$ to the realization as a Goldie rank polynomial representation of the special W-representation corresponding to the associated nilpotent orbit via the Springer correspondence ([Vogan 1978], [King 1981]). In fact this ϕ_γ factors the cell $\mathrm{Cell}(\gamma)$. An important fact is $\phi_\gamma(\bar{\Theta}^G_\eta)$ is nonzero and proportional to the Goldie rank polynomial of the annihilator of $\bar{\pi}(\eta)$ in $U(\mathfrak{g})$ for all $\eta \sim \gamma$ ([King 1981], [Joseph 1980]). Hence, the multiplicity in $\mathrm{Cell}(\gamma)$ of the special W-representation corresponding to the associated nilpotent orbit via the Springer correspondence is at least one.

1.9 Induction of a coherent family

We retain notations as above. Let P be a parabolic subgroup of G with θ-stable Levi part L such that ${}^sH \subseteq L$. (We remark that all the Cartan subgroups of L are connected.) We fix a regular character ${}^s\lambda \in {}^s\mathfrak{h}^*$ as above. Put $\Lambda_L = {}^s\lambda + \mathcal{P}_L$ and $\Lambda_G = {}^s\lambda + \mathcal{P}_G$. Then, we easily see $\Lambda_G \subseteq \Lambda_L$. Let Θ be a coherent family on Λ_L. For fixed $\nu \in \Lambda_G$, we write $\Theta(\nu) = \sum_{i=1}^n a_i[V_i]$, where V_i are certain Harish-Chandra $(\mathfrak{l}, K \cap L)$-modules with infinitesimal character ν and a_i are complex numbers. We write $\mathsf{Ind}^G_L(\Theta)(\nu) = \sum_{i=1}^n a_i[\mathsf{Ind}^G_P(V_i)]$. The above definition is independent of the choice of the linear combination, since the parabolic induction is exact. From a property of induction, the above definition depends only on L and does not depend on P. Moreover, $\nu \rightsquigarrow \mathsf{Ind}_L^G(\Theta)(\nu)$ forms a coherent family on Λ_G, thanks to a version of MacKey's tensor product theorem ([Speh-Vogan 1980] Lemma 5.8) for induction and the exactness of the induction.

Let H be a θ-stable Cartan subgroup of L. Hence H is also a Cartan subgroup of G. Let $\gamma = (H, \Gamma.\lambda)$ be a regular character for L with an infinitesimal character ${}^s\lambda$. Then γ is also a regular character for G. We easily see $\mathsf{Ind}_L^G(\Theta_\gamma^L) = \Theta_\gamma^G$.

1.10 Comparison of Hecke module structures

Let G be any connected real reductive linear Lie group whose Cartan subgroups are all connected. We define θ, K, ${}^sH$, $\mathfrak{g}$, $\mathfrak{k}$, ${}^s\mathfrak{h}$, etc. as in § 1.

Besides G we also consider another real reductive linear Lie group G' whose Cartan subgroups are all connected. We denote the objects with respect to G' by attaching the "prime" to the notations for the corresponding objects for G. For example, we fix a Cartan involution θ' for G' and fix a θ'-invariant maximally split Cartan subgroup ${}^sH'$, etc. We fix a regular weights ${}^s\lambda \in {}^s\mathfrak{h}^*$ and put $\Lambda = {}^s\lambda + \mathcal{P}_G$. Moreover, we assume the following conditions on G and G'.

(C1) There is a linear isomorphism $\psi : {}^s\mathfrak{h}^* \to ({}^s\mathfrak{h}')^*$ such that $\psi(\Delta_{{}^s\lambda}) = \Delta'$. Here, Δ' means the root system with respect to $(\mathfrak{g}', {}^s\mathfrak{h}')$. Moreover, $\psi({}^s\lambda)$ is regular integral with respect to Δ' and $\psi(\mathcal{P}_G) \subseteq \mathcal{P}_{G'}$. ψ induces an isomorphism $\psi_\sharp : W_{{}^s\lambda} \to W'$. Here, W' is the Weyl group for Δ'.

(C2) There is a bijection Ψ of the K-conjugacy classes of ${}^s\lambda$-integral θ-invariant Cartan subgroups of G to the K'-conjugacy classes of $\psi({}^s\lambda)$-integral θ'-invariant Cartan subgroups of G'.

(C3) There is a bijection $\tilde{\Psi} : \mathbf{St}_G({}^s\lambda) \to \mathbf{St}_{G'}(\psi({}^s\lambda))$ which is compatible with Ψ in (C2).

(C4) For $\Theta \in \mathbf{St}_G({}^s\lambda)$, we have $\psi \circ \theta_\Theta = \theta_{\tilde{\Theta}} \circ \psi$. Hence, for $\alpha \in \Delta_{{}^s\lambda}$, we have α is imaginary, complex, real with respect to Θ if and only if $\psi(\alpha)$ is imaginary, complex, real, respectively, with respect to $\tilde{\Psi}(\Theta)$.

(C5) Let $\alpha \in \Delta_{{}^s\lambda}$ and $\Theta \in \mathbf{St}_G({}^s\lambda)$. If α is imaginary, we have α is compact with respect to Θ if and only if $\psi(\alpha)$ is compact with respect to $\tilde{\Psi}(\Theta)$. If α is real, we have α satisfies the parity condition with respect to Θ if and only if $\psi(\alpha)$ satisfies the parity condition with respect to $\tilde{\Psi}(\Theta)$.

(C6) $\tilde{\Psi}$ is compatible with the cross actions. Namely, for $w \in W_{^s\lambda}$ and $\Theta \in \mathbf{St}_G({}^s\lambda)$ we have $\psi_\sharp(w) \times \tilde{\Psi}(\Theta) = \tilde{\Psi}(w \times \Theta)$.

(C7) $\tilde{\Psi}$ is compatible with the Cayley transform. Namely, if $\Theta \in \mathbf{St}_G({}^s\lambda)$ and if $\alpha \in \Delta_{^s\lambda}$ is noncompact imaginary with respect to Θ, then we have $\tilde{\Psi}(c_\alpha(\Theta)) = c_{\psi(\alpha)}(\tilde{\Psi}(\Theta))$. Moreover, if $\Theta \in \mathbf{St}_G({}^s\lambda)$ and if $\alpha \in \Delta_{^s\lambda}$ is real and satisfies the parity condition with respect to Θ, we have $\tilde{\Psi}(c^\alpha_\pm(\Theta)) = c^{\psi(\alpha)}_\pm(\tilde{\Psi}(\Theta))$.

Put $\Lambda' = \psi({}^s\lambda) + \mathcal{P}_{G'}$. Since $\mathbf{St}_G({}^s\lambda)$ (resp. $\mathbf{St}_G(\psi({}^s\lambda))$) forms a basis of $\mathcal{C}(\Lambda)$ (resp. $\mathcal{C}(\Lambda')$), $\tilde{\Psi}$ in (C3) extends to a linear (resp. $\mathbb{C}[q]$-module) isomorphism of $\mathcal{C}(\Lambda)$ (resp. $\mathcal{C}(\Lambda)_q$) onto $\mathcal{C}(\Lambda')$ (resp. $\mathcal{C}(\Lambda')$). We denote these isomorphisms of complex vector spaces and $\mathbb{C}[q]$-modules by the same letter $\tilde{\Psi}$. If we identify $W_{^s\lambda}$ and W' via the isomorphism $\psi_\sharp$ in(C1) above, we can regard $\mathcal{C}(\Lambda')$ (resp. $\mathcal{C}(\Lambda')_q$) as a $W_{^s\lambda}$-representation (resp. a $H(W_{^s\lambda})$-module).

Examining the description on the Hecke algebra module structures in 1.6, we easily see the conditions (C4)-(C7) imply $\tilde{\Psi}$ is $H(W_{^s\lambda})$-module isomorphism of $\mathcal{C}(\Lambda)_q$ onto $\mathcal{C}(\Lambda')_q$. From Theorem 1.7.1, we also see $\tilde{\Psi} : \mathcal{C}(\Lambda) \to \mathcal{C}(\Lambda')$ is an isomorphism between coherent continuation representations.

From Theorem 1.7.2 (Kazhdan-Lusztig type algorithm for Harish-Chandra modules), we see $\overline{\tilde{\Psi}(\Theta)} = \tilde{\Psi}(\bar{\Theta})$ for all $\Theta \in \mathbf{St}_G({}^s\lambda)$. Here, $\Theta \leadsto \bar{\Theta}$ is a bijection of $\mathbf{St}_G({}^s\lambda)$ (resp. $\mathbf{St}_G(\psi({}^s\lambda))$) onto $\mathbf{Irr}_G({}^s\lambda)$ (resp. $\mathbf{Irr}_G(\psi({}^s\lambda))$) defined in 1.5.

Moreover, we have:

Lemma 1.10.1. *Under the above setting, we have the followings. Let $\eta \in \Lambda$ and let $\Xi \in \mathcal{C}(\Lambda)$. Assume that there exists an irreducible Harish-Chandra $(\mathfrak{g}', K')$-module V' such that $\tilde{\Psi}(\Xi)(\psi(\eta)) = [V']$.*

Then, there is some irreducible Harish-Chandra $(\mathfrak{g}, K)$-module V with the infinitesimal character η such that $\Xi(\eta) = [V]$.

Proof There is some $w \in W_{^s\lambda}$ such that $\langle \alpha, w\eta \rangle \geq 0$ for all $\alpha \in \Delta^+_{^s\lambda}$. We write $w\Xi = \sum_{\bar{\Theta} \in \mathbf{Irr}_G({}^s\lambda)} c_{\bar{\Theta}} \bar{\Theta}$. Since $\tilde{\Psi}(\Xi)(\psi(\eta)) = \tilde{\Psi}(w\Xi)(\psi(w\eta))$, we have $[V'] = \sum_{\bar{\Theta} \in \mathbf{Irr}_G({}^s\lambda)} c_{\bar{\Theta}} \tilde{\Psi}(\bar{\Theta})(\psi(w\eta))$. It is known that there is a unique $\bar{\Upsilon}_0 \in \mathbf{Irr}_{G'}({}^s\lambda)$ such that $\bar{\Upsilon}_0(\psi(w\eta)) = [V']$ (cf. [Vogan (green)] Theorem 7.2.7). Put $\bar{\Theta}_0 = \tilde{\Psi}^{-1}(\bar{\Upsilon})$. For any $\bar{\Theta} \in \mathbf{Irr}_{G'}({}^s\lambda)$ either $\bar{\Theta}(\psi(w\eta)) = 0$ or $\bar{\Theta}(\psi(w\eta)) = [X]$ for some irreducible Harish-Chandra module X (cf. [Vogan 1983] Theorem 7.6). Hence, we have $c_{\bar{\Theta}_0} = 1$ and if $c_{\bar{\Theta}} \neq 0$ and $\bar{\Theta} \neq \bar{\Theta}_0$ then $\tilde{\Psi}(\bar{\Theta})(\psi(w\eta)) = 0$. From [Vogan 1983] Theorem 7.6 (also see [Vogan 1983] Definition 5.3), the above (C1)-(C7) imply that

$\tilde{\Psi}(\bar{\Theta})(\psi(w\eta)) = 0$ if and only if $\bar{\Theta}(w\eta) = 0$ for all $\bar{\Theta} \in \mathbf{Irr}_G({}^s\lambda)$. Hence, we have $\bar{\Theta}(w\eta) = 0$ if $c_{\bar{\Theta}} \neq 0$ and $\bar{\Theta} \neq \bar{\Theta}_0$. Moreover, there is an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module V such that $\bar{\Theta}_0(w\lambda) = [V]$. Therefore $\Xi(\eta) = (w\Xi)(w\eta) = \sum_{\bar{\Theta} \in \mathbf{Irr}_G({}^s\lambda)} c_{\bar{\Theta}} \bar{\Theta}(\psi(w\eta)) = \bar{\Theta}_0(w\eta) = [V]$. Q.E.D.

§ 2. Change of polarization

2.1 $\sigma\theta$ pair

We consider here the following setting.

Let G be a real reductive linear Lie group which is contained in the complexification $G_{\mathbb{C}}$. We fix a maximal compact subgroup K of G and let θ be the corresponding Cartan involution. We denote by $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$)the Lie algebra of G (resp. K) and denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$) its complexification. We denote also by the same letter θ the complexified Cartan involution on $\mathfrak{g}$. We denote by σ he complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

Definition 2.1.1. *A pair $(\mathfrak{p}, \mathfrak{q})$ is called a $\sigma\theta$ pair of parabolic subalgebras, if it satisfies the following conditions (S1-2)*

(S1) $\mathfrak{q}$ *(resp. $\mathfrak{p}$) is a θ-stable (resp. σ-stable) parabolic subalgebra of $\mathfrak{g}$.*

(S2) *There exists a θ and σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{p} \cap \mathfrak{q}$.*

Hereafter, we fix a $\sigma\theta$ pair $(\mathfrak{p}, \mathfrak{q})$. Let $\mathfrak{h}$ be any θ and σ-stable Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p} \cap \mathfrak{q}$. For $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}))$, we denote by $\mathfrak{g}_\alpha$ (resp. s_α) the root space (resp. the reflection) corresponding to α. Since $\mathfrak{h}$ is θ-stable, θ and σ induce actions on $\Delta(\mathfrak{g}, \mathfrak{h})$. We easily see $\theta\alpha = -\sigma\alpha$ for any $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$.

For a subspace U in $\mathfrak{g}$, we denote by $\Delta(U)$ the set of roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ whose root space is contained in U. We put

$$\mathfrak{m} = \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{p}) \cap (-\Delta(\mathfrak{p}))} \mathfrak{g}_\alpha, \quad \mathfrak{n} = \sum_{\alpha \in \Delta(\mathfrak{p}) - \Delta(\mathfrak{m})} \mathfrak{g}_\alpha, \quad \bar{\mathfrak{n}} = \sum_{\alpha \in \Delta(\mathfrak{n})} \mathfrak{g}_{-\alpha},$$

$$\mathfrak{l} = \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{q}) \cap (-\Delta(\mathfrak{q}))} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \sum_{\alpha \in \Delta(\mathfrak{q}) - \Delta(\mathfrak{l})} \mathfrak{g}_\alpha, \quad \bar{\mathfrak{u}} = \sum_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}_{-\alpha}.$$

We immediately see $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ (resp. $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$) is an orderly Levi decomposition of $\mathfrak{q}$ (resp. $\mathfrak{p}$) and the nilradical satisfies $\sigma(\mathfrak{u}) = \bar{\mathfrak{u}}$ (resp. $\theta(\mathfrak{n}) = \bar{\mathfrak{n}}$). Moreover, $\bar{\mathfrak{u}}$ (resp. $\bar{\mathfrak{n}}$) is the opposite nilradical to $\mathfrak{u}$ (resp. $\mathfrak{n}$).

We denote by $L_{\mathbb{C}}$, $P_{\mathbb{C}}$, and $M_{\mathbb{C}}$ the analytic subgroups of $G_{\mathbb{C}}$ with respect to $\mathfrak{l}$, $\mathfrak{p}$, and $\mathfrak{m}$, respectively. We put $L = L_{\mathbb{C}} \cap G$, $P = P_{\mathbb{C}} \cap G$, $M = M_{\mathbb{C}} \cap G$.

We easily have:

Proposition 2.1.2. *Under the above setting, we have the followings.*

(S3) $\mathfrak{l} \cap \mathfrak{p}$ *is a parabolic subalgebra of* $\mathfrak{l}$ *and* $L \cap P$ *is a parabolic subgroup of* L.

(S4) $\mathfrak{m} \cap \mathfrak{q}$ *is a parabolic subalgebra of* $\mathfrak{m}$.

(S5) $\mathfrak{l} \cap \mathfrak{m}$ *is a* θ *and* σ*-stable Levi subalgebra of the both* $\mathfrak{l} \cap \mathfrak{p}$ *and* $\mathfrak{m} \cap \mathfrak{q}$.

For a subspace U in $\mathfrak{g}$, we denote by $\Delta(U)$ the set of roots in Δ whose root space is contained in U. We also write $\rho(U) = \frac{1}{2}\sum_{\alpha \in \Delta(U)} \in \mathfrak{h}^*$. For a Borel subalgebra $\mathfrak{b}$, we write $\Delta^+_{\mathfrak{b}}$ for $\Delta(\mathfrak{b})$. $\Delta^+_{\mathfrak{b}}$ is a positive system of $\Delta(\mathfrak{g}, \mathfrak{h})$

Put $\tilde{\mathfrak{n}} = \mathfrak{u} \cap \mathfrak{m} + \mathfrak{n}$, $\tilde{\mathfrak{u}} = \mathfrak{n} \cap \mathfrak{l} + \mathfrak{u}$, $\tilde{\mathfrak{p}} = \mathfrak{l} \cap \mathfrak{m} + \tilde{\mathfrak{n}}$, and $\tilde{\mathfrak{q}} = \mathfrak{l} \cap \mathfrak{m} + \tilde{\mathfrak{u}}$. Then $\tilde{\mathfrak{p}}$ (resp. $\tilde{\mathfrak{q}}$) is a parabolic subalgebra of $\mathfrak{g}$ with a Levi part $\mathfrak{l} \cap \mathfrak{m}$ and the nilradical $\tilde{\mathfrak{n}}$ (resp. $\tilde{\mathfrak{u}}$).

We fix any Borel subalgebra $\mathfrak{b}^0$ of $\mathfrak{l} \cap \mathfrak{m}$ containing $\mathfrak{h}$. We put $\mathfrak{b}_1 = \mathfrak{b}^0 + \tilde{\mathfrak{n}}$ and $\mathfrak{b}_2 = \mathfrak{b}^0 + \tilde{\mathfrak{u}}$. Obviously, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are Borel subalgebras of $\mathfrak{g}$. Let $\mathfrak{v}$, $\mathfrak{v}_1$, and $\mathfrak{v}_2$ be the nilradical of $\mathfrak{b}^0$, $\mathfrak{b}_1$, and $\mathfrak{b}_2$, respectively. Put $\mathfrak{d} = \mathfrak{v} + \mathfrak{n} \cap \mathfrak{l} + \mathfrak{u} \cap \mathfrak{m} + \mathfrak{u} \cap \mathfrak{n}$. Then, we easily see $\mathfrak{v}_1 = \mathfrak{d} \oplus \mathfrak{n} \cap \bar{\mathfrak{u}}$ and $\mathfrak{v}_2 = \mathfrak{d} \oplus \bar{\mathfrak{n}} \cap \mathfrak{u}$.

Lemma 2.1.3. *We have*

$$\dim \mathfrak{u} \cap \mathfrak{k} - \dim \mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k} = \dim \mathfrak{u} \cap \bar{\mathfrak{n}}.$$

Proof Since $\mathfrak{g} = \mathfrak{m} \oplus \bar{\mathfrak{n}} \oplus \mathfrak{n}$ and $\mathfrak{u}$ is θ-stable, we have $\dim \mathfrak{u} \cap \mathfrak{k} - \dim \mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k} = \dim((\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n})) \cap \mathfrak{k}$. Let $p : (\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n}) \to \mathfrak{u} \cap \bar{\mathfrak{n}}$ be the projection to the first factor. Since $\mathfrak{n} \cap \mathfrak{k} = 0$, the restriction of p to $((\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n})) \cap \mathfrak{k}$ is an injection. On the other hand, for any $X \in \mathfrak{u} \cap \bar{\mathfrak{n}}$, we have $X \oplus \theta X \in ((\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n})) \cap \mathfrak{k}$. So, the restriction of p to $((\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n})) \cap \mathfrak{k}$ is onto. ■

Lemma 2.1.4. *Put* $d = \dim \mathfrak{u} \cap \bar{\mathfrak{n}}$. *There exits a sequence of complex roots* $\alpha_1, \ldots .\alpha_d \in \Delta(\mathfrak{g}, \mathfrak{h})$ *satisfying the following conditions (1)-(4). For* $1 \leqslant k \leqslant d$, *we put* $\Delta^+_k = s_{\alpha_k} \cdots s_{\alpha_1} \Delta^+_{\mathfrak{b}_1}$. *We also put* $\Delta^+_0 = \Delta^+_{\mathfrak{b}_1}$.

(1) *For* $1 \leqslant k \leqslant d$, $\alpha_k \in \Delta(\mathfrak{n} \cap \bar{\mathfrak{u}})$.

(2) *For* $1 \leqslant k \leqslant d$, $\Delta(\mathfrak{d}) \subseteq \Delta^+_k$.

(3) *For* $1 \leqslant k \leqslant d$, α_k *is simple with respect to* Δ^+_{k-1}.

(4) *For* $1 \leqslant k \leqslant d$, $\theta\alpha_k \notin \Delta^+_{k-1}$.

(5) *For* $1 \leqslant k \leqslant d$, $\alpha_k \in \Delta^+_{k-1}$ *and* $-\theta\alpha_k \in \Delta^+_{k-1}$.

(6) $\Delta^+_d = \Delta^+_{\mathfrak{b}_2}$.

Proof (cf. [Knapp-Vogan 1995], Lemma 11.128)

For a positive system Δ^+ of $\Delta(\mathfrak{g},\mathfrak{h})$, we define $\mathrm{ht}(\Delta^+) = \mathrm{card}(\Delta^+ \cap \Delta(\bar{\mathfrak{n}} \cap \mathfrak{u}))$. We immediately see $\mathrm{ht}(\Delta^+_{\mathfrak{b}_1}) = 0$ and $\mathrm{ht}(\Delta^+_{\mathfrak{b}_2}) = d$.

We construct the sequence $\alpha_1, ..., \alpha_d$ inductively as follows. Let $1 \leqslant k \leqslant d$ and assume that $\alpha_1, ..., \alpha_{k-1}$ are already defined so that the conditions in (1)-(5) above hold. First, (1) and (3) imply $\mathrm{ht}(\Delta_{k-1}) = k-1$.

We have a disjoint union $\Delta(\mathfrak{g},\mathfrak{h}) = \Delta(\mathfrak{n} \cap \bar{\mathfrak{u}}) \sqcup \Delta(\mathfrak{v}_2) \sqcup -\Delta(\mathfrak{d})$. So, (2) implies $\Delta^+_{k-1} \subseteq \Delta(\mathfrak{n} \cap \bar{\mathfrak{u}}) \sqcup \Delta(\mathfrak{v}_2)$. If there is no simple root for Δ^+_{k-1} contained in $\mathfrak{n} \cap \bar{\mathfrak{u}}$, we have any simple root for Δ^+_{k-1} is contained in $\Delta(\mathfrak{v}_2) = \Delta^+_{\mathfrak{b}_2}$. Hence we have $\Delta^+_{k-1} = \Delta^+_{\mathfrak{b}_2}$. However, it contradicts $\mathrm{ht}(\Delta^+_{k-1}) = k-1 < d = \Delta^+_{\mathfrak{b}_2}$. So, there exists some simple root α_k for Δ^+_{k-1} such that $\alpha_k \in \Delta(\mathfrak{n} \cap \bar{\mathfrak{u}})$. Since $\theta(\mathfrak{n}) = \bar{\mathfrak{n}}$ and $\theta(\bar{\mathfrak{u}}) = \bar{\mathfrak{u}}$, we see α_k is complex and $\theta\alpha_k \in \Delta(\bar{\mathfrak{n}} \cap \bar{\mathfrak{u}}) \subseteq -\Delta(\mathfrak{d}) \subseteq -\Delta^+_{k-1}$. Hence, we see α_k satisfies the conditions in the above (1)-(5). If $\Delta(\mathfrak{d}) \subseteq \Delta^+$ and $\mathrm{ht}(\Delta^+) = d$, then clearly $\Delta^+ = \Delta_{\mathfrak{b}_2}$. So, we have $\Delta^+_d = \Delta_{\mathfrak{b}_2}$, since $\mathrm{ht}(\Delta^+_d) = d$. Thus, we have (6). ■

We immediately see:

Corollary 2.1.5. *The complex roots $\alpha_1, ...\alpha_d$ in the above Lemma 1.2.4 are all distinct and we have $\Delta(\mathfrak{n} \cap \bar{\mathfrak{u}}) = \{\alpha_1, ...\alpha_d\}$.*

CAUTION The above numeration $\{\alpha_1, ..., \alpha_d\}$ of $\Delta(\mathfrak{n} \cap \bar{\mathfrak{u}})$ may depend on the choice of $\mathfrak{b}^0$.

2.2 Change of polarization

In this section, we fix a $\sigma - \theta$ pair $(\mathfrak{p}, \mathfrak{q})$. Let $\mathfrak{m}, \mathfrak{l}, ...$ be as in 1.2.

Let $L^\sim$ (resp. $(L \cap M)^\sim$) be the metaplectic double covering of L (resp. $L \cap M$) with respect to $\delta(\mathfrak{u})$ (resp. $\delta(\mathfrak{u} \cap \mathfrak{m})$).

Lemma 2.2.1. *On $\mathfrak{l} \cap \mathfrak{m}$, we have*

$$\delta(\mathfrak{u}) - \delta(\mathfrak{u} \cap \mathfrak{m}) = \delta(\bar{\mathfrak{n}} \cap \mathfrak{l}) + \delta(\mathfrak{n}) + 2\delta(\mathfrak{u} \cap \bar{\mathfrak{n}}).$$

Proof Remark that $\delta(\bar{\mathfrak{n}} \cap \mathfrak{u}) = -\delta(\mathfrak{n} \cap \bar{\mathfrak{u}})$, $\delta(\bar{\mathfrak{n}} \cap \mathfrak{l}) = -\delta(\mathfrak{n} \cap \mathfrak{l})$, etc. So, we have the lemma from the computation below.

$$\begin{aligned}\delta(\mathfrak{u})+\delta(\mathfrak{n}\cap\mathfrak{l}) &= \delta(\mathfrak{u}\cap\mathfrak{m})+\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})+\delta(\mathfrak{u}\cap\mathfrak{n})+\delta(\mathfrak{n}\cap\mathfrak{l})\\ &= \delta(\mathfrak{u}\cap\mathfrak{m})+2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})+\delta(\bar{\mathfrak{u}}\cap\mathfrak{n})+\delta(\mathfrak{n}\cap\mathfrak{l})+\delta(\mathfrak{u}\cap\mathfrak{n})\\ &= \delta(\mathfrak{u}\cap\mathfrak{m})+2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})+\delta(\mathfrak{n}) \quad \blacksquare\end{aligned}$$

We define a one dimensional representation $\xi_{\mathfrak{p},\mathfrak{q}}$ of $L\cap M$ on a space $\mathbb{C}_{\mathfrak{p},\mathfrak{q}}$ by

$$\xi_{\mathfrak{p},\mathfrak{q}}(\ell)=\xi_{\delta(\bar{\mathfrak{n}}\cap\mathfrak{l})}(\ell)\xi_{\delta(\mathfrak{n})}(\ell)\xi_{2\delta(\bar{\mathfrak{n}}\cap\mathfrak{u})'}(\ell) \quad (\ell\in L\cap M).$$

From Lemma 1.3.1, we easily see:

Lemma 2.2.2. *Assigning $(\ell,z)\in(L\cap M)^\sim$ to $(\ell,z\xi_{\mathfrak{p},\mathfrak{q}}(\ell))$, we have an embedding of the group $(L\cap M)^\sim\hookrightarrow L^\sim$.*

Let $(P\cap L)^\sim$ be the parabolic subgroup of $L^\sim$ which is the pull-back of $P\cap L$ to $L^\sim$. Under the identification by the embedding in Lemma 2.2.2, we can regard $(L\cap M)^\sim$ as a Levi subgroup of $(P\cap L)^\sim$.

Following is the main result of the section.

Theorem 2.2.3. *(1) Let Z be a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$-module with an infinitesimal character $\lambda\in\mathfrak{h}^*$. assume $\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m})-\delta(\mathfrak{n}),\alpha\rangle\geqslant 0$ for all $\alpha\in\Delta(\mathfrak{u})$ such that $2\frac{\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m})-\delta(\mathfrak{n}),\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$.*

Then, we have

$$(*)\qquad [{}^u\mathsf{Ind}_P^G(({}^u\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(Z))]=[({}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{\dim\mathfrak{u}\cap\mathfrak{k}}({}^u\mathsf{Ind}^L_{P\cap L}(Z\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'}))]$$

(2) Let Z be a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$-module with an infinitesimal character $\lambda\in\mathfrak{h}^*$. We assume $\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m}),\alpha\rangle\geqslant 0$ for all $\alpha\in\Delta(\mathfrak{u})$ such that $2\frac{\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m}),\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$. Then, we have

$$(**)\qquad [\mathsf{Ind}_P^G(({}^u\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(Z))]=[({}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{\dim\mathfrak{u}\cap\mathfrak{k}}(\mathsf{Ind}^L_{P\cap L}(Z\otimes\mathbb{C}_{\mathfrak{p},\mathfrak{q}}))]$$

(3) Let $\tilde{Z}$ be a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m},(L\cap M\cap K)^\sim)$-module with an infinitesimal character $\lambda\in\mathfrak{h}^*$ such that $Z=\tilde{Z}\otimes\mathbb{C}_{\delta(\mathfrak{u}\cap\mathfrak{m})'}$ is reduced to a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$-module. We assume $\langle\lambda,\alpha\rangle\geqslant 0$ for all $\alpha\in\Delta(\mathfrak{u})$ such that $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$. Then,

$$(***)\qquad [\mathsf{Ind}_P^G(({}^n\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(\tilde{Z}))]=[({}^n\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{\dim\mathfrak{u}\cap\mathfrak{k}}(\mathsf{Ind}^{L^\sim}_{(P\cap L)^\sim}(\tilde{Z}))]$$

Proof

(2),(3) are rephrasements of (1). We remark that characters of standard modules form a basis of the Grothendieck group of the category of Harish-Chandra modules. Taking account of additivity of cohomological inductions, it suffices to show $(*)$ in the case of Z is a standard module.

As in 1.2, we fix a θ and σ stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}\cap\mathfrak{m}$ and a Borel subalgebra $\mathfrak{b}^0$ of $\mathfrak{l}\cap\mathfrak{m}$ containing $\mathfrak{h}$. We denote by $\mathfrak{v}$ the nilradical of $\mathfrak{b}^0$. Let $H_{\mathbb{C}}$ be the analytic subgroup of $G_{\mathbb{C}}$ and put $H = H_{\mathbb{C}}\cap G$. Let Y be a one-dimensional H-representation whose differential is just λ. We consider the case of $Z = (\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{l}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{v}\cap\mathfrak{k}}(Y)$. Put $\mathfrak{b}_1 = \mathfrak{b}+\mathfrak{u}\cap\mathfrak{m}+\mathfrak{n}$ and $\mathfrak{b}_2 = \mathfrak{b}+\mathfrak{n}\cap\mathfrak{l}+u$. Then, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ areBorelsubalgebra of $\mathfrak{g}$. From [Knapp-Vogan] Corollary 11.86 (Induction-by-stage formula), we have

$$ {}^u\mathsf{Ind}_P^G(({}^u\mathcal{R}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K}^{\mathfrak{m},M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}((\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{l}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{v}\cap\mathfrak{k}}(Y))) \cong (({}^u\mathcal{R}_{\mathfrak{b}_1,T}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}+\dim\mathfrak{v}\cap\mathfrak{k}}(Y), $$

$$ ({}^u\mathcal{R}_{\mathfrak{q},L\cap K}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}}({}^u\mathsf{Ind}_{P\cap L}^{L}((\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{l}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{v}\cap\mathfrak{k}}(Y)\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'})) \cong ({}^u\mathcal{R}_{\mathfrak{b}_2,T}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}+\dim\mathfrak{v}\cap\mathfrak{k}}(Y\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'}). $$

So, we have only to show

$$ (\circ)\qquad (({}^u\mathcal{R}_{\mathfrak{b}_1,T}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}+\dim\mathfrak{v}\cap\mathfrak{k}}(Y) \cong ({}^u\mathcal{R}_{\mathfrak{b}_2,T}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}+\dim\mathfrak{v}\cap\mathfrak{k}}(Y\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'}). $$

However, we have Lemma 1.2.3 and 1.2.4. So, $(\circ)$ can be obtained by the successive application of the transfer theorem ([Knapp-Vogan] Theorem 11.87). $\blacksquare$

We also give a variation of Theorem 2.2.3. Proof is just the same.

Theorem 2.2.4. *(1) Let Z be a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$-module with an infinitesimal character $\lambda\in\mathfrak{h}^*$. assume $\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m})-\delta(\mathfrak{n}),\alpha\rangle\geqslant 0$ for all $\alpha\in\Delta(\bar{\mathfrak{n}}\cap\mathfrak{u})$ such that $2\frac{\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m})-\delta(\mathfrak{n}),\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$. Moreover, we assume that*

$$ {}^u\mathcal{R}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K}^{\mathfrak{m},M\cap K})^i(Z)) = 0 \quad \textit{for all } i\neq\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k} $$
$$ ({}^u\mathcal{R}_{\mathfrak{q},L\cap K}^{\mathfrak{g},K})^i({}^u\mathsf{Ind}_{P\cap L}^{L}(Z\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'})) = 0 \quad \textit{for all } i\neq\dim\mathfrak{u}\cap\mathfrak{k} $$

Then, we have

$$ (*)\qquad [{}^u\mathsf{Ind}_P^G(({}^u\mathcal{R}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K}^{\mathfrak{m},M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(Z))] = [({}^u\mathcal{R}_{\mathfrak{q},L\cap K}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}}({}^u\mathsf{Ind}_{P\cap L}^{L}(Z\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'}))] $$

(2) *Let Z be a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$-module with an infinitesimal character $\lambda\in\mathfrak{h}^*$. We assume $\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m}),\alpha\rangle\geqslant 0$ for all $\alpha\in\Delta(\bar{\mathfrak{n}}\cap\mathfrak{u})$ such that $2\frac{\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m}),\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$. Moreover, we assume that*

$$ {}^{u}\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K})^i(Z)) = 0 \quad \textit{for all } i\neq \dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}$$
$$({}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^i(\mathsf{Ind}^L_{P\cap L}(Z\otimes\mathbb{C}_{\mathfrak{p},\mathfrak{q}})) = 0 \quad \textit{for all } i\neq \dim\mathfrak{u}\cap\mathfrak{k}$$

Then, we have

$$(**)\qquad [\mathsf{Ind}^G_P(({}^{u}\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(Z))] = [({}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{\dim\mathfrak{u}\cap\mathfrak{k}}(\mathsf{Ind}^L_{P\cap L}(Z\otimes\mathbb{C}_{\mathfrak{p},\mathfrak{q}}))]$$

(3) *Let $\tilde{Z}$ be a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m},(L\cap M\cap K)^{\sim})$-module with an infinitesimal character $\lambda\in\mathfrak{h}^*$ such that $Z=\tilde{Z}\otimes\mathbb{C}_{\delta(\mathfrak{u}\cap\mathfrak{m})'}$ is reduced to a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$-module. We assume $\langle\lambda,\alpha\rangle\geqslant 0$ for all $\alpha\in\Delta(\bar{\mathfrak{n}}\cap\mathfrak{u})$ such that $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$. Moreover, we assume that*

$$ {}^{n}\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K})^i(\tilde{Z}) = 0 \quad \textit{for all } i\neq \dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}$$
$$({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^i(\mathsf{Ind}^{L^{\sim}}_{(P\cap L)^{\sim}}(\tilde{Z})) = 0 \quad \textit{for all } i\neq \dim\mathfrak{u}\cap\mathfrak{k}$$

Then,

$$(***)\qquad [\mathsf{Ind}^G_P(({}^{n}\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(\tilde{Z}))] = [({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{\dim\mathfrak{u}\cap\mathfrak{k}}(\mathsf{Ind}^{L^{\sim}}_{(P\cap L)^{\sim}}(\tilde{Z}))]$$

2.3 Derived functor modules; complex case

For complex connected reductive groups, irreducible unitary representations with regular integral infinitesimal character is a parabolic induction from a one-dimensional unitary representation ([Enright]). Moreover, Enright proved they have non-trivial $(\mathfrak{g},K)$-cohomologies. On the other hands, for general reductive Lie groups, Vogan-Zuckerman proved that if irreducible unitary representation with regular integral infinitesimal characters and with non-trivial $(\mathfrak{g},K)$-cohomologies are nothing but derived functor modules. Here, we give an explanation of such phenomenon in viewpoint of the change of polarization.

Let G be a complex connected reductive Lie group and we fix a Cartan involution θ. Here, we denote by $\mathfrak{g}_0$ the real Lie algebra of G. Then the complexification of $\mathfrak{g}_0$ can be identified with $\mathfrak{g}_0\times\mathfrak{g}_0$. Let $\mathfrak{p}_0$ be any parabolic subalgebra of $\mathfrak{g}_0$ with a Levi decomposition $\mathfrak{p}_0=\mathfrak{m}_0+\mathfrak{n}_0$ such that m_0 is θ stable. If we choose the identification adequately, then the complexification $\mathfrak{p}$ of $\mathfrak{p}_0$ can be identified with $\mathfrak{p}_0\times\mathfrak{p}_0\subseteq\mathfrak{g}_0\times\mathfrak{g}_0$. On the other hand, if we put $\mathfrak{q}=\mathfrak{p}_0\times\bar{\mathfrak{p}}_0$,

$\mathfrak{q}$ is a θ-stable parabolic algebra. Here, $\bar{\mathfrak{p}}_0$ means the opposite parabolic subalgebra to $\mathfrak{p}_0$. We immediately see $(\mathfrak{p}, \mathfrak{q})$ is a $\sigma\theta$-pair and $\mathfrak{p}$ and $\mathfrak{q}$ have a common Levi part $\mathfrak{m}_0 \times \mathfrak{m}_0$. Applying the Theorem 1.3.3, we see that, for complex connected reductive groups, derived functor modules are actually certain irreducible degenerate principal series representations.

2.4 Derived functor modules; general case

For $G = \mathrm{GL}(n, \mathbb{R})$, derived functor modules are parabolic induction from the external tensor product of some copies of distinguished derived functor modules so-called Speh representations and possibly a one-dimensional representation. ([Speh])

We examine such phenomenon in viewpoint of the change of polarization. Here, we use notations as in 1.2, such as G, $G_{\mathbb{C}}$, K, $K_{\mathbb{C}}$, $\mathfrak{g}$, $\mathfrak{g}_0$, θ, σ, etc. Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra with an orderly Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. Let L be the Levi subgroup corresponding to $\mathfrak{l}$ defined as in 1.1.

Let $\mathfrak{a}$ be the -1-eigenspace with respect to θ in the center of $\mathfrak{l}$. We call $\mathfrak{q}$ pure imaginary if $\mathfrak{a}$ is contained the center of $\mathfrak{g}$.

Let $\mathfrak{m}$ be the centerizer of $\mathfrak{a}$ in $\mathfrak{g}$. Then $\mathfrak{m}$ is a Levi subalgebra of a σ-stable parabolic subgroup $\mathfrak{p}$. Obviously $(\mathfrak{p}, \mathfrak{q})$ is a $\sigma\theta$-pair and $\mathfrak{l} \subseteq \mathfrak{m}$. $\mathfrak{q}$ is imaginary if and only if $\mathfrak{m} = \mathfrak{g}$ holds.

Conversely, we assume that there is a σ-stable parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $(\mathfrak{p}, \mathfrak{q})$ is a $\sigma\theta$-pair and there is an orderly Levi decomposition $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ such that $\mathfrak{l} \subseteq \mathfrak{m} \neq \mathfrak{g}$. Then, we have $\mathfrak{q}$ is not pure imaginary since the -1-eigenspace with respect to θ in the center of $\mathfrak{m}$ also centerize $\mathfrak{l}$.

From Theorem 2.2.3, we have:

Proposition 2.4.1. *Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra with an orderly Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. Assume that $\mathfrak{q}$ is not pure imaginary. Then, there is a σ-stable parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with an orderly Levi decomposition $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ such that the derived functor modules of $\mathfrak{g}$ with respect to $\mathfrak{q}$ is isomorphic to the parabolic induction of a derived functor module of $\mathfrak{m}$.*

Obviously, if G has a compact Cartan subgroup, any θ-stable parabolic subalgbra is pure imaginary.

We interpret Speh's result as follows. So, for a while, we put $G = \mathrm{GL}(n\mathbb{R})$ We fix a Cartan involution $\theta(g) = {}^t g^{-1}$ of G. So, we put $K = \mathrm{O}(n)$ here. For a positive integer k, we put

$J_k = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$. First, we assume n is even and write $n = 2k$, Put

$$\mathfrak{l}(k) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{gl}(2k, \mathbb{C}) \middle| A, B \in M_k(\mathbb{C}) \right\},$$

$$\mathfrak{u}(k) = \left\{ \begin{pmatrix} \sqrt{-1}S & S \\ S & -\sqrt{-1}S \end{pmatrix} \in \mathfrak{gl}(2k, \mathbb{C}) \middle| S \in M_k(\mathbb{C}) \right\},$$

$$\mathfrak{q}(k) = \mathfrak{l}(k) + \mathfrak{u}(k).$$

Then, $\mathfrak{q}(k)$ is a θ-stable parabolic subalgebra of $\mathfrak{gl}(2k, \mathbb{C})$ and $\mathfrak{q}(k) = \mathfrak{l}(k) + \mathfrak{u}(k)$ is a Levi decomposition such that $\mathfrak{l}(k)$ is a θ and σ-stable Levi part. The derived functor module with respect to $\mathfrak{q}(k)$ is a Speh representations of $\mathrm{GL}(2k, \mathbb{C})$. Actually, we have:

Proposition 2.4.2. *If n is odd, there is no proper pure imaginary θ-stable parabolic subalgebra.*

If n is even, any proper pure imaginary θ-stable parabolic subalgebra is $SO(n)$-conjugate to $\mathfrak{q}(\frac{n}{2})$.

Next, we consider general θ-stable parabolic subalgebras. For a sequence of positive integers $\vec{n} = (n_1, ..., n_\ell)$ such that $0 \leqslant n - 2n_1 + \cdots + 2n_\ell$, we put $q = n - 2n_1 + \cdots + 2n_\ell$

$$\mathfrak{t}(\vec{n}) = \{\mathrm{diag}(t_1 J_{n_1}, ..., t_\ell J_{n_\ell}, 0_q) \in \mathfrak{gl}(n, \mathbb{C}) | t_1, ..., t_\ell \in \mathbb{C}\}$$

. We denote by $\mathfrak{l}(\vec{n})$ the centerizer of $\mathfrak{t}(\vec{n})$ in $\mathfrak{gl}(n, \mathbb{C})$.

Then we have

$$\mathfrak{l}(\vec{n}) = \{\mathrm{diag}(A_1, ..., A_\ell, D) \in \mathfrak{gl}(n, \mathbb{C}) | A_i, \in \mathfrak{l}(n_i) \ \ (1 \leqslant i \leqslant \ell), D \in \mathfrak{gl}(q, \mathbb{C})\}.$$

and $\mathfrak{l}_0(\vec{n}) = \mathfrak{l}(\vec{n}) \cap \mathfrak{gl}(n, \mathbb{R})$ is a real form of $\mathfrak{l}(\vec{n})$ and

$$\mathfrak{l}_0(\vec{n}) \cong \mathfrak{gl}(n_1, \mathbb{C}) \times \cdots \times \mathfrak{gl}(n_\ell, \mathbb{C}) \times \mathfrak{gl}(q, \mathbb{R}).$$

Put

$$\mathfrak{m}(\vec{n}) = \{\mathrm{diag}(A_1, ..., A_\ell, D) \in \mathfrak{gl}(n, \mathbb{C}) | A_i \in GL(2n_i, \mathbb{C}) \ \ (1 \leqslant i \leqslant \ell), D \in \mathfrak{gl}(q, \mathbb{C})\}.$$

There a θ-stable parabolic subalgebra $\mathfrak{q}(\vec{n})$ such that

$$\mathfrak{m}(\vec{n}) \cap \mathfrak{q}(\vec{n}) = \{\mathrm{diag}(A_1, ..., A_\ell, D) \in \mathfrak{gl}(n, \mathbb{C}) | A_i, \in \mathfrak{q}(n_i) \ \ (1 \leqslant i \leqslant \ell), D \in \mathfrak{gl}(q, \mathbb{C})\}.$$

Any θ-stable parabolic subalgebra in $\mathfrak{gl}(n,\mathbb{C})$ is $\mathrm{O}(n,\mathbb{C})$-conjugate to some $\mathfrak{q}(\vec{n})$. Let $\mathfrak{n}$ be the Lie algebra of the upper triangular matrices in $\mathfrak{gl}(n,\mathbb{C})$ and put $\mathfrak{p}(\vec{n}) = \mathfrak{m}(\vec{n}) + \mathfrak{n}$. We denote by $\mathfrak{n}(\vec{n})$ the nilradical of $\mathfrak{p}(\vec{n})$. Then, $(\mathfrak{p}(\vec{n}), \mathfrak{q}(\vec{n}))$ is a $\sigma\theta$-pair. If we apply Theorem 2.2.3 to the $\sigma\theta$-pair, we get Speh's result.

Next, we consider the case of $G = \mathrm{GL}(k,\mathbb{H})$. Write $\mathbb{H} = \mathbb{C} + j\mathbb{C}$. This case we put $K = \mathrm{Sp}(n) = \{g \in \mathrm{GL}(k,\mathbb{H}) | {}^t\bar{g}g = I_k\}$. Then we regard $\mathfrak{gl}(k,\mathbb{C})$ as a real Lie subalgebra of $\mathfrak{gl}(k,\mathbb{H})$. For $\ell \in \mathbb{Z}$ and $t \in \sqrt{-1}\mathbb{R}$, we define a one-dimensional unitary representation $\xi_{\ell,t}$ of $\mathrm{GL}(k,\mathbb{C})$ as follows.

$$\xi_{\ell,t}(g) = \left(\frac{\det(g)}{|\det(g)|}\right)^{\ell} |\det(g)|^t.$$

Let $\mathfrak{q}(k)$ be a θ-stable parabolic subalgebra with an orderly Levi decomposition $\mathfrak{q}(k) = \mathfrak{l}(k) + \mathfrak{u}(k)$. We choose the nilradical $\mathfrak{n}(k)$ so that $\xi_{\ell,t}$ is good with respect to $\mathfrak{q}(k)$ for sufficient large ℓ. Derived functor modules with respect to $\mathfrak{q}(k)$ is called quarternionic Speh representations.

For $t \in \sqrt{-1}\mathbb{R}$, there is a one-dimensional unitary representation $\tilde{\xi}_t$ of $\mathrm{GL}(k,\mathbb{H})$ whose restriction to $\mathrm{GL}(k,\mathbb{C})$ is $\xi(0,t)$.

We put

Definition 2.4.3.

$$(*) \qquad A_k(\ell,t) = ({}^u\mathcal{R}_{\mathfrak{q}(k),O(k)}^{\mathfrak{gl}(k,\mathbb{H})\otimes_{\mathbb{R}}\mathbb{C},Sp(k)})^{k(k+1)}(\xi_{\ell+2k,t}) \quad (\ell \in \mathbb{Z}).$$

We also put

$$A_k(-\infty,t) = \tilde{\xi}_t$$

For $\ell \in \mathbb{Z}$, $A_k(\ell,t)$ is derived functor module in the good (resp. weakly fair) range in the sense of [Vogan 1988] if and only if $\ell \geqslant 0$ (resp. $\ell \geqslant -k$).

We immediately see:

$$A_k(\ell,t) \cong A_k(\ell,0) \otimes \tilde{\xi}_t.$$

We easily have:

Proposition 2.4.4. *Any proper pure imaginary θ-stable parabolic subalgebra is $Sp(k)$-conjugate to $\mathfrak{q}(k)$.*

As in the case of $\mathrm{GL}(k, \mathbb{R})$, any derived functor modules of $\mathrm{GL}(k, \mathbb{H})$ is a parabolic induction from the external tensor product of some copies of quartrnionic Speh representations and possibly a one-dimensional representation. (cf. [Vogan 1986])

Next, we consider the case of $G = \mathrm{SO}_0(2p+1, 2q+1)$. This case, a Levi part of a non-pure imaginary θ-stable parabolic subalgebra $\mathfrak{q}$ is isomorphic to $\mathfrak{so}(1,1) \oplus \mathfrak{u}(p_1, q_1) \oplus \cdots \mathfrak{u}(p_k, q_k)$. Here, $p_1 + \cdots + p_k = p$ and $q_1 + \cdots + q_k = q$.

Let $\mathfrak{p}$ be a maximal cuspidal parabolic subalgebra whose Levi part is isomorphic to $\mathfrak{so}(2p, 2p) \oplus \mathfrak{so}(1,1)$. The derived functor module with respect to the above $\mathfrak{q}$ is a parabolic induction with respect to $\mathfrak{p}$ from a derived funcctor module of $\mathfrak{so}(2p, 2q)$ with respect to a θ-stable parabolic subalgebra whose Levi part is isomorphic to $\mathfrak{u}(p_1, q_1) \oplus \cdots \mathfrak{u}(p_k, q_k)$.

Among the exceptional real simple Lie algebras, only E I and E IV have non pure imaginary θ-stable parabolic subalgebras.

§ 3. Application of change-of-polarization to $\mathbf{SO}^*(2n)$ and $\mathbf{Sp}(p,q)$

Throughout this section, we assume G is either $\mathrm{SO}^*(2n)$ or $\mathrm{Sp}(n-q,q)$. Put $p=n-q$. For $G=\mathrm{SO}^*(2n)$, we put $q=\left[\frac{n}{2}\right]$. So, in the both cases $G=\mathrm{Sp}(p,q)$ and $G=\mathrm{SO}^*(2n)$, q is the real rank of G.

3.1 Root systems

We fix a maximal compact subgroup K of $SO^*(2n)$ (resp. $\mathrm{Sp}(p,q)$), which is isomorphic to $U(n)$ (resp. $\mathrm{Sp}(p)\times\mathrm{Sp}(q)$). We denote by $G_{\mathbb{C}}$ the complexification of G as in 1.2. So, $G_{\mathbb{C}}$ is isomorphic to $\mathrm{SO}(2n,\mathbb{C})$ or $\mathrm{Sp}(n,\mathbb{C})$. We denote by θ the Cartan involution corresponding to K as in 1.2. We fix a θ-stable maximally split Cartan subgroup ${}^sH$ of G. We remark that all the Cartan subgroup of G is connected. We stress that we use notations introduced in § 1.

First, we consider the root system $\Delta(\mathfrak{g},{}^s\mathfrak{h})$ for $G=\mathrm{SO}^*(2n)$. Then we can choose an orthonormal basis $e_1,...,e_n$ of ${}^s\mathfrak{h}^*$ such that

$$\Delta(\mathfrak{g},{}^s\mathfrak{h})=\{\pm e_i\pm e_j \mid 1\leqslant i<j\leqslant n\}.$$

If n is even, we write $n=2m$. In this case, we choose the above $e_1,...,e_n$ so that $\theta(e_{2i-1})=-e_{2i}$ and $\theta(e_{2i})=-e_{2i-1}$ for all $1\leqslant i\leqslant m$. If n is odd, we write $n=2m+1$. In this case, we choose the above $e_1,...,e_n$ so that $\theta(e_{2i-1})=-e_{2i}$ and $\theta(e_{2i})=-e_{2i-1}$ for all $1\leqslant i\leqslant m$ and $\theta(e_{2m+1})=e_{2m+1}$.

We immediately see that $\pm(e_{2i-1}-e_{2i})$ (resp. $\pm(e_{2i-1}+e_{2i})$) $\quad(1\leqslant i\leqslant m)$ are compact imaginary (resp. real) and all other roots are complex.

If $G=\mathrm{Sp}(n-q,q)$, put $p=n-q$ and choose $e_1,...e_n$ such that

$$\Delta(\mathfrak{g},{}^s\mathfrak{h})=\{\pm e_i\pm e_j \mid 1\leqslant i<j\leqslant n\}\cup\{\pm 2e_i \mid 1\leqslant i\leqslant n\},$$

and

$$\theta(e_{2i-1})=-e_{2i},\theta(e_{2i})=-e_{2i-1}\quad(1\leqslant i\leqslant q),$$

$$\theta(e_i)=e_i\quad(2q<i\leqslant n).$$

We fix an simple system for $\Delta(\mathfrak{g},{}^s\mathfrak{h})$ as follows. If $G=\mathrm{SO}^*(2n)$, then put $\Pi=\{e_1-e_2,...,e_{n-1}-e_n,e_{n-1}+e_n\}$. If $G=\mathrm{Sp}(p,q)$, then put $\Pi=\{e_1-e_2,...,e_{n-1}-e_n,2e_n\}$.

We denote by Δ^+ the corresponding positive system of $\Delta(\mathfrak{g},{}^s\mathfrak{h})$. Let $E_1,....,E_n$ be the dual basis of ${}^s\mathfrak{h}$ to $e_1,....,e_n$.

3.2 Square Quadruplets

One of a famous realizations of $\mathrm{Sp}(p,q)$ is the automorphism group of an indefinite Hermitian form on a $\mathbb{H}$-vector space. Namely,

$$(\bigstar) \qquad \mathrm{Sp}(p,q) = \left\{ g \in \mathrm{GL}(p+q,\mathbb{H}) | {}^t\bar{g} I_{p,q} g = I_{p,q} \right\}.$$

Here, $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Similarly, we consider complex indefinite unitary group.

$$\mathrm{U}(p,q) = \left\{ g \in \mathrm{GL}(p+q,\mathbb{C}) | {}^t\bar{g} I_{p,q} g = I_{p,q} \right\}.$$

$\mathrm{U}(p,q)$ is regarded as a subgroup of $\mathrm{Sp}(p,q)$ in the obvious way. We fix a maximal compact subgroup of $\mathrm{Sp}(p+q)$ as follows.

$$K = \mathrm{Sp}(p) \times \mathrm{Sp}(q) = \left\{ \left. \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right| A \in \mathrm{Sp}(p), B \in \mathrm{Sp}(q) \right\}.$$

We denote by θ the corresponding Cartan involution.

For the case of $p = q$, we also consider another realization:

$$\mathrm{Sp}(k,k) = \left\{ g \in \mathrm{GL}(2m,\mathbb{H}) | {}^t\bar{g} J_k g = J_k \right\}.$$

We put $n = 2k$. Here, $J_k = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}$. Then, identifying $\mathrm{GL}(k,\mathbb{H})$ with the following group, we regard $\mathrm{GL}(k,\mathbb{H})$ as a subgroup of $\mathrm{Sp}(k,k)$:

$$\left\{ \left. \begin{pmatrix} A & 0 \\ 0 & {}^t\bar{A} \end{pmatrix} \right| A \in \mathrm{GL}(k,\mathbb{H}) \right\}.$$

We consider $\mathrm{U}(k,k) \cap \mathrm{GL}(k,\mathbb{H})$ as a subgroup of $\mathrm{Sp}(k,k)$. This group is

$$\left\{ \left. \begin{pmatrix} A & 0 \\ 0 & {}^t\bar{A} \end{pmatrix} \right| A \in \mathrm{GL}(k,\mathbb{C}) \right\}.$$

We identify it with $\mathrm{GL}(k,\mathbb{C})$ and obtain the following "square quadruplet."

$$(\mathrm{A}) \qquad \begin{array}{ccc} \mathrm{GL}(k,\mathbb{H}) & \subseteq & \mathrm{Sp}(k,k) \\ \cup| & & \cup| \\ \mathrm{GL}(k,\mathbb{C}) & \subseteq & \mathrm{U}(k,k) \end{array}$$

We easily see $\mathrm{U}(k,k)$, $\mathrm{GL}(k,\mathbb{H})$, and $\mathrm{GL}(k,\mathbb{C})$ are the centralizers of their centers in $\mathrm{Sp}(k,k)$. Since $\mathrm{GL}(k,\mathbb{C})$ has the same rank and the same real rank as $\mathrm{Sp}(k,k)$, we can choose θ-stable maximally split Cartan subgroup ${}^sH$ of $\mathrm{Sp}(k,k)$. We denote by ${}^s\mathfrak{h}$ the complexified Lie algebra of ${}^sH$. We may apply the notations on the root system for $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$

First, we choose the standard Borel subalgebra $\mathfrak{b}_1(k)$ of $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$ corresponding to Δ^+ in 3.1. We define a subset $S(k) = \{e_i - e_{i+1} \mid 1 \leqslant i \leqslant n\}$ of Π. We denote by $\tilde{\mathfrak{p}}(k)$ the standard parabolic subalgebra corresponding to $S(k)$, namely $\mathfrak{b}_1(k) \subseteq \tilde{\mathfrak{p}}(k)$ and $\Delta(\tilde{\mathfrak{p}}(k), {}^s\mathfrak{h}) = \Delta^+ \cup (\mathbb{Z}S(k) \cap \Delta(\mathfrak{g}, {}^s\mathfrak{h}))$ Then, we easily see $\mathrm{GL}(k,\mathbb{H})$ is the θ-stable Levi subgroup for $\tilde{\mathfrak{p}}(k)$.

Next, we consider another simple system Π_u of $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$ as follows.

$$\Pi_u = \{e_i - e_{i+2} \mid 1 \leqslant i \leqslant n-2\} \cup \{e_{n-1} + e_n\} \cup \{-2e_2\}.$$

We also put $S_u(k) = \Pi_u - \{-2e_2\}$. We we choose the standard Borel subalgebra $\mathfrak{b}_2(k)$ of $\mathfrak{g} = {}^s\mathfrak{p}(n,\mathbb{C})$ corresponding to Π_u and denote by $\tilde{\mathfrak{q}}(k)$ the parabolic subalgbra of $\mathfrak{g}$ containing $\mathfrak{b}_2(k)$ and corresponding to $S_u(k)$. Since, $\theta(S_u(k)) = -S_u(k)$ and $\theta(-2e_2) \equiv -2e_2 \pmod{\mathbb{Z}S_u(k)}$, $\tilde{\mathfrak{q}}(k)$ is θ-stable. We easily see $\mathrm{U}(k,k)$ is a Levi subgroup for $\tilde{\mathfrak{q}}(k)$. $\mathrm{U}(k,k)$, $\mathrm{GL}(k,\mathbb{H})$, and $\mathrm{GL}(k,\mathbb{C})$ are the centralizers of their centers in $\mathrm{Sp}(k,k)$. In fact, the Lie algebra of the center of $\mathrm{U}(k,k)$ (resp. $\mathrm{GL}(k,\mathbb{H})$) is spanned by $\sum_{i=1}^{k}(E_{2i-1} - E_{2i})$ (resp. $E_1 + \cdots + E_n$). Here, $n = 2k$ and we follows the notations in 3.1. The center of $\mathrm{U}(k,k)$ (resp. $\mathrm{GL}(k,\mathbb{H})$) is compact (resp. real split) and θ-stable, and $\mathrm{U}(k,k)$ (resp. $\mathrm{GL}(k,\mathbb{H})$) is a Levi subgroup for a maximal θ-stable (resp. σ-stable) parabolic subalgebra (say $\tilde{\mathfrak{q}}(k)$ (resp. $\tilde{\mathfrak{q}}(k)$)) of $\mathfrak{sp}(2k,\mathbb{C}) = \mathfrak{sp}(k,k) \otimes_{\mathbb{R}} \mathbb{C}$.

Since ${}^s\mathfrak{h} \subseteq \tilde{\mathfrak{p}}(k) \cap \tilde{\mathfrak{q}}(k)$, we see $(\tilde{\mathfrak{p}}(k), \tilde{\mathfrak{q}}(k))$ forms a $\sigma\theta$-pair. Put $\mathfrak{p}(k) = \tilde{\mathfrak{p}}(k) \cap (\mathfrak{u}(k,k) \otimes_{\mathbb{R}} \mathbb{C})$ and $\mathfrak{q}(k) = \tilde{\mathfrak{q}}(k) \cap (\mathfrak{gl}(k,\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C})$.

Similarily, $\mathrm{GL}(k,\mathbb{C})$ is the centerizer of the split (resp. compact) part of its center in $\mathrm{U}(k,k)$ (resp. $\mathrm{GL}(k,\mathbb{H})$). $\mathrm{GL}(k,\mathbb{C})$ is a Levi subgroup for a maximal σ-stable (resp. θ-stable) parabolic subalgebra $\mathfrak{p}(k)$ (resp. $\mathfrak{q}(k)$) of $\mathfrak{gl}(2k,\mathbb{C}) = \mathfrak{u}(k,k) \otimes_{\mathbb{R}} \mathbb{C}$ (resp. $\mathfrak{gl}(2k,\mathbb{C}) = \mathfrak{gl}(k,\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$).

$\mathfrak{p}(k)$ is usually called a Siegel parabolic subalgebra and $\mathfrak{q}(k)$ is the one defined in 3.2, the unique (up to $\mathrm{Sp}(k)$-conjugacy) pure imaginary θ-stable parabolic subalgebra. We denote by $P(k)$ the Siegel parabolic subgroup of G corresponding to $\mathfrak{p}(k)$. For $\ell \in \mathbb{Z}$ and $t \in \sqrt{-1}\mathbb{R}$, we define a one-dimensional unitary representation $\xi_{\ell,t}$ of $\mathrm{GL}(n,\mathbb{C})$ as follows.

$$\xi_{\ell,t}(g) = \left(\frac{\det(g)}{|\det(g)|}\right)^{\ell} |\det(g)|^t.$$

We define the degenerate unitary principal series with respect to $P(k)$ as follows:

$$(\dagger) \qquad I_k(\ell,t) = \mathsf{Ind}_{P(k)}^{\mathrm{U}(k,k)}(\xi_{\ell,t}) \quad (k \in \mathbb{Z}, t \in \sqrt{-1}\mathbb{R})$$

In (A), each inclusion gives a symmetric pair. Moreover, except $\mathrm{GL}(k,\mathbb{C}) \subseteq \mathrm{GL}(k,\mathbb{H})$, they give symmetric pairs of G/K_ε-type ([Oshima-Sekiguchi 1980]).

We introduce similar structure for $\mathrm{SO}^*(4k)$ as follows.

$$\text{(B)} \qquad \begin{array}{ccc} \mathrm{GL}(k,\mathbb{H}) & \subseteq & \mathrm{SO}^*(4k) \\ \cup\,| & & \cup\,| \\ \mathrm{GL}(k,\mathbb{C}) & \subseteq & \mathrm{U}(k,k) \end{array}$$

In fact, as in the case of $\mathrm{Sp}(k,k)$, $\mathrm{U}(k,k)$ (resp. $\mathrm{GL}(k,\mathbb{H})$) above is the centerilzer of $\sum_{i=1}^{k}(E_{2i-1}-E_{2i})$ (resp. $E_1+\cdots+E_n$) in $\mathrm{SO}^*(4k)$. (Here, $n=2k$.) $GL(k,\mathbb{C})$ is the intersection of $\mathrm{U}(k,k)$ and $\mathrm{GL}(k,\mathbb{H})$. For $k\geqslant 2$, we define

$$\Pi_u=\{e_i-e_{i+2}\mid 1\leqslant i\leqslant n-2\}\cup\{e_{n-1}+e_n\}\cup\{-e_2-e_4\},$$

$$S_u(k)=\Pi_u-\{-e_2-e_4\}.$$

If $k=1$, put $\Pi_u=\{e_1+e_2,e_1-e_2\}$ and $S_u(1)=\{e_1+e_2\}$. We define $\tilde{\mathfrak{p}}(k)$ and $\tilde{\mathfrak{q}}(k)$ in a similar manner to the case of $G=\mathrm{Sp}(k,k)$. In this case, situation is quite similar to the case of $\mathrm{Sp}(k,k)$.

3.3 Maximal parabolic subgroups

Let k be a positive integer such that $k\leq q$. If $G=\mathrm{Sp}(p,q)$, put $p'=p-k$ and $q'=q-k$. If $G=\mathrm{SO}^*(2n)$, put $r=n-2k_s^*$.

We put

$$A=\sum_{j=1}^{k}E_j.$$

Then we have $\theta(A)=-A$. We denote by $\mathfrak{a}_{(k)}$ the one-dimensional Lie subalgebra of ${}^s\mathfrak{h}$ spanned by A.

We define a subset $S(k)$ of Π as follows. If $G=\mathrm{Sp}(p,q)$, we define

$$S(k)=\begin{cases}\Pi-\{e_{2k}-e_{2k+1}\} & \text{if } p'>0,\\ \Pi-\{2e_n\} & \text{if } p'=0\end{cases}.$$

If $G=\mathrm{SO}^*(2n)$, we define

$$S(\kappa)=\begin{cases}\Pi-\{e_{2k}-e_{2k+1}\} & \text{if } r>0,\\ \Pi-\{e_{n-1}+e_n\} & \text{if } r=0\end{cases}.$$

We denote by $M_{(k)}$ (resp. $\mathfrak{m}_{(k)}$) the standard maximal Levi subgroup (resp. subalgebra) of G (resp. $\mathfrak{g}$) corresponding to $S(k)$. Namely $M_{(k)}$ is the centerizer of $\mathfrak{a}_{(k)}$ in G.

We denote by $P_{(k)}$ the parabolic subgroup of G whose θ-invariant Levi part is $M_(k)$. We choose $P_{(k)}$ so that the roots in Δ whose root spaces are contained in the complexified Lie algebra of the nilradical of $P_{(k)}$ are all in Δ^+. We denote by $N_{(k)}$ the nilradical of $P_{(k)}$.

Formally, we denote by $\mathrm{Sp}(0,0)$ and $\mathrm{SO}^*(0)$ the trivial group $\{1\}$. Then, we have

$$M_{(k)} \cong \begin{cases} \mathrm{GL}(k,\mathbb{H}) \times \mathrm{Sp}(p',q') & \text{if } G = \mathrm{Sp}(p,q) \\ \mathrm{GL}(k,\mathbb{H}) \times \mathrm{SO}^*(2r) & \text{if } G = \mathrm{SO}^*(2n) \end{cases}.$$

Often, we identify $\mathrm{GL}(k,\mathbb{H})$, $\mathrm{Sp}(p',q')$, $\mathrm{SO}^*(2r)$ with subgroups of $M_{(k)}$ in obvious ways. We call such identifications the standard identifications. The Cartan involution θ induces Cartan involutions on $M_{(k)}$, $\mathrm{GL}(k,\mathbb{H})$, $\mathrm{Sp}(p',q')$, and $SO^*(2r)$ and we denote them by the same letter θ. We put $M^\circ_{(k)} = \mathrm{Sp}(p',q')$ if $G = \mathrm{Sp}(p,q)$ and put $M^\circ_{(k)} = \mathrm{SO}^*(2r)$ if $G = \mathrm{SO}^*(2n)$.

We denote by $\mathfrak{p}_(k)$, $\mathfrak{m}_(k)$, $\mathfrak{m}^\circ_\kappa$ and $\mathfrak{n}_\kappa$ the complexified Lie algebra of P_κ, M_κ, M°_κ, and N_κ, respectively.

Later, we treat various $\mathrm{Sp}(p,q)$'s and $\mathrm{SO}^*(2n)$'s at the same time. So, sometimes we write $P_(k)(p,q)$ (resp. $P^*_{(k)}(2n)$) for $P_{(k)}$ if $G = \mathrm{Sp}(p,q)$ (resp. $G = \mathrm{SO}^*(2n)$).

We define a basis $\Pi^{(k)}_u$ of $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$ as follows. If $2k = n$, then we put $\Pi^{(k)}_u = \Pi_u$, where Π_u is defined in 3.2. If $2k < n$, then we put

$$\Pi^{(k)}_u = \{e_i - e_{i+2} \mid 1 \leqslant i \leqslant 2k-2\} \cup \{e_{2k-1} + e_{2k}, -e_2 - e_{2k+1}\} \cup \{\gamma \in \Pi \mid \gamma(E_i) = 0 (1 \leqslant i \leqslant 2k)\}.$$

Here, Π is the basis of $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$ defined in 3.1. We denote by $\mathfrak{b}_{(k)}$ the standard Borel subaalgebra of $\mathfrak{g}$ Put $S^{(k)}_u = \Pi^{(k)}_u - \{-e_2 - e_{2k+1}\}$. Let $\mathfrak{q}_{(k)}$ be the parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}_{(k)}$ corresponding to $S^{(k)}_u$. Since, $\theta(\mathbb{Z}S^{(k)}_u) = \mathbb{Z}S^{(k)}_u$ and $\theta(-e_2 - e_{2k+1}) \equiv -e_2 - e_{2k+1} \pmod{\mathbb{Z}S^{(k)}_u}$, $\mathfrak{q}_{(k)}$ is θ-stable. We easily see that $U(k,k) \times M^\circ_{(k)}$ is a Levi subgroup (say $L_{(k)}$)for $\mathfrak{q}_{(k)}$. We denote by $\mathfrak{l}_{(k)}$ the complexifed Lie algebra of $L_{(k)}$.

Since ${}^s\mathfrak{h} \subseteq \mathfrak{p}_{(k)} \cap \mathfrak{q}_{(k)}$, $(\mathfrak{p}_{(k)}, \mathfrak{q}_{(k)})$ is a $\sigma\theta$-pair.

We denote by $G_{(k)}$ the centerizer of $\{E_i \mid 2k < i \leqslant n\}$ in G. (If $2k = n$, we put $G_{(k)} = G$.) Then, we have

$$G_{(k)} \cong \begin{cases} \mathrm{Sp}(k,k) & \text{if } G = \mathrm{Sp}(p,q) \\ \mathrm{SO}^*(4k) & \text{if } G = \mathrm{SO}^*(2n) \end{cases}.$$

Let $G_{(k)}M^\circ_{(k)}$ be the subgroup of G generated by $G_{(k)}$ and $M^\circ_{(k)}$. Since $G^1_{(k)}$ commutes with $M^\circ_{(k)}$, we have $G_{(k)}M^\circ_{(k)} \cong G_{(k)} \times M^\circ_{(k)}$.

We have the following diagram:

$$
\text{(C)} \qquad \begin{array}{ccc} M_{(k)} & \subseteq & G_{(k)}M^{\circ}_{(k)} \\ \cup | & & \cup | \\ M_{(k)} \cap L_{(k)} & \subseteq & L_{(k)} \end{array}
$$

Taking intersection of $G_{(k)}$ and each term of (C), we have a square quadruplet in the sense of 3.2:

$$
\text{(D)} \qquad \begin{array}{ccc} \mathrm{GL}(k, \mathbb{H}) & \subseteq & G_{(k)} \\ \cup | & & \cup | \\ \mathrm{GL}(k, \mathbb{C}) & \subseteq & \mathrm{U}(k,k) \end{array}
$$

We have :

$$
\Delta(\mathfrak{u}_{(k)} \cap \mathfrak{m}_{(k)}) = \{e_{2i-1} - e_{2j} \mid 1 \leqslant i, j \leqslant k\},
$$

$$
\Delta(\mathfrak{n}_{(k)} \cap \mathfrak{l}_{(k)}) = \{e_{2i-1} + e_{2j} \mid 1 \leqslant i, j \leqslant k\}.
$$

If $G = \mathrm{SO}^*(2n)$, we have

$$
\Delta(\mathfrak{n}_{(k)} \cap \mathfrak{u}_{(k)}) = \{e_{2i-1} + e_{2j-1} \mid 1 \leqslant i, j \leqslant k, i \neq j\} \cup \{e_{2i-1} \pm e_j \mid 1 \leqslant i, \leqslant k, 2k < j \leqslant n\},
$$

$$
\Delta(\mathfrak{n}_{(k)} \cap \bar{\mathfrak{u}}_{(k)}) = \{e_{2i} + e_{2j} \mid 1 \leqslant i, j \leqslant k, i \neq j\} \cup \{e_{2i} \pm e_j \mid 1 \leqslant i, \leqslant k, 2k < j \leqslant n\}.
$$

If $G = \mathrm{Sp}(p,q)$, we have

$$
\Delta(\mathfrak{n}_{(k)} \cap \mathfrak{u}_{(k)}) = \{e_{2i-1} + e_{2j-1} \mid 1 \leqslant i, j \leqslant k\} \cup \{e_{2i-1} \pm e_j \mid 1 \leqslant i, \leqslant k, 2k < j \leqslant n\}.
$$

$$
\Delta(\mathfrak{n}_{(k)} \cap \bar{\mathfrak{u}}_{(k)}) = \{e_{2i} + e_{2j} \mid 1 \leqslant i, j \leqslant k\} \cup \{e_{2i} \pm e_j \mid 1 \leqslant i, \leqslant k, 2k < j \leqslant n\}.
$$

Put $a_G = 1$ (resp. $a_G = -1$), if $G = \mathrm{Sp}(p,q)$ (resp. $G = \mathrm{SO}^*(2n)$). For $1 \leqslant i \leqslant n$, we have

$$
\delta(\mathfrak{u}_{(k)} \cap \mathfrak{m}_{(k)})(E_i) = \begin{cases} (-1)^{i+1}\frac{k}{2} & \text{if } 1 \leqslant i \leqslant 2k, \\ 0 & \text{otherwise} \end{cases},
$$

$$
2\delta(\bar{\mathfrak{u}}_{(k)} \cap \mathfrak{n}_{(k)})(E_i) = \begin{cases} 3k - 2n - a_G & \text{if } i \in 2\mathbb{Z}, 1 \leqslant i \leqslant 2k, \\ 0 & \text{otherwise} \end{cases},
$$

$$
\delta(\bar{\mathfrak{n}}_{(k)} \cap \mathfrak{l}_{(k)})(E_i) = \begin{cases} -\frac{k}{2} & \text{if } 1 \leqslant i \leqslant 2k, \\ 0 & \text{otherwise} \end{cases},
$$

$$
\delta(\mathfrak{n}_{(k)})(E_i) = \begin{cases} -\frac{2n-2k+a_G}{2} & \text{if } 1 \leqslant i \leqslant 2k \\ 0 & \text{otherwise} \end{cases}.
$$

Hence, we have:

Lemma 3.3.1. *Define $\xi_{\mathfrak{p}_{(k)},\mathfrak{q}_{(k)}}$ as in 2.2. For $1 \leqslant i \leqslant n$, we have:*

$$\xi_{\mathfrak{p}_{(k)},\mathfrak{q}_{(k)}}(E_i) = \begin{cases} (-1)^{i+1}\frac{2n-3k+a_G}{2} & \text{if } 1 \leqslant i \leqslant 2k \\ 0 & \text{otherwise} \end{cases}.$$

We denote by ${}^s\mathfrak{h}_{(k)}$ (resp. ${}^s\mathfrak{h}^{(k)}$) the $\mathbb{C}$-linear span of $\{E_i \mid 2k < i \leqslant n\}$ (resp. $\{E_i \mid 1 \leqslant i \leqslant 2k\}$). Using the direct sum docomposition ${}^s\mathfrak{h} = {}^s\mathfrak{h}^{(k)} \oplus {}^s\mathfrak{h}_{(k)}$, we have ${}^s\mathfrak{h}^* = ({}^s\mathfrak{h}^{(k)})^* \oplus {}^s\mathfrak{h}^*_{(k)}$.

Let π be an irreducible unitary representation of $M^\circ_{(k)}$. Since ${}^s\mathfrak{h}_{(k)} = {}^s\mathfrak{h} \cap \mathfrak{m}^\circ_{(k)}$, ${}^s\mathfrak{h}_{(k)}$ is a Cartan subalgebra of $\mathfrak{m}^\circ_{(k)}$. Let $\lambda_\pi \in {}^s\mathfrak{h}^*_{(k)}$ be the infinitesimal character of π. (λ_π is determined up to the Weyl group action.)

For $\ell \in \mathbb{Z}$ and $t \in \sqrt{-1}\mathbb{R}$, we conder the one-dimensional representation $\xi_{\ell,t}$ of $\mathrm{GL}(k,\mathbb{C})$ defined in 3.2. We consider the representation $\xi_{\ell,t} \boxtimes \pi$ of $\mathrm{GL}(k,\mathbb{C}) \times M^\circ_{(k)}$. Let $\lambda_{\ell,t,\pi}$ be the infinitesimal character of $\xi_{\ell,t} \boxtimes \pi$. Then we have:

$$\begin{aligned} \lambda_{\ell,t,\pi}(E_{2i-1}) &= \frac{k-1+\ell+t}{2} - i + 1 \quad (1 \leqslant i \leqslant k), \\ \lambda_{\ell,t,\pi}(E_{2i}) &= \frac{k-1-\ell+t}{2} - i + 1 \quad (1 \leqslant i \leqslant k), \\ \lambda_{\ell,t,\pi}|_{{}^s\mathfrak{h}_{(k)}} &= \lambda_\pi \end{aligned}$$

We define:

$$c_{\ell,t,\pi} = \max\left(\{0\} \cup \left\{ |\lambda_\pi(E_i)| \,\middle|\, n - 2k < i \leqslant n, \left\{\pm\lambda(E_i) - \frac{\ell+t-1}{2}\right\} \cap \mathbb{Z} \neq \emptyset \right\}\right).$$

Applying Theorem 2.2.3 (2) to the $\sigma\theta$-pair $(\mathfrak{p}_{(k)}, \mathfrak{q}_{(k)})$, we have:

Proposition 3.3.2. *Let π be an irreducible unitary representation of $M^\circ_{(k)}$. Let $\ell \in \mathbb{N}$ and $t \in \sqrt{-1}\mathbb{R}$. We assume $\ell \geqslant 2c_{\ell,t,\pi} - 1$. Put $S = k(n-2k+1)$ (resp. $S = k(2n-3k)$), if $G = Sp(p,q)$ (resp. if $G = SO^*(2n)$).*

Then,

$$\mathsf{Ind}^G_{P_{(k)}}(A_k(\ell,t) \boxtimes \pi) \cong (\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q}_{(k)},K\cap L_{(k)}})^S(I_k(\ell+2n-k+a_G,t) \boxtimes \pi).$$

Here, $A_k(\ell,t)$ (resp. $I_k(\ell,t)$) is a quarternionic Speh representation (resp. a degenerate principal series representatiion) defined in 2.4 (*) (resp. 3.2 (†)).

3.4 θ-stable parabolic subalgebras

We retain the notations in 3.1 and 3.3. The classifications of K-conjugate class of θ-stable parabolic subalgebras with respect to real classical groups are more or less well-known. Here, we review the classification for $G = \mathrm{U}(p,q), \mathrm{Sp}(p,q), \mathrm{SO}^*(2n)$. First, we discuss θ-stable parabolic subalgebras with respect to $\mathrm{U}(p,q)$ (cf. [Vogan 1996] Example 4.5).

Let ℓ be a positive integer. Put

$$\mathbb{P}_\ell(p,q) = \left\{ ((p_1, ..., p_\ell), (q_1, ..., q_\ell)) \in \mathbb{N}^\ell \times \mathbb{N}^\ell \mid \sum_{i=1}^{\ell} p_i = p, \sum_{i=1}^{\ell} q_i = q, p_j + q_j > 0 \text{ for all } 1 \leq j \leq \ell \right\},$$

We also put

$$\mathbb{P}(p,q) = \bigcup_{\ell > 0} \mathbb{P}_\ell(p,q),$$

$$\mathbb{P}(0,0) = \mathbb{P}_0(0,0) = \{((\emptyset), (\emptyset))\}.$$

If $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}(p,q)$ satisfies $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_\ell(p,q)$, we call ℓ the length of $(\underline{\mathbf{p}}, \underline{\mathbf{q}})$. For $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}(p,q)$, we define

$$I_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})} = \mathrm{diag}(I_{p_1,q_1}, ..., I_{p_\ell,q_\ell})$$

Then we have

$$\mathrm{U}(p,q) = \left\{ g \in \mathrm{GL}(p+q, \mathbb{C}) | {}^t\bar{g} I_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})} g = I_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})} \right\}.$$

Let θ be the Cartan involution given by the conjugation by $I_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})}$. In this realization, we denote by $\mathfrak{q}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$ the block-upper-triangular parabolic subalgebra of $\mathfrak{gl}(p+q, \mathbb{C}) = \mathfrak{u}(p,q) \otimes_{\mathbb{R}} \mathbb{C}$ with blocks of zizes $p_1 + q_1, ..., p_\ell + q_\ell$ along the diagonal. Then, $\mathfrak{q}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$ is a θ-stable parabolic subalgebra. The corresponding Levi subgroup $\mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$ consists of diagonal blocks.

$$\mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \cong \mathrm{U}(p_1, q_1) \times \cdots \times \mathrm{U}(p_\ell, q_\ell).$$

We denote by $\mathfrak{u}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$ the Lie algebra of $\mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$. Via the above construction of $\mathfrak{q}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$, K-conjugate class of θ-stable parabolic subalgebras with respect to $\mathrm{U}(p,q)$ is classified by $\mathbb{P}(p,q)$.

For $G = \mathrm{Sp}(p,q), \mathrm{SO}^*(2n)$, we put

$$\mathbb{P}_G = \begin{cases} \bigcup_{\substack{p' \leqslant p \\ q' \leqslant q}} \mathbb{P}(p', q') & \text{if } G = \mathrm{Sp}(p,q), \\ \bigcup_{p'+q' \leqslant n} \mathbb{P}(p', q') & \text{if } G = \mathrm{SO}^*(2n) \end{cases},$$

K-conjugate class of θ-stable parabolic subalgebras with respect to G is classified by $\mathbb{P}_G$. We give a construction of θ-stable parabolic subalgebra $\tilde{\mathfrak{q}}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$ for $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_G$.

First, we assume $G = \mathrm{Sp}(p,q)$, $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_\ell(p',q')$, $0 \leqslant p' \leqslant p$, and $0 \leqslant q' \leqslant q$. Put $p_0 = p - p'$ and $q_0 = q - q'$. Then we have a symmetric pair $(\mathrm{Sp}(p,q), \mathrm{Sp}(p',q') \times \mathrm{Sp}(p_0,q_0))$. Taking account of the realization of $\mathrm{Sp}(p',q')$ as the automorphism group of an indefinite Hermitian form on a $\mathbb{H}$-vector space (3.2 ($\bigstar$)), we see that $\mathrm{U}(p',q') \subseteq \mathrm{Sp}(p',q')$. Hence we have $\mathrm{U}(p',q') \times \mathrm{Sp}(p_0,q_0) \subseteq \mathrm{Sp}(p,q)$. Put $L_{(p',q')}(p,q) = \mathrm{U}(p',q') \times \mathrm{Sp}(p_0,q_0)$ Since the centerizer in $\mathrm{Sp}(p,q)$ of the center of $\mathrm{U}(p',q')$ is $L_{(p',q')}(p,q)$, $L_{(p',q')}(p,q)$ is a Levi subgroup of a θ-stable maximal parabolic subalgebra $\tilde{\mathfrak{q}}_{(p',q')}(p,q)$ of $\mathfrak{sp}(p+q,\mathbb{C})$. We denote by $\tilde{\mathfrak{u}}_{(p',q')}(p,q)$ the nilradical of $\tilde{\mathfrak{q}}_{(p',q')}(p,q)$. In fact there are two possibilities of the choices of $\tilde{\mathfrak{u}}_{(p',q')}(p,q)$. Our choice should be compatible with the construction in 3.4. Namely, we should choose $\tilde{\mathfrak{u}}_{(p',q')}(p,q)$ so that $\tilde{\mathfrak{q}}_{(p',q')}(p,q) = \tilde{\mathfrak{q}}_{(k)}$, if $p' = q' = k$. Such a choice is determined as follows. For $\ell \in \mathbb{Z}$, we define a character η_ℓ of $\mathrm{U}(p',q')$ by

$$\eta_\ell(g) = \det(g)^\ell \quad (g \in \mathrm{U}(p',q')).$$

Let π be any irreducible unitary representation of $\mathrm{Sp}(p_0,q_0)$. Then, we choose $\tilde{\mathfrak{u}}_{(p',q')}(p,q)$ so that $\eta_\ell \boxtimes \pi$ is good with respect to $\tilde{\mathfrak{q}}_{(p',q')}(p,q)$ for a sufficiently large ℓ.

We denote by $\mathfrak{l}_{(p',q')}(p,q)$ the complexfied Lie algebra of $L_{(p',q')}(p,q)$. Let $\mathfrak{q}(\underline{\mathbf{p}}, \underline{\mathbf{q}})(p,q)$ be the θ-stable parabolic subgroup of $\mathfrak{u}(p',q') \otimes_{\mathbb{R}} \mathbb{C}$ defined as above. Since $L_{(p',q')}(p,q) = \mathrm{U}(p',q') \times \mathrm{Sp}(p_0,q_0)$, $\mathfrak{q}(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \oplus \mathfrak{sp}(p_0+q_0,\mathbb{C})$ is a θ-stable parabolic subalgebra of $\mathfrak{l}_{(p',q')}$. Define

$$\tilde{\mathfrak{q}}_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})}(p,q) = (\mathfrak{q}(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \oplus \mathfrak{sp}(p_0+q_0,\mathbb{C})) + \tilde{\mathfrak{u}}_{(p',q')}(p,q).$$

Then $\tilde{\mathfrak{p}}_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})}(p,q)$ is a θ-stable parabolic subalgebra of $\mathfrak{sp}(p+q,\mathbb{C})$. The corresponding Levi subgroup is $L_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})}(p,q) = \mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \times \mathrm{Sp}(p_0,q_0)$.

Next, we consider the case $G = \mathrm{SO}^*(2n)$. Assume $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_\ell(p',q')$, $p'+q' \leqslant n$. Put $n' = p'+q'$ and $n_0 = n - n'$. Then we have a symmetric pair $(\mathrm{SO}^*(2n), \mathrm{SO}^*(2n') \times \mathrm{SO}^*(2n_0))$. There is a symmetric pair $(\mathrm{U}(p',q'), \mathrm{SO}^*(2n'))$. This is of G/K_ε-type except for $p'q'+1 \in 2\mathbb{Z}$. (cf. [Oshima-Sekiguchi 1984]) Put $L^*_{(p',q')}(2n) = \mathrm{U}(p',q') \times \mathrm{SO}^*(2n_0)$. Since the centerizer in $\mathrm{SO}^*(2n)$ of the center of $\mathrm{U}(p',q')$ is $L^*_{(p',q')}(2n)$, $L^*_{(p',q')}(2n)$ is a Levi subgroup of a θ-stable maximal parabolic subalgebra $\tilde{\mathfrak{q}}^*_{(p',q')}(2n)$ of $\mathfrak{so}(2n,\mathbb{C})$. Now that we can construct a θ-stable parabolic subalgebra $\tilde{\mathfrak{p}}^*_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})}(2n)$ of $\mathfrak{so}(2n,\mathbb{C})$ in the same way as the case of $G = \mathrm{Sp}(p,q)$. In this case the Levi subgroup $L^*_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})}(2n)$ is isomorphic to $\mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \times \mathrm{SO}^*(2n_0)$.

3.5 A rearrangement formula

First, we consider the case of $G = \mathrm{Sp}(p,q)$. Let p' and q' be non-negative integers such that $p'+q'>0$. Moreover, we assume that $p' \leqslant p$ and $q' \leqslant q$. Put $p_0 = p - p'$ and $q_0 = q - q'$. We consider θ-stable maximal parabolic subalgebra $\tilde{\mathfrak{q}}_{(p',q')}(p,q)$ defined in 3.4.

Let $\mathfrak{h}(p_0,q_0)$ (resp. $\mathfrak{h}_{(p',q')}$) be a θ and σ-stable compact Cartan subalgebra for $\mathrm{Sp}(p_0,q_0)$ (resp. $\mathrm{U}(p',q')$).

Taking account of $L_{(p',q')}(p,q) = \mathrm{U}(p',q') \times \mathrm{Sp}(p_0,q_0)$, we put

$$\mathfrak{h}(p,q) = \mathfrak{h}_{(p',q')} \oplus \mathfrak{h}(p_0,q_0) \subseteq \mathfrak{l}_{(p',q')}(p,q) \subseteq \mathfrak{sp}(p+q,\mathbb{C}).$$

Then, $\mathfrak{h}(p,q)$ is a θ and σ-stable compact Cartan subalgebra for $\mathrm{Sp}(p,q)$. Using the above direct sum decomposition, we regard $\mathfrak{h}^*_{(p',q')}$ and $\mathfrak{h}(p_0,q_0)^*$ as a subspace of $\mathfrak{h}(p,q)^*$. We introduce an orthonormal basis $\{f_1,...,f_{p'+q'}\}$ (resp. $\{f_{p'+q'+1},...,f_{p+q}\}$) of $\mathfrak{h}^*_{(p',q')}$ (resp. $\mathfrak{h}(p_0,q_0)^*$) such that

$$\begin{aligned}
\Delta(\mathfrak{sp}(p+q,\mathbb{C}),\mathfrak{h}(p,q)) &= \{\pm f_i \pm f_j \mid 1 \leqslant i < j \leqslant p+q\} \cup \{\pm 2f_i \mid 1 \leqslant i \leqslant p+q\},\\
\Delta(\mathfrak{u}(p',q') \otimes_{\mathbb{R}} \mathbb{C}, \mathfrak{h}_{(p',q')}) &= \{f_i f_j \mid 1 \leqslant i,j \leqslant p'+q', i \neq j\},\\
\Delta(\mathfrak{sp}(p_0+q_0,\mathbb{C}),\mathfrak{h}(p_0,q_0)) &= \{\pm f_i \pm f_j \mid p'+q' < i < j \leqslant p+q\} \cup \{\pm 2f_i \mid p'+q' < i \leqslant p+q\},\\
\Delta(\tilde{\mathfrak{u}}_{(p',q')}(p,q),\mathfrak{h}(p,q)) &= \{f_i \pm f_j \mid 1 \leqslant i \leqslant p'+q' < j \leqslant p+q\} \cup \{f_i + f_j \mid 1 \leqslant i \leqslant j \leqslant p'+q'\},
\end{aligned}$$

We denote by $F_1,...,F_{p+q}$ the basis of $\mathfrak{h}(p,q)$ dual to $f_1,...,f_{p+q}$. We have

$$\delta(\tilde{\mathfrak{u}}_{(p',q')}(p,q))(F_i) = \begin{cases} \frac{2p+2q-p'-q'+1}{2} & \text{if } 1 \leqslant i \leqslant p'+q', \\ 0 & \text{otherwise} \end{cases}$$

For $\ell \in \mathbb{Z}$, we consider the one-dimensional unitary representation η_ℓ of $\mathrm{U}(p',q')$ defined in 3.5.

Let Z be any Harish-Chandra module for $\mathrm{Sp}(p_0,q_0)$ with an infinitesimal character $\lambda \in \mathfrak{h}(p_0,q_0)^* \subseteq \mathfrak{h}(p,q)^*$. λ is unique up to the Weyl group action. Put $||\lambda|| = \max(\{0\} \cup \{|\lambda(F_i)| \mid p'+q' < i \leqslant p+q, \lambda(F_i) \in \mathbb{Z}\})$. $||\lambda||$ is invariant under the Weyl group action on λ, so we write $||Z|| = ||\lambda||$.

$\eta_\ell \boxtimes Z$ has an infinitesimal character $[\ell,\lambda] \in \mathfrak{h}(p,q)^*$ such that

$$[\ell,\lambda](F_i) = \begin{cases} \ell + \frac{p'+q'+1}{2} - i & \text{if } 1 \leqslant i \leqslant p'+q', \\ \lambda(F_i) & \text{if } p'+q' < i \leqslant p+q \end{cases}$$

We denote by $\mathcal{H}(\mathrm{Sp}(p,q))_\mu$ the category of Harish-Chandra modules for $\mathrm{Sp}(p,q)$ with an infinitesimal character μ.

Definition 3.5.1. *For $\ell \in \mathbb{Z}$ and $Z \in \mathcal{H}(Sp(p_0, q_0))_\lambda$, put*

$$\mathcal{R}^{p,q}_{p',q'}(\ell)(Z) = \left(\mathcal{R}^{\mathfrak{sp}(p+q,\mathbb{C}),K}_{\tilde{\mathfrak{q}}_{(p',q')}(p,q),L_{(p',q')}(p,q)\cap K}\right)^S ((\eta_\ell \boxtimes Z) \otimes \mathbb{C}_{2\delta(\tilde{\mathfrak{u}}_{(p',q')}(p,q))}),$$

where $S = \frac{p'(4p-3p'+1)+q'(4q-3q'+1)}{2}$. If $\ell \geqslant ||\lambda|| - (p_0+q_0)$, then the above cohomological induction is in good range and we have an exact functor

$$\mathcal{R}^{p,q}_{p',q'}(\ell) : \mathcal{H}(Sp(p_0, q_0))_\lambda \to \mathcal{H}(Sp(p, q))_{[\ell,\lambda]+\delta(\tilde{\mathfrak{u}}_{(p',q')}(p,q))}$$

Next, we consider the following setting. Let k be a positive integer such that $k \leqslant p$ and $k \leqslant q$. Let p' and q' be non-negative integers such that $p' + q' > 0$. Moreover, we assume that $p' + k \leqslant p$ and $q' + k \leqslant q$. We consider θ-stable parabolic subalgebra $\tilde{\mathfrak{q}}_{(p',q')}(p-k, q-k)$ of $\mathfrak{m}^\circ_{(k)} = \mathrm{Sp}(p-k, q-k)$ defined in 3.5.

So, the Levi subgroup $L_{(p',q')}(p-k, q-k)$ of $\tilde{\mathfrak{q}}_{(p',q')}(p-k, q-k)$ is written as $U(p', q') \times \mathrm{Sp}(p-p'-k, q-q'-k)$. Put ${}^sH^{(k)} = \exp({}^s\mathfrak{h}^{(k)}) \cap G$. Then ${}^sH^{(k)}$ is a maximally split Cartan subgroup of $\mathrm{GL}(k, \mathbb{H})$. (Here, we consider the decomposition $M_{(k)} = \mathrm{GL}(k, \mathbb{H}) \times M^\circ_{(k)}$.) We fix a compact Cartan subgroup ${}^uH_{(k,p',q')}$ of $L_{(p',q')}(p-k, q-k)$ and put $H(k, p', q') = {}^sH^{(k)} \times {}^uH_{(k,p',q')}$. We denote by $\mathfrak{h}(k, p', q')$ the complexified Cartan subalgebra of $H(k, p', q')$. We denote by $L'_{(k,p',q')}$ the centerizer of the center of $\mathrm{U}(p', q')$ in G. Then, we have $L'(k, p', q') \cong \mathrm{U}(p', q') \times \mathrm{Sp}(p-p', q-q')$. Let $\tilde{\mathfrak{q}}'(k, p', q')$ be a θ-stable parabolic subalgebra of $\mathrm{Sp}(p, q)$ with the Levi subgroup $L'_{(k,p',q')}$. Let ϕ be any irreducible unitary representation of $L^\star_{(k,p',q')}$. We choose $\tilde{\mathfrak{q}}'(k, p', q')$ so that $\eta_\ell \boxtimes \phi$ is good with respect to $\tilde{\mathfrak{q}}'(k, p', q')$ for sufficiently large ℓ. $\tilde{\mathfrak{q}}'(k, p', q')$ is K-conjugate to $\tilde{\mathfrak{q}}_{(p',q')}(p, q)$ defined in 3.4.

Since $\mathfrak{h}(k, p', q') \subseteq \tilde{\mathfrak{q}}'(k, p', q') \cap \mathfrak{p}_{(k)}(p, q)$, $(\mathfrak{p}_{(k)}(p, q), \tilde{\mathfrak{q}}'(k, p', q'))$ is a $\sigma\theta$-pair.

Since ${}^s\mathfrak{h}^{(k)} \subseteq {}^s\mathfrak{h}$ has a basis $E_1, ..., E_{2k}$, for any $\lambda \in ({}^s\mathfrak{h}^{(k)})^*$, we define

$$||\lambda|| = \max(\{0\} \cup \{|\lambda(E_i)| \mid 1 \leqslant i \leqslant 2k, |\lambda(E_i)| \in \mathbb{Z}\}).$$

For any Harish-Chandra module V with an infinitesimal character $\lambda \in ({}^s\mathfrak{h}^{(k)})^*$, we put $||V|| = ||\lambda||$. This is well-defined, since $||\lambda||$ is invariant under the Weyl group action. For example, we easily have:

Lemma 3.5.2.

(1) *If χ is a one-dimensional unitary representation of $GL(k, \mathbb{H})$, then $||\chi|| = 0$.*

(2) *For $\ell \in \mathbb{Z}$ and $t \in \sqrt{-1}\mathbb{R}$, we have*

$$||A_k(\ell, t)|| = \begin{cases} \frac{2k+\ell-1}{2} & \text{if } \ell \text{ is odd and } t = 0, \\ 0 & \text{otherwise} \end{cases}$$

Applying Theorem 2.2.3 to the $\sigma\theta$-pair $(\mathfrak{p}_{(k)}(p,q), \tilde{\mathfrak{q}}'(k,p',q'))$, we have:

Theorem 3.5.3. *(a rearrangement formula for $Sp(p,q)$)*

Let k be a positive integer such that $k \leqslant p$ and $k \leqslant q$. Let p' and q' be non-negative integers such that $p' + q' > 0$. Moreover, we assume that $p' + k \leqslant p$ and $q' + k \leqslant q$. Let V (resp. Z) be a Harish-Chandra module with an infinitesimal character for $GL(k, \mathbb{H})$ (resp. $Sp(p-p'-k, q-q'-k)$). Let ℓ be an integer such that $\ell \geqslant \max\{||V||, ||Z||\} - (p-p'-k) - (q-q'-k)$. Then we have

$$\left[\mathsf{Ind}_{P_{(k)}(p,q)}^{Sp(p,q)}\left(V \boxtimes \mathcal{R}_{p',q'}^{p-k,q-k}(\ell)(Z)\right)\right] = \left[\mathcal{R}_{p',q'}^{p,q}(\ell - 2k)\left(\mathsf{Ind}_{P_{(k)}(p-p',q-q')}^{Sp(p-p',q-q')}(V \boxtimes Z)\right)\right].$$

The above cohomological inductions are in the good region.

Next, we consider the case of $\mathrm{SO}^*(2n)$.

Put $n_0 = n - p' - q'$. We consider θ-stable maximal parabolic subalgebra $\tilde{\mathfrak{q}}^*_{(p',q')}(2n)$ defined in 3.5.

Let $\mathfrak{h}(2n_0)$ (resp. $\mathfrak{h}_{(p',q')}$) be a θ and σ-stable compact Cartan subalgebra for $\mathrm{SO}^*(2n_0)$ (resp. $\mathrm{U}(p',q')$).

Taking account of $L^*_{(p',q')}(2n) = \mathrm{U}(p',q') \times \mathrm{SO}^*(2n_0)$, we put

$$\mathfrak{h}(2n) = \mathfrak{h}_{(p',q')} \oplus \mathfrak{h}(2n_0) \subseteq \mathfrak{l}^*_{(p',q')}(2n) \subseteq \mathfrak{so}(2n, \mathbb{C}).$$

Then, $\mathfrak{h}(2n)$ is a θ and σ-stable compact Cartan subalgebra for $\mathrm{SO}^*(2n)$. Using the above direct sum decomposition, we regard $\mathfrak{h}^*_{(p',q')}$ and $\mathfrak{h}(2n_0)^*$ as a subspace of $\mathfrak{h}(2n)^*$. We introduce an orthonormal basis $\{f_1, ..., f_{p'+q'}\}$ (resp. $\{f_{p'+q'+1}, ..., f_{2n}\}$) of $\mathfrak{h}^*_{(p',q')}$ (resp. $\mathfrak{h}(2n_0)^*$) such that

$$\Delta(\mathfrak{so}(2n,\mathbb{C}), \mathfrak{h}(2n)) = \{\pm f_i \pm f_j \mid 1 \leqslant i < j \leqslant p+q\},$$
$$\Delta(\mathfrak{u}(p',q') \otimes_{\mathbb{R}} \mathbb{C}, \mathfrak{h}_{(p',q')}) = \{f_i f_j \mid 1 \leqslant i,j \leqslant p'+q', i \neq j\},$$
$$\Delta(\mathfrak{so}(2n_0,\mathbb{C}), \mathfrak{h}(2n_0)) = \{\pm f_i \pm f_j \mid p'+q' < i < j \leqslant p+q\},$$
$$\Delta(\tilde{\mathfrak{u}}^*_{(p',q')}(2n), \mathfrak{h}(2n)) = \{f_i \pm f_j \mid 1 \leqslant i \leqslant p'+q' < j \leqslant p+q\} \cup \{f_i + f_j \mid 1 \leqslant i < j \leqslant p'+q'\},$$

We denote by $F_1, ..., F_{p+q}$ the basis of $\mathfrak{h}(2n)$ dual to $f_1, ..., f_{2n}$.

We have

$$\delta(\tilde{\mathfrak{u}}^*_{(p',q')}(2n))(F_i) = \begin{cases} \frac{2n-p'-q'-1}{2} & \text{if } 1 \leqslant i \leqslant p'+q', \\ 0 & \text{otherwise} \end{cases}$$

Let Z be any Harish-Chandra module for $\mathrm{SO}^*(2n_0)$ with an infinitesimal character $\lambda \in \mathfrak{h}(2n_0)^* \subseteq \mathfrak{h}(2n)^*$. λ is unique up to the Weyl group action. Put $||\lambda|| = \max(\{0\} \cup \{|\lambda(F_i)| \mid p'+q' < i \leqslant 2n, \lambda(F_i) \in \mathbb{Z})\})$. $||\lambda||$ is invariant under the Weyl group action on λ, so we write $||Z|| = ||\lambda||$.

$\eta_\ell \boxtimes Z$ has an infinitesimal character $[\ell, \lambda] \in \mathfrak{h}(2n)^*$ such that

$$[\ell,\lambda](F_i) = \begin{cases} \ell + \frac{p'+q'+1}{2} - i & \text{if } 1 \leqslant i \leqslant p'+q', \\ \lambda(F_i) & \text{if } p'+q' < i \leqslant p+q \end{cases}$$

We denote by $\mathcal{H}(\mathrm{SO}^*(2n))_\mu$ the category of Harish-Chandra modules for $\mathrm{SO}^*(2n)$ with an infinitesimal character μ.

Definition 3.5.4. *For $\ell \in \mathbb{Z}$ and $Z \in \mathcal{H}(SO^*(2n_0))_\lambda$, put*

$$\mathcal{R}^{2n}_{p',q'}(\ell)(Z) = \left(\mathcal{R}^{\mathfrak{so}(2n,\mathbb{C}),K}_{\tilde{\mathfrak{q}}^*_{(p',q')}(2n), L^*_{(p',q')}(2n)\cap K}\right)^S ((\eta_\ell \boxtimes Z) \otimes \mathbb{C}_{2\delta(\tilde{\mathfrak{u}}^*_{(p',q')}(2n))}),$$

where $S = (p'+q')(n-p'-q') + p'q'$. If $\ell \geqslant ||\lambda|| - n_0 + 1$, then the above cohomological induction is in good range and we have an exact functor

$$\mathcal{R}^{2n}_{p',q'}(\ell) : \mathcal{H}(SO^*(2n_0))_\lambda \to \mathcal{H}(SO^*(2n))_{[\ell,\lambda]+\delta(\tilde{\mathfrak{u}}^*_{(p',q')}(2n))}$$

In the similar way to the case of $\mathrm{Sp}(p,q)$, we have:

Theorem 3.5.5. *(a rearrangement formula for $SO^*(2n)$)*

Let k be a positive integer such that $k \leqslant p$ and $k \leqslant q$. Let p' and q' be non-negative integers such that $p'+q' > 0$. Moreover, we assume that $p'+q'+2k \leqslant n$. Let V (resp. Z) be a Harish-Chandra module with an infinitesimal character for $GL(k,\mathbb{H})$ (resp. $SO^*(2(n-p'-q'-2k))$). Let ℓ be an integer such that $\ell \geqslant \max\{||V||, ||Z||\} - (n-p'-q'-2k) - 1$. Then we have

$$\left[\mathsf{Ind}^{SO^*(2n)}_{P^*_{(k)}(2n)}\left(V \boxtimes \mathcal{R}^{2(n-2k)}_{p',q'}(\ell)(Z)\right)\right] = \left[\mathcal{R}^{2n}_{p',q'}(\ell-2k)\left(\mathsf{Ind}^{SO^*(2(n-p'-q'))}_{P^*_{(k)}(2(n-p'-q'))}(V \boxtimes Z)\right)\right].$$

The above cohomological inductions are in the good region.

3.6 Decomposition formulas

We define:

Definition 3.6.1. *Let k be a positive inter and ℓ be an integer such that $\ell + k \in 2\mathbb{Z}$. Let i be an integer such that $0 \leqslant i \leqslant k$. We define the following derived functor module for $U(k,k)$.*

$$B^{(i)}_k(\ell) = \left({}^u\mathcal{R}^{\mathfrak{u}(k,k)\otimes_{\mathbb{R}}\mathbb{C},\, U(k)\times U(k)}_{\mathfrak{q}((i,k-i),(k-i,i)),\, U(i)\times U(k-i)\times U(k-i)\times U(i)}\right)^{2i(k-i)} (\eta_{\frac{\ell+k}{2}} \boxtimes \eta_{\frac{\ell-k}{2}}).$$

$B_k^{(i)}(\ell)$ is not in the good region. In fact, it is an irreducible unitary representation located at the end of the weakly fair region in the sense of [Vogan 1988].

We quote the following reducibility result of the degenerate principal series.

Theorem 3.6.2. (Kashiwara-Vergne, Johnson,...)

Let $\ell \in \mathbb{Z}$ and $t \in \sqrt{-1}\mathbb{R}$.

(1) If $\ell + k \in 2\mathbb{Z}$, then

$$I_k(\ell, 0) = \bigoplus_{i=0}^{k} B_k^{(i)}(\ell)$$

(2) If $t \neq 0$ or $\ell + k + 1 \in 2\mathbb{Z}$, then $I_k(\ell, t)$ is irreducible.

Some remarks are in order. The reducibilty of $I_k(\ell, 0)$ is established by [Kashiwara-Vergne, 1979]. The irreducibilty result is due to [Johnson 1990]. Identifying irreducible components in (1) as derived functor modules is easy conclusion from [Barbasch-Vogan 1983] and it has been more or less known by experts. For example, a proof is given in [Matumoto 1996] 3.4.

Combining Theorem 3.6.2 and Proposition 3.3.2, we have:

Proposition 3.6.3.

(1) Let p, q be positive integers such that $q \leqslant p$. Let $G = Sp(p, q)$ and let k be a positive integer such that $k \leq q$. Let V be an irreducible unitary representation of $Sp(p-k, q-k)$. Let m be an integer such that $m \geqslant ||V|| + k - 1$. Then we have

$$\mathsf{Ind}_{P_{(k)}(p,q)}^{Sp(p,q)}(A_k(2m+1, 0) \boxtimes V) \cong \bigoplus_{i=0}^{k} \mathcal{R}_{i,k-i}^{p,q}(m-n+k)\left(\mathcal{R}_{k-i,i}^{p-i,q-k+i}(m-n+2k)(V)\right).$$

(2) Let n be a positive integer. Let $G = SO^*(2n)$ and let k be a positive integer such that $2k \leq n$. Let V be an irreducible unitary representation of $SO^*(2(n-2k))$. Let m be an integer such that $m \geqslant ||V|| + k - 1$. Then we have

$$\mathsf{Ind}_{P^*_{(k)}(2n)}^{SO^*(2n)}(A_k(2m+1, 0) \boxtimes V) \cong \bigoplus_{i=0}^{k} \mathcal{R}_{i,k-i}^{2n}(m-n+k+1)\left(\mathcal{R}_{k-i,i}^{2(n-k)}(m-n+2k+1)(V)\right).$$

We introduce notations for derived functor modules.

First, we assume $G = \mathrm{Sp}(p, q)$, $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_m(p', q')$, $0 \leqslant p' \leqslant p$, and $0 \leqslant q' \leqslant q$. Put $p_0 = p - p'$ and $q_0 = q - q'$. We consider the derived functor modules with respect to $\tilde{\mathfrak{p}}_{(\underline{\mathbf{p}}, \underline{\mathbf{q}})}(p, q)$.

For $1 \leqslant i \leqslant m$, we put $p_i^* = p_1 + \cdots + p_i$ and $q_i^* = q_1 + \cdots + q_i$. Let $\ell, ..., \ell_m$ be integers and put

$$A^{p,q}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1, ..., \ell_m) =$$
$$\mathcal{R}^{p,q}_{p_1,q_1}(\ell_1)\left(\mathcal{R}^{p-p_1,q-q_1}_{p_2,q_2}(\ell_2)\left(\cdots\left(\mathcal{R}^{p-p^*_{i-1},q-q^*_{i-1}}_{p_i,q_i}(\ell_i)\left(\cdots\mathcal{R}^{p_0+p_m,q_0+q_m}_{p_m,q_m}(\ell_m)(1_{\mathrm{Sp}(p_0,q_0)})\right)\cdots\right)\cdots\right)\right).$$

Here, $1_{\mathrm{Sp}(p_0,q_0)}$ is the trivial representation of $\mathrm{Sp}(p_0, q_0)$. In this setting, we define as follows.

$$\delta_i = p + q - p_i^* - q_i^* - \frac{p_i + q_i - 1}{2} \qquad (1 \leqslant i \leqslant m),$$
$$\tilde{\ell}_i = \ell_i + \delta_i \qquad (1 \leqslant i \leqslant m).$$

Then, $A^{p,q}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1, ..., \ell_m)$ is in good (resp. weakly fair) region if and only if $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_m \geqslant 0$ (resp. $\tilde{\ell}_1 \geqslant \tilde{\ell}_2 \geqslant \cdots \geqslant \tilde{\ell}_m \geqslant 0$).

Next, we assume $G = \mathrm{SO}^*(2n)$, $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_m(p', q')$, $0 \leqslant p' + q' \leqslant n$. Put $n_0 = n - p' - q'$. We consider the derived functor modules with respect to $\tilde{\mathfrak{p}}^*_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(2n)$. For $1 \leqslant i \leqslant m$, we put $p_i^* = p_1 + \cdots + p_i$ and $q_i^* = q_1 + \cdots + q_i$. Let $\ell, ..., \ell_m$ be integers and put

$$A^{2n}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1, ..., \ell_m) =$$
$$\mathcal{R}^{2n}_{p_1,q_1}(\ell_1)\left(\mathcal{R}^{2n-p_1-q_1}_{p_2,q_2}(\ell_2)\left(\cdots\left(\mathcal{R}^{2n-p^*_{i-1}-q^*_{i-1}}_{p_i,q_i}(\ell_i)\left(\cdots\mathcal{R}^{n_0+p_m+q_m}_{p_m,q_m}(\ell_m)(1_{\mathrm{SO}^*(2n_0)})\right)\cdots\right)\cdots\right)\right).$$

Here, $1_{\mathrm{SO}^*(2n_0)}$ is the trivial representation of $\mathrm{SO}^*(2n_0)$. In this setting, we define as follows.

$$\delta_i = p + q - p_i^* - q_i^* - \frac{p_i + q_i - 1}{2} - 1 \qquad (1 \leqslant i \leqslant m),$$
$$\tilde{\ell}_i = \ell_i + \delta_i \qquad (1 \leqslant i \leqslant m).$$

Then, $A^{2n}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1, ..., \ell_m)$ is in good (resp. weakly fair) region if and only if $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_m \geqslant 0$ (resp. $\tilde{\ell}_1 \geqslant \tilde{\ell}_2 \geqslant \cdots \geqslant \tilde{\ell}_m \geqslant 0$).

Combining Theorem 3.5.3, Theorem 3.5.5, and Proposition 3.6.3 , we have:

Theorem 3.6.4.

(1) Let p, q be positive integers such that $q \leqslant p$. We consider the setting of $G = Sp(p, q)$. We assume $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_m(p', q')$, $0 \leqslant p' \leqslant p$, and $0 \leqslant q' \leqslant q$. Let k be a positive integer. Put $n = p + q$ and put $n'_j = (p_j + q_j) + \cdots + (p_m + q_m) + 2k$ for $1 \leqslant i \leqslant m$. Let s be a non-negative integer.

Let $\ell_1, ..., \ell_m$ be integers such that $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_m \geqslant 0$. Moreover, we assume there is some $1 \leqslant j \leqslant m$ such that $\ell_{j-1} \geqslant s - n'_j + 3k$ and $s - n'_j + 2k \geqslant \ell_j$. (Here, we put, formally,

$\ell_0 = +\infty$.) Put $\underline{\mathbf{p}}'_i = (p_1, ..., p_{j-1}, i, k-i, p_j ..., p_m)$ and $\underline{\mathbf{q}}'_i = (q_1, ..., q_{j-1}, k-i, i, q_j, ..., q_m)$ for $1 \leqslant i \leqslant k$. Then we have

$$(A) \quad \mathrm{Ind}_{P_{(k)}(p+k,q+k)}^{Sp(p+k,q+k)}(A_k(2s+1) \boxtimes A_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}^{p,q}(\ell_1, ..., \ell_m))$$
$$\cong \bigoplus_{1=0}^{k} A_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}^{p+k,q+k}(\ell_1 - 2k, ..., \ell_{j-1} - 2k, s - n'_j + k, s - n'_j + 2k, \ell_j, ..., \ell_1).$$

(2) Let n be positive integer and we consider the setting of $G = SO^*(2n)$. We assume $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_m(p', q')$, $0 \leqslant p' + q' \leqslant n$. Let k be a positive integer. Put $n'_j = (p_j + q_j) + \cdots + (p_m + q_m) + 2k$ for $1 \leqslant i \leqslant m$. Let s be a non-negative integer.

Let $\ell_1, ..., \ell_m$ be integers such that $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_m \geqslant 0$. Moreover, we assume there is some $1 \leqslant j \leqslant m$ such that $\ell_{j-1} \geqslant s - n'_j + 3k + 1$ and $s - n'_j + 2k + 1 \geqslant \ell_j$. (Here, we put, formally, $\ell_0 = +\infty$.) Put $\underline{\mathbf{p}}'_i = (p_1, ..., p_{j-1}, i, k-i, p_j ..., p_m)$ and $\underline{\mathbf{q}}'_i = (q_1, ..., q_{j-1}, k-i, i, q_j, ..., q_m)$ for $1 \leqslant i \leqslant k$. Then we have

$$(A') \quad \mathrm{Ind}_{P^*_{(k)}(2(n+2k))}^{SO^*(2(n+2k))}(A_k(2s+1) \boxtimes A_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}^{2n}(\ell_1, ..., \ell_m))$$
$$\cong \bigoplus_{1=0}^{k} A_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}^{2(n+2k)}(\ell_1 - 2k, ..., \ell_{j-1} - 2k, s - n'_j + k + 1, s - n'_j + 2k + 1, \ell_j, ..., \ell_1).$$

(3) The derived functor modules in the right hand side of (A) and (A') are all non-zero and irreducible. (Actually, they are good-range cohomological induction form non-zero irreducible modules.)

Here, we apply the translation principle in weakly fair range in [Vogan 1988] to the above result and obtain the following.

Theorem 3.6.5.

(1) Let p, q be positive integers such that $q \leqslant p$. We consider the setting of $G = Sp(p, q)$. We assume $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_m(p', q')$, $0 \leqslant p' \leqslant p$, and $0 \leqslant q' \leqslant q$. Let k be a positive integer. Put $n = p + q$ and put $n'_j = (p_j + q_j) + \cdots + (p_m + q_m) + 2k$ for $1 \leqslant i \leqslant m$. Let s be an integer such that $2s + 1 \geqslant -k$. Let $\ell_1, ..., \ell_m$ be integers such that $\tilde{\ell}_1 \geqslant \tilde{\ell}_2 \geqslant \cdots \geqslant \tilde{\ell}_m \geqslant 0$. We choose any $1 \leqslant j \leqslant m$ such that $\tilde{\ell}_{j-1} \geqslant s + \frac{k+1}{2} \geqslant \tilde{\ell}_j$. (Here, we put, formally, $\ell_0 = +\infty$.) Put $\underline{\mathbf{p}}'_i = (p_1, ..., p_{j-1}, i, k-i, p_j ..., p_m)$ and $\underline{\mathbf{q}}'_i = (q_1, ..., q_{j-1}, k-i, i, q_j, ..., q_m)$ for $1 \leqslant i \leqslant k$. Then

we have

$$(B)\quad \mathsf{Ind}_{P_{(k)}(2(p+k,q+k)}^{Sp(p+k,q+k)}(A_k(2s+1)\boxtimes A^{p,q}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1,....,\ell_m))$$
$$\cong\bigoplus_{1=0}^{k} A^{p+k,q+k}_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}(\ell_1-2k,...,\ell_{j-1}-2k,s-n'_j+k,s-n'_j+2k,\ell_j,...,\ell_1).$$

(2) Let n be positive integer and we consider the setting of $G=SO^*(2n)$. We assume $(\underline{\mathbf{p}},\underline{\mathbf{q}})\in\mathbb{P}_m(p',q')$, $0\leqslant p'+q'\leqslant n$. Let k be a positive integer. Put $n'_j=(p_j+q_j)+\cdots+(p_m+q_m)+2k$ for $1\leqslant i\leqslant m$.

Let s be an integer such that $2s+1\geqslant -k$. Let $\ell_1,...,\ell_m$ be integers such that $\tilde{\ell}_1\geqslant\tilde{\ell}_2\geqslant\cdots\geqslant\tilde{\ell}_m\geqslant 0$. We choose any $1\leqslant j\leqslant m$ such that $\tilde{\ell}_{j-1}\geqslant s+\frac{k+1}{2}\geqslant\tilde{\ell}_j$. (Here, we put, formally, $\ell_0=+\infty$.) Put $\underline{\mathbf{p}}'_i=(p_1,...,p_{j-1},i,k-i,p_j...,p_m)$ and $\underline{\mathbf{q}}'_i=(q_1,...,q_{j-1},k-i,i,q_j,...,q_m)$ for $1\leqslant i\leqslant k$. Then we have

$$(B')\quad \mathsf{Ind}_{P^*_{(k)}(2(n+2k))}^{SO^*(2(n+2k))}(A_k(2s+1)\boxtimes A^{2n}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1,....,\ell_m))$$
$$\cong\bigoplus_{1=0}^{k} A^{2(n+2k)}_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}(\ell_1-2k,...,\ell_{j-1}-2k,s-n'_j+k+1,s-n'_j+2k+1,\ell_j,...,\ell_1).$$

Proof The proof is similar to the arguments in [Matumoto 1996] 3.3. We consider the case of $G=\mathrm{Sp}(p,q)$. (The case of $G=\mathrm{SO}^*(2n)$ is similar.) For an integer a, we denote by η_a one dimensional representation of $\mathrm{GL}(h,\mathbb{C})$ defined by

$$\eta_a(g)=\det(g)^a$$

Let $a_1,...,a_m$ and b be non-negative integers and consider a one dimensional representation $\eta=\eta_{a_1}\boxtimes\cdots\boxtimes\eta_{a_{j-1}}\boxtimes\eta_b\boxtimes\eta_b\boxtimes\eta_{a_j}\cdots\boxtimes\eta_{a_m}\boxtimes 1_{\mathrm{Sp}(p_0+q_0,\mathbb{C})}$ of $\mathrm{GL}(p_1+q_1,\mathbb{C})\times\cdots\times\mathrm{GL}(p_{j-1}+q_{j-1},\mathbb{C})\times\mathrm{GL}(k,\mathbb{C})\times\mathrm{GL}(k,\mathbb{C})\times\mathrm{GL}(p_j+q_j,\mathbb{C})\times\cdots\times\mathrm{GL}(p_m+q_m,\mathbb{C})\times\mathrm{Sp}(p_0+q_0,\mathbb{C})$. If $\eta_1\geqslant\cdots\geqslant a_{j-1}\geqslant b\geqslant a_j\geqslant\cdots\geqslant a_m$, then there is an irreducible finite dimemensional representation V of $G_{\mathbb{C}}$ which contains η as the highest weight space . If we choose $a_1\geqslant a_2\geqslant\cdots\geqslant a_{j-1}\gg b\gg a_j\geqslant\cdots\geqslant a_m$ suitably, we have $s'=s+b$, $\ell'_r=\ell_r+a_r$ $(1\leqslant r\leqslant m)$ satisfy the regularity assumption in Theorem 3.6.4. So, we have:

$$(C)\quad \mathsf{Ind}_{P_{(k)}(p+k,q+k)}^{\mathrm{Sp}(p+k,q+k)}(A_k(2s'+1)\boxtimes A^{p,q}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell'_1,....,\ell'_m))$$
$$\cong\bigoplus_{1=0}^{k} A^{p+k,q+k}_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}(\ell'_1-2k,...,\ell'_{j-1}-2k,s'-n'_j+k,s'-n'_j+2k,\ell'_j,...,\ell'_1).$$

Let T the translation functor from the infinitesimal character of the modules in (C) to that of (B). If we apply T to the both sides of C, we obtain (B) above. The argument is the same as [Matumoto 1996] Lemma3.3.3. The main ingredient is [Vogan 1988] Proposition 4.7 . (We may apply similar argument to non-elliptic cohomological induction by [Vogan (green)] Lemma 7.2.9 (b).)

Q.E.D.

Remark In Theorem3.6.5, a choice of j need not be unique. So, depending on the choices of j, we have apparently different formulas. Their compatibility is assured by [Matumoto 1996] Theorem 3.3.4, which is an easy conclusion of [Barbasch-Vogan 1983] Theorem 4.2. The derived functor modules in the right hand side of (B) and $(B')$are all in the weakly fair region.

§ 4. Reduction of irreducibilities

4.1 Standard parabolic subgroups

In this section, Let G be either $\mathrm{Sp}(n-q,q)$ with $2q \leqslant n$ or $\mathrm{SO}^*(2n)$. Fix θ, ${}^sH$, etc. as in 3.1.

We also fix some particular orthonormal basis $e_1, ..., e_n$ of ${}^s\mathfrak{h}^*$, as in 3.1. We fix a simple system Π of $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$ as in 3.1.

Let $\kappa = (k_1, ..., k_s)$ be a finite sequence of positive integers such that

$$k_1 + \cdots + k_s \leqslant \begin{cases} q & \text{if } G = \mathrm{Sp}(p,q), \\ \frac{n}{2} & \text{if } G = \mathrm{SO}^*(2n). \end{cases}.$$

We put $k_i^* = k_1 + \cdots k_i$ for $1 \leqslant i \leqslant s$ and $k_0^* = 0$. If $G = \mathrm{Sp}(p,q)$, put $p' = p - k_s^*$ and $q' = q - k_s^*$. If $G = \mathrm{SO}^*(2n)$, put $r = n - 2k_s^*$.

We put

$$A_i = \sum_{j=1}^{2k_i} E_{k_{i-1}^*+j} \quad (1 \leqslant i \leqslant s)),$$

Then we have $\theta(A_i) = -A_i$ for $1 \leqslant i \leqslant s$. We denote by $\mathfrak{a}_\kappa$ the Lie subalgebra of ${}^s\mathfrak{h}$ spanned by $\{A_i \mid 1 \leqslant i \leqslant s\}$.

We define a subset $S(\kappa)$ of Π as follows. If $G = \mathrm{Sp}(p,q)$, we define

$$S(\kappa) = \begin{cases} \Pi - \{e_{2k_i^*} - e_{2k_i^*+1} | 1 \leqslant i \leqslant s\} & \text{if } p' > 0, \\ \Pi - (\{e_{2k_i^*} - e_{2k_i^*+1} | 1 \leqslant i \leqslant s-1\} \cup \{2e_n\}) & \text{if } p' = 0 \end{cases}.$$

If $G = \mathrm{SO}^*(2n)$, we define

$$S(\kappa) = \begin{cases} \Pi - \{e_{2k_i^*} - e_{2k_i^*+1} | 1 \leqslant i \leqslant s\} & \text{if } r > 0, \\ \Pi - (\{e_{2k_i^*} - e_{2k_i^*+1} | 1 \leqslant i \leqslant s-1\} \cup \{e_{n-1} + e_n\}) & \text{if } r = 0 \end{cases}.$$

We denote by M_κ (resp. $\mathfrak{m}_\kappa$) the standard Levi subgroup (resp. subalgebra) of G (resp. $\mathfrak{g}$) corresponding to $S(\kappa)$. Namely M_κ is the centerizer of $\mathfrak{a}_\kappa$ in G.

We denote by P_κ the parabolic subgroup of G whose θ-invariant Levi part is M_κ. We choose P_κ so that the roots in Δ whose root spaces are contained in the complexified Lie algebra of the nilradical of P_κ are all in Δ^+. We denote by N_κ the nilradical of P_κ.

Formally, we denote by $\mathrm{Sp}(0,0)$ and $\mathrm{SO}^*(0)$ the trivial group $\{1\}$ and we denote by $\mathrm{GL}(\kappa, \mathbb{H})$ a product group $\mathrm{GL}(k_1, \mathbb{H}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{H})$. Then, we have

$$M_\kappa \cong \begin{cases} \mathrm{GL}(\kappa, \mathbb{H}) \times \mathrm{Sp}(p', q') & \text{if } G = \mathrm{Sp}(p,q) \\ \mathrm{GL}(\kappa, \mathbb{H}) \times \mathrm{SO}^*(2r) & \text{if } G = \mathrm{SO}^*(2n) \end{cases}.$$

Often, we identify $\mathrm{GL}(\kappa, \mathbb{H})$, $\mathrm{Sp}(p', q')$, $\mathrm{SO}^*(2r)$ with subgroups of M_κ in obvious ways. We call such identifications the standard identifications. The Cartan involution θ induces Cartan involutions on M_κ, $\mathrm{GL}(\kappa, \mathbb{H})$, $\mathrm{Sp}(p', q')$, and $SO^*(2r)$ and we denote them by the same letter θ. We put $M_\kappa^\circ = \mathrm{Sp}(p', q')$ if $G = \mathrm{Sp}(p, q)$ and put $M_\kappa^\circ = \mathrm{SO}^*(2r)$ if $G = \mathrm{SO}^*(2n)$.

We denote by $\mathfrak{p}_\kappa$, $\mathfrak{m}_\kappa$, $\mathfrak{m}_\kappa^\circ$ and $\mathfrak{n}_\kappa$ the complexified Lie algebra of P_κ, M_κ, M_κ°, and N_κ, respectively.

For $\tau \in \mathfrak{S}_s$ and $\kappa = (k_1, ..., k_s)$, we define $\kappa^\tau = (k_{\tau(1)}, ..., k_{\tau(s)})$.

Let ξ be an irreducible unitary representation of M_κ. ξ can be written as $\xi = \xi_1 \boxtimes \cdots \boxtimes \xi_s \boxtimes \xi_0$, where for $1 \leqslant i \leqslant s$ (resp. for $i = 0$) ξ_i is an irreducible unitary representation of $\mathrm{GL}(k_i, \mathbb{H})$ (resp. M_κ°). For $\tau \in \mathfrak{S}_s$, we denote by ξ^τ an irreducible unitary representation of L_{κ^τ}, $\xi_{\tau(1)} \boxtimes \cdots \boxtimes \xi_{\tau(s)} \boxtimes \xi_0$. The following is well-known.

Lemma 4.1.1. (Harish-Chandra)

Let $\kappa = (k_1, ..., k_s)$ and $\tau \in \mathfrak{S}_s$ be as above. Let ξ be an irreducible unitary representation of M_κ. Then we have

$$\mathsf{Ind}_{P_\kappa}^G(\xi) \cong \mathsf{Ind}_{P_{\kappa^\tau}}^G(\xi^\tau)$$

Sometimes, we treat various $\mathrm{Sp}(p, q)$'s and $\mathrm{SO}^*(2n)$'s at the same time. So, we write sometimes $P_\kappa(p, q)$ (resp. $P_\kappa^*(2n)$)for P_κ if $G = \mathrm{Sp}(p, q)$ (resp. $G = \mathrm{SO}^*(2n)$).

Let $A_k(\ell, t)$ $\quad (\ell \in \{\ell' \in \mathbb{Z} \mid \ell' \geqslant -k\} \cup \{-\infty\})$ be the representation of $\mathrm{GL}(n\mathbb{H})$ defined in Definition 2.4.3. If $\ell \geqslant -k$, $A_k(\ell, t)$ is a quarternionic Speh representation in the weakly fair range. $A_k(-\infty, t)$ is a unitary one-dimensional representation.

Any derived functor module is a parabolic induction from an external tensor product of some $A_k(\ell, t)$'s. So, the unitarily induced module from a derived functor module (in weakly fair range) can be written as:

$$(\circledast) \qquad \mathsf{Ind}_{P_\kappa}^G(A_{k_1}(\ell_1, t_1) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s, t_s) \boxtimes Z).$$

Here, Z is a derived functor module of M_κ° in the weakly fair range. Moreover, $\ell_i \in \{\ell \in \mathbb{Z} \mid \ell \geqslant -k_i\} \cup \{-\infty\}$, and $t_i \in \sqrt{-1}\mathbb{R}$ for $1 \leqslant i \leqslant s$. Using well-known Harish-Chandra's result,we may assume $\sqrt{-1}t_i \geqslant 0$ for all $1 \leqslant i \leqslant s$.

We assume that $\ell_i + 1 \in 2\mathbb{Z}$ and $t_i = 0$ for some $1 \leqslant i \leqslant s$. Then, using Lemma 4.1.1, we may assume $i = s$. Let $\kappa' = (k_1, ..., k_{s-1})$. Then from the induction-by-stage, we have

$$\begin{aligned}&\mathsf{Ind}_{P_\kappa}^G(A_{k_1}(\ell_1, t_1) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s, t_s) \boxtimes Z)\\ &\qquad\cong \mathsf{Ind}_{P'_\kappa}^G(A_{k_1}(\ell_1, t_1) \boxtimes \cdots \boxtimes A_{k_{s-1}}(\ell_{s-1}, t_{s-1}) \boxtimes \mathsf{Ind}_{P_{(k_s)}}^{M_{\kappa'}^\circ}(A_{k_s}(\ell_s, 0) \boxtimes Z)).\end{aligned}$$

Applying the decomposition formula Theorem 3.7.5, we see that the above induced module is a direct sum of the induced modules of the form like

$$\mathsf{Ind}_{P'_\kappa}^G(A_{k_1}(\ell_1, t_1) \boxtimes \cdots \boxtimes A_{k_{s-1}}(\ell_{s-1}, t_{s-1}) \boxtimes Z')).$$

Here, Z' is a derived functor module of $M_{\kappa'}^\circ$ in the weakly fair range. Assume that we understand the reducibility of Z''s. Then, applying the above argument, we can reduce the irreducible decomposition of the above $\circledast$ to the following.

$$(\Diamond) \qquad \mathsf{Ind}_{P_\kappa}^G(A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0) \boxtimes A_{k_{h+1}}(\ell_{h+1}, t_{h+1}) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s, t_s) \boxtimes Z).$$

Here, ℓ_i is not odd integer if $1 \leqslant i \leqslant h$, $\sqrt{-1}t_i > 0$ if $h < i \leqslant s$, and Z is an irreducible representation of M_κ° whose infinitesimal character is in $\mathcal{P}_{M_\kappa^\circ}$. Put $\tau = (k_1, ..., k_h)$ and $\tau' = (k_{h+1}, ..., k_s)$. Also put $a = k_1 + \cdots + k_h$ and $b = k_{h+1} + \cdots + k_s$.

We state the main result of §4.

Theorem 4.1.2. *The following is equivalent.*

(1) The above $\Diamond$ is irreducible.

(2) The following induced module $\divideontimes$ is irreducible.

$$(\divideontimes) \qquad \mathsf{Ind}_{P_\tau}^{SO^*(4a)}(A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0)).$$

Remark Under an adequate regularity condition on $\ell_1, ..., \ell_h$, we may apply Proposition 3.3.2 to $\divideontimes$ successively, and we obtain that $\divideontimes$ is a good-range elliptic cohomological induction from an irreducible module like $I_{k_1}(\ell'_1, 0) \boxtimes \cdots \boxtimes I_{k_h}(\ell'_h, 0)$. Hence $\divideontimes$ is irreducible for such parameters.

In §5, we show $\divideontimes$ is irreducible if $\ell_1, ..., \ell_h$ are all $-\infty$.

4.2 Proof of Theorem 4.1.2

We denote by ${}^s\mathfrak{h}_\kappa$ (resp. ${}^s\mathfrak{h}^\kappa$) the $\mathbb{C}$-linear span of $E_1, ..., E_{2k_s^*}$ (resp. $E_{2k_s^*+1}, ..., E_n$. Then, we can regard ${}^s\mathfrak{h}_\kappa$ (resp. ${}^s\mathfrak{h}^\kappa$) as the complexified Lie algebra of a θ-invariant maximally split Cartan subgroup of $\mathrm{GL}(\kappa, \mathbb{H})$ (resp. $\mathrm{Sp}(p', q')$ or $\mathrm{SO}^*(2r)$) via the standard identification. We have a direct sum decomposition ${}^s\mathfrak{h} = {}^s\mathfrak{h}_\kappa \oplus {}^s\mathfrak{h}^\kappa$ and it induces ${}^s\mathfrak{h}^* = {}^s\mathfrak{h}^*_\kappa \oplus ({}^s\mathfrak{h}^\kappa)^*$. Namely, we identify ${}^s\mathfrak{h}^*_\kappa$ (resp. ${}^s\mathfrak{h}^*_r$) the $\mathbb{C}$-linear span of $e_1, ..., e_{2k_s^*}$ (resp. $e_{2k_s^*+1}, ..., e_n$).

We denote by ρ the half sum of the positive roots in Δ^+. Let $\eta \in {}^s\mathfrak{h}^*$ be the infinitesimal character of $A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0) \boxtimes A_{k_{h+1}}(\ell_{h+1}, t_{h+1}) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s, t_s) \boxtimes Z$. We may (and do) assume $\Re\eta$ is in the closed Weyl chamber with respect to $\Delta^+ \cap \Delta(\mathfrak{m}_\kappa, {}^s\mathfrak{h})$.

We fix a sufficiently large integer N and we put ${}^s\lambda = 2N\rho + \eta$ and $\Lambda = {}^s\lambda + \mathcal{P}_G$. Then, we have $\eta \in \Lambda$. Hence, $\Delta_\eta = \Delta_{{}^s\lambda}$. Moreover, we have ${}^s\lambda$ is regular and $\Delta^+_{{}^s\lambda} = \Delta^+ \cap \Delta_{{}^s\lambda}$.

We construct a surgroup G' of M_κ as follows. As a Lie group G' is a product group $\mathrm{SO}^*(4a) \times \mathrm{GL}(b, \mathbb{H}) \times M_\kappa^\circ$. The embedding of $M_\kappa = \mathrm{GL}(\tau, \mathbb{H}) \times \mathrm{GL}(\tau', \mathbb{H}) \times M_\kappa^\circ$ into G' is induced from the inclusions $\mathrm{GL}(\tau, \mathbb{H}) \subseteq \mathrm{SO}^*(4a)$ and $\mathrm{GL}(\tau', \mathbb{H}) \subseteq \mathrm{GL}(b, \mathbb{H})$. Easily see that we may fix a Cartan involution whose restriction to M_κ is θ. We denote such a Cartan involution on G' by the same letter θ and denote by K' the corresponding maximal compact subgroup. Since M_κ is a Levi subgroup of both G and G', ${}^sH$ is a θ-stable maximally split Cartan subgroup of G' as well as G.

We denote by $\mathfrak{g}'$ the complexified Lie algebra of G' and denote by Δ' the root system for $(\mathfrak{g}', {}^s\mathfrak{h})$. From the construction of G', we have the integral root system $\Delta_{{}^s\lambda}$ coincides with $\Delta'_{{}^s\lambda}$.

We want to apply Lemma 1.10.1 to G, G', and ${}^s\lambda$ above. In our setting, we put ${}^sH' = {}^sH$ and ${}^s\mathfrak{h}' = {}^s\mathfrak{h}$ and put ψ in (C1) to be the identity map. Hereafter, we denote by $G^\sharp$ any of G and G'. Similarly, we write $K^\sharp$, etc.

In order to define Ψ and $\tilde{\Psi}$, we describe conjugacy classes of Cartan subgroups in G and G'.

First, we remark that there is one to one correspondence between $G^\sharp$-conjugacy classes of Cartan subgroups in $G^\sharp$ and $K^\sharp$-conjugacy classes of θ-stable Cartan subgroups in $G^\sharp$ ([Matsuki 1979]). Second, a G-conjugacy class of Cartan subgroups of G is determined by the dimension of the split part and $GL(k, \mathbb{H})$ has a unique G-conjugacy class of Cartan subgroups (cf. [Sugiura 1959]). Hence, we see a K-conjugacy class (resp. a K'-conjugacy class) of θ-stable (resp. θ'-stable) Cartan subgroups of G (resp. G') is determined by the dimension of the split part. We also see the same statement holds for M_κ.

Since there is obvious one to one correspondence between the conjugacy classes of Cartan subgroups and the conjugacy classes of the Cartan subgroups which is stable with respect to the complex conjugation, hereafter we consider Cartan subalgebras rather than Cartan subgroups.

In order to understand the Cayley transforms on Cartan subalgebras, we examine some particular Cartan subalgebras as follows. Let m be the greatest positive integer which is equal to or less than $\frac{h}{2}$. For $1 \leq i \leq m$, we put $\alpha_i = e_{2i-1} + e_{2i}$. Then, $\{\alpha_1, ..., \alpha_m\}$ is the entire collection of real roots in Δ^+. We define $c^{\alpha_i} \in G_{\mathbb{C}}$ as in 1.6. Since $\alpha_1, ..., \alpha_m$ are mutually orthogonal, we may regard α_i as a real root for $\mathrm{Ad}(c^{\alpha_j})({}^s\mathfrak{h})$. So, we can regard $\mathrm{Ad}(c^{\alpha_i})(\mathrm{Ad}(c^{\alpha_j})({}^s\mathfrak{h})$ as a result of successive applications of Cayley transforms to ${}^s\mathfrak{h}$. Because of the orthogonality of α_i and α_j, we see $\mathrm{Ad}(c^{\alpha_i})(\mathrm{Ad}(c^{\alpha_j})({}^s\mathfrak{h})) = \mathrm{Ad}(c^{\alpha_j})(\mathrm{Ad}(c^{\alpha^i})({}^s\mathfrak{h}))$.

Let $J = \{\alpha_{r_1}, ..., \alpha_{r_k}\} \subseteq \{\alpha_1, ..., \alpha_m\}$. (We assume $r_i \neq r_j$ for $i \neq j$. Similarly as above. we can define successive applications of Cayley transforms as follows.

$$\mathfrak{h}_J = \mathrm{Ad}(c^{\alpha_{r_k}})(\mathrm{Ad}(c^{\alpha_{r_{k-1}}})(\cdots(\mathrm{Ad}(c^{\alpha_{r_1}})({}^s\mathfrak{h})\cdots).$$

$\mathfrak{h}_J$ only depends on J and it is θ and complex conjugacy invariant. We denote by H_J the corresponding Cartan subgroup of G to $\mathfrak{h}_J$.

Put $J_0 = \{\alpha_{k_s^*+1}, ..., \alpha_m\}$. If $J \subseteq J_0$, then $H_J \subseteq M_\kappa$ and H_J is a θ-stable Cartan subgroup of M_κ.

Since a $K^\sharp$-conjugacy (resp. $K \cap M_\kappa$-conjugacy) class of θ-stable Cartan subgroups of $G^\sharp$ (resp. M_κ) is determined by the dimension of the split part, for $J_1, J_2 \subseteq J_0$, the followings are equivalent.

(1) H_{J_1} is K-conjugate to H_{J_2}.

(2) H_{J_1} is K'-conjugate to H_{J_2}.

(3) H_{J_1} is $K \cap M_\kappa$-conjugate to H_{J_2}.

(4) $\mathrm{card} J_1 = \mathrm{card} J_2$.

If $J \subseteq J_0$, H_J is ${}^s\lambda$-integral with respect to both G and G'. Conversely, it is easy to check any ${}^s\lambda$-integral θ-stable Cartan subgroup of $G^\sharp$ is $K^\sharp$-conjugate to H_J for some $J \subseteq J_0$. (For example, using a criterion for the parity condition ([Vogan (green)]), we may check α_i satisfies the parity condition with respect to ${}^s\lambda$ if and only if $\alpha_i \in J_0$. The statement is deduced from this fact.) We also remark that any θ-stable Cartan subgroup of M_κ is $K \cap M_\kappa$-conjugate to some H_J with $J \subseteq J_0$. Hence, there is a bijection Φ (resp. Φ') of the set of the $K \cap M_\kappa$-conjugacy classes of θ-stable Cartan subgroups of M_κ to the set of K-conjugacy (resp. K'-conjugacy) classes of ${}^s\lambda$-integrable θ-stable Cartan subgroups of G (resp. G'). In fact Φ (resp. Φ') is defined such that the image of the K-conjugacy class of H_J under Ψ is the K'-conjugacy class of H_J for any $J \subseteq J_0$. We put $\Psi = \Phi' \circ \Phi^{-1}$. Ψ is a bijection of the set of the K-stable conjugacy classes of

${}^s\lambda$-integral θ-stable Cartan subgroups of G to the set of the K'-conjugacy classes of ${}^s\lambda$-integral θ'-stable Cartan subgroups of G'. Ψ is compatible with Cayley transforms on (conjugacy classes of) Cartan subgroups, since Φ and Φ' are.

Next, we consider the lift of Ψ to the standard coherent families.

We put $J(i) = \{\alpha_m, \alpha_{m-1}, ..., \alpha_{m-i+1}\}$ for $1 \leq i \leq m - k_s^*$ and $J(0) = \emptyset$. Put $H_i = H_{J(i)}$ for $0 \leq i \leq m - k_s^*$. Then, we easily see $H_1, ..., H_{m-k_s^*}$ form a complete system of representatives of the $K^\sharp$-conjugacy (resp. $K \cap M_\kappa$-conjugacy) classes of θ-stable Cartan subgroups of $G^\sharp$ (resp. M_κ). We denote by $\mathfrak{h}_i$ the complexified Lie algebra of H_i and by $W(\mathfrak{g}^\sharp, \mathfrak{h}_i)$ the Weyl group for $(\mathfrak{g}^\sharp, \mathfrak{h}_i)$. We denote by $W(G^\sharp; H_i)$ the subgroup of $W(\mathfrak{g}^\sharp, \mathfrak{h}_i)$ consisting the elements of $W(\mathfrak{g}^\sharp, \mathfrak{h}_i)$ whose representatives can be chosen in $G^\sharp$. We collect some of the useful facts:

Lemma 4.2.1. *For $1 \leq i \leq m - k_s^*$, we have*

(1) $W(\mathfrak{m}_\kappa, \mathfrak{h}_i) \subseteq W(\mathfrak{g}', \mathfrak{h}_i) \subseteq W(\mathfrak{g}, \mathfrak{h}_i)$,

(2) $W(G'; H_i) = W(\mathfrak{g}', \mathfrak{h}_i) \cap W(G; H_i)$,

(3) $R_{M_\kappa}(H_i, {}^s\lambda) \subseteq R_{G'}(H_i, {}^s\lambda) \subseteq R_G(H_i, {}^s\lambda)$.

(1) is easy to see from our construction of G'. (2) is easily checked using [Vogan 1982] Proposition 4.16. (3) follows from (1).

We define $\tilde{\Omega} : \mathbf{St}_{G'}({}^s\lambda) \to \mathbf{St}_G({}^s\lambda)$ by $\tilde{\Omega}(\Theta_\gamma^{G'}) = \Theta_\gamma^G$ for $\gamma \in R_{G'}(H_i, {}^s\lambda)$ for $1 \leq i \leq m - k_s^*$.

We have remarked in section 1 that for $\gamma_1 = (H_i, \lambda_1), \gamma_2 = (H_i, \lambda_2) \in R_{G^\sharp}(H_i, {}^s\lambda)$, the followings are equivalent:

(a) γ_1 and γ_2 are $K^\sharp$-conjugate.

(b) There is some $w \in W(G^\sharp; H_i)$ such that $\lambda_1 = w\lambda_2$.

(c) $\Theta_{\gamma_1}^{G^\sharp} = \Theta_{\gamma_2}^{G^\sharp}$.

Hence, from (2) and (3) of lemma 4.2.1, we see $\tilde{\Omega}$ is well-defined.

We have:

Lemma 4.2.2. $\tilde{\Omega}$ *is bijective.*

Proof From lemma 4.2.1 (2) and the above remark, we see that the regularity of ${}^s\lambda$ implies the injectivity of $\tilde{\Omega}$. So, we show the surjectivity.

First, we fix some $1 \leq i \leq m - k_s^*$. Then $\mathrm{Ad}(c^{\alpha_{k_s^*+i}}) \circ \mathrm{Ad}(c^{\alpha_{k_s^*+i-1}}) \circ \cdots \circ \mathrm{Ad}(c^{\alpha_{k_s^*+1}})$ induces an linear isomorphism of $\mathfrak{h}$ onto $\mathfrak{h}_i$. So, we also have an isomorphism $\mathfrak{h}^* \cong \mathfrak{h}_i^*$. We denote by

$\bar{e}_1, ..., \bar{e}_n \in \mathfrak{h}_i^*$ the image of $e_1, ..., e_n \in \mathfrak{h}^*$ under this isomorphism. Then the Cartan involution acts on $\bar{e}_1, ..., \bar{e}_n$ as follows.

$$\theta(\bar{e}_{2i-1}) = -\bar{e}_{2i}, \theta(\bar{e}_{2i}) = -\bar{e}_{2i-1} \quad (1 \leq i \leq m-i),$$

$$\theta(\bar{e}_i) = \bar{e}_i \quad (2(m-i) < i \leq n).$$

We also denote by $\lambda \in \mathfrak{h}_i^*$ the image of ${}^s\lambda$ under this isomorphism. Write $\lambda = \sum_{j=1}^n \ell_j \bar{e}_j$. Let $w \in W(\mathfrak{g}, \mathfrak{h}_i)$ and write $\lambda = \sum_{j=1}^n \bar{\ell}_j \bar{e}_j$. Then $\bar{\ell}_1, ..., \bar{\ell}_n$ is made from $\ell_1, ..., \ell_n$ by a permutation of their indices and sign flips. We assume that $\gamma_w = (H_i, w\lambda) \in R_G({}^s\lambda)$. Then, it should satisfy the condition (R5) in 1.4. So, we easily see:

(d1) $\bar{\ell}_j \in \mathbb{Z}$ $(2(m-i) < j \leq n)$,

(d2) $\bar{\ell}_{2j-1} - \bar{\ell}_{2j} \in \mathbb{Z}$ $(1 < j \leq m-i)$.

We write $\sum_{k=1}^n a_i \bar{e}_i \in \mathfrak{h}_i^*$ by $(a_1, ..., a_n)$.

From [Vogan 1982] Proposition 4.86, we easily see the following elements in $W(\mathfrak{g}, \mathfrak{h}_i)$ are contained in $W(G; H_i)$.

$$w_j(a_1, ..., a_n) = (a_1, .., a_{2j-2}, -a_{2j-1}, -a_{2j}, a_{2j+2}, ..., a_n) \quad (1 \leq j \leq m-i),$$

$$w_{b,c}(a_1, ..., a_n) = (a_1, ..., a_{2b-2}, a_{2c-1}, a_{2c}, a_{2b+1}, ..., a_{2c-2}, a_{2b-1}, a_{2b}, a_{2c+1}, ..., a_n) \quad (1 \leq b < c \leq m-i).$$

If we choose the product w^* of suitable w_j's and $w_{b,c}$'s above, we may have $w^*\lambda = (d_1, ..., d_n)$ satisfies:

(e1) $d_j \in \mathbb{Z}$ for all $k_s^* < j \leqslant n$.

(e2) $d_j \notin \mathbb{R}$ for all $k_h^* < j \leqslant k_s^*$.

(e3) $d_j - \frac{1}{2} \in \mathbb{Z}$ for all $1 \leqslant j \leqslant k_h^*$.

This means that $\gamma' = (H_i, w^* w\lambda) \in R_{G'}({}^s\lambda)$ and $\Theta^G_{\gamma_w} = \Theta^G_{\gamma'}$. Hence $\tilde{\Omega}$ is surjective. □

We define $\tilde{\Psi} : \mathbf{St}_{G'}({}^s\lambda) \to \mathbf{St}_G({}^s\lambda)$ by the inverse of $\tilde{\Omega}$. From the above constructions, we easily see:

Lemma 4.2.3. *ψ, Ψ, and $\tilde{\Psi}$ defined above satisfy (C1)-(C7) in 3.1.*

Now, we finish the proof of Theorem 4.2.1. If $\gamma \in R_{M_\kappa}({}^s\lambda)$, then, taking account of $\gamma \in R_{G^\sharp}({}^s\lambda)$, we easily see the followings:

(f1) $\Theta_\gamma^{G'} = \mathsf{Ind}_{M_\kappa}^{G'}(\Theta_\gamma^{M_\kappa})$,

(f2) $\Theta_\gamma^{G} = \mathsf{Ind}_{M_\kappa}^{G}(\Theta_\gamma^{M_\kappa})$,

(f3) $\tilde{\Psi}(\Theta_\gamma^G) = \Theta_\gamma^{G'}$.

Taking account of the additivity of induction, we see that for all $\bar{\Theta} \in \mathbf{Irr}_{M_\kappa}({}^s\lambda)$ we have $\tilde{\Psi}(\mathsf{Ind}_{M_\kappa}^{G}(\bar{\Theta})) = \mathsf{Ind}_{M_\kappa}^{G'}(\bar{\Theta})$. It is easy to see that there is some $\bar{\Theta} \in \mathbf{Irr}_{M_\kappa}({}^s\lambda)$ such that $\bar{\Theta}(\eta) = [A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0) \boxtimes A_{k_{h+1}}(\ell_{h+1}, t_{h+1}) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s, t_s) \boxtimes Z]$. Hence, lemma 1.10.1 implies that the irreducibility of $\lozenge$ is reduced to the irreducibility of a Harish-Chandra module which is the external product of the followings:

(g1) Z, which is an irreducible Harish-Chandra module for M_κ°,

(g2) $\mathsf{Ind}_{P_\tau}^{SO^*(4a)}(A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0))$,

(g3) Harish-Chandra modules for $\mathrm{GL}(b, \mathbb{H})$ induced from irreducible unitary representations of their parabolic subgroups.

The irreducibilities of (g3) is found in [Vogan 1986] p502. Q.E.D.

§ 5. Irreducibility representations $SO^*(2n)$ and $\mathbf{Sp}(p,q)$ paarabolically induced from one-diemsional unitary representations

5.1 Some induced representations of $SO^*(4m)$

In this section we retain the notations in 3.1. and 4.1, and consider the case of $G = \mathrm{SO}^*(2n)$. Moreover, we assume n is even. So, we write $n = 2m$.

Since the universal covering group of $G_{\mathbb{C}}$ is a double cover, $\mathcal{P}_G$ (cf. 1.2) is a subgroup of $\mathcal{P}$ of index two. Put $\Lambda = \mathcal{P} - \mathcal{P}_G$ (set theoretical difference). Λ is the other $\mathcal{P}_G$ coset in $\mathcal{P}$ than $\mathcal{P}_G$ itself. We fix a regular weight ${}^s\lambda \in \Lambda$ as follows

$$ {}^s\lambda = \sum_{i=1}^{n} \frac{2n-2i+1}{2} e_i. $$

Hereafter, we simply write $W = W(\mathfrak{g}, {}^s\mathfrak{h})$ and $\Delta = \Delta(\mathfrak{g}, {}^s\mathfrak{h})$. We have $W = W_{{}^s\lambda}$, $\Delta = \Delta_{{}^s\lambda}$, and

$$ \Pi_{{}^s\lambda} = \{e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n, e_{n-1} + e_n\}. $$

Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ such that ${}^s\mathfrak{h} \subseteq \mathfrak{b}$ and the nilradical of $\mathfrak{b}$ is the sum of the root spaces corresponding to the roots in $\Delta^+_{{}^s\lambda}$. We denote by ρ the half sum of the positive roots in $\Delta^+_{{}^s\lambda}$.

We consider a partition $\pi = (p_1, ..., p_k)$ of a positive integer m $(p_1, ..., p_k)$ such that $0 < p_1 \geq p_2 \geq \cdots \geq p_k$ and $p_1 + p_2 + \cdots + p_k = m$. Let $PT(m)$ be the set of partitions of m. As in 4.1, we consider the standard parabolic subgroup $P_\pi$and its Levi subgroup M_π of G corresponding to π.

Let $(\sigma^\pi_\lambda, \mathbb{C}^\pi_\lambda)$ be a one dimensional unitary representation of M_π (or $\mathfrak{m}_\pi$) such that the restriction to ${}^s\mathfrak{h}$ of the differential of σ^π_λ is $\lambda \in {}^s\mathfrak{h}^*$.

We denote by ρ_π the half sum of all the positive roots whose root space is in $\mathfrak{m}_\pi$. We put $\rho^\pi = \rho - \rho_\pi$. The infinitesimal character of $\mathsf{Ind}^G_{P_\pi}(\mathbb{C}^\pi_\lambda)$ is $\rho_\pi + \lambda$.

It is easy to construct a nondegenerate $\mathfrak{g}$-invariant pairing between $\mathsf{Ind}^G_{P_\pi}(\mathbb{C}^\pi_\lambda)$ and a generalized Verma module $M_{\mathfrak{p}_\pi}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\pi)} \mathbb{C}^\pi_{-\lambda-\rho^\pi}$.

We are going to show the following our main result.

Lemma 5.1.1. *Let π be any partition of m. Then, $Ind^G_{P_\pi}(\mathbb{C}^\pi_0)$ is irreducible.*

We prove this lemma in 5.3.

Combining Lemma 5.1.1 and Theorem 4.1.2, we have:

Corollary 5.1.2. *Representations of $SO^*(2n)$ and $Sp(p,q)$ induced from one-dimensional unitary representations of thier parabolic subgroups are irreducible.*

5.2 Coherent continuation representation for $SO^*(4m)$ with respect to Λ

We retain the notations in 5.1.

For a partition $\pi = (p_1, ..., p_k) \in PT(m)$ of m, put $p_i^* = \sum_{j=1}^{i} p_j$ for $1 \leq i \leq k$ and define a subset S_π of $\Pi = \Pi_{^s\lambda}$ as follows. (P_π is the sstandard parabolic subgroup corresponding to S_π.)

$$S_\pi = \Pi - (\{e_{2p_i^*} - e_{2p_i^*+1})|1 \leq i \leq k-1\} \cup \{e_{2m-1} + e_{2m}\}.$$

For $\pi \in PT(m)$, we denote by σ_π the MacDonald representation (cf. [Carter 1985] p368) of W with respect to $S_\pi \subseteq \Pi$. From [Lusztig-Spaltenstein 1979], σ_π is a special representation ([Lusztig 1979, 1982] also see [Carter 1985] p374), which corresponds to the Richardson orbit in $\mathfrak{g}$ with respect to the parabolic subalgebra $\mathfrak{p}_\pi$ via the Springer correspondence.

There is another description of σ_π. Since W is the Weyl group of type D_{2m}, it is embedded into the Weyl group W' of type B_{2m}. It is well known that the irreducible representations of W' is parameterized by the pairs of partitions (κ, ω) such that $\kappa \in PT(k)$ and $\omega \in PT(2m-k)$ for some $0 \leq k \leq 2m$. Here, we regard $PT(0)$ consists of the empty partition $\emptyset$. If $\kappa \neq \omega$, then the restriction of the representation corresponding to (κ, ω) is irreducible. However, the restriction of the irreducible W'-representation corresponding to (π, π) $\quad (\pi \in PT(m))$ to W is decomposed into two irreducible W-representation, which are equidimensional. From [Carter 1985] p423 line 11-33, σ_π is one of the irreducible constituent.

For each partition $\kappa \in PT(k)$, we denote by $\dim(\kappa)$ the dimension of the irreducible representation of $\mathfrak{S}_k$ corresponding to κ. It is well-known that the dimension of the irreducible W'-representation corresponding to (κ, ω) ($\kappa \in PT(k)$ and $\omega \in PT(2m-k)$) is $\frac{(2m)! \dim(\kappa) \dim(\omega)}{k!(2m-k)!}$. (For example, see [Kerber 1971/75].) So, we have :

Lemma 5.2.1. *For $\pi \in PT(m)$,*

$$\dim(\sigma_\pi) = \frac{(2m)! \dim(\pi)^2}{2(m!)^2}.$$

We shall show:

Theorem 5.2.2. *As a W-module the coherent continuation representation $\mathcal{C}(\Lambda)$ is decomposed* *as follows.*

$$\mathcal{C}(\Lambda) \cong \bigoplus_{\pi \in PT(m)} \sigma_\pi$$

First, we prove:

Lemma 5.2.3. *For each $\pi \in PT(m)$, the multiplicity of σ_π in $\mathcal{C}(\Lambda)$ is at least one.*

Proof We have only to show that there is an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module V such that the infinitesimal character of V is in Λ and the character polynomial of V ([King 1981]) generates a W-representation isomorphic to σ_π. First, we remark that $\mathsf{Ind}_{P_\pi}^G(\mathbb{C}^\pi_{{}^s\lambda-\rho_\pi})$ has a nondegenerate pairing with an irreducible generalized Verma module $M_{\mathfrak{p}_\pi}(-{}^s\lambda - \rho_\pi)$ with the infinitesimal character $-{}^s\lambda$. Easily see that there is at least one irreducible constituent V of $\mathsf{Ind}_{P_\pi}^G(\mathbb{C}^\pi_{{}^s\lambda-\rho_\pi})$ whose annihilator I in $U(\mathfrak{g})$ is the dual of the annihilator of the generalized Verma module. So the associated variety of I is the closure of the Richardson orbit (say $\mathcal{O}_\pi$) corresponding to $\mathfrak{p}_\pi$. The character polynomial with respect to V is proportional to the Goldie rank polynomial of I ([King 1981]) and the W-representation generated by the Goldie rank polynomial is $\mathcal{O}_\pi$. So, we the lemma. Q.E.D.

Proof of Theorem 5.2.2 From Lemma 5.2.3, it suffices to show that

$$\dim \mathcal{C}(\Lambda) = \sum_{\pi \in PT(m)} \dim(\sigma_\pi).$$

From Lemma 5.2.1, the right hand side is $\sum_{\pi \in PT(m)} \frac{(2m)!(\dim(\pi)^2}{2(m!)^2} = \frac{(2m)!}{2\cdot m!}$, since we have

$$\sum_{\pi \in PT(m)} (\dim(\pi))^2 = \mathrm{card}\mathfrak{S}_m = m!.$$

So, we have to show $\dim \mathcal{C}(\Lambda) = \frac{(2m)!}{2\cdot m!}$.

$\dim \mathcal{C}(\Lambda)$ is clearly, the number of K-conjugacy classes in the regular characters in $R_G({}^s\lambda)$. Since only maximally split Cartan subgroups are ${}^s\lambda$-integral, each K-conjugacy class has a representative in $R_G({}^sH, {}^s\lambda)$. We denote by $W(G; {}^sH)$ the subgroup of W consisting the elements w of W such that some representative of w in $G_\mathbb{C}$ is in G (or equivalently in K) and normalizes ${}^sH$. Examining elements in K which preserves ${}^sH$, we easily see $\dim \mathcal{C}(\Lambda) = \mathrm{card}(W/W(G; {}^sH))$. From [Knapp (1975)] (also see [Vogan (1982)], Proposition 4.16), $W(G; {}^sH)$ is generated by the following elements in W:

(1) $s_{e_{2i-1}-e_{2i}}$ $(1 \leq i \leq m)$: reflections with respect to compact imaginary roots in $\Delta = \Delta(\mathfrak{g}, {}^s\mathfrak{h})$.

(2) $s_{e_{2i-1}+e_{2i}}$ $(1 \leq i \leq m)$: reflections with respect to real roots in Δ.

(3) $s_{e_{2i-1}-e_{2j-1}} s_{e_{2i}-e_{2j}}$ $(1 \leq i < j \leq m)$

So, we can easily see $W(G; {}^sH)$ is isomorphic to $\mathfrak{S}_m \times ((\mathbb{Z}/2\mathbb{Z})^m \times (\mathbb{Z}/2\mathbb{Z})^m)$.

So, we have $\dim \mathcal{C}(\Lambda) = \frac{\mathrm{card} W}{\mathrm{card} W(G;{}^sH)} = \frac{(2m)! \cdot 2^{2m-1}}{m! \cdot 2^m \cdot 2^m} = \frac{(2m)!}{2 \cdot m!}$ as desired. Q.E.D.

We can interpret in terms of cell structure of the coherent continuation representation $\mathcal{C}(\Lambda)$. In our particular setting, the proof of Lemma 2.3.3 tells us for each $\pi \in PT(m)$ there is at least one cell whose associated nilpotent orbit is $\mathcal{O}_\pi$. From Theorem 5.2.2, we have:

Corollary 5.2.4.

(1) There is an one to one correspondence between the set of cells for $\mathcal{C}(\Lambda)$ and $PT(m)$ induced from the above association of nilpotent orbits to cells.

(2) Any cells for $\mathcal{C}(\Lambda)$ is irreducible and isomorphic to the special representation corresponding to the associated nilpotent orbit via the Springer correspondence.

From Corollary 5.2.4, we have:

Corollary 5.2.5. *Let $\lambda \in \Lambda$ and let V_i $(i = 1, 2)$ be irreducible Harish-Chandra $(\mathfrak{g}, K)$-modules with an infinitesimal character λ. Assume that the annihilator of V_1 in $U(\mathfrak{g})$ coincides with that of V_2. Then, V_1 is isomorphic to V_2.*

Proof We may assume that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+_{{}^s\lambda}$. It is known that for each $i = 1, 2$ there is unique coherent family $\bar{\Theta}^G_{\gamma_i} \in \mathbf{Irr}_G({}^s\lambda)$ such that $[V_i] = \bar{\Theta}^G_{\gamma_i}(\lambda)$. We show $\bar{\Theta}^G_{\gamma_1} = \bar{\Theta}^G_{\gamma_2}$.

First, we remark that the Goldie rank polynomial and the associated variety of the annihilator of V_i in $U(\mathfrak{g})$ coincide with those of $\bar{\pi}(\gamma_i)$ for each i. Hence, we have $\gamma_1 \sim \gamma_2$ since there is at most one cell whose associated nilpotent orbit is the unique dense orbit in the associated variety of V_i. We consider the homomorphism $\phi_{\gamma_1}(= \phi_{\gamma_2})$ mentioned in 1.8. Since $\phi_{\gamma_1}(\bar{\Theta}^G_{\gamma_i})$ is nonzero and proportional to the Goldie rank polynomial of the annihilator of V_i in $U(\mathfrak{g})$ for each $i = 1, 2$, $\bar{\Theta}^G_{\gamma_1}$ is proportional to $\bar{\Theta}^G_{\gamma_2}$ modulo the kernel of ϕ_{γ_1}. Since the cell $\mathrm{Cell}(\gamma_1) = \mathrm{Cell}(\gamma_2)$ is irreducible, ϕ_{γ_1} induces an isomorphism of the cell $\mathrm{Cell}(\gamma_1)$ to the corresponding Goldie rank polynomial representation. This means that $\bar{\Theta}^G_{\gamma_1}$ is proportional to $\bar{\Theta}^G_{\gamma_2}$ modulo the subspace of $\mathrm{Cone}(\gamma_1)$ generated as a $\mathbb{C}$-vector space by $\bar{\Theta}^G_\eta$ such that $\eta \geq \gamma_1$ and $\eta \not\sim \gamma_1$. Since $\mathbf{Irr}_G({}^s\lambda)$ is a basis of $\mathcal{C}(\Lambda)$, we have $\bar{\Theta}^G_{\gamma_1} = \bar{\Theta}^G_{\gamma_2}$ as desired. Q.E.D.

5.3 Proof of Lemma 5.1.1

We need:

Lemma 5.3.1. *The annihilator of* $\mathsf{Ind}_{P_\pi}^G(\mathbb{C}_0^\pi)$ *in* $U(\mathfrak{g})$ *is a maximal ideal for all* $\pi \in PT(m)$.

Remark In fact, a more general result holds. So, we consider more general setting temporally. Let G be any connected real semisimple Lie group and P be any parabolic subgroup of G. We denote by M a Levi subgroup of P. We denote by $\mathfrak{g}$, $\mathfrak{m}$, and $\mathfrak{p}$ the complexified Lie algebras of G, M, and P, respectively. We denote by 1_M the trivial representation of M.

Lemma 5.3.2. *The annihilator of* $\mathsf{Ind}_P^G(1_M)$ *in* $U(\mathfrak{g})$ *is a maximal ideal. We denote by* $\mathfrak{n}$ *the nilradical of* $\mathfrak{p}$

As far as I know, such a result has not been published but is known by experts (at least including D. A. Vogan). For the combinience for the readers, we give a proof here.

Proof We denote by $\mathbb{C}_{-\rho^P}$ a one-dimensional representation of $\mathfrak{p}$ defined by $\mathfrak{p} \ni X \rightsquigarrow -\frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{n}})$. From the existence of nondegenerate pairing, it suffices to show that the annihilator of a generalized Verma module $M_{\mathfrak{p}}(0) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{-\rho^P}$ is maximal. We denote by I the annihilator of $M_{\mathfrak{p}}(0)$ in $U(\mathfrak{g})$. We define

$$L(M_{\mathfrak{p}}(0), M_{\mathfrak{p}}(0)) = \{\phi \in \mathrm{End}_{\mathbb{C}}(M_{\mathfrak{p}}(0)) \mid \dim \mathrm{ad}(U(\mathfrak{g}))\phi < \infty\}.$$

$L(M_{\mathfrak{p}}(0), M_{\mathfrak{p}}(0))$ has a obvious $U(\mathfrak{g})$-bimodule structure. Then, $L(M_{\mathfrak{p}}(0), M_{\mathfrak{p}}(0))$ is isomorphic to a Harish-Chandra module of an induced representation of $G_{\mathbb{C}}$ from a unitary one-dimensional representation of $P_{\mathbb{C}}$. Hence, $L(M_{\mathfrak{p}}(0), M_{\mathfrak{p}}(0))$ is completely reducible as a $U(\mathfrak{g})$-bimodule. Considering the action of $U(\mathfrak{g})$ on $M_{\mathfrak{p}}(0)$, we have an embedding of a $U(\mathfrak{g})$-bimodule $\mathrm{U}(\mathfrak{g})/I \hookrightarrow L(M_{\mathfrak{p}}(0), M_{\mathfrak{p}}(0))$. Hence, $\mathrm{U}(\mathfrak{g})/I$ is also completely reducible as a $U(\mathfrak{g})$-bimodule. We consider the unit element 1 of $U(\mathfrak{g})/I$. Then, $\mathbb{C}1$ is the unique trivial $\mathrm{ad}(U(\mathfrak{g}))$-type in $\mathrm{U}(\mathfrak{g})/I$. So, the unit 1 must contained in some irreducible component of $\mathrm{U}(\mathfrak{g})/I$ Since $\mathrm{U}(\mathfrak{g})/I$ is generated by 1 as a $U(\mathfrak{g})$-bimodule, $\mathrm{U}(\mathfrak{g})/I$ is irreducible as a $U(\mathfrak{g})$-bimodule. This means that I is maximal.

Q.E.D.

Proof of Lemma 5.1.1 From Corollary 5.2.5 and Lemma 5.3.1, we see that all the irreducible constituent of $\mathsf{Ind}_{P_\pi}^G(\mathbb{C}_0^\pi)$ is isomorphic to each other. However, the multiplicity of the trivial K-representation in $\mathsf{Ind}_{P_\pi}^G(\mathbb{C}_0^\pi)$ is just one. Hence $\mathsf{Ind}_{P_\pi}^G(\mathbb{C}_0^\pi)$ is irreducible as we desired.
Q.E.D.

References

[Adams-Barbasch-Vogan 1992] J. Adams, D. Barbasch, and D. A. Vogan Jr., "The Langlands Classification and Irreducible Characters for Real Reductive Groups," Progress in Mathematics Vol. 104, Birkhäuser, Boston-Basel-Stuttgart, 1992.

[Barbasch-Vogan 1983] D. Barbasch and D. A. Vogan Jr., Weyl group representations and nilpotent orbits, in: P. C. Trombi, editor, "Representation Theory of Reductive Groups," Progress in Mathematics Vol. 40, 21-33, Birkhäuser, Boston-Basel-Stuttgart, 1983.

[Carter 1985] R, W. Carter, "Finite groups of Lie type," Wiley series in pure and applied mathematics, John Wiley and Sons, Chichester, New York, Brisbane, Toronto, Singapore, 1985.

[Helgason 1970] S. Helgason, A duality for symmetric spaces with applications to group representations, *Adv. in Math.* **5** (1970), 1-154.

[Howe-Tan 1993] R. Howe and E. Tan, Homogeneous functions on light cones: The infinitesimal structure of some degenerate principal series representations, *Bull. Amer. Math. Soc.* **28** (1993), 1-74.

[Johnson 1990] Degenerate principal series and compact groups, *Math. Ann.* **287** (1990), 703-718.

[Kashiwara-Vergne 1979] M. Kashiwara and M. Vergne, Functions on the Shilov boundary of the generalized half plane, in: Non-Commutative Harmonic Analysis, *Lecture Notes in Mathematics* No. 728, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[Kerber 1971/75] A. Kerber, "Representations of Permutation Group 1,2," *Lecture Notes in Mathematics* No. 240 and No. 495, Springer-Verlag, Berlin-Heidelberg-New York, 1971, 1975.

[King 1981] D. King, The character polynomial of the annihilator of an irreducible Harish-Chandra module, *Amer. J. Math.* **103** (1981), 1195-1240.

[Knapp 1975] A. W. Knapp, Weyl group of a cuspidal parabolic, *Ann. Sci. École Norm. Sup.* **8** (1975), 275-294.

[Knapp (brown)] A. W. Knapp, "Representation Theory of Semisimple Groups, An Overview Based on Examples" Princeton Mathematical series **36**, Princeton University Press, Lawrenceville, New Jersey, 1986.

[Knapp-Vogan 1995] A. W. Knapp, "Cohomological Induction and Unitary Representations" Princeton Mathematical series **45**, Princeton University Press, Lawrenceville, New Jersey, 1995.

[Knapp-Zuckerman 1982] A. W. Knapp and G. J. Zuckerman, Classification of irreducible tempered representations of semisimple Lie groups, *Ann. of Math.* **116**(1982), 389-455.

[Kobayasi 1992] T. Kobayashi, Singular unitary representations and discrete series for indefinite Stiefel manifolds $U(p,q;\mathbb{F})/U(p-m,q;\mathbb{F})$, *Memoirs of AMS* vol. 462

[Kostant 1969] B. Kostant, On the existence and irreducibility of certain series of representations, *Bull. Amer. Math. Soc.* **75** (1969), 627-642.

[Lusztig 1979] G. Lusztig, A class of irreducible representations of a Weyl group, *Indag. Math.* **41** (1979), 323-335.

[Lusztig 1982] G. Lusztig, A class of irreducible representations of a Weyl group II, *Indag. Math.* **44** (1982), 219-226.

[Lusztig-Spaltenstein 1979] G. Lusztig and N. Spaltenstein, Induced unipotent classes, *J. London Math. Soc.* **19** (1979), 41-52.

[Matsuki 1979] T. Matsuki, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, *J. Math. Soc. Japan* **31** (1979), 331-357.

[Matumoto 1996] H. Matumoto, On the representations of $U(m,n)$ unitarily induced from derived functor modules, *Comp. Math.* **100** (1996), 1-39.

[Sahi 1993] S. Sahi, Unitary representations on the Shilov boundary of a symmetric tube domain, in: "Representation Theory of Groups and Algebras" *Contemporary Math.* vol. 145, American Mathematical Society 1993, 275-286.

[Schmid 1988] W. Schmid, Geometric constructions of representations, *Adv. Stud. in Pure Math.* vol. 14, Kinokuniya Book Store, 349-368, 1988.

[Speh-Vogan 1980] B. Speh and D. A. Vogan Jr., Reducibility of generalized principal series representations, *Acta Math.* **145** (1980), 227-299.

[Sugiura 1959] M. Sugiura, Conjugate classes of Cartan subalgebras in real semisimple Lie algebras, *J. Math. Soc. Japan* **11** (1959), 374-434.

[Trapa 2001] P. E. Trapa, Annihilators and associated varieties of $A_{\mathfrak{q}}(\lambda)$ modules for $U(p,q)$, *Comp. Math.* **129** (2001), 1-45.

[Vogan (green)] D. A. Vogan Jr., "Representations of Real reductive Lie Groups", *Progress in Mathematics,* Birkhäuser, 1982.

[Vogan 1982] D. A. Vogan Jr., Irreducible characters of semisimple Lie groups IV character-multiplicity duality, *Duke Math. J.* **49** (1982), 943-1073.

[Vogan 1983] D. A. Vogan Jr., Irreducible characters of semisimple Lie groups III proof of Kazhdan-Lusztig conjectures in the integral case, *Invent Math.* **71** (1983), 381-417.

[Vogan 1984] D. A. Vogan Jr., Unitarizability of certain series of representations, *Ann. of Math.* **120** (1984), 141-187.

[Vogan 1986] D. A. Vogan Jr., The unitary dual of $GL(n)$ over an archimedean field, *Invent. Math.* **83** (1986), 449-505.

[Vogan (orange)] D. A. Vogan Jr., "Unitary Representations of reductive Lie Groups", *Annals of Mathematics Studies,* Princeton University Press, 1987.

[Vogan 1988] D. A. Vogan Jr., Irreducibilities of discrete series representations for semisimple symmetric spaces, *Adv. Stud. in Pure Math.* vol. 14, Kinokuniya Book Store, 1988, 381-417.

[Vogan-Zuckerman 1984] D. A. Vogan Jr. and G. Zuckerman, Unitary representations with non-zero cohomology, *Compositio Math.* **53** (1984), 51-90.

[Wallach (dark-green)] N. R. Wallach, "Real reductive Groups I," Academic press, 1987.

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