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Abstract

In this paper, we study the existence and the regularity of local strong solutions for the Cauchy problem of nonlinear Schrödinger equations with time-dependent potentials and magnetic fields. We consider these equations when the nonlinear term is the power type which is, for example, equal to $\lambda|u|^{p-1}u$ with some $1 \leq p < \infty$, $\lambda \in \mathbf{R}$. We prove local well-posedness of strong solutions under the additional assumption $1 \leq p < 1 + 4/(n-4)$ for space dimension $n \geq 5$, and local smoothing effects of it under the additional assumption $1 \leq p \leq 1 + 2/(n-4)$ for $n \geq 5$ without any restrictions on n.

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1 Introduction

We study local well-posedness and smoothing effects of the following nonlinear Schrödinger equation with magnetic fields:

$$\begin{cases} i\partial_t u = \dfrac{1}{2}\displaystyle\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 u \\ \qquad\qquad + V(t,x)u + F(u), \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0,x) = \phi(x), \quad x \in \mathbf{R}^n, \end{cases} \tag{NLS}$$

where u is a complex valued unknown function on $\mathbf{R} \times \mathbf{R}^n$, the initial data ϕ is a complex valued given function on $\mathbf{R}^n$, the components of the vector potential A_j ($j = 1, \ldots, n$) are real valued given functions on $\mathbf{R} \times \mathbf{R}^n$, the linear scalar potential V is a real valued given function on $\mathbf{R} \times \mathbf{R}^n$, and the nonlinear function F is a complex valued given function on $\mathbf{C}$. We can find this type equation, for example, in the Maxwell Schrödinger equations, which are the classical approximation to the quantum field equations for an electrodynamical nonrelativistic many body system (see, e.g., Tsutsumi [23]).

This paper is the sequel to the paper [13] by one of the authors. We will construct the strong solutions by using the contraction methods. For (NLS), the corresponding time-dependent linear Schrödinger equation is as follows:

$$i\partial_t u = H(t)u, \tag{LS}$$

where

$$H(t) = \frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,\cdot))^2 + V(t,\cdot),$$

is time-dependent Schrödinger operator acting in $L^2(\mathbf{R}^n)$. In [27], Yajima constructed the fundamental solution generated by this Hamiltonian as an extension of Fujiwara's results [3, 4]. We will solve the integral equation corresponding to (NLS) by using some properties of the propagator to this Hamiltonian.

The Cauchy problem for nonlinear Schrödinger equations with power nonlinearity and linear potentials or magnetic fields has been investigated by many authors. Well-posedness of weak solutions to (NLS) is well known (see also [5, 6, 8, 9, 21] for the case $A = V = 0$, [9, 14] for the case $A = 0$ and $V \neq 0$, [1, 13] for the case $A \neq 0$ and $V \neq 0$ and references therein). In particular, we mention that well-posedness of strong solutions to (NLS) was proved in [9, 22] when $A = V = 0$, in [9] when $A = 0$ and $V \neq 0$ (cf. [15] for the the Zakharov equations). In this paper, we will study well-posedness of strong solutions to (NLS) when $A \neq 0$ and $V \neq 0$.

For the proof of well-posedness for nonlinear Schrödinger equations, we usually employ the Strichartz estimate, which is an estimate on a space-time integral of solutions to the linear problem. For the free Schrödinger group, this was proved by Strichartz [20] (see also [6, 9, 25]). It is well-known that this estimate also holds for $A = 0$ and $V \neq 0$ with some conditions (cf. [7, 9, 10]). In this paper, we use an Strichartz estimate with $A \neq 0$ and $V \neq 0$ which is obtained by Yajima [27] (Lemma 2.6). We also use so-called the endpoint Strichartz estimates obtained by Keel and Tao [11].

On the other hand, solutions of the Schrödinger type equations have smoothing effects, that is, the solution is smoother than the initial data for almost all time t. For the free Schrödinger group, Sjölin [17] has proved the following inequality to exhibit this property

$$\int_{\mathbf{R}} \int_{\mathbf{R}^n} |\phi(t,x)(1-\Delta)^{\frac{1}{4}} e^{it\Delta} f|^2 dx dt \leq C \|f\|_{L^2}^2, \quad \forall f \in L^2, \phi \in C_0^\infty(\mathbf{R}^{n+1})$$

(cf. [2, 24]). Yajima [26] has proved it for the equation (LS), which we will quote as Lemma 4.1 below. Recently, Yajima and Zhang [28, 29, 30, 31] have proved this property for (LS) and well-posedness for (NLS), when $A = 0$ and V is superquadratic at infinity. When $1 \leq n \leq 7$, one of the authors [12] and Sjölin [18] showed this property for the strong solutions to (NLS) with $A = V = 0$. We will prove the smoothing effects of the strong solutions to (NLS) with scalar potentials and magnetic fields for all space dimensions, time-locally. This property for the weak solutions to (NLS) with potentials and magnetic fields was studied in the previous paper [13] (cf. [19] for the case $A = V = 0$).

We assume the following assumptions on the vector potential and the scalar potential, which are introduced by Yajima [26, 27].

Assumption (A). For $j = 1, \ldots, n$, A_j is a continuous function of $(t,x) \in \mathbf{R} \times \mathbf{R}^n$ and a C^∞ class function of x for each t. $\partial_x^\alpha A_j$ is a C^1 class function of $(t,x) \in \mathbf{R} \times \mathbf{R}^n$ for any multi-index α. A satisfies for $|\alpha| \geq 1$,

$$|\partial_x^\alpha B_{jk}(t,x)| \leq C_\alpha \langle x \rangle^{-1-\varepsilon}, \tag{1.1}$$

$$|\partial_x^\alpha A(t,x)| + |\partial_t \partial_x^\alpha A(t,x)| \leq C_\alpha, \tag{1.2}$$

with some $\varepsilon > 0$ where $A(t,x) = (A_1(t,x), \ldots, A_n(t,x))$, $B_{jk}(t,x) = \partial_j A_k(t,x) - \partial_k A_j(t,x)$.

Assumption (V). V is a continuous function of $(t,x) \in \mathbf{R} \times \mathbf{R}^n$ and a C^∞ class function of x for each t. $\partial_x^\alpha V$ is a continuous function of $(t,x) \in \mathbf{R} \times \mathbf{R}^n$ for any multi-index α. V satisfies

$$|\partial_x^\alpha V(t,x)| \le C_\alpha, \tag{1.3}$$

for $|\alpha| \ge 2$.

We also assume the following assumptions on the nonlinear function F. (cf.[8, 9])

Assumption (F1). $F \in C^1(\mathbf{C}, \mathbf{C})$ in the real sense with $F(0) = 0$.

Let $F \in C^1(\mathbf{C}, \mathbf{C})$. For $z \in \mathbf{C}$, we define the linear operator $F'(z)$ on $\mathbf{C}$ by

$$F'(z)\omega = \partial_z F(z)\omega + \partial_{\bar{z}} F(z)\bar{\omega}, \qquad \text{for } \omega \in \mathbf{C},$$

where $\partial_z = \frac{1}{2}(\partial_\xi - i\partial_\eta)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_\xi + i\partial_\eta)$ and where ξ and η are real and imaginary parts of $z \in \mathbf{C}$, respectively.

Assumption (F2). There exists $M > 0$ such that for $|z| > 1$,

$$|F'(z)| \equiv \max\{|\partial_z F(z)|, |\partial_{\bar{z}} F(z)|\} \le M|z|^{p-1},$$

with some $1 \le p < \infty$.

We introduce the following function spaces. We set for $k = 0, 1, \dots,$

$$\Sigma(k) = \{f \in L^2 \colon \|f\|_{\Sigma(k)} < \infty\},$$
$$\|f\|_{\Sigma(k)} = \sum_{|\alpha+\beta| \le k} \|x^\alpha \partial^\beta f\|_2,$$

and let $\Sigma(-k)$ be a dual space of $\Sigma(k)$. Then $\Sigma(k)$ is a Banach space with the norm $\|\cdot\|_{\Sigma(k)}$.

Definition. We call the components (q,r) *an admissible pair* if they satisfy

$$\frac{2}{r} = n\left(\frac{1}{2} - \frac{1}{q}\right), \tag{1.4}$$

and $2 \le q \le \infty$ if $n = 1$, $2 \le q < \infty$ if $n = 2$, $2 \le q \le \frac{2n}{n-2}$ if $n \ge 3$.

Let

$$\mathcal{X}_T = \bigcap_{(q,r)\colon \text{admissible pair}} L^r(I_T, L^q), \tag{1.5}$$

and let

$$\bar{\mathcal{X}}_T = \mathcal{X}_T \cap C(I_T, L^2), \tag{1.6}$$

where $I_T = [0, T]$.

Remark 1.1. For this definition, we take the results of Keel and Tao [11] into consideration.

We claim the main results of this paper.

Theorem 1. *Assume Assumptions (A), (V) and (F1). In addition, if $n \geq 5$, assume Assumption (F2) with $1 \leq p < 1+4/(n-4)$. Then for any $\phi \in \Sigma(2)$, there exists $T > 0$ depending only on $\|\phi\|_{\Sigma(2)}$ such that* (NLS) *has a unique solution u with $u(0) = \phi$ in $C(I_T, \Sigma(2))$. Furthermore $\partial_t u \in \bar{\mathcal{X}}_T$, in particular $u \in C(I_T, \Sigma(2)) \cap C^1(I_T, L^2)$.*

Theorem 2. *Let $\phi \in \Sigma(2)$ and let $u \in C(I_{T_0}, \Sigma(2))$ be a solution of* (NLS) *with $u(0) = \phi$. If $\phi_k \to \phi$ in $\Sigma(2)$, then for $k \in \mathbf{N}$ sufficiently large, there exists a solution $u_k \in C(I_{T_0}, \Sigma(2))$ of* (NLS) *with $u_k(0) = \phi_k$, and $u_k \to u$ in $C(I_{T_0}, \Sigma(2))$.*

Theorem 3. *Assume Assumptions (A), (V) and (F1). In addition, if $n \geq 5$, assume Assumption (F2) with $1 \leq p \leq 1 + 2/(n-4)$. Let u be the solution of* (NLS) *obtained in Theorem 1. Then for $\mu > 1/2$,*

$$\int_{I_T} \|\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2}u(t)\|_2^2\,dt < \infty,$$

where $\langle D_x\rangle = (I - \triangle)^{1/2}$.

Remark 1.2. When $n \geq 5$, we can prove Theorems 1 and 2, that is, local well-posedness of (NLS) under the assumption $1 \leq p < 1 + 4/(n-4)$. On the other hand, we can obtain local smoothing effects only in the case of $1 \leq p \leq 1 + 2/(n-4)$, because we have to show that the nonlinear term, the time derivative of it and so forth belong to $L^1(I_T; L^2(\mathbf{R}^n))$ by using the Sobolev embedding theorem (see Lemma 4.2).

Remark 1.3. In Assumption (V), we assume continuity for the scalar potentials. In fact the fundamental solution of (LS) can be constructed for the scalar potentials with singularities under the suitable conditions (see Theorem 7 in Yajima [27]). Thus using this property, we can show the local well-posedness of (NLS) for these scalar potentials with singularities. But since we do not have the local smoothing property of the propagator of (LS) for singular potentials even when $A = F = 0$, we nned to assume continuity for the scalar potentials to prove the local smoothing effects of (NLS).

Remark 1.4. When A and V are independent of t, that is, $H(t) = H$, it is rather easy to prove Theorems 1 and 2 because ∂_t is commutable with H.

If V is bounded from below, H defined on C_0^∞ is essentially self-adjoint in $L^2(\mathbf{R}^n)$ (see, e.g., Theorem X.34 in [16]). Therefore by Stone's theorem we can prove theorems by using $e^{-it\bar{H}}$ instead of $U(t,0)$, where $\bar{H}$ is the self-adjoint realization of H. We note that $e^{-it\bar{H}}$ satisfy the Strichartz estimate if $|t|$ is small enough (cf. [1]).

Notations. Let $L^q(\mathbf{R}^n) = \left\{\psi \colon \|\psi\|_q = \left(\int_{\mathbf{R}^n} |\psi(x)|^q\, dx\right)^{1/q} < \infty\right\}$ for $1 \le q < \infty$, and let $L^\infty(\mathbf{R}^n) = \{\psi \colon \|\psi\|_\infty = \operatorname{ess.sup}_{x\in\mathbf{R}^n}|\psi(x)| < \infty\}$. Let the Sobolev space $H^k(\mathbf{R}^n) = \left\{\psi \colon \|\psi\|_{H^k} = \sum_{|\alpha|\le k} \|\partial^\alpha\psi\|_2 < \infty\right\}$, for positive integer k. For simplicity, we denote the space $L^q(\mathbf{R}^n)$ by L^q and the space $H^k(\mathbf{R}^n)$ by H^k, respectively. For the Banach space X and the interval I, let $C(I, X)$ be a set of X-valued strong continuous functions on I, and let $L^q(I, X)$ be a set of X-valued L^q-functions on I. We put $L^{q,r} = L^r(I, L^q)$ with the norm

$$\|f\|_{q,r} = \left(\int_I \|f(t,\cdot)\|_q^r\, dt\right)^{1/r}, \qquad \text{if } 1 \le r < \infty,$$
$$\|f\|_{q,\infty} = \operatorname{ess.sup}_{t\in I}\|f(t,\cdot)\|_q.$$

We denote the set of rapidly decreasing functions on $\mathbf{R}^n$ by $\mathcal{S}(\mathbf{R}^n)$. We denote various constants by C, M and so forth. They may differ from line to line, when it does not cause any confusion.

We use the following symbols:

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_k = \frac{\partial}{\partial x_k}, \text{ for } k = 1, \dots, n,$$
$$\partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}, \quad x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \quad \text{for any multi-index } \alpha = (\alpha_1, \dots, \alpha_n),$$
$$\nabla = (\partial_1, \dots, \partial_n), \quad \triangle = \partial_1^2 + \cdots + \partial_n^2,$$
$$\langle x\rangle = (1 + |x|^2)^{1/2},$$
$$a \vee b = \max\{a, b\}.$$

Outline of this paper is as follows. In Section 2, we introduce some results of (LS) obtained in Yajima [27]. In Section 3, we prove Theorems 1 and 2, that is, the local well-posedness of the strong solutions to (NLS) by the contraction method in the suitable function spaces. In Section 4, we prove Theorem 3, that is, the local smoothing effects of the strong solutions to (NLS) by using the smoothing property of (LS) obtained in Yajima [26].

2 Preliminaries

We introduce some results for the linear equation (LS) in Yajima [27].

Lemma 2.1 (Yajima [27]). *Assume Assumptions (A) and (V). Then there exists a unique propagator* $\{U(t,s)\}_{t,s\in\mathbf{R}}$ *for* (LS) *satisfying the following properties:*

1. *For any* $t \neq s$, $U(t,s)$ *maps* $\mathcal{S}(\mathbf{R}^n)$ *into* $\mathcal{S}(\mathbf{R}^n)$ *continuously and extends a unitary operator in* $L^2(\mathbf{R}^n)$ *which satisfies* $U(t,r)U(r,s) = U(t,s)$.

2. *For* $\psi \in \Sigma(2)$, $U(\cdot,\cdot)\psi \in C(\mathbf{R}^2, \Sigma(2)) \cap C^1(\mathbf{R}^2, L^2)$, *and the following equations hold:*

$$i\partial_t U(t,s)\psi = H(t)U(t,s)\psi,$$
$$i\partial_t U(t,s)\psi = -U(t,s)H(s)\psi.$$

Lemma 2.2 (Yajima [27]). *Assume Assumptions (A) and (V). Then there exists* $\tilde{T} > 0$ *such that for* $0 < |t-s| < \tilde{T}$, $U(t,s)$ *can be represented as in the form of oscillatory integral*

$$(U(t,s)f)(x) = (2\pi i(t-s))^{-n/2} \int_{\mathbf{R}^n} e^{iS(t,s,x,y)} b(t,s,x,y) f(y)\, dy.$$

Then $\{U(t,s)\colon |t-s| < \tilde{T},\ t,s \in \mathbf{R}\}$ *is strongly continuous in* $L^2(\mathbf{R}^n)$. *Here* $S(t,s,x,y)$ *and* $b(t,s,x,y)$ *are uniquely determined functions satisfy the following properties:*

1. $S(t,s,x,y)$ *is* C^1 *in* (t,s,x,y), C^∞ *in* (x,y), *and satisfies*

$$(\partial_t S)(t,s,x,y) + \frac{1}{2}((\partial_x S)(t,s,x,y) - A(t,x)) + V(t,x) = 0,$$
$$(\partial_s S)(t,s,x,y) - \frac{1}{2}((\partial_y S)(t,s,x,y) - A(s,y)) + V(s,y) = 0.$$

Furthermore, if $0 < |t-s| < \tilde{T}$,

$$\left|\partial_x^\alpha \partial_y^\beta \left[S(t,s,x,y) - \frac{|x-y|^2}{2(t-s)}\right]\right| \le C_{\alpha,\beta},$$

for $|\alpha+\beta| \le 2$.

2. $b(t,s,x,y)$ *is* C^∞ *in* (x,y), *and for any multi-indices* α, β, $\partial_x^\alpha \partial_y^\beta b(t,s,x,y)$ *is* C^1 *in* (t,s,x,y), *and satisfies*

$$|\partial_x^\alpha \partial_y^\beta [b(t,s,x,y) - 1]| \le C_{\alpha,\beta}$$

for $0 < |t-s| < \tilde{T}$ *and for any* α, β.

The following lemma is the L^p-L^q estimate for $U(t,s)$.

Lemma 2.3 (Yajima [27]). *Let $2 \le q \le \infty$ and let q' satisfy $\frac{1}{q} + \frac{1}{q'} = 1$. Then there exist $\tilde{T} > 0$ sufficiently small and $C_q > 0$ such that for $0 < |t-s| < \tilde{T}$,*

$$\|U(t,s)f\|_q \le C_q |t-s|^{-n(\frac{1}{2}-\frac{1}{q})}\|f\|_{q'}.$$

We define linear operators L and G as follows:

$$(L\phi)(t) = U(t,0)\phi,$$
$$(Gf)(t) = \int_0^t U(t,s)f(s)\,ds,$$

for $t \in \mathbf{R}$.

These operators have the following properties (see Yajima [26], [27]). Let I be a compact subinterval of $[0, \tilde{T}]$.

Lemma 2.4. *L is a bounded operator from L^2 into $C(I, L^2) \cap AC(I, \Sigma(-2))$ satisfying*

$$i\partial_t L\phi = H(t)L\phi,$$

for $\phi \in L^2$ and a.e. $t \in I$ in $\Sigma(-2)$, where AC is the class of absolutely continuous functions.

Lemma 2.5. *If $f \in L^1(I, L^2)$, then $Gf \in C(I, L^2) \cap AC(I, \Sigma(-2))$ and it follows that*

$$i\partial_t Gf = H(t)Gf + if, \tag{2.1}$$

for a.e. $t \in I$ in $\Sigma(-2)$.

The following Strichartz estimates are obtained in Yajima [27]. Let $I_T = [0,T] \subset [0, \tilde{T}]$.

Lemma 2.6. *Assume that the components (q_i, r_i) are arbitrary admissible pairs, where $i = \emptyset, 1, 2$, and let (q_i', r_i') be dual of (q_i, r_i), namely $1/q_i + 1/q_i' = 1$ and $1/r_i + 1/r_i' = 1$. Then L is a bounded operator from L^2 into $L^{q,r}$, and G is a bounded operator from $L^{q_2', r_2'}$ into L^{q_1, r_1}, the bounds are independent of T. Namely, there exist $C, C' > 0$ independent of T such that*

$$\|L\phi\|_{q,r} \le C\|\phi\|_2, \tag{2.2}$$
$$\|Gf\|_{q_1,r_1} \le C'\|f\|_{q_2',r_2'}. \tag{2.3}$$

Furthermore, $L\phi \in C(I_T, L^2)$ and $Gf \in C(I_T, L^2)$ for any $\phi \in L^2$ and $f \in L^{q_2', r_2'}$.

Remark 2.1. We can obtain the endpoint estimate since $U(t,s)$ satisfies both the energy and the decay estimates introduced by Keel and Tao (see p.955 in [11]).

Remark 2.2. Under Assumptions (A) and (V), it is easily seen that there exists $C > 0$, depending on $\tilde{T}$, such that

$$|A(t,x)| \leq C\langle x\rangle, \tag{2.4}$$

$$\begin{aligned} |V(t,x)| &\leq C\langle x\rangle^2, \\ |\nabla V(t,x)| &\leq C\langle x\rangle, \end{aligned} \tag{2.5}$$

for any $t \in I$ and $x \in \mathbf{R}^n$.

Remark 2.3. Under Assumptions (F1) and (F2), it is easily seen that F can be decomposed in the form

$$\begin{gathered} F = F_1 + F_2, \quad F_1, F_2 \in C^1(\mathbf{C},\mathbf{C}), \quad F_1(0) = F_2(0) = 0, \\ |F_1(z)| \leq M_1|z|, \quad |F_1'(z)| \leq M_1, \\ |F_2(z)| \leq M_2|z|^p, \quad |F_2'(z)| \leq M_2|z|^{p-1}, \end{gathered}$$

for $z \in \mathbf{C}$ (see Kato [9]).

3 Proof of Theorem 1 and Theorem 2

First we consider $n \geq 4$. We assume that $1 \leq p < 1 + 4/(n-4)$. Note that $1 \leq p < \infty$ when $n = 4$.

We introduce the following function spaces and their norms. Let $I_T = [0,T]$ for $0 < T \leq \tilde{T}$, where $\tilde{T}$ is introduced in Lemma 2.3.

$$\begin{gathered} X_T = L^{2,\infty} \cap L^{\frac{4p}{p+1},\frac{2}{l}}, \\ \|u\|_{X_T} = \|u\|_{2,\infty} \vee \|u\|_{\frac{4p}{p+1},\frac{2}{l}}, \\ X_T' = L^{2,1} + L^{\frac{4p}{3p-1},\frac{2}{2-l}}, \\ \|v\|_{X_T'} = \inf\{\|v_1\|_{2,1} + \|v_2\|_{\frac{4p}{3p-1},\frac{2}{2-l}} : v = v_1 + v_2\} \\ \bar{X_T} = C(I_T, L^2) \cap L^{\frac{4p}{p+1},\frac{2}{l}}, \end{gathered}$$

where $l = \frac{n}{4}(1 - \frac{1}{p})$ so that $0 \leq l < 1$. Then X_T and X_T' are Banach spaces.

Remark 3.1. The pairs $(2,\infty)$ and $(\frac{4p}{p+1}, \frac{2}{l})$ are admissible. The pairs $(2,1)$ and $(\frac{4p}{3p-1}, \frac{2}{2-l})$ are dual of $(2,\infty)$ and $(\frac{4p}{p+1}, \frac{2}{l})$, respectively.

We define the function space Z_T as follows:

$$Z_T = \{u\colon \|u\|_{Z_T} < \infty\},$$
$$\bar{Z}_T = \{u \in Z_T\colon u \in C(I_T, \Sigma(2)),\ \partial_t u \in C(I_T, L^2)\}$$

where

$$\|u\|_{Z_T} = \|u\|_{2,\infty} \vee \|\triangle u\|_{2,\infty} \vee \|x \cdot \nabla u\|_{X_T} \vee \||x|^2 u\|_{X_T} \vee \|\partial_t u\|_{X_T}.$$

Then Z_T is a Banach space.

Remark 3.2. Since $n \geq 4$ and $1 \leq p < 1 + 4/(n-4)$, it follows from the Sobolev embedding theorem that

$$\Sigma(2) \hookrightarrow H^2 \hookrightarrow H^{2l} \hookrightarrow L^{2p}. \tag{3.1}$$

Lemma 3.1. *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$, $f \in \mathcal{S}(\mathbf{R}^{n+1})$, and let $v = L\phi - iGf$. Then*

$$x_k{}^2 v = L(x_k{}^2\phi) - iG[2x_k(\partial_k - iA_k)v + v + x_k{}^2 f], \tag{3.2}$$

$$\begin{aligned} x_k\partial_k v =& L(x_k\partial_k\phi) - iG[(i/2)x_k(\partial_k\nabla\cdot A)v + ix_k\partial_k A\cdot\nabla v \\ &+ (x_k A\cdot\partial_k A + x_k\partial_k V)v + \partial_k^2 v - iA_k\partial_k v + x_k\partial_k f], \end{aligned} \tag{3.3}$$

$$\begin{aligned} \partial_k^2 v =& L(\partial_k^2\phi) - iG[\{(i/2)(\partial_k^2(\nabla\cdot A)) + |\partial_k A|^2 + A\cdot\partial_k^2 A + \partial_k V\}v \\ &+ \{i\partial_k\nabla\cdot A + 2(A\cdot\partial_k A + \partial_k V)\}\partial_k v \\ &+ i\partial_k^2 A\cdot\nabla v + 2i\partial_k A\cdot\partial_k\nabla v + \partial_k^2 f]. \end{aligned} \tag{3.4}$$

Proof. We differentiate the equations $v = L\phi - iGf$. Then we have

$$i\partial_t v = H(t)v + f,$$

and hence,

$$\begin{aligned} i\partial_t(x_k^2 v) =& x_k^2 H(t)v + x_k^2 f \\ =& H(t)(x_k^2 v) - [H(t), x_k^2]v + x_k^2 f \\ =& H(t)(x_k^2 v) + 2x_k(\partial_k - iA_k)v + v + x_k^2 f. \end{aligned}$$

Noting that $(x_k^2 v)(0) = x_k^2\phi$, we have (3.2). In the exactly similar way, we can prove (3.3) and (3.4). $\square$

Lemma 3.2. *If $T > 0$ is sufficiently small, L is a bounded operator from $\Sigma(2)$ into Z_T, the bound is independent of T. Namely, if $T > 0$ is sufficiently small, there exists $C > 0$ independent of T such that*

$$\|L\phi\|_{Z_T} \leq C\|\phi\|_{\Sigma(2)}. \tag{3.5}$$

Furthermore, $L\phi \in \bar{Z}_T$ for any $\phi \in \Sigma(2)$.

Proof. We first assume $\phi \in \mathcal{S}(\mathbf{R}^n)$. By (2.2), we have

$$\|L\phi\|_{2,\infty} \leq c\|\phi\|_2. \tag{3.6}$$

By the application of Lemma 2.6 to the equalities in Lemma 3.1, we have the following estimates

$$\begin{aligned}
&\|\partial_k^2 L\phi\|_{X_T}\\
\leq&\|L(\partial_k^2\phi) - iG[\{(i/2)(\partial_k^2(\nabla\cdot A)) + |\partial_k A|^2 + A\cdot\partial_k^2 A + \partial_k V\}L\phi\\
&+ \{i\partial_k\nabla\cdot A + 2(A\cdot\partial_k A + \partial_k V)\}\partial_k L\phi\\
&+ i\partial_k^2 A\cdot\nabla L\phi + 2i\partial_k A\cdot\partial_k\nabla L\phi]\|_{X_T}\\
\leq&c\|\partial_k^2\phi\|_2 + c\|\{(i/2)(\partial_k^2(\nabla\cdot A)) + |\partial_k A|^2 + A\cdot\partial_k^2 A + \partial_k V\}L\phi\\
&+ \{i\partial_k\nabla\cdot A + 2(A\cdot\partial_k A + \partial_k V)\}\partial_k L\phi\\
&+ i\partial_k^2 A\cdot\nabla L\phi + 2i\partial_k A\cdot\partial_k\nabla L\phi\|_{2,1}\\
\leq&c\|\partial_k^2\phi\|_2 + cT(\|L\phi\|_{2,\infty} + \|{x_k}^2 L\phi\|_{2,\infty}\\
&+ \|x_k\partial_k L\phi\|_{2,\infty} + \|\partial_k^2 L\phi\|_{2,\infty}),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\|x_k\partial_k L\phi\|_{X_T}\\
\leq&\|L(x_k\partial_k\phi) - iG[(i/2)x_k(\partial_k\nabla\cdot A)L\phi + ix_k\partial_k A\cdot\nabla L\phi\\
&+ (x_k A\cdot\partial_k A + x_k\partial_k V)L\phi + \partial_k^2 L\phi - iA_k\partial_k L\phi]\|_{X_T}\\
\leq&\|x_k\partial_k\phi\|_2 + c\|(i/2)x_k(\partial_k\nabla\cdot A)L\phi + ix_k\partial_k A\cdot\nabla L\phi\\
&+ (x_k A\cdot\partial_k A + x_k\partial_k V)L\phi + \partial_k^2 L\phi - iA_k\partial_k L\phi\|_{2,1}\\
\leq&c\|x_k\partial_k\phi\|_2 + cT(\|L\phi\|_{2,\infty} + \|{x_k}^2 L\phi\|_{2,\infty}\\
&+ \|x_k\partial_k L\phi\|_{2,\infty} + \|\partial_k^2 L\phi\|_{2,\infty}),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\|{x_k}^2 L\phi\|_{X_T}\\
\leq&\|L({x_k}^2\phi) - iG[2x_k(\partial_k - iA_k)L\phi + L\phi]\|_{X_T}\\
\leq&c\|{x_k}^2\phi\|_2 + c\|2x_k(\partial_k - iA_k)L\phi + L\phi\|_{2,1}\\
\leq&c\|x_k\partial_k\phi\|_2 + cT(\|L\phi\|_{2,\infty} + \|{x_k}^2 L\phi\|_{2,\infty} + \|x_k\partial_k L\phi\|_{2,\infty}).
\end{aligned} \tag{3.9}$$

We have used Remark 2.2. By Lemmas 2.4 and 2.6, we obtain

$$\begin{aligned}
\|\partial_t L\phi\|_{X_T} \leq&\|H(t)L\phi\|_{X_T}\\
\leq&c(\|L\phi\|_{X_T} + \||x|^2 L\phi\|_{X_T} + \|x\cdot\nabla L\phi\|_{X_T} + \|\triangle L\phi\|_{X_T}).
\end{aligned} \tag{3.10}$$

From (3.6), (3.7), (3.8), (3.9) and (3.10), it follows that

$$\|L\phi\|_{Z_T} \leq c\|\phi\|_{\Sigma(2)} + cT\|L\phi\|_{Z_T}.$$

Therefore, if $T > 0$ is small enough, (3.5) holds for $\phi \in \mathcal{S}(\mathbf{R}^n)$. By the density argument, we see that if $T > 0$ is small enough, (3.5) holds for any $\phi \in \Sigma(2)$.

Actually, $L\phi \in \bar{Z}_T$ for any $\phi \in \Sigma(2)$. This follows from Lemmas 2.4 and 2.6 immediately. $\square$

Remark 3.3. According to the proof of Lemma 3.2, we see that $\triangle L\phi \in X_T$ for any $\phi \in \Sigma(2)$.

Lemma 3.3. *Let $f \in L^{2,\infty}$ and $\partial_t f,\ |x|^2 f,\ x\cdot\nabla f \in X_T'$. Assume that $T>0$ is sufficiently small and that $f(0) \in L^2$ exists. Then $Gf \in Z_T$. Furthermore there exists $C>0$ independent of T such that*

$$\|Gf\|_{Z_T} \le C(\|f\|_{2,\infty} + \||x|^2 f\|_{X_T'} + \|x\cdot\nabla f\|_{X_T'} + \|\partial_t f\|_{X_T'}). \tag{3.11}$$

In particular, if $f \in C(I_T, L^2)$, then $Gf \in \bar{Z}_T$.

Proof. First we assume that $f \in \mathcal{S}(\mathbf{R}^{n+1})$. By Lemma 2.6, we see that

$$\|Gf\|_{2,\infty} \le c\|f\|_{2,1}, \tag{3.12}$$

and that $Gf \in C(I_T, L^2)$. By the application of Lemma 2.6 to the equalities in Lemma 3.1, we have the following estimates

$$\begin{aligned}
&\|x_k\partial_k Gf\|_{X_T}\\
\le&\|G[(i/2)x_k(\partial_k\nabla\cdot A)Gf + ix_k\partial_k A\cdot\nabla Gf\\
&+(x_kA\cdot\partial_kA + x_k\partial_kV)Gf + \partial_k^2 Gf - iA_k\partial_k Gf + x_k\partial_k f]\|_{X_T}\\
\le&c\|(i/2)x_k(\partial_k\nabla\cdot A)Gf + ix_k\partial_k A\cdot\nabla Gf\\
&+(x_kA\cdot\partial_kA + x_k\partial_kV)Gf + \partial_k^2 Gf - iA_k\partial_k Gf\|_{2,1}\\
&+c\|x_k\partial_k f\|_{X_T'}\\
\le&cT(\|Gf\|_{2,\infty} + \|\partial_k^2 Gf\|_{2,\infty} + \|x_k\partial_k Gf\|_{2,\infty} + \|{x_k}^2 Gf\|_{2,\infty})\\
&+c\|x_k\partial_k f\|_{X_T'},
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\|{x_k}^2 Gf\|_{X_T} \le&\|G[2x_k(\partial_k - iA_k)Gf + Gf + {x_k}^2 f]\|_{X_T}\\
\le&c\|2x_k(\partial_k - iA_k)Gf + Gf\|_{2,1} + c\|{x_k}^2 f\|_{X_T'}\\
\le&cT(\|Gf\|_{2,\infty} + \|x_k\partial_k Gf\|_{2,\infty} + \|{x_k}^2 Gf\|_{2,\infty})\\
&+c\|{x_k}^2 f\|_{X_T'}.
\end{aligned} \tag{3.14}$$

We have used Remark 2.2. Since

$$\begin{aligned}
\partial_t Gf =&G\partial_t f + Lf(0) + iGH(\cdot)f - iH(t)Gf\\
=&G\partial_t f + Lf(0) + i[\frac{1}{2}(\triangle Gf - G\triangle f)\\
&+G\{(i(\nabla\cdot A) + \frac{i}{2}A\cdot\nabla + \frac{1}{2}|A|^2 + V)f\}\\
&-(i(\nabla\cdot A) + \frac{i}{2}A\cdot\nabla + \frac{1}{2}|A|^2 + V)Gf],
\end{aligned} \tag{3.15}$$

it follows from Lemma 2.6 that

$$\begin{aligned}\|\partial_t Gf\|_{X_T} \le & c(\|\partial_t f\|_{X_T'} + \|f\|_{2,1} + \|f(0)\|_2 + \||x|^2 f\|_{X_T'} + \|x\cdot\nabla f\|_{X_T'} \\ & + \||x|^2 Gf\|_{X_T} + \|x\cdot\nabla Gf\|_{X_T}) + cT\|\triangle Gf\|_{2,\infty}.\end{aligned} \tag{3.16}$$

It follows from (2.1) that

$$\triangle Gf = -2i\partial_t Gf + i(\nabla\cdot A)Gf + 2iA\cdot\nabla Gf + |A|^2 Gf + 2VGf + 2if. \tag{3.17}$$

By Lemma 2.6, we see

$$\begin{aligned}& \|\triangle Gf\|_{2,\infty} \\ \le & \|\partial_t Gf - i(\nabla\cdot A)Gf - 2iA\cdot\nabla Gf - |A|^2 Gf - 2VGf - f\|_{2,\infty} \\ \le & c(\|x\cdot\nabla Gf\|_{2,\infty} + \||x|^2 Gf\|_{2,\infty} + \|\partial_t f\|_{X_T'} + \|f\|_{2,1} \\ & + \|f(0)\|_2 + \||x|^2 f\|_{X_T'} + \|x\cdot\nabla f\|_{X_T'}) + cT\|\triangle Gf\|_{2,\infty},\end{aligned}$$

and hence for $T > 0$ sufficiently small,

$$\begin{aligned}\|\triangle Gf\|_{2,\infty} \le & c(\|x\cdot\nabla Gf\|_{2,\infty} + \||x|^2 Gf\|_{2,\infty} + \|\partial_t f\|_{X_T'} \\ & + \|f\|_{2,1} + \|f(0)\|_2 + \||x|^2 f\|_{X_T'} + \|x\cdot\nabla f\|_{X_T'}).\end{aligned} \tag{3.18}$$

From the above estimates, we have

$$\begin{aligned}\|Gf\|_{Z_T} \le & c(\|f\|_{2,1} + \|f(0)\|_2 + \||x|^2 f\|_{X_T'} + \|x\cdot\nabla f\|_{X_T'} + \|\partial_t f\|_{X_T'}) \\ & + cT\|Gf\|_{Z_T}.\end{aligned}$$

Therefore, if $T > 0$ is sufficiently small,

$$\|Gf\|_{Z_T} \le c(\|f\|_{2,1} + \|f(0)\|_2 + \||x|^2 f\|_{X_T'} + \|x\cdot\nabla f\|_{X_T'} + \|\partial_t f\|_{X_T'}), \tag{3.19}$$

for $f \in \mathcal{S}(\mathbf{R}^{n+1})$. By the density argument, (3.19) holds for any f satisfying the assumptions of this lemma. Since $\|f\|_{2,1} \le T\|f\|_{2,\infty}$, $\|f(0)\|_2 \le \|f\|_{2,\infty}$, this implies (3.11) for $T > 0$ sufficiently small.

In view of Lemma 2.6 and (3.15), it is easy to see that $\partial_t Gf \in C(I_T, L^2)$. Therefore we see that $Gf \in \bar{Z}_T$ if in addition $f \in C(I_T, L^2)$. $\square$

Remark 3.4. Note that $\triangle Gf$ does not always belong to the auxiliary space X_T for f satisfying the assumptions of Lemma 3.3. On the other hand, $\triangle L\phi \in X_T$ for $\phi \in \Sigma(2)$ (see Remark 3.3). We will set $f = F(u)$ for $u \in Z_T$ in the proof of Theorem 1. On the other hand, in the proof of existence of $\Sigma(1)$-solution, $\nabla GF(u)$ belongs to the auxiliary space. (cf. [13]).

To estimate the nonlinear term, we need the following two lemmas which are immediate consequences of Lemma 4.1 and Proposition 7.5 in Kato [9], respectively.

Lemma 3.4. *F maps L^{2p} into L^2 continuously, and maps bounded sets of L^{2p} into bounded sets of L^2.*

Let $m = 1 - l$.

Lemma 3.5. *There exists $C > 0$ independent of T such that*

$$\|u(t) - u(s)\|_2 \le C|t-s|\|u\|_{Z_T},$$
$$\|u(t) - u(s)\|_{2p} \le C|t-s|^m\|u\|_{Z_T},$$

for any $u \in Z_T$ and $t, s \in I_T$.

We prove Theorem 1 by the contraction method. We introduce the following integral equation

$$u(t) = (L\phi)(t) - i(GF(u))(t). \tag{3.20}$$

We define the linear operator

$$K(u) = L\phi - iGF(u),$$

and the ball in Z_T

$$B_{T,R} = \{u \in Z_T \colon \|u\|_{Z_T} \le R,\ u(0) = \phi\},$$

for $T, R > 0$.

Remark 3.5. $B_{T,R}$ is a complete metric space in X_T metric. We can prove this property following, e.g., the proof of Proposition 6.6 in Kato [9].

Proposition 3.1. *Let $\phi \in \Sigma(2)$. K maps $B_{T,R}$ into $B_{T,R}$ if R is sufficiently large and T is sufficiently small, depending only on $\|\phi\|_{\Sigma(2)}$.*

Proof. Let $u \in B_{T,R}$. Note that by the Hölder inequality and (3.1), the following inequalities hold.

$$\|f\|_{\frac{4p}{3p-1},\frac{2}{2-l}} \le T^m\|f\|_{\frac{4p}{3p-1},\frac{2}{l}}, \tag{3.21}$$
$$\|fg^{p-1}\|_{\frac{4p}{3p-1},\frac{2}{l}} \le \|f\|_{\frac{4p}{p+1},\frac{2}{l}}\|g\|_{2p,\infty}^{p-1}, \tag{3.22}$$
$$\|g\|_{2p,\infty} \le c\|g\|_{L^\infty(I_T,H^2)} \le c\|g\|_{Z_T}. \tag{3.23}$$

Recall Remark 2.3. We have the following estimates:

$$
\begin{aligned}
\||x|^2F(u)\|_{X_T'} \le& M_1\||x|^2u\|_{2,1} + M_2\||x|^2|u|^p\|_{\frac{4p}{3p-1},\frac{2}{2-l}}\\
\le& M_1T\||x|^2u\|_{2,\infty} + M_2T^m\||x|^2|u|^p\|_{\frac{4p}{3p-1},\frac{2}{l}}\\
\le& M_1T\||x|^2u\|_{2,\infty} + M_2T^m\||x|^2u\|_{\frac{4p}{p+1},\frac{2}{l}}\|u\|_{2p,\infty}^{p-1}\\
\le& M_1T\|u\|_{Z_T} + cM_2T^m\|u\|_{Z_T}^p,
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
\|x\cdot\nabla F(u)\|_{X_T'} \le& M_1\|x\cdot\nabla u\|_{2,1} + M_2\|x\cdot\nabla u|u|^{p-1}\|_{\frac{4p}{3p-1},\frac{2}{2-l}}\\
\le& M_1T\|x\cdot\nabla u\|_{2,\infty} + M_2T^m\|x\cdot\nabla u|u|^{p-1}\|_{\frac{4p}{3p-1},\frac{2}{l}}\\
\le& M_1T\|x\cdot\nabla u\|_{2,\infty} + M_2T^m\|x\cdot\nabla u\|_{\frac{4p}{p+1},\frac{2}{l}}\|u\|_{2p,\infty}^{p-1}\\
\le& M_1T\|u\|_{Z_T} + cM_2T^m\|u\|_{Z_T}^p,
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
\|\partial_tF(u)\|_{X_T'} \le& M_1\|\partial_tu\|_{2,1} + M_2\||u|^{p-1}\partial_tu\|_{\frac{4p}{3p-1},\frac{2}{2-l}}\\
\le& M_1T\|\partial_tu\|_{2,\infty} + M_2T^m\||u|^{p-1}\partial_tu\|_{\frac{4p}{3p-1},\frac{2}{l}}\\
\le& M_1T\|\partial_tu\|_{2,\infty} + M_2T^m\|\partial_tu\|_{\frac{4p}{p+1},\frac{2}{l}}\|u\|_{2p,\infty}^{p-1}\\
\le& M_1T\|u\|_{Z_T} + cM_2T^m\|u\|_{Z_T}^p.
\end{aligned}
\tag{3.26}
$$

By Remark 2.3, we have for $z_1, z_2 \in \mathbf{C}$,

$$
\begin{aligned}
|F_1(z_1) - F_1(z_2)| \le& M_1|z_1 - z_2|,\\
|F_2(z_1) - F_2(z_2)| \le& M_2|z_1 - z_2|(|z_1|^{p-1} + |z_2|^{p-1}).
\end{aligned}
\tag{3.27}
$$

From Lemma 3.5, (3.1) and the Hölder inequality, we have

$$
\begin{aligned}
\|F_1(u(t)) - F_1(u(s))\|_2 \le& M_1\|u(t) - u(s)\|\\
\le& cM_1|t-s|\|u\|_{Z_T},
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
\|F_2(u(t)) - F_2(u(s))\|_2 \le& M_2\||u(t) - u(s)|(|u(t)|^{p-1} + |u(s)|^{p-1})\|_2\\
\le& M_2\|u(t) - u(s)\|_{2p}(\|u(t)\|_{2p}^{p-1} + \|u(s)\|_{2p}^{p-1})\\
\le& cM_2|t-s|^m\|u\|_{Z_T}^p,
\end{aligned}
\tag{3.29}
$$

for any $t, s \in I_T$. Therefore from Lemma 3.4, we obtain

$$
\begin{aligned}
\|F(u)\|_{2,\infty} \le& \|F(\phi)\|_{2,\infty} + \|F(u) - F(\phi)\|_{2,\infty}\\
\le& c\|\phi\|_{\Sigma(2)} + cM_1T\|u\|_{Z_T} + cM_2T^m\|u\|_{Z_T}^p,
\end{aligned}
\tag{3.30}
$$

where $u(0) = \phi$. From these estimates, if $T > 0$ is small enough, we can apply Lemma 3.3 with $f = F(u)$. Then we have

$$
\|GF(u)\|_{Z_T} \le c\|\phi\|_{\Sigma(2)} + cM_1T\|u\|_{Z_T} + cM_2T^m\|u\|_{Z_T}^p,
$$

for T sufficiently small. By (2.2), we see that if $T > 0$ is small enough,

$$\|K(u)\|_{Z_T} \leq c\|\phi\|_{\Sigma(2)} + cM_1T\|u\|_{Z_T} + cM_2T^m\|u\|_{Z_T}^p. \tag{3.31}$$

The fact $u \in B_{T,R}$ implies

$$\|K(u)\|_{Z_T} \leq c\|\phi\|_{\Sigma(2)} + cM_1TR + cM_2T^mR^p.$$

Hence we can choose $R > 0$ sufficiently large and $T > 0$ sufficiently small so that

$$c\|\phi\|_{\Sigma(2)} + cM_1TR + cM_2T^mR^p \leq R.$$

It follows from (3.31) that

$$\|K(u)\|_{Z_T} \leq R.$$

□

Proposition 3.2. *K is a contraction mapping on $B_{T,R}$ in X_T metric if R is sufficiently large and T is sufficiently small, depending only on $\|\phi\|_{\Sigma(2)}$.*

Proof. Let $u, v \in B_{T,R}$. By the definition of K, we see

$$K(u) - K(v) = -i(GF(u) - GF(v)).$$

Then we have

$$\begin{aligned}
&\|K(u) - K(v)\|_{X_T} \\
=&\|GF(u) - GF(v)\|_{X_T} \\
\leq&c\|F_1(u) - F_1(v)\|_{2,1} + c\|F_2(u) - F_2(v)\|_{\frac{4p}{3p-1},\frac{2}{2-l}} \\
\leq&cT\|F_1(u) - F_1(v)\|_{2,\infty} + cT^m\|F_2(u) - F_2(v)\|_{\frac{4p}{3p-1},\frac{2}{l}} \\
\leq&cM_1T\|u - v\|_{2,\infty} + cM_2T^m\||u - v|(|u|^{p-1} + |v|^{p-1})\|_{\frac{4p}{3p-1},\frac{2}{l}} \\
\leq&cM_1T\|u - v\|_{2,\infty} + cM_2T^m(\|u\|_{2p,\infty}^{p-1} + \|v\|_{2p,\infty}^{p-1})\|u - v\|_{\frac{4p}{p+1},\frac{2}{l}} \\
\leq&(cM_1T + cM_2T^mR^{p-1})\|u - v\|_{X_T}.
\end{aligned} \tag{3.32}$$

As in Proposition 3.1, we can choose R sufficiently large and T sufficiently small, depending only on $\|\phi\|_{\Sigma(2)}$, so that

$$cM_1T + cM_2T^mR^{p-1} \leq \frac{1}{2}.$$

The proof of this proposition completes. □

Now we prove Theorem 1 and Theorem 2.

Proof of Theorem 1. First we assume that $n \geq 4$ and that $1 \leq p < 1+4/(n-4)$. We show the existence argument. By Remark 3.5, Propositions 3.1 and 3.2, if R is sufficiently large and T is sufficiently small, K has a unique fixed point u in $B_{T,R}$. Namely, u is a unique solution of (3.20) in $B_{T,R}$. By (3.28) and (3.29), we see that $F(u) \in C(I_T, L^2)$. Therefore $u \in \bar{Z}_T$ follows from Lemmas 3.2 and 3.3. Since $F(u) \in C(I_T, L^2)$, u is a solution of (NLS) by Lemmas 2.4 and 2.5.

We next prove the uniqueness argument. Let $u, v \in C(I_T, \Sigma(2))$ be solutions of (NLS) with $u(0) = v(0) = \phi$. Then u, v satisfy the integral equation (3.20). Therefore, as in the proof of Proposition 3.2,

$$\begin{aligned}
&\|u-v\|_{X_T} \\
=&\|Gu - Gv\|_{X_T} \\
\leq& cM_1T\|u-v\|_{2,\infty} + cM_2T^m(\|u\|_{2p,\infty}^{p-1} + \|v\|_{2p,\infty}^{p-1})\|u-v\|_{\frac{4p}{p+1},\frac{2}{l}} \\
\leq& \{cM_1T + cM_2T^m(\|u\|_{L^\infty(I_T,\Sigma(2))}^{p-1} + \|v\|_{L^\infty(I_T,\Sigma(2))}^{p-1})\}\|u-v\|_{X_T}.
\end{aligned}$$

We can choose $T > 0$ sufficiently small, depending only on $\|\phi\|_{\Sigma(2)}$, so that

$$cM_1T + cM_2T^m(\|u\|_{L^\infty(I_T,\Sigma(2))}^{p-1} + \|v\|_{L^\infty(I_T,\Sigma(2))}^{p-1}) \leq \frac{1}{2}.$$

Hence, if $T > 0$ sufficiently small, we have

$$\|u-v\|_{X_T} = 0.$$

Finally we show that $\partial_t u \in \bar{\mathcal{X}}_T$. From (3.15), we note that

$$\begin{aligned}
i\partial_t u =& H(t)L\phi - i\partial_t GF(u) \\
=& H(t)L\phi - iG\partial_t F(u) - iLF(\phi) + \frac{1}{2}(\triangle GF(u) - G\triangle F(u)) \\
&+ G\{(i(\nabla\cdot A) + \frac{i}{2}A\cdot\nabla + \frac{1}{2}|A|^2 + V)F(u)\} \\
&- (i(\nabla\cdot A) + \frac{i}{2}A\cdot\nabla + \frac{1}{2}|A|^2 + V)GF(u).
\end{aligned} \tag{3.33}$$

Since terms in RHS of (3.33) except the 1st and the last terms are images of L or G, they are in $\mathcal{X}_T$. For the 4th term, we have also used (3.4). For the first and the last terms, by Lemma 2.6, for any (q, r) satisfying (1.4), we can see that

$$\|H(t)L\phi\|_{q,r} \leq c\|\phi\|_{\Sigma(2)},$$

and that

$$\begin{aligned}
&\|GF(u)\|_{q,r} \vee \||x|^2 GF(u)\|_{q,r} \vee \|x \cdot \nabla GF(u)\|_{q,r} \\
&\vee \|\partial_t GF(u)\|_{q,r} \vee \|\triangle GF(u)\|_{2,\infty} \\
\leq& C(\|F(u)\|_{2,\infty} + \||x|^2 F(u)\|_{X'_T} + \|x \cdot \nabla F(u)\|_{X'_T} + \|\partial_t F(u)\|_{X'_T}),
\end{aligned}$$

in the exactly same way as the proof of (3.5), (3.11), respectively. Thus we obtain $\partial_t u \in \mathcal{X}_T$. Since $u \in \bar{Z}_T$, we have $\partial_t u \in \bar{\mathcal{X}}_T$. The proof of Theorem 1 is complete for $n \geq 4$.

When $n \leq 3$, the proof is similar but much simpler. In this case, we set $F_1 = F$, $F_2 = 0$ and $X_T = L^{2,\infty}$ in the proof above. Using the fact $Z_T \hookrightarrow L^\infty(I_T \times \mathbf{R}^n)$, we can show this theorem in the same way as above. We omit the details.

□

Proof of Theorem 2. First we assume that $n \geq 4$ and that $1 \leq p < 1 + 4/(n-4)$. Assume that $\phi \in \Sigma(2)$ and that $u \in C(I_{T_0}, \Sigma(2))$ is a solution of (NLS) with $u(0) = \phi$. We also assume that $\phi_n \in \Sigma(2)$ for $n = 1, 2, \ldots,$ and that $\phi_n \to \phi$ as $n \to \infty$ in $\Sigma(2)$. By standard continuation argument, it is sufficient to prove that $u_n \to u$ in Z_T for $T > 0$ sufficiently small depending only on $\|\phi\|_{\Sigma(2)}$. Let

$$K_n(v) = L\phi_n - iGF(v),$$

for $n = 1, 2, \ldots.$ According to the proof of Theorem 1, if $T > 0$ is sufficiently small and $R > 0$ is sufficiently large, depending only on $\|\phi\|_{\Sigma(2)}$, u is a unique fixed point of K in $B_{T,R}$, and K_n has a unique fixed point u_n in $B_{T,R}$ for n sufficiently large. Then u_n is a unique solution of (NLS) in $C(I_T, \Sigma(2))$ with $u_n(0) = \phi_n$.

As in the proof of Proposition 3.2, we see

$$\begin{aligned}
\|u_n - u\|_{X_T} &= \|K_n(u_n) - K(u)\|_{X_T} \\
&\leq \|L\phi_n - L\phi\|_{X_T} + \|GF(u_n) - GF(u)\|_{X_T} \\
&\leq c\|\phi_n - \phi\|_2 + (cM_1 T + cM_2 T^m R^{p-1})\|u_n - u\|_{X_T}.
\end{aligned}$$

This implies that if $T > 0$ is small enough,

$$\|u_n - u\|_{X_T} \leq c\|\phi_n - \phi\|_{\Sigma(2)}.$$

We obtain that $u_n \to u$ in X_T. Since $\|u_n(t)\|_{H^2} \leq \|u_n\|_{Z_T} \leq R$, it follows that $u_n \to u$ in $L^\infty(I_T, H^{2l})$ for any $l < 1$.

By Lemmas 3.2 and 3.3, we see that

$$
\begin{aligned}
&\|u_n - u\|_{Z_T}\\
=&\|K_n(u_n) - K(u)\|_{Z_T}\\
\le&\|L\phi_n - L\phi\|_{Z_T} + \|GF(u_n) - GF(u)\|_{Z_T}\\
\le&c\|\phi_n - \phi\|_{\Sigma(2)} + c(\|F(u_n) - F(u)\|_{2,\infty} + \||x|^2(F(u_n) - F(u))\|_{X'_T}\\
&+ \|x \cdot \nabla(F(u_n) - F(u))\|_{X'_T} + \|\partial_t(F(u_n) - F(u))\|_{X'_T})\\
\le&c\|\phi_n - \phi\|_{\Sigma(2)} + c(\|F(u_n) - F(u)\|_{2,\infty} + \||x|^2(F(u_n) - F(u))\|_{X'_T}\\
&+ \|F'(u_n)x \cdot \nabla(u_n - u)\|_{X'_T} + \|(F'(u_n) - F'(u))x \cdot \nabla u\|_{X'_T}\\
&+ \|F'(u_n)(\partial_t u_n - \partial_t u)\|_{X'_T} + \|(F'(u_n) - F'(u))\partial_t u\|_{X'_T}.
\end{aligned}
\tag{3.34}
$$

As in the proof of Proposition 3.1, we have

$$
\begin{aligned}
\|F(u_n) - F(u)\|_{2,\infty} \le& M_1T\|u_n - u\|_{2,\infty}\\
&+ M_2T^m R^{p-1}\|u_n - u\|_{\frac{4p}{p+1},\frac{2}{l}}\\
\le&(M_1T + M_2T^m R^{p-1})\|u_n - u\|_{X_T},
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
\|{x_k}^2(F(u_n) - F(u))\|_{X'_T} \le& M_1T\|{x_k}^2(u_n - u)\|_{2,\infty}\\
&+ M_2T^m R^{p-1}\|{x_k}^2(u_n - u)\|_{\frac{4p}{p+1},\frac{2}{l}}\\
\le&(M_1T + M_2T^m R^{p-1})\|{x_k}^2(u_n - u)\|_{X_T}.
\end{aligned}
\tag{3.36}
$$

On the other hand, by Assumption (F2), it is easily seen

$$\|F'(u_n)x \cdot \nabla(u_n - u)\|_{X'_T} \le (M_1T + M_2T^m R^{p-1})\|x \cdot \nabla(u_n - u)\|_{X_T}, \tag{3.37}$$

$$\|F'(u_n)(\partial_t u_n - \partial_t u)\|_{X'_T} \le (M_1T + M_2T^m R^{p-1})\|\partial_t u_n - \partial_t u\|_{X_T}. \tag{3.38}$$

From (3.34)-(3.38), we have

$$
\begin{aligned}
\|u_n - u\|_{Z_T} \le& c\|\phi_n - \phi\|_{\Sigma(2)} + c\|(F'(u_n) - F'(u))x \cdot \nabla u\|_{X'_T}\\
&+ c\|(F'(u_n) - F'(u))\partial_t u\|_{X'_T},
\end{aligned}
$$

for $T > 0$ sufficiently small.

Therefore it remains to prove

$$(F'(u_n) - F'(u))\partial_t u \to 0, \tag{3.39}$$
$$(F'(u_n) - F'(u))x \cdot \nabla u \to 0, \tag{3.40}$$

as $n \to \infty$ in X'_T.

By Lemma 2.6 and Hölder's inequality, it is easily seen

$$
\begin{aligned}
&\|(F'(u_n)-F'(u))\partial_t u\|_{X'_T}\\
\le&\|(F'_1(u_n)-F'_1(u))\partial_t u\|_{2,1}+\|(F'_2(u_n)-F'_2(u))\partial_t u\|_{\frac{4p}{3p-1},\frac{2}{2-l}}\\
\le&T\|(F'_1(u_n)-F'_1(u))\partial_t u\|_{2,\infty}+T^m\|(F'_2(u_n)-F'_2(u))\partial_t u\|_{\frac{4p}{3p-1},\frac{2}{l}}\\
\le&T\|(F'_1(u_n)-F'_1(u))\partial_t u\|_{2,\infty}\\
&+T^m\|(F'_2(u_n)-F'_2(u))\|_{\frac{2p}{p-1},\infty}\|\partial_t u\|_{\frac{4p}{p+1},\frac{2}{l}}\\
\le&T\|(F'_1(u_n)-F'_1(u))\partial_t u\|_{2,\infty}\\
&+T^m\|(F'_2(u_n)-F'_2(u))\|_{\frac{2p}{p-1},\infty}\|\partial_t u\|_{X_T}.
\end{aligned}
$$

Then noting Remark 2.3, we see that

$$\|(F'_1(u_n)-F'_1(u))\partial_t u\|_{2,\infty}\to 0,$$

as $n\to\infty$, by Remark 4.3 in Kato [9], the dominated convergence theorem and the fact $\partial_t u\in X_T\subset L^{2,\infty}$, and that

$$\|F'_2(u_n)-F'_2(u)\|_{\frac{2p}{p-1},\infty}\to 0,$$

as $n\to\infty$, by Lemma 4.2 in Kato [9]. We note that $u_n\to u$ in $L^\infty(I_T,H^{2l})$ for any $l<1$ and Remark 4.2. These imply (3.39). Similarly, we can prove (3.40) since $x\cdot\nabla u\in X_T\subset L^{2,\infty}$.

When $n\le 3$, the proof is similar but much simpler as in the proof of Theorem 1. Hence we omit the proof in that case.

□

4 Proof of Theorem 3

Before the proof of Theorem 3, We introduce the following results of Yajima [26].

Lemma 4.1 (Yajima [26]). *Assume Assumptions (A) and (V). Let $T>0$ be sufficiently small, $\mu>1/2$ and $\rho>0$. There exists a constant $C>0$, depending on μ and ρ, such that for $s\in\mathbf{R}$*

$$\int_{s-T}^{s+T}\|\langle x\rangle^{-\mu-\rho}\langle D_x\rangle^{\rho}U(t,s)f\|_2^2\,dt\le C\|\langle D_x\rangle^{\rho-1/2}f\|_2^2.$$

Using Lemma 4.1, we prove Theorem 3.

Proof of Theorem 3. By (3.20), (3.15) and (3.17), we have

$$\begin{aligned}
&\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2}u \\
=&\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2}L\phi - i\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2}GF(u) \\
=&\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2}L\phi + 2\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2}LF(\phi) \\
&+ 2\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2}G\partial_t F(u) \\
&+ 2i\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2}(\triangle GF(u) - G\triangle F(u)) \\
&+ i\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2}G[i(\nabla\cdot A)F(u) + \frac{i}{2}A\cdot\nabla F(u) \\
&+ \frac{1}{2}|A|^2F(u) + VF(u) - F(u)] \\
&+ 2\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2}F(u).
\end{aligned} \tag{4.1}$$

First we estimate the 1st and 2nd term in the RHS of (4.1). By Lemma 4.1, it is easily seen that

$$\int_{I_T}\|\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2}(L\phi)(t)\|_2^2\,dt \le c\|\langle D_x\rangle^2\phi\|_2^2 < \infty, \tag{4.2}$$

and

$$\begin{aligned}
\int_{I_T}\|\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2}(LF(\phi))(t)\|_2^2\,dt \le& c\|F(\phi)\|_2^2 \\
\le& c(\|\phi\|_2^2 + \|\phi\|_{2p}^{2p}) \\
\le& c(\|\phi\|_2^2 + \|\phi\|_{H^2}^{2p}) \\
<&\infty.
\end{aligned} \tag{4.3}$$

We have used Remark 2.3 and (3.1) in the second estimate.

To estimate the 3rd, the 4th and the 5th terms in the RHS of (4.1), we need the following lemma.

Lemma 4.2. *For $\mu > 1/2$, there exists $C > 0$, depending on μ such that*

$$\left(\int_{I_T}\|\langle x\rangle^{-\mu-1/2}\langle D_x\rangle^{1/2}(Gf)(t)\|_2^2\,dt\right)^{1/2} \le C\|f\|_{2,1},$$

Proof. Let $g \in C_0^\infty(I_T \times \mathbf{R}^n)$. By Lemma 4.1 and the Schwarz inequality, we obtain

$$\begin{aligned}
&\left| \int_{I_T} \left(\langle x\rangle^{-\mu-1/2} \langle D_x\rangle^{1/2} (Gf)(t), g(t) \right) \, dt \right| \\
\leq & \int_{I_T} \int_0^t |(\langle x\rangle^{-\mu-1/2} \langle D_x\rangle^{1/2} U(t,s) f(s), g(t))| \, ds \, dt \\
\leq & \int_{I_T} \int_{I_T} \|\langle x\rangle^{-\mu-1/2} \langle D_x\rangle^{1/2} U(t,s) f(s)\|_2 \|g(t)\|_2 \, dt \, ds \\
\leq & \left(\int_{I_T} \left(\int_{I_T} \|\langle x\rangle^{-\mu-1/2} \langle D_x\rangle^{1/2} U(t,s) f(s)\|_2^2 \, dt \right)^{1/2} ds \right) \|g\|_{2,2} \\
\leq & c \left(\int_{I_T} \|f(s)\|_2 \, ds \right) \|g\|_{2,2},
\end{aligned}$$

where $(\cdot,\cdot)$ is the $L^2(\mathbf{R}^n)$ scalar product. By the duality argument, we have this lemma. □

For the 4th term in the RHS of (4.1), we have

$$\begin{aligned}
&\left(\int_{I_T} \|\langle x\rangle^{-\mu-5/2} \langle D_x\rangle^{1/2} ((\triangle GF(u))(t) - (G\triangle F(u))(t))\|_2^2 \, dt \right)^{1/2} \\
\leq & c(\|GF(u)\|_{2,1} \vee \||x|^2 GF(u)\|_{2,1} \vee \|x\cdot\nabla GF(u)\|_{2,1} \\
& \vee \|\triangle GF(u)\|_{2,1}) \\
\leq & cT(\|GF(u)\|_{2,\infty} \vee \||x|^2 GF(u)\|_{2,\infty} \vee \|x\cdot\nabla GF(u)\|_{2,\infty} \\
& \vee \|\triangle GF(u)\|_{2,\infty}) \\
\leq & cT\|GF(u)\|_{Z_T} \\
\leq & cT(\|F(u)\|_{2,\infty} + \||x|^2 F(u)\|_{X_T'} + \|x\cdot\nabla F(u)\|_{X_T'} + \|\partial_t F(u)\|_{X_T'}).
\end{aligned} \tag{4.4}$$

Since $u \in Z_T$, it follows from (3.24), (3.25), (3.26) and (3.30) that the RHS of above inequality is finite.

According to Lemma 4.2, to estimate the $L^{2,2}$-norm of the 3rd term in the RHS of (4.1), it is sufficient to show

$$\partial_t F(u) = F'(u)\partial_t u \in L^{2,1}, \tag{4.5}$$

and to estimate the $L^{2,2}$-norm of the 5th term in the RHS of (4.1), it is enough to prove

$$F(u) \in L^{2,1}, \tag{4.6}$$

$$|x|^2 F(u) \in L^{2,1}, \tag{4.7}$$

$$x\cdot\nabla F(u) = F'(u)(x\cdot\nabla u) \in L^{2,1}. \tag{4.8}$$

On the other hand, to estimate the $L^{2,2}$-norm of the 6th term in the RHS of (4.1), it is sufficient to prove

$$F(u) \in L^{2,2}, \tag{4.9}$$
$$\nabla F(u) = F'(u)\nabla u \in L^{2,2}. \tag{4.10}$$

(4.6) and (4.9) follow from (3.30).

We show (4.5). When $n \le 3$, we have already proved in the end of the proof of Theorem 1. Actually, $\partial_t u \in L^{2,\infty}$ implies $\partial_t F(u) \in L^{2,1}$ when $n \le 3$. When $n \ge 4$, since $u \in L^\infty(I_T, H^2)$, we see that $u \in L^{q,\infty}$ for $2 \le q < \infty$ when $n = 4$ and for $2 \le q \le 2n/(n-4)$ when $n \ge 5$. For proving in the case of $n = 4$, we note that there exist the real constants a', b' satisfying

$$\frac{p-1}{a'} + \frac{1}{b'} = \frac{1}{2},$$
$$0 < \frac{1}{a'} \le \frac{1}{2}, \quad \frac{1}{4} \le \frac{1}{b'} \le \frac{1}{2}.$$

On the other hand, when $n \ge 5$, the assumption $1 \le p \le 1 + 2/(n-4)$ implies that there exist the real numbers a, b satisfying

$$\frac{p-1}{a} + \frac{1}{b} = \frac{1}{2},$$
$$\frac{1}{2} - \frac{2}{n} \le \frac{1}{a} \le \frac{1}{2}, \quad \frac{1}{2} - \frac{1}{n} \le \frac{1}{b} \le \frac{1}{2}.$$

Therefore we obtain that when $n = 4$, by Hölder's inequality and the fact $\partial_t u \in \mathcal{X}_T$,

$$\begin{aligned}
\|F'(u)\partial_t u\|_{2,1} \le & cM_1 T \|\partial_t u\|_{2,\infty} + cM_2 T^{1/2} \||u|^{p-1}\partial_t u\|_{2,2} \\
\le & cM_1 T \|\partial_t u\|_{2,\infty} + cM_2 T^{1/2} \|u\|_{a',\infty}^{p-1} \|\partial_t u\|_{b',2} \\
\le & cM_1 T \|\partial_t u\|_{2,\infty} + cM_2 T^{1-(1/r')} \|u\|_{a',\infty}^{p-1} \|\partial_t u\|_{b',r'} \\
< & \infty,
\end{aligned}$$

where r' is a constant such that (b', r') is an admissible pair, and that when $n \ge 5$, by Hölder's inequality and the fact $\partial_t u \in \mathcal{X}_T$,

$$\begin{aligned}
\|F'(u)\partial_t u\|_{2,1} \le & cM_1 T \|\partial_t u\|_{2,\infty} + cM_2 T^{1/2} \||u|^{p-1}\partial_t u\|_{2,2} \\
\le & cM_1 T \|\partial_t u\|_{2,\infty} + cM_2 T^{1/2} \|u\|_{a,\infty}^{p-1} \|\partial_t u\|_{b,2} \\
\le & cM_1 T \|\partial_t u\|_{2,\infty} + cM_2 T^{1-(1/r)} \|u\|_{a,\infty}^{p-1} \|\partial_t u\|_{b,r} \\
< & \infty,
\end{aligned}$$

where r is a constant such that (b, r) is an admissible pair. These imply (4.5).

By (3.20), (3.2) and (3.3), we see that $|x|^2 u$ and $x \cdot \nabla u$ are represented as the sum of the images of L or G. Thus $|x|^2 u, x \cdot \nabla u \in \mathcal{X}_T$. In the exactly same way as the proof of (4.5), we can show (4.7) and (4.8).

It remains to show (4.10). When $n \leq 3$, the proof of (4.10) is easy. In fact, noting $H^2(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$ for $n \leq 3$, we see that $\nabla u \in L^{2,\infty}$ implies $\nabla F(u) \in L^{2,2}$. When $n \geq 4$, since $\nabla u \in L^\infty(I_T, H^1)$, we see that $\nabla u \in L^\infty(I_T, L^q)$ for $2 \leq q \leq 2n/(n-2)$, by the Sobolev embedding theorem. Therefore by Hölder's inequality, we have when $n = 4$,

$$\begin{aligned}
\|F'(u)\nabla u\|_{2,2} \leq & cM_1 T^{1/2} \|\nabla u\|_{2,\infty} + cM_2 T^{1/2} \||u|^{p-1} \nabla u\|_{2,\infty} \\
\leq & cM_1 T^{1/2} \|\nabla u\|_{2,\infty} + cM_2 T^{1/2} \|u\|_{a',\infty}^{p-1} \|\nabla u\|_{b',\infty} \\
< & \infty,
\end{aligned}$$

and when $n \geq 5$,

$$\begin{aligned}
\|F'(u)\nabla u\|_{2,2} \leq & cM_1 T^{1/2} \|\nabla u\|_{2,\infty} + cM_2 T^{1/2} \||u|^{p-1} \nabla u\|_{2,\infty} \\
\leq & cM_1 T^{1/2} \|\nabla u\|_{2,\infty} + cM_2 T^{1/2} \|u\|_{a,\infty}^{p-1} \|\nabla u\|_{b,\infty} \\
< & \infty.
\end{aligned}$$

These imply (4.10).

□

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