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A GLOBAL CONFORMAL UNIQUENESS IN THE ANISOTROPIC INVERSE BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we discuss the two dimensional inverse problem of determining the anisotropic conductivity from the Dirichlet to Neumann map. The global conformal uniqueness is proved. The key of the proof is the global uniqueness result for the inverse problem of determining the convection terms from the Dirichlet to Neumann map.

1. Introduction

Suppose that Ω is a simply connected domain in $\mathcal{R}^2$ with the Lipschitz boundary $\partial\Omega$. Let $W^{1,p}(\Omega)$, $W^{2,p}(\Omega)$ denote the usual Sobolev spaces for $p > 2$ and $C^\alpha(\partial\Omega)$, $C^{1,\alpha}(\partial\Omega)$ denote the Hölder continuous space on $\partial\Omega$ with $\alpha = \frac{p-2}{p}$ ([17]).

We consider the following Dirichlet problem for the time independent electrical potential $u = u(x_1, x_2)$:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = \sum_{j,k=1}^2 \frac{\partial}{\partial x_j}\left(\sigma^{jk}\frac{\partial u}{\partial x_k}\right) = 0 \qquad in \quad \Omega \\ u = f \qquad on \quad \partial\Omega. \end{cases} \tag{1.1}$$

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where $f(x) \in C^{1,\alpha}(\partial\Omega)$, $\sigma(x) = (\sigma^{jk}(x)) \in W^{1,p}(\Omega)$ ($p > 2$) is a positive-definite symmetric matrix and there exists a constant $c > 0$ such that

$$\psi \cdot (\sigma\psi) \geq c\,|\psi|^2 \qquad for \quad \psi = (\psi_1, \psi_2) \in \mathcal{R}^2. \tag{1.2}$$

By the theory of the generalized analytic functions and complex partial differential equations ([17], [18]), we know that there exists a unique solution $u \in W^{2,p}(\Omega)$. Therefore we can define the Dirichlet to Neumann map Λ_σ by

$$\begin{aligned} \Lambda_\sigma : C^{1,\alpha}(\partial\Omega) \quad &\longrightarrow \quad C^\alpha(\partial\Omega) \\ f \quad &\longrightarrow \quad \sum_{j,k=1}^{2} \nu_j \sigma^{jk} \frac{\partial u}{\partial x_k} \end{aligned} \tag{1.3}$$

where $\nu = (\nu_1, \nu_2)$ is the outer unit normal to $\partial\Omega$.

The inverse problem we discuss in this paper is determination of the conductivity matrix σ from the Dirichlet to Neumann map.

It is well known that the isotropic conductivity can be determined by the Dirichlet to Neumann map when dimension is greater than 2 (e.g. [8], [15]). Moreover we can refer to Isakov [7]. However, an anisotropic conductivity can not be uniquely determined by the Dirichlet to Neumann map since a diffeomorphism fixing points on the boundary will not change the Dirichlet to Neumann map. There are papers establishing the uniqueness modulo diffeomorphism in determining anisotropic conductivity: [11] in the two dimensional case and [9], [10], [14], [16]. Since this uniqueness contains a diffeomorphism which is fixed on the boundary, there are still unknown factors which can not be determined uniquely by the Dirichlet to Neumann map. For example, such uniqueness modulo diffeomorphism does not give answers when we want to determine a scalar factor function provided that anisotropic conductivity is known up to such an unknown factor. More precisely, by [11], [14], we know that there exists a diffeomorphism Ψ such that $\sigma_1 = \Psi * \sigma_2$. However, to authors' knowledge, it is difficult to obtain the global uniqueness result from this equality. Because we can only obtain a nonlinear equation whose solution seems difficult.

The uniqueness in determining such a factor is called conformal uniqueness. In [10] and [14], some conformal uniqueness results are proved under some analytic or smallness assumptions for anisotropic conductivity. In this paper, we will prove the global conformal uniqueness in two dimensions for the conductivity in $W^{1,p}(\Omega)(p > 2)$. Our class of the conductivity is more general than in [10], [14]. The key of our proof is the global uniqueness for the inverse problem of determining the convection terms from the Dirichlet to Neumann map which is proved in [3], [4] by the inverse scattering method for first order elliptic systems.

This paper is organized as:

- Section 2: Main result
- Section 3: Proof of the main results
- Section 4: Conclusion and remarks

2. Main results

Suppose that $p > 2$ and $\sigma_0 \in W^{1,p}(\Omega)$ is a positive symmetric matrix and satisfies

$$\psi \cdot (\sigma_0(x)\psi) \geq c_0|\psi|^2 \qquad for \quad \psi = (\psi_1, \psi_2) \in \mathcal{R}^2,\, x \in \Omega \tag{2.1}$$

for some constant $c_0 > 0$.

Let R_0 be a positive constant such that

$$\Omega \subset B_{\frac{R_0}{2}}(0) \equiv \left\{ (x_1, x_2) \,|\, x_1^2 + x_2^2 < \frac{R_0^2}{4} \right\}.$$

Without loss of generality, we assume that σ_0 can extended to $B_{R_0}(0)$ such that $\sigma_0 \in W^{1,p}(B_{R_0}(0))$ and (2.1) still holds for $x \in B_{R_0}(0)$(e.g Chapter VI, §3 in [13]). Throughout this paper, we fix σ_0.

For the conductivity equation

$$\nabla \cdot (\sigma \nabla u) = 0, \tag{2.2}$$

we look for the conductivity σ in the admissible set

$$\mathcal{F} = \left\{ \sigma \mid \sigma = \beta\sigma_0,\, \beta \in W^{1,p}(\Omega),\, \beta > c_1 \right\} \tag{2.3}$$

where $c_1 > 0$ is a fixed constant.

Now we can state our main result:

Theorem 2.1. *Suppose that $\sigma_j \in \mathcal{F}$, $j = 1, 2$. If the Dirichlet to Neumann maps Λ_{σ_j}, $j = 1, 2$, satisfy*

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2}, \tag{2.4}$$

then we have

$$\sigma_1(x) = \sigma_2(x), \qquad x \in \Omega. \tag{2.5}$$

Remark 2.2. If we take the 2×2 identity matrix as σ_0, then the result in Theorem 2.1 coincides with the global uniqueness proved in [2], [11].

3. Proof of the main result

3.1. **Some Lemmas.** For the proof of the main result, we need the following global uniqueness results for determining the convection coefficients from the Dirichlet to Neumann map.

Let us recall that $p > 2$ and $\alpha = \frac{p-2}{p}$. We set $b(x) = (b_1(x), b_2(x)) \in L^p(\Omega) \times L^p(\Omega)$. The Dirichlet to Neumann map can be defined as

$$\begin{aligned} \widetilde{\Lambda}_b : C^{1,\alpha}(\partial\Omega) &\longrightarrow C^{\alpha}(\partial\Omega) \\ f &\longrightarrow \frac{\partial v}{\partial \nu}|_{\partial\Omega} \end{aligned} \tag{3.1}$$

where $v \in W^{2,p}(\Omega)$ is the solution of the following Dirichlet problem:

$$\begin{cases} \Delta v + b \cdot \nabla v = 0 \qquad in \quad \Omega \\ v = f \qquad on \quad \partial\Omega. \end{cases}$$

Lemma 3.1. *Suppose that $b^j \in L^p(\Omega) \times L^p(\Omega)$, $j = 1, 2$. If $\widetilde{\Lambda}_{b^1} = \widetilde{\Lambda}_{b^2}$, then we have*

$$b^1(x) = b^2(x), \qquad x \in \Omega.$$

The proof of this lemma is based on the theory of generalized analytic function and the inverse scattering method for the first order elliptic systems. The readers can find the proof in [3], [4].

Henceforth we identify $x = (x_1, x_2)$ with $z = x_1 + ix_2 \in \mathcal{C}$. Moreover we set $\partial_{\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$ and $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$.

Next we state a result about the quasi-confomal mapping which we will use later.

Lemma 3.2. *Suppose that $q \in C^{\alpha}(\mathcal{R}^2) \cap L^{p'}(\mathcal{R}^2)$ with $0 < p' < 2$ and satisfies*

$$\|q\|_{L^\infty} \le q_0 < 1 \tag{3.2}$$

where $q_0 > 0$ is a constant.

Then there is a homeomorphism solution $\zeta = \zeta(z)$ *of the following Beltrami system*

$$\partial_{\overline{z}}\zeta - q\partial_z\zeta = 0, \tag{3.3}$$

which satisfies $\zeta - z \in C^{1,\alpha}(\mathcal{R}^2)$.

Moreover, if $q \in W^{1,p}(\Omega)$, *then* $\zeta \in W^{2,p}(\Omega)$.

This lemma can be found in Chapter II, §5 in [17].

3.2. **Uniqueness of the boundary value of β.** There are several ways for proving the uniqueness of the boundary value of β ([1], [6], [8], [12]). Here we follow the approach by singular solutions in [1].

Lemma 3.3. *Suppose that* $\sigma_j = \beta_j\sigma_0 \in \mathcal{F}$, $j = 1, 2$. *If the Dirichlet to Neumann maps* Λ_{σ_j}, $j = 1, 2$, *satisfy*

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2},$$

then we have

$$\beta_1(x) = \beta_2(x), \qquad x \in \partial\Omega.$$

Proof. Without of loss generality, we extend β_j from Ω to $B_{R_0}(0)$ such that $\beta_j \in W^{1,p}(B_{R_0}(0))$. This is possible since Ω is a simply connected domain with Lipschitz boundary $\partial\Omega$(e.g. [13]).

Let $u_j \in W^{2,p}(\Omega)$, $j = 1, 2$ be the solution of

$$\begin{cases} \nabla \cdot (\sigma_j \nabla u_j) = 0 & in \quad \Omega \\ u_j = \phi_j & on \quad \partial\Omega. \end{cases}$$

Then we have

$$\begin{aligned}\int_\Omega \nabla u_1\cdot(\sigma_2\nabla u_2)dx &= \int_{\partial\Omega}\phi_1\Lambda_{\sigma_2}\phi_2 ds\\ \int_\Omega \nabla u_2\cdot(\sigma_1\nabla u_1)dx &= \int_{\partial\Omega}\phi_2\Lambda_{\sigma_1}\phi_1 ds.\end{aligned}$$

Since σ_j, $j=1,2$, are the symmetric matrices and Λ_{σ_j}, $j=1,2$, are self-adjoint, by $\Lambda_{\sigma_1}=\Lambda_{\sigma_2}$, we obtain that

$$\int_\Omega \nabla u_1\cdot\left((\sigma_2-\sigma_1)\nabla u_2\right)dx=0. \tag{3.4}$$

Assume that $\beta_1\neq\beta_2$. Then there exists a point $x^*\in\partial\Omega$ such that

$$\delta=|\beta_1(x^*)-\beta_2(x^*)|\neq 0.$$

Without loss of generality, we assume that $\delta=\beta_1(x^*)-\beta_2(x^*)$. Since $\beta_j\in W^{1,p}(\Omega)\hookrightarrow C^\alpha(\overline{\Omega})$, there exists $\varepsilon>0$ such that

$$\beta_1(x)-\beta_2(x)>\frac{\delta}{2},\qquad x\in B_\varepsilon(x^*)\cap\overline{\Omega} \tag{3.5}$$

where $B_\varepsilon(x^*)=\{x\in\mathcal{R}^2\,|\,|x-x^*|<\varepsilon\}$.

We define a vector $\widetilde{\nu}$ at x^*, which is non-tangential to $\partial\Omega$, such that $x^\tau\equiv x^*+\tau\widetilde{\nu}\in B_{R_0}(0)\setminus\overline{\Omega}$ for $\tau\in(0,\tau_0)$. Here $\tau_0>0$ is a constant.

It is easy to verify that there exists a constant $c_2>0$ such that

$$\frac{\tau}{c_2}\leq|x^*-x^\tau|\leq c_2\tau. \tag{3.6}$$

By the result in [1] (Theorem 1.1), there exists $\phi_j\in C^\alpha(\partial\Omega)$ such that the solutions u_j, $j=1,2$ of the problem (1.1) corresponding to $\sigma_j=\beta_j\sigma_0$ can be expressed as

$$u_j(x)=\ln|J(x-x^\tau)|+w_j(x),\qquad x\in B_{R_0}(0)\setminus\{x^\tau\} \tag{3.7}$$

and

$$|\nabla w_j(x)| \le C|x - x^\tau|^{\alpha-1}, \qquad x \in B_{R_0}(0) \setminus \{x^\tau\} \tag{3.8}$$

where $J = (\sigma_0(x^\tau))^{-\frac{1}{2}}$ is a symmetric matrix and $C > 0$ is independent of x^τ.

Let $0 < c_2\,\tau < \frac{\varepsilon}{2}$ and $\mathcal{U} = B_\varepsilon(x^*) \cap \Omega$.

By (2.1), it can be verified directly that there exists a constant $c_3 > 0$ such that

$$\int_{\mathcal{U}} \nabla \ln |J(x - x^\tau)| \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla \ln |J(x - x^\tau)|)\, dx \ge \int_{\mathcal{U}} \frac{c_3 dx}{|x - x^\tau|^2}$$

where $0 < \tau < \tau_0$ and c_3 is independent of x^τ.

Then, by (3.5) and (3.7), we have

$$\begin{aligned} c_3 \int_{\mathcal{U}} \frac{dx}{|x - x^\tau|^2} \quad &\le \quad \int_{\mathcal{U}} \nabla \ln |J(x - x^\tau)| \cdot (\beta_1 - \beta_2)\sigma_0 \nabla \ln |J(x - x^\tau)| dx \\ &\le \quad \left| \int_{\mathcal{U}} \nabla u_1 \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla u_2)\, dx \right| \\ &\quad + \left| \int_{\mathcal{U}} \nabla w_1 \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla u_2)\, dx \right| \\ &\quad + \left| \int_{\mathcal{U}} \nabla u_1 \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla w_2)\, dx \right| \\ &\quad + \left| \int_{\mathcal{U}} \nabla w_1 \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla w_2)\, dx \right|. \end{aligned}$$

By (3.4), we have

$$\int_{\mathcal{U}} \nabla u_1 \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla u_2)\, dx = -\int_{\Omega \setminus \mathcal{U}} \nabla u_1 \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla u_2)\, dx.$$

Noting that $|x - x^*| > \frac{\varepsilon}{2}$ for $x \in \Omega \setminus \mathcal{U}$, we can obtain the estimate

$$\left| \int_{\mathcal{U}} \nabla u_1 \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla u_2)\, dx \right| \le C_0, \tag{3.9}$$

where $C_0 > 0$ is a constant which depends on ε, σ_0, β_j, $j = 1, 2$ and $\partial\Omega$, but is independent of τ.

By (3.8), we have that

$$\left|\int_{\mathcal{U}} \nabla w_1 \cdot ((\beta_1-\beta_2)\sigma_0 \nabla u_2)\, dx\right| \leq C_1' \int_{|x-x^*|<\varepsilon} |x-x^\tau|^{\alpha-2} dx \leq C_1, \tag{3.10}$$

where $C_1' > 0$ and $C_1 > 0$ are constants which depend on ε, σ_0, β_j, $j=1,2$ and $\partial\Omega$, but is independent of τ.

By a similar argument, we have

$$\left|\int_{\mathcal{U}} \nabla u_1 \cdot ((\beta_1-\beta_2)\sigma_0 \nabla w_2)\, dx\right| \leq C_2 \tag{3.11}$$

and

$$\left|\int_{\mathcal{U}} \nabla w_1 \cdot ((\beta_1-\beta_2)\sigma_0 \nabla w_2)\, dx\right| \leq C_3, \tag{3.12}$$

where $C_2 > 0$, $C_3 > 0$ are constants which depend on ε, σ_0, β_j, $j=1,2$ and $\partial\Omega$, but are independent of τ.

Combining (3.9) – (3.12), we can obtain that

$$\int_{\mathcal{U}} \frac{dx}{|x-x^\tau|^2} \leq \frac{1}{c_4}(C_0+C_1+C_2+C_3). \tag{3.13}$$

It is easy to verify that

$$\int_{\mathcal{U}} \frac{dx}{|x-x^\tau|^2} \longrightarrow \infty \qquad as \quad \tau \to 0^+.$$

This is a contradiction to (3.13) since the right hand of (3.13) is independent of τ. Therefore we have $\beta_1(x^*) = \beta_2(x^*)$. The proof is complete. □

3.3. **Transform the differential equation to the canonical form.** For applying Lemma 3.1, we will transform the elliptic equation

$$\nabla \cdot (\beta \sigma_0 \nabla u) = 0 \tag{3.14}$$

to an elliptic equation whose principal part is the Laplace operator.

We set

$$q(z) = \frac{\sigma_0^{11} - \sqrt{H} + i\sigma_0^{12}}{i\sigma_0^{12} - \sigma_0^{11} - \sqrt{H}} \tag{3.15}$$

where $H(z) = \sigma_0^{11}(z)\sigma_0^{22}(z) - \sigma_0^{12}(z)\sigma_0^{21}(z)$ and $z = x_1 + ix_2 \in \Omega$.

By (2.1), it is easy to verify that there exists a constant $0 < q_0 < 1$ such that

$$|q| \leq q_0 < 1. \tag{3.16}$$

Since $W^{1,p}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$, we can extend q to $\widetilde{q}$ in the whole complex plane such that

$$\widetilde{q}(z) = q(z), \qquad z \in \Omega$$

and

$$|\widetilde{q}(z)| \leq q_0 < 1, \qquad \widetilde{q} \in C_0^\alpha(\mathcal{R}^2),$$

where $C_0^\alpha(\mathcal{R}^2)$ denotes the space of Hölder continuous functions with compact supports in $\mathcal{R}^2$.

Now we consider the homeomorphism solution $\zeta = \zeta(z)$ of the Beltrami equation:

$$\partial_{\bar{z}}\zeta - \widetilde{q}\partial_z \zeta = 0. \tag{3.17}$$

By Lemma 3.2, we know that there exists a unique homeomorphism solution $\zeta(z)$ such that $\zeta(z) - z \in C^{1,\alpha}(\mathcal{R}^2)$ and $\zeta \in W^{2,p}(\Omega)$ since $q \in W^{1,p}(\Omega)$.

We denote $\widetilde{\Omega} = \zeta(\Omega)$. Then $\widetilde{\Omega}$ is a simply bounded domain in the ζ-plane with the Lipschitz boundary boundary $\partial\widetilde{\Omega}$.

Let $\zeta = \xi_1 + i\xi_2$. Since ζ is a homeomorphism, we can consider the following coordinate transform:

$$\begin{cases} \xi_1 = \xi_1(x_1, x_2) \\ \\ \xi_2 = \xi_2(x_1, x_2). \end{cases}$$

By direct calculations and $\sigma_j = \beta_j \sigma_0$, $j = 1, 2$, we see that the elliptic equation (2.2) with $\sigma = \beta_j \sigma_0$ can be transformed to

$$\Delta_\xi v_j + b^j \cdot \nabla_\xi v_j = 0 \qquad in \quad \widetilde{\Omega}, \tag{3.18}$$

where we set $v_j(\xi_1(x_1, x_2), \xi_2(x_1, x_2)) = u_j(x_1, x_2)$ and $J = \frac{\partial \xi_1}{\partial x_1}\frac{\partial \xi_2}{\partial x_2} - \frac{\partial \xi_1}{\partial x_2}\frac{\partial \xi_2}{\partial x_1} \neq 0$, $b^j = (b_1^j, b_2^j)$, $j = 1, 2$. Here

$$\begin{aligned} b_1^j &= \frac{J}{4\sqrt{H}} \left(\nabla_x \cdot (\beta_j \sigma_0 \nabla_x \xi_1) \right) \\ b_2^j &= \frac{J}{4\sqrt{H}} \left(\nabla_x \cdot (\beta_j \sigma_0 \nabla_x \xi_2) \right). \end{aligned}$$

For the details of this transform, we can refer to [17], Chapter II, §7.

Now we can complete the proof of Theorem 2.1:

Proof of Theorem 2.1:

First we note that the coordinate transform $\xi_j = \xi_j(x_1, x_2)$, $j = 1, 2$, is independent of β_k, $k = 1, 2$. For $j = 1, 2$, we consider the following Dirichlet problem

$$\begin{cases} \Delta_\xi v_j + b^j \cdot \nabla_\xi v_j = 0 \qquad in \quad \widetilde{\Omega} \\ \\ v_j = f \qquad on \quad \partial\widetilde{\Omega}. \end{cases} \tag{3.19}$$

Then we have that $u_j(x_1,x_2) = v_j(\xi_1(x_1,x_2),\xi_2(x_1,x_2))$ is the solution of the following problem

$$\begin{cases} \nabla\cdot(\beta_j\sigma_0\nabla u_j) = 0 \qquad in \quad \Omega \\ u_j = \phi \qquad on \quad \partial\Omega \end{cases}$$

where $\phi(x_1,x_2) = f(\xi_1(x_1,x_2),\xi_2(x_1,x_2))$.

Since $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, by Lemma 3.3, we have

$$\sum_{j,k=1}^{2} \nu_j\sigma_1^{jk}\frac{\partial}{\partial x_k}(u_1 - u_2) = 0 \qquad on \quad \partial\Omega,$$

with which we combine $u_1 = u_2$ on $\partial\Omega$ to obtain

$$\nabla_x u_1 = \nabla_x u_2 \qquad on \quad \partial\Omega.$$

Therefore we can obtain that

$$\frac{\partial v_1}{\partial\widetilde{\nu}_\xi} = \frac{\partial v_2}{\partial\widetilde{\nu}_\xi} \qquad on \quad \partial\widetilde{\Omega} \tag{3.20}$$

where $\widetilde{\nu}_\xi$ is the outer unit normal to $\partial\widetilde{\Omega}$.

The equality (3.20) means that the Dirichlet to Neumann map $\widetilde{\Lambda}_{b^1}$ and $\widetilde{\Lambda}_{b^2}$, which are defined by (3.1) with $b = b^j$, $j = 1,2$, satisfy

$$\widetilde{\Lambda}_{b^1} = \widetilde{\Lambda}_{b^2}.$$

By Lemma 3.1, we can conclude that

$$\begin{aligned} \frac{J}{4\sqrt{H}}\left(\nabla_x\cdot(\beta_1\sigma_0\nabla_x\xi_1)\right) &= \frac{J}{4\sqrt{H}}\left(\nabla_x\cdot(\beta_2\sigma_0\nabla_x\xi_1)\right) \qquad in \quad \widetilde{\Omega} \\ \frac{J}{4\sqrt{H}}\left(\nabla_x\cdot(\beta_1\sigma_0\nabla_x\xi_2)\right) &= \frac{J}{4\sqrt{H}}\left(\nabla_x\cdot(\beta_2\sigma_0\nabla_x\xi_2)\right) \qquad in \quad \widetilde{\Omega}. \end{aligned}$$

Hence

$$\nabla_x \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla_x \xi_1) = 0 \quad in \quad \Omega \tag{3.21}$$

$$\nabla_x \cdot ((\beta_1 - \beta_2)\sigma_0 \nabla_x \xi_2) = 0 \quad in \quad \Omega. \tag{3.22}$$

We set $\delta\beta = \beta_1 - \beta_2$. Then we can rewrite (3.21) and (3.22) as

$$\nabla(\delta\beta) \cdot (\sigma_0 \nabla \xi_1) + \delta\beta \nabla \cdot (\sigma_0 \nabla \xi_1) = 0 \quad in \quad \Omega \tag{3.23}$$

$$\nabla(\delta\beta) \cdot (\sigma_0 \nabla \xi_2) + \delta\beta \nabla \cdot (\sigma_0 \nabla \xi_2) = 0 \quad in \quad \Omega. \tag{3.24}$$

Since σ_0 is a symmetric positive matrix and $\nabla\xi_1$, $\nabla\xi_2$ are linearly independent, from (3.23) and (3.24), we can have

$$\nabla(\delta\beta) + \delta\beta D = 0, \qquad in \quad \Omega \tag{3.25}$$

where $D = D(x)$ is a vector which depends on ξ_j and σ_0.

By Lemma 3.3, we have

$$(\delta\beta)(x) = 0, \qquad for \quad x \in \partial\Omega.$$

Therefore $\delta\beta$ satisfies the first order partial differential equation (3.25) with zero boundary condition. By the uniqueness of the boundary value problem for (3.25), we obtain that

$$\delta\beta(x) = \beta_1(x) - \beta_2(x) = 0, \qquad x \in \Omega. \tag{3.26}$$

The proof is complete.

4. Conclusion and remarks

We discuss the global uniqueness for the inverse problem of determining the anisotropic conductivity from the Dirichlet to Neumann map. The key of our proof is the global uniqueness for the inverse problem of determining the convection term in an elliptic partial differential equation by the Dirichlet to Neumann map. The quasi-conformal map is also used to transform the conductivity equation to an elliptic equation whose principal part is the Laplace operator. We do not need the smallness assumption or analytic assumption on the conductivity.

It is well known that the anisotropic conductivity can not be uniquely determined by the Dirichlet to Neumann map. For establishing the uniqueness, we have to restrict the class in which we want to find the conductivity. This is the reason why we discuss this conformal uniqueness.

Next we give several remarks about our results and ways.

Remark 4.1. It is also possible to define the Dirichlet to Neumann Λ_σ from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$ ([12]) and to discuss the conformal uniqueness.

Remark 4.2. By the techniques in [3], [4], it is possible to prove the conditional stability for the inverse problem which we discuss in this paper. From [5], we know that this kind conditional stability is very important for guaranteeing stable numerical reconstruction algorithm based on the Tikhonov regularization.

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