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Homotopy Algebra Morphism and Geometry of Classical String Field Theories

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Abstract

We discuss general properties of classical string field theories with symmetric vertices in the context of deformation theory. For a given conformal background there are many string field theories corresponding to different decomposition of moduli space of Riemann surfaces. It is shown that any classical open string field theories on a fixed conformal background are A_∞-quasi-isomorphic to each other. This indicates that they have isomorphic moduli space of classical solutions. The minimal model theorem in A_∞-algebras plays a key role in these results. Its natural and geometric realization on formal supermanifolds is also given. The same results hold for classical closed string field theories, whose algebraic structure is governed by L_∞-algebras.

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Contents

1 Introduction and Summary

The present paper is motivated to make clear the complicated structures of string field theories (SFTs) in terms of homotopy algebras. We assume the existence of well-defined SFTs on a conformal background, and discuss the general properties which they should possess. We show that any classical open SFTs, which are constructed so as to reproduce the open string correlation functions on-shell, are quasi-isomorphic to each other. Moreover, when such SFT actions are given, A_∞-quasi-isomorphisms between them are constructed. These results guarantee that there is one-to-one correspondence of the equations of motions corresponding to marginal deformation between such family of SFTs on the same conformal background. This gives an answer about the issue for the relation between SFTs with different decomposition of moduli spaces. These arguments also give the minimal model theorem in deformation theory a geometric insight. These arguments are applicable for classical closed SFT similarly, and it can be shown that all such consistent classical closed SFTs are L_∞-quasi-isomorphic to each other.

SFT has been investigated as a candidate for string theory which describes nonperturbative effects. SFT gives one of the way which extends on-shell two dimensional string theory to off-shell theories. Many Lorentz covariant SFTs have been constructed. The covariant open or closed SFT with light cone type vertices (HIKKO's SFT)[1], a very simple open SFT which consists of only a three-point vertex (Witten's open SFT or cubic SFT)[2], and so on. Witten's SFT can be treated in the context of Batalin-Vilkovisky (BV)-formalism[3]. HIKKO's closed SFT is also extended to quantum SFT by employing the quantum BV-master equation[4]. The quantum master equation is moreover applied to construct quantum closed SFT with symmetric vertices[5]. Though this theory has infinite sort of vertices of higher punctures and higher genus, it has a very beautiful algebraic structure. For instance for the classical part, the set of the tree vertices has the structure of a L_∞-algebra. Open-closed SFT is also considered in this direction [6]. The open-closed symmetric vertices have relations from quantum BV-master equation, where 'symmetric' means cyclic for open string punctures and commutative for closed string punctures. Several subalgebras of subsets of the vertices can be considered : disk (tree) vertices with punctures only on the boundary, which has the structure of an A_∞-algebra, sphere vertices with punctures (an L_∞-algebra), disk vertices with both open and closed vertex insertions (though the algebraic structure of which does not have a particular name), all vertices with no boundaries (the theory of which is the above quantum closed SFT[5]), and so on. Recently a classical open SFT, which possesses the A_∞-structure, is constructed explicitly[7] by deforming the cubic SFT[2].

All the above SFTs satisfy the (classical or quantum) BV-master equation. In constructing a SFT action, any types of vertices, which are written by the powers of string fields and their coefficients, are considered and the master equation are used in order to decide the coefficients [1]. Moreover, in the two-dimensional world sheet picture, the fact that the SFT action satisfies the BV-master equation corresponds to that the moduli spaces of Riemann surfaces are single-covered[13]. Hereafter in this paper, we treat only (bosonic) SFTs with the symmetric vertices and those algebraic structures are discussed.

There have been mainly two issues for the SFTs constructed as above : i) the realization about the relation between SFTs constructed by different decomposition of moduli space of Riemann surfaces on a fixed conformal background ; and ii) the background independence [14, 15, 16]. In order to assert that the SFT gives an nonperturbative definition of string theory the second issue is necessary. For the first issue, it might be believed that the SFTs derived with different decomposition of the moduli space are physically equivalent in some sense. Indeed in [17] for quantum closed SFT it is shown that any infinitesimal variation of the decomposition leads infinitesimal field redefinition preserving the SFT actions and the BV-symplectic struc-

[1]The use of BV-formalism is different from the original use of BV-formalism for gauge theories (see subsection 4.2), but the similar treatment is done for instance for topological theories in superfield formalisms [8, 9, 10, 11, 12].

tures. Analogous consequence is expected for open SFTs. In fact in [7] an one parameter family of the classical open SFT is discussed and the infinitesimal field redefinition preserving the action is found. However the relation between SFTs which differ finitely in the decomposition of the moduli space has not ever been discussed.

Roughly speaking, these issues are equivalent to that of looking for the map between SFTs preserving the BV-symplectic structures. Restricting the arguments to the classical theory, as were mentioned above, the classical closed SFT satisfying the classical BV-master equation has the structure of a L_∞-algebra, and similarly the classical open SFT has the structure of an A_∞-algebra. As will be explained in section 4, the algebraic structure of classical open (resp. closed) SFT is an A_∞-algebra (resp. L_∞-algebra) which possesses the graded cyclic (resp. commutative) symmetry through the BV-symplectic structure. A_∞-algebra has appeared for the first time in [18]. It is a deformation of an associative graded algebra with differential (DGA), and consists of a differential Q, a product $\bullet$, and higher products. L_∞-algebra is its graded commutative-symmetrized version. It is given as a deformation of a differential graded Lie algebra (DGLA), and consists of a differential Q, a Lie bracket $[\ ,\ ]$, and higher brackets[19]. In SFT, Q is the BRST-operator [20], the product $\bullet$ (resp. the Lie bracket $[\ ,\ ]$)corresponds to the trivalent vertex of classical open SFT (resp. classical closed SFT), and higher products (resp. higher brackets) correspond to higher vertices of classical open (resp. closed) SFT.

In SFT or others, one of the important advantage to finding out the A_∞ or L_∞-structures is presumably that they have A_∞ or L_∞-morphisms which transform an A_∞ or L_∞-algebra to another one. For example, the existence of the deformation quantization[21] on general Poisson manifolds is proved as a consequence of constructing a L_∞-morphism between certain two L_∞-algebras [2] [22]. Here when a Poisson structure on a manifold M is given, the deformation quantization on M means that the associative product is constructed as the power expansion of a deformation parameter $\hbar$, the leading term of which is the usual commutative product of functions on M, and next term of which is the Poisson bracket. The constructed L_∞-morphism induces the isomorphism between the cohomologies of these two algebras with respect to the differentials Q. Such morphism is called a *quasi-isomorphism*. The fact that the above two algebras are quasi-isomorphic to each other is called *formality*[22], which has been conjectured originally in [23]. Moreover when an A_∞ or L_∞-algebra is given, one can define its Maurer-Cartan equation, the solution space of which gives moduli space in the context of deformation theory. In addition, an A_∞ or L_∞-morphism preserves the solution space of the Maurer-Cartan equations. In the above case of the deformation quantization problem[22], the solution space of the Maurer-Cartan equation for one side of the two L_∞-algebras gives the space of the Poisson structures, and that for another side gives the space of the structures of the associative products. Therefore constructing the L_∞-morphism has led the existence of the deformation quantization [3].

[2]DGLAs are L_∞-algebras whose higher brackets are set to be zero. In fact, the two L_∞-algebras considered in [22] are both DGLAs. The reason that these DGLAs have to be treated in the context of L_∞-algebras is that a L_∞-morphism, which is a *nonlinear map* between these two DGLAs which preserves the solutions of the Maurer-Cartan equations, is needed.

[3]In [24], the L_∞-morphism defined in [22] is explicitly derived as a BV-quantization of Poisson-sigma model

For classical SFT, the Maurer-Cartan equation is nothing but the equation of motion, and the A_∞ or L_∞-morphisms, which, by definition, preserve the Maurer-Cartan equation, correspond to the field transformation. Thus the problem of considering the family of SFTs satisfying the classical BV-formalism is translated to that of considering the family of A_∞ or L_∞-algebras (with the BV-symplectic structure), and one can employ the A_∞ or L_∞-morphisms in order to realize the relation between two different SFTs in the family. In such reason, we want to find A_∞ or L_∞-morphisms in various situations in SFT. However unfortunately, in general when any two A_∞ or L_∞-algebras are given, the canonical way of constructing the morphism between them does not exist, though in the deformation quantization problem, the L_∞-morphism is exactly found (from the insight of the two-dimensional topological field theory).

Recently there is a development at this point. In [23, 22] the following fact is mentioned : when an A_∞ (resp. L_∞)-algebra $\mathcal{H}$ with nonvanishing Q is given, by restricting the algebra to the kernel (or the cohomology class) of Q, the vector space $\mathcal{H}^p$ (with $Q=0$) is obtained, and then there exists an A_∞ (resp. L_∞)-structure on $\mathcal{H}^p$ (with $Q=0$) which is quasi-isomorphic to the original A_∞ (L_∞)-algebra $\mathcal{H}$. This is called *minimal model theorem* in [23, 22]. Moreover in [25] the canonical A_∞-structure on $\mathcal{H}^p$ and a canonical A_∞-quasi-isomorphism from $\mathcal{H}^p$ to $\mathcal{H}$ are constructed explicitly using Feynman graphs. The same argument holds for L_∞-algebra. In [25] this is applied to the homological mirror conjecture, which states that the derived category of A_∞-category [26] in A-model and the derived category of coherent sheaves in B-model are equivalent[27]. Recently, for example, the minimal theorem is applied in [28, 29, 30] to transform the topological Chern-Simons SFT action [31] to so-called D-brane superpotential.

However the minimal model theorem is very important not only for topological theories but for usual SFT, where Q is the BRST-operator and $\mathcal{H}^p$ is the physical state space. The fact that an A_∞-structure is constructed on $\mathcal{H}^p$ means that the two-dimensional string theory has the structure of an A_∞-algebra. This statement is essentially already known. In [32] it is described that the tree closed string theory has the structure of the L_∞-algebra (and extended this result to quantum case [33]). The fact holds similar for open string theory and which implies that the A_∞-structure of a classical open SFT on a conformal background is A_∞-quasi-isomorphic to the A_∞-structure of the two-dimensional theory.

Using the above argument, we get the results stated at the beginning. We will explain in the classical *open* SFT case in the following two reason. First, open SFT has the cyclic symmetry of its vertices and closed SFT has the graded commutative symmetry, which includes the cyclic symmetry but is much larger symmetry than it. Therefore essentially we can get the algebraic structure of closed SFT by graded commutative- symmetrizing the cyclic open string vertices. Second, since A_∞-morphisms transform the classical solutions of classical open SFT to those of another classical open SFT, it may be applicable to the problem of tachyon condensation[34] in SFT[35, 36], though directly can not as will be commented in Discussions.

This paper is organized as follows.

on a disk. The condition that the morphism is actually a L_∞-morphism is identified with the Ward identity in BV-formalism. However the relation between the L_∞-algebra and BV-formalism is different from that in SFT. In the former case the BV-bracket corresponds to a Lie bracket in the L_∞-algebra on one side.

In section 2, the definitions and some known facts about A_∞-algebras are summarized.

In section 3, the way of constructing consistent SFT which satisfies classical BV-master equation is reviewed, because the main idea of them is essential for the later arguments. We give only the formal construction of SFT, and its concrete realization is omitted. For more details, see for example [5, 7].

In section 4, the algebraic structures of SFT constructed in section 3 are discussed. In subsection 4.1 it is clarified that the algebraic structure of the SFTs, which has cyclic vertices and satisfies classical BV-master equation, is an A_∞-algebra with cyclic symmetry through the BV-symplectic structure. Moreover, we describe that the gauge transformation for the A_∞-structure is the gauge transformation of BV-formalism[37, 38, 39] in subsection 4.2. Finally in subsection 4.3 the notations and definitions are summarized for later arguments including those already used in section 3 and 4. The result in this section might also be known essentially, but there might not be literatures which gives the similar explanations.

In section 5, the minimal model theorem[25] is introduced and applied to classical open SFTs. In order to understand its meaning, we demonstrate that it arises from the issue of finding the solutions of the Maurer-Cartan equation for an A_∞-algebra. For the original A_∞-algebra, another canonical A_∞-algebra and an A_∞-quasi-isomorphism between them are derived naturally, and they are expressed using some Feynman diagrams. Moreover we give a proof of the minimal model theorem in this direction. Its geometrical realization on formal noncommutative supermanifolds is also given. In subsection 5.2, it is clarified that the Feynman graph defined in the previous subsection is actually the Feynman graph in SFT. In order to see the propagator explicitly, we discuss mainly the case of Siegel gauge and clarify that the usual propagator in SFT can be applied to the procedure in the previous subsection. In subsection 5.3, it is shown that the canonical A_∞-algebra generated graphically gives the correlation functions of open strings (Lem.5.1). Moreover in subsection 5.4 we show that all SFTs on a fixed conformal field theory are quasi-isomorphic to each other(Thm.5.1). This immediately follows from the fact that any SFT constructed consistently as will be explained in section 3 coincides with the open string correlation functions on-shell. The quasi-isomorphism is described in terms of Feynman diagram in SFT, and it gives a finite field transformation on certain subspace. Furthermore, a boundary SFT like action which is isomorphic to the original SFT action is proposed.

Since all the above arguments are very formal, in section 6 we apply them to the classical open SFT explicitly constructed in [7]. In [7] an one parameter family of the classical open SFTs is discussed from the viewpoint of the renormalization group[40], which states that the variation of the fields and the coefficients of the vertices with respect to the renormalization scale cancel each other and the action is invariant. The flow of the fields is then derived, which gives an infinitesimal field redefinition between the SFTs with different renormalization scales. After reviewing a part of the arguments in [7], we show that the infinitesimal field redefinition gives an A_∞-isomorphism on the Siegel gauge. Moreover it is observed that the infinitesimal version of the finite A_∞-quasi-isomorphism discussed in the above section coincides with that given in [7] on the subspace. Finally various viewpoints in this paper is summarized in this explicit model.

In Conclusions and Discussions, the issue of the background independence[41, 42] is rearranged. The tachyonic solution in cubic SFT[35] is also argued from the viewpoint of this paper. The relation to the boundary SFT[43] and the application to other SFTs are also commented.

In Appendix A, the precise meaning of taking the dual of the A_∞-algebras is presented. The correspondence between an A_∞-algebra and its dual means the correspondence between a SFT in the operator language and its field theory representation, and it is used implicitly in the body of this paper in order to simplify some explanations. In Appendix A.1, the dual is defined with an inner product, and its graphical explanation is also described. The dual picture is used in many literatures, but there are less literatures where the explicit relation between them is presented. In Appendix A.2, A_∞-algebras are realized geometrically in the dual picture. It will be seen that those are described in terms of noncommutative formal supermanifolds.

In Appendix B some detail calculations for string vertices are presented.

2 A_∞-algebra

It is known that classical open (resp. closed) SFTs have the structure of A_∞-algebras[6, 44, 7] (resp. L_∞-algebras[5]). Here summarizes some basic facts about A_∞-algebras ((strong) homotopy associative algebras). The facts are applicable for L_∞-algebras ((strong) homotopy Lie algebras).

A_∞-algebras are defined in terms of coalgebras. As will be defined below, for $\mathcal{H}$ a graded vector space, we consider $C(\mathcal{H}) := \oplus_{k\geq 1}\mathcal{H}^{\otimes k}$ as a coalgebra. In the terminology of SFT, $\mathcal{H}$ is the Hilbert space of string states and the degree (grading) of $\mathcal{H}$ is related to the ghost number. Though coalgebras may be unfamiliar, it seems natural to the many body system (of strings), and it is useful to control A_∞- (or L_∞-)algebras formally very simple. Intuitively, or geometrically, the dual picture of coalgebras is suitable to realize them. For $\Phi \in \mathcal{H}$ a string field of degree zero, splitting $|\Phi\rangle$ as $|\Phi\rangle = |\mathbf{e}_i\rangle\phi^i$ where $\{|\mathbf{e}_i\rangle\}$ are the basis of the string state $\mathcal{H}$ and $\{\phi^i\}$ are the corresponding supercoordinate [4] , the 'dual picture' means the picture on the supercoordinates. The supercoordinates are the string fields in SFT. Later we define a 'coproduct' on a coalgebra. The 'coproduct' is natural structure for field theory because, in the dual picture, it is equivalent to that the fields $\{\phi^i\}$ possesses an associative product as an algebra.

2.1 Coalgebra, coderivation, and cohomomorphism

Since A_∞- (and L_∞-) algebras are coalgebras with some additional structures, here introduce it.

Definition 2.1 (coalgebra, coassociativity) Let C be (generally infinite dimensional) graded vector space. When a *coproduct* $\triangle : C \longrightarrow C\otimes C$ is defined on C and it is *coassociative*, *i.e.*

$$(\triangle\otimes\mathbf{1})\triangle = (\mathbf{1}\otimes\triangle)\triangle$$

then C is called a *coalgebra.*

[4]Here 'super' means 'graded', and 'graded' means having degree.

Definition 2.2 (coderivation) A linear operator $\mathfrak{m} : C \to C$ raising the degree of C by one is called *coderivation* when

$$\triangle\mathfrak{m} = (\mathfrak{m} \otimes \mathbf{1})\triangle + (\mathbf{1} \otimes \mathfrak{m})\triangle$$

is satisfied. Here, for $x, y \in C$, the sign is defined $(\mathbf{1} \otimes \mathfrak{m})(x \otimes y) = (-1)^x (x \otimes \mathfrak{m}(y))$ through this operation where the x on (-1) denotes the degree of x.

Definition 2.3 (cohomomorphism) Given two coalgebras C and C', an *cohomomorphism (coalgebra homomorphism)* $\mathcal{F}$ from C to C' is a map of degree zero which satisfies the condition

$$\triangle\mathcal{F} = (\mathcal{F} \otimes \mathcal{F})\triangle \ . \tag{2.1}$$

Remark 2.1 Coassociativity of $\triangle$, the condition of coderivations and cohomomorphisms are equal to that the following diagrams commute

$$\begin{array}{ccc} C & \xrightarrow{\triangle} & C \otimes C \\ {\scriptstyle\triangle}\downarrow & & \downarrow{\scriptstyle\triangle\otimes\mathbf{1}} \\ C \otimes C & \xrightarrow{\mathbf{1}\otimes\triangle} & C \otimes C \otimes C \end{array} \ , \quad \begin{array}{ccc} C & \xrightarrow{\mathfrak{m}} & C \\ {\scriptstyle\triangle}\downarrow & & \downarrow{\scriptstyle\triangle} \\ C \otimes C & \xrightarrow{\mathbf{1}\otimes\mathfrak{m}+\mathfrak{m}\otimes\mathbf{1}} & C \otimes C \end{array} \ , \quad \begin{array}{ccc} C & \xrightarrow{\mathcal{F}} & C' \\ {\scriptstyle\triangle}\downarrow & & \downarrow{\scriptstyle\triangle} \\ C \otimes C & \xrightarrow{\mathcal{F}\otimes\mathcal{F}} & C' \otimes C' \end{array} \ .$$

If the orientation of these map are reversed and the coproduct is replaced by a product, then the coassociativity, the coderivation, and the cohomomorphism take place to associativity, a derivation, and a homomorphism of the corresponding algebra, respectively.

Reversing the orientation of the maps corresponds to taking the dual of the coalgebra. The precise meaning of the dual in the present paper is given in Appendix A.1.

Here, for any graded vector space $\mathcal{H}$, on can consider its tensor coalgebra

$$C(\mathcal{H}) = \oplus_{k\geq 1}\mathcal{H}^{\otimes k}$$

as a coalgebra. Note that $C(\mathcal{H})$ does not contain the summand $\mathcal{H}^{\otimes 0} = \mathbb{C}$. In particular, it does not have a counit.

For this coalgebra, the coassociative coproduct on $C(\mathcal{H})$ is uniquely determined as

$$\triangle(\mathbf{e}_1 \cdots \mathbf{e}_n) = \sum_{k=1}^{n-1} (\mathbf{e}_1 \cdots \mathbf{e}_k) \otimes (\mathbf{e}_{k+1} \cdots \mathbf{e}_n) \ . \tag{2.2}$$

The form of the coderivation corresponding to this coproduct is also given as follows : let $\{m_k\}_{k\geq 1}$ be the set of multilinear maps of degree one

$$\begin{array}{cccc} m_k \ : & \mathcal{H}^{\otimes k} & \longrightarrow & \mathcal{H} \\ & \mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_k & \mapsto & m_k(\mathbf{e}_1 \cdots \mathbf{e}_k) \end{array} \ , \tag{2.3}$$

and define

$$\mathfrak{m}_k(\mathbf{e}_1 \cdots \mathbf{e}_n) = \sum_{p=1}^{n-k} (-1)^{\mathbf{e}_1+\cdots+\mathbf{e}_{p-1}} \mathbf{e}_1 \cdots \mathbf{e}_{p-1} m_k(\mathbf{e}_p \cdots \mathbf{e}_{p+k-1}) \mathbf{e}_{p+k} \cdots \mathbf{e}_n \ , \quad \mathbf{e}_i \in \mathcal{H} \ .$$

Here $\mathbf{e}_1 + \cdots + \mathbf{e}_{p-1}$ on (-1) denotes the degree of $\mathbf{e}_1 \cdots \mathbf{e}_{p-1}$. The sign factor appears when m_k, which has degree one, passes through the $\mathbf{e}_1 \cdots \mathbf{e}_{p-1}$. Then summing up this $\mathfrak{m}_k$ for $k \geq 1$,

$$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + \cdots \ , \tag{2.4}$$

and this $\mathfrak{m}$ is the coderivative. The coderivative on the coalgebra $C(\mathcal{H})$ is always written in this form.

Moreover, the form of a cohomomorphism $\mathcal{F} : C(\mathcal{H}) \to C(\mathcal{H}')$ is determined by a collection of degree zero multilinear maps $f_n : \mathcal{H}^{\otimes n} \to \mathcal{H}'(n \geq 1)$ which are homogeneous of degree zero as the following form

$$\mathcal{F}(\mathbf{e}_1 \cdots \mathbf{e}_n) = \sum_{1 \leq k_1 < k_2 \cdots < k_i = n} f_{k_1}(\mathbf{e}_1 \cdots \mathbf{e}_{k_1}) \otimes f_{k_2 - k_1}(\mathbf{e}_{k_1+1} \cdots \mathbf{e}_{k_2}) \otimes \cdots \otimes f_{n-k_{i-1}}(\mathbf{e}_{k_{i-1}+1} \cdots \mathbf{e}_n) \ , \tag{2.5}$$

where each $f(\cdots)$ belongs to $\mathcal{H}'$.

2.2 A_∞-algebra and A_∞-morphism

Definition 2.4 (A_∞-algebra) Let $\mathcal{H}$ be a graded vector space and $C(\mathcal{H}) = \oplus_{k \geq 1} \mathcal{H}^{\otimes k}$ be its tensor coalgebra. An A_∞-algebra is a coalgebra $C(\mathcal{H})$ with a coderivation $\mathfrak{m}$ which satisfies

$$(\mathfrak{m})^2 = 0 \ .$$

If we act $(\mathfrak{m})^2 = (\mathfrak{m}_1 + \mathfrak{m}_2 + \cdots)^2$ on $\mathbf{e}_1 \cdots \mathbf{e}_n \in C(\mathcal{H})$, its image belongs to $\mathcal{H}^{\otimes 1} \oplus \cdots \oplus \mathcal{H}^{\otimes n}$, and the condition that the $\mathcal{H}$ part of the image equal zero is sufficient to the condition $(\mathfrak{m})^2 = 0$ [5]. The equation becomes

$$\sum_{\substack{k+l=n+1 \\ j=0,\cdots,k-1}} (-1)^{\mathbf{e}_1 + \cdots + \mathbf{e}_j} m_k(\mathbf{e}_1, \cdots, \mathbf{e}_j, m_l(\mathbf{e}_{j+1}, \cdots, \mathbf{e}_{j+l}), \mathbf{e}_{j+l+1}, \cdots, \mathbf{e}_n) = 0 \tag{2.6}$$

for $n \geq 1$, and $\mathbf{e}_i$ on (-1) denotes the degree of $\mathbf{e}_i$.

The first three constraints in eq.(2.6) read:

$$\begin{aligned}
&m_1^2 = 0 \quad , \\
&m_1(m_2(\mathbf{e}_1, \mathbf{e}_2)) + m_2(m_1(\mathbf{e}_1), \mathbf{e}_2) + (-1)^{\mathbf{e}_1} m_2(\mathbf{e}_1, m_1(\mathbf{e}_2)) = 0 \quad , \\
&m_2(\mathbf{e}_1, m_2(\mathbf{e}_2, \mathbf{e}_3)) + m_2(m_2(\mathbf{e}_1, \mathbf{e}_2), \mathbf{e}_3) \\
&\qquad + m_1(m_3(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)) + m_3(m_1(\mathbf{e}_1), \mathbf{e}_2, \mathbf{e}_3) + (-1)^{\mathbf{e}_1} m_3(\mathbf{e}_1, m_1(\mathbf{e}_2), \mathbf{e}_3) \\
&\qquad + (-1)^{\mathbf{e}_1 + \mathbf{e}_2} m_3(\mathbf{e}_1, \mathbf{e}_2, m_1(\mathbf{e}_3)) = 0 \quad .
\end{aligned} \tag{2.7}$$

The first equation indicates m_1 is nilpotent and $(\mathcal{H}, m_1)$ makes a complex. The second equation implies differential m_1 satisfies Leibniz rule for the product m_2. The third equation means product m_2 is associative up to the term including m_3.

[5] in the same reason that the differential d on differential forms or BRST-operator δ on polynomials of fields and ghosts (and antifields) is nilpotent.

Remark 2.2 In the case $m_n = 0$ for $n \geq 3$, an A_∞-algebra reduces to a differential graded (associative) algebra (DGA). The differential d and the product $\bullet$ of DGA correspond to m_1 and m_2, respectively. However, the product $\bullet$ of DGA preserves the degree and m_2 in A_∞-algebras raises the degree by one. In this reason, when a DGA $\mathfrak{g}$ is included in an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$, the degree in the A_∞-algebra is defined as the degree of the DGA minus one. Let $s : \mathfrak{g}^k \to \mathcal{H}^{k-1}[1]$ be the inclusion map. The [1] 'eats' one degree of $\mathcal{H}$, and the degree of $\mathcal{H}^{k-1}[1]$ is defined as $k-1$ through the operation. Then the following diagram commutes

$$\begin{array}{ccc}
\mathfrak{g}^k \otimes \mathfrak{g}^l & \xrightarrow{\quad\bullet\quad} & \mathfrak{g}^{k+l} \\
{\scriptstyle s}\Big\downarrow & & {\scriptstyle s}\Big\downarrow \\
\mathcal{H}^{k-1}[1] \otimes \mathcal{H}^{l-1}[1] & \xrightarrow{m_2(\ ,\ )} & \mathcal{H}^{k+l-1}[1] \ .
\end{array}$$

The degree of Witten's open SFT[2] is usually defined as the degree of DGA explained above. There are many literatures where the degree of A_∞-algebras are defined with the DGA degree. However, when higher products $m_3, m_4, \cdots$ are introduced, the degree given in (Def.2.2) is simpler for A_∞-algebras. In this reason, we use this convention in the present paper. The precise relation between these two conventions can be found in [45].

Remark 2.3 We have mentioned in (Rem.2.1) about the dual picture of coalgebras. In this dual picture, the nilpotent coderivation $\mathfrak{m}$ is replaced to a differential on a formal (noncommutative) supermanifold. Let $\Phi = \mathbf{e}_i \phi^i \in \mathcal{H}$ be an elements of $\mathcal{H}$ with supercoordinates $\{\phi^i\}$. The degree of ϕ^i is minus the degree of $\mathbf{e}_i$ so that the degree of Φ is zero. Define

$$m_k(\mathbf{e}_1, \cdots, \mathbf{e}_k) = \mathbf{e}_j c^j_{1\cdots k} \ , \qquad c^j_{1\cdots k} \in \mathbb{C} \ . \tag{2.8}$$

In this representation, $m_k(\Phi, \cdots, \Phi) = \Phi \frac{\overleftarrow{\partial}}{\partial \phi^j} c^j_{i_1\cdots i_k} \phi^{i_k} \cdots \phi^{i_1}$, and the differential δ in the dual picture is written as

$$\delta = \sum_{k=1}^{\infty} \frac{\overleftarrow{\partial}}{\partial \phi^j} c^j_{i_1\cdots i_k} \phi^{i_k} \cdots \phi^{i_1} \ , \tag{2.9}$$

where the operation of $\frac{\overleftarrow{\partial}}{\partial \phi^j} \cdots$ is defined as $(\phi^3\phi^2\phi^1)\frac{\overleftarrow{\partial}}{\partial \phi^j} c^j_{i_1\cdots i_k} \phi^{i_k} \cdots \phi^{i_1} = \phi^3\phi^2 c^1_{i_1\cdots i_k} \phi^{i_k} \cdots \phi^{i_1} + (-1)^{\mathbf{e}_1} \phi^3 c^2_{i_1\cdots i_k} \phi^{i_k} \cdots \phi^{i_1} \phi^1 + (-1)^{\mathbf{e}_1+\mathbf{e}_2} c^3_{i_1\cdots i_k} \phi^{i_k} \cdots \phi^{i_1} \phi^2 \phi^1$. The sign arises when the δ with degree one passes through some elements which have their degree. The consistency of this operation is explained in Appendix A.1. The condition that δ is differential *i.e.* $(\delta)^2 = 0$ is equal to the A_∞-condition(2.6) rewritten using (2.8). This actually corresponds to the BV-BRST transformation as will be seen in subsection 4.1. Note that in this paper we denote $\{\phi^i\}$ as both fields and antifields in the terminology of the BV-formalism. The geometry on this dual picture is explained in Appendix A.2.

Definition 2.5 (A_∞-morphism) Given two A_∞ algebras $(\mathcal{H}, \mathfrak{m})$ and $(\mathcal{H}', \mathfrak{m}')$, *an A_∞-morphism* $\mathcal{F} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$ as a cohomomorphism from $C(\mathcal{H})$ to $C(\mathcal{H}')$ satisfying

$$\mathcal{F}\mathfrak{m} = \mathfrak{m}'\mathcal{F} \ .$$

If we act this equation on $\mathbf{e}_1\cdots\mathbf{e}_n\in C(\mathcal{H})$ for $n\geq 1$, its image belongs to $\sum_{n'=1}^{n}\mathcal{H}'^{\otimes n'}$, and picking up the $\mathcal{H}'^{\otimes 1}$ part of the equation yields

$$\begin{aligned}\sum_{1\leq k_1<k_2\cdots<k_i=n} & m'_i(f_{k_1}(\mathbf{e}_1,\cdots,\mathbf{e}_{k_1}),f_{k_2-k_1}(\mathbf{e}_{k_1+1},\cdots,\mathbf{e}_{k_2})\cdots f_{n-k_{i-1}}(\mathbf{e}_{k_{i-1}+1},\cdots,\mathbf{e}_n)) \\ &=\sum_{k+l=n+1}\sum_{j=0}^{k-1}(-1)^{\mathbf{e}_1+\cdots+\mathbf{e}_j}f_k(\mathbf{e}_1,\cdots,\mathbf{e}_j,m_l(\mathbf{e}_{j+1},\cdots,\mathbf{e}_{j+l}),\mathbf{e}_{j+l+1},\cdots,\mathbf{e}_n)\ .\end{aligned} \tag{2.10}$$

The first two constraints in (2.10) read:

$$\begin{aligned} m'_1(f_1(\mathbf{e}_1)) &= f_1(m_1(\mathbf{e}_1))\ , \\ m'_2(f_1(\mathbf{e}_1),f_1(\mathbf{e}_2)) &= f_1(m_2(\mathbf{e}_1,\mathbf{e}_2)) \\ &\quad +m'_1(f_2(\mathbf{e}_1,\mathbf{e}_2))+f_2(m_1(\mathbf{e}_1),\mathbf{e}_2)+(-1)^{\mathbf{e}_1}f_2(\mathbf{e}_1,m_1(\mathbf{e}_2))\ .\end{aligned}$$

In particular, the first equation implies f_1 induces a degree zero linear map f_{1*} between the cohomologies $H_{m_1}(\mathcal{H})$ and $H_{m'_1}(\mathcal{H}')$. The dual representation of $\mathcal{F}$ will be mentioned in the next subsection. In SFT this $\mathcal{F}$ corresponds to a field transformation between two different SFTs and the condition of the A_∞-morphism (Def.2.5) indicates that the field transformation $\mathcal{F}$ is compatible with the BV-BRST transformations.

Definition 2.6 (quasi-isomorphism) An A_∞-morphism $\mathcal{F}$ is called a *quasi-isomorphism* if f_1 is a degree zero isomorphism between the cohomology spaces $H_{m_1}(\mathcal{H})$ and $H_{m'_1}(\mathcal{H}')$.

It is known that if $\mathcal{F}$ is quasi-isomorphism, there is a inverse quasi-isomorphism $\mathcal{F}^{-1}:(\mathcal{H}',\mathfrak{m}')\to(\mathcal{H},\mathfrak{m})$ [23, 22], which will be shown in (Rem.5.4).

2.3 Maurer-Cartan equation

Here we define Maurer-Cartan equations for A_∞-algebras. It corresponds to the equation of motions in SFT (eq.(5.1)). Consider formally the following exponential map [6] of $\Phi\in\mathcal{H}$

$$e^\Phi:=\mathbf{1}+\Phi+\Phi\otimes\Phi+\Phi\otimes\Phi\otimes\Phi+\cdots\ . \tag{2.11}$$

$e^\Phi-\mathbf{1}\in C(\mathcal{H})$ satisfies $\triangle(e^\Phi-\mathbf{1})=(e^\Phi-\mathbf{1})\otimes(e^\Phi-\mathbf{1})$ and such element is called an *grouplike element*. If we define

$$\mathfrak{m}_*(e^\Phi):=m_1(\Phi)+m_2(\Phi\otimes\Phi)+m_3(\Phi\otimes\Phi\otimes\Phi)+\cdots\ ,$$

then $\mathfrak{m}(e^\Phi)=e^\Phi\mathfrak{m}_*(e^\Phi)\cdot e^\Phi$, and $\mathfrak{m}(e^\Phi)=0$ is equivalent to $\mathfrak{m}_*(e^\Phi)=0$, where $\mathbf{1}$ is defined as $\mathcal{H}^{\otimes m}\otimes\mathbf{1}\otimes\mathcal{H}^{\otimes n}=\mathcal{H}^{\otimes(m+n)}$ for $m,n\geq 0$ and $m+n\geq 1$. $\mathfrak{m}_*(e^\Phi)=0$ is called Maurer-Cartan equation for A_∞-algebras. When an A_∞-algebra is DGA, *i.e.* $m_3=m_4=\cdots=0$, its Maurer-Cartan equation takes the form $m_1(\Phi)+m_2(\Phi\otimes\Phi)=0$. It is nothing but the condition of a flat connection.

[6]Note that this e^Φ does not belong to $C(\mathcal{H})$ because $\mathbf{1}\in\mathcal{H}^{\otimes 0}$. $e^\Phi-\mathbf{1}$ is then an element of $C(\mathcal{H})$. However as will be seen it is convenient to include $\mathbf{1}$ for some formulation.

Now we explain that any A_∞-morphisms preserve the solution of two Maurer-Cartan equations. The fact means that the A_∞-morphisms preserve the equations of motions for SFTs. Let $(\mathcal{H}, \mathfrak{m})$ and $(\mathcal{H}', \mathfrak{m}')$ be two A_∞-algebras and $\mathcal{F} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$ be a A_∞-morphism. Φ' is constructed from Φ as the pushforward of $\mathcal{F}$

$$\Phi' = \mathcal{F}_*(\Phi) = \sum_{n=1}^{\infty} f_n(\Phi \cdots \Phi) \ . \tag{2.12}$$

Direct calculation using eq.(2.5) then yields that $\mathcal{F}$ satisfies

$$\mathcal{F}(e^{\Phi} - \mathbf{1}) = e^{\Phi'} - \mathbf{1} \ . \tag{2.13}$$

To show that $\mathcal{F}$ preserves the solutions of Maurer-Cartan equations, it is sufficient to say that $\mathfrak{m}'(e^{\Phi'}) = 0$ if $\mathfrak{m}(e^{\Phi}) = 0$, which can be immediately shown because

$$\mathfrak{m}'(e^{\Phi'}) = \mathfrak{m}'\mathcal{F}(e^{\Phi}) = \mathcal{F}\mathfrak{m}(e^{\Phi}) = 0 \ .$$

Remark 2.4 In the dual picture of these coalgebras, for $\Phi = \mathbf{e}_i\phi^i \in \mathcal{H}$ and $\Phi' = \mathbf{e}_{i'}\phi^{i'} \in \mathcal{H}'$, define

$$f_k(\mathbf{e}_1 \cdots \mathbf{e}_k) = \mathbf{e}_{j'} f^{j'}_{1\cdots k} \ , \qquad f^{j'}_{1\cdots k} \in \mathbb{C} \ , \quad \mathbf{e}_{j'} \in \mathcal{H}'$$

and eq.(2.12) can be expressed as

$$\phi^{i'} = f^{i'}_j \phi^j + f^{i'}_{j_1 j_2}\phi^{j_2}\phi^{j_1} + f^{i'}_{j_1 j_2 j_3}\phi^{j_3}\phi^{j_2}\phi^{j_1} + \cdots \ .$$

This can be regarded as a nonlinear coordinate transformation between the two formal noncommutative supermanifolds. This statement is also explained in Appendix A.2. Moreover in this expression one can easily seen that when the f^i_j has its inverse, this transformation is locally diffeomorphism, and the map $\Phi' = \mathcal{F}_*(\Phi)$ has its inverse.

Remark 2.5 There are gauge transformations which preserve the Maurer-Cartan equations. Its infinitesimal representation is of the form

$$\delta_\alpha \Phi = \mathfrak{m}_*(e^{\Phi}\alpha e^{\Phi})$$

where $\alpha = \mathbf{e}_i\alpha^i$ is a gauge parameter of degree minus one, therefore the degree of α^i is minus the degree of $\mathbf{e}_i$ minus one. The fact that this transformation preserves the space of the solution of the Maurer-Cartan equation can be directly checked as

$$\delta_\alpha \mathfrak{m}_*(e^{\Phi}) = \mathfrak{m}_*(e^{\Phi}(\delta_\alpha\Phi)e^{\Phi}) = \mathfrak{m}_*(e^{\Phi}\mathfrak{m}_*(e^{\Phi}\alpha e^{\Phi})e^{\Phi}) = \mathfrak{m}_*(\mathfrak{m}(e^{\Phi}\alpha e^{\Phi})) = 0 \ .$$

In the third equality, the Maurer-Cartan equation $\mathfrak{m}_*(e^{\Phi}) = 0$ is used. Moreover, if any two A_∞-algebras $(\mathcal{H}, \mathfrak{m})$, $(\mathcal{H}', \mathfrak{m}')$ and an A_∞-morphism $\mathcal{F}$ between them are given, then A_∞-morphism restricted to the spaces of the solutions of the Maurer-Cartan equations is equivariant under the gauge transformations on both sides. In other words $\delta_{\alpha'}\Phi' = \mathcal{F}(\delta_\alpha\Phi)$ holds on the Maurer-Cartan equations where α' is defined by $\mathcal{F}$ as $\alpha' = \mathcal{F}_*(e^{\Phi}\alpha e^{\Phi}) := f_1(\alpha) + f_2(\alpha, \Phi) + f_2(\Phi, \alpha) + \cdots$. This obeys from the condition of the A_∞-morphism (Def.2.5)

$$\mathfrak{m}'\mathcal{F}(e^{\Phi}\alpha e^{\Phi})|_{\mathcal{H}'\otimes 1} = \mathcal{F}\mathfrak{m}(e^{\Phi}\alpha e^{\Phi})|_{\mathcal{H}'\otimes 1} \ .$$

The space of Maurer-Cartan equation over this gauge action is considered as the moduli space in the terminology of deformation theory. The above fact then means that the moduli space is transformed by A_∞-morphisms. In particular, if the A_∞-morphism is quasi-isomorphism, the moduli space is isomorphically transformed by the quasi-isomorphism. This gauge transformation exactly corresponds to the gauge transformation in SFT as will be explained in subsection 4.2.

3 Moduli space of Riemann surfaces and BV-formalism

In this section, the relevance of BV-formalism in SFT will be reviewed. When SFTs are controlled by BV-formalism, classical open SFTs[6, 44, 7] have A_∞-structures and classical closed SFTs[13, 5] have L_∞-structures. These algebraic structure of SFT is summarized in subsection 4.1 and 4.2. We attempt to explain those as simple as possible, and in order to do so we will transfer various representations for SFT or A_∞-algebras to each other. The precise definition and convention used there are summarized in subsection 4.3.

When constructing SFT, the sum of the Feynman graphs of the same topology in the sense of Riemann surfaces must reproduce the correlation function of corresponding Riemann surface on-shell. The correlation functions in string theory are calculated by integrating out over the moduli space of Riemann surfaces. Each vertex in SFT is constructed by integrating out over the subspace of the moduli space of Riemann surfaces. In order that the Feynman rule reproduces the correlation function of string theory on-shell, the sum of each Feynman graphs with the same topology fill all the moduli space of Riemann surface without crevices and without double covered. As will be seen below, this condition restricts the way of creating the vertices of SFT, and produces recursion relations for vertices, which is often called the string factorization equations[13] (equation (3.8)).

Here we review them briefly in the case of classical open SFT[6, 7]. The argument is similar for the classical closed SFT[13, 5].

Let $\Phi \in \mathcal{H}$ be a string field. More precisely, $\Phi = \mathbf{e}_i \phi^i$ where $\{\mathbf{e}_i\}$ are the basis of the string Hilbert space $\mathcal{H}$ and $\{\phi^i\}$ are the string fields. The aim is constructing an open SFT action of the following form,

$$S = S_0 + \mathcal{V}\ , \qquad S_0 := \frac{1}{2}\omega(\Phi, Q\Phi), \quad \mathcal{V} = \sum_{k\geq 3} \mathcal{V}_k\ , \tag{3.1}$$

where S_0 is the kinetic term, and $\mathcal{V}_k$ is the k-point vertex (the term of k powers of string fields Φ). $\omega(\ ,\ )$ denotes BPZ-inner product (in CFT). In order for the action S to be consistent for string theory, the n point amplitude which is calculated by using the Feynman rule with the action S must reproduce the corresponding n-point correlation function of string theory when the n external states $\mathbf{e}_i, (n = 1, \cdots, n)$ are physical, *i.e.* $Q|\mathbf{e}_i\rangle = 0$ [7]. Here we concentrate to review for classical open SFT, so the n point amplitude which should be considered is the n

[7]Here we assume for simplicity that the basis $\mathbf{e}_i$ are taken so that the subbasis of $\{\mathbf{e}_i\}$ span the physical Hilbert space.

point tree graph amplitude for open string, and the corresponding correlation function in string theory is the disk amplitude with n external states on the boundary of the disk. (The arguments are same for other case like classical and quantum closed SFT[5], quantum open-closed SFT[6]).

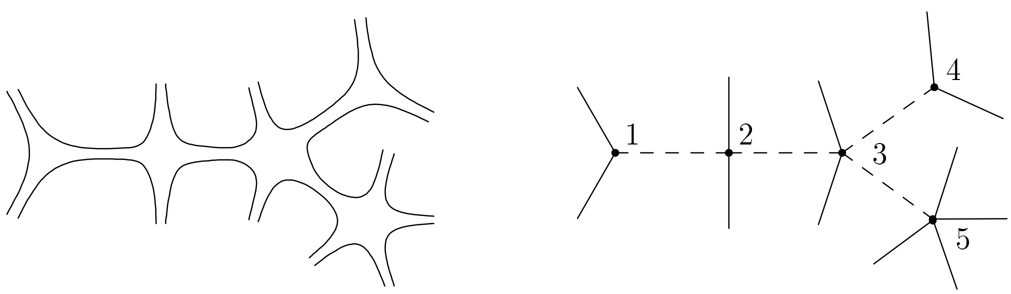
Let $\mathcal{M}_n$ be the compactified moduli space of disk with n punctures. The dimension of $\mathcal{M}_n$ is $\dim\mathcal{M}_n = n-3$. Suppose that the vertices are now constructed, and consider the n point tree amplitude. It consists of the sum of every Feynman graph of n point tree graphs. Here n point tree graphs are the graphs which are produced by connecting the vertices with the propagators and the topology of which are n point tree graph of open strings. As has been explained briefly, in order for the vertices to be made consistently, the sum of this Feynman graphs must reproduce the single-covered moduli space $\mathcal{M}_n$

$$\mathcal{M}_n = \mathcal{M}_n^0 \cup \mathcal{M}_n^1 \cup \mathcal{M}_n^2 \cup \cdots \ , \tag{3.2}$$

where $\mathcal{M}_n^I$ denotes the subspace of the moduli space $\mathcal{M}_n$ which corresponds to the Feynman graphs with I propagators. Now the tree graphs are considered, therefore the number of the vertices of the n point tree Feynman graphs with I propagators is $I+1$. Because

$$n = \sum_{m=1}^{I+1} v_m - 2I \tag{3.3}$$

where $v_m \geq 3$ are the numbers of the external legs of the vertices, $\cdots$ in eq.(3.2) does not continue infinitely.



(a) (b)

Figure 1: Consider for example the diagram of the string interaction as Fig.(a). We represent such diagrams as Fig.(b). The dashed lines denote the propagators. Here the vertices are labeled by $1\cdots5$. The numbers of legs for the vertices are $v_1 = 3$, $v_2 = 4$, $v_3 = 5$, $v_4 = 3$, $v_5 = 5$. The number of the propagators equal $I = 4$. The graph has twelve external legs, and eq.(3.3) holds because $12 = 3+4+5+3+5-2\cdot4$.

Let us consider to construct the vertices inductively and suppose that the vertices $\mathcal{V}_k$ with $3 \leq k \leq n-1$ are constructed in the following form

$$\mathcal{V}_k := \frac{1}{k}\int_{\mathcal{M}_k^0} \langle\Omega|_{1\cdots k}|\Phi\rangle_1 \cdots |\Phi\rangle_k =: \frac{1}{k}\langle V_k||\Phi\rangle\cdots|\Phi\rangle \ . \tag{3.4}$$

Here Ω denotes the volume form on the Moduli space $\mathcal{M}_k$. A point on $\mathcal{M}_k$ characterizes a conformal structure of the disk with k punctures at its boundary. The k boundary insertions are symmetrized cyclic, *i.e.* $\langle\Omega|_{1\cdots k} = \langle\Omega|_{2\cdots k1}$. Thus, when $\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_k}$ are physical states, $\langle\Omega|_{1\cdots k}|\mathbf{e}_{i_1}\rangle_1\cdots|\mathbf{e}_{i_k}\rangle_k$ on a point on $\mathcal{M}_k$ is the correlation function of the k-point disk with corresponding conformal structure, and $\int_{\mathcal{M}_k}\langle\Omega|_{1\cdots k}|\mathbf{e}_{i_1}\rangle_1\cdots|\mathbf{e}_{i_k}\rangle_k$ gives the on-shell S-matrix element. By construction, the dimension which the vertex $\mathcal{V}_k$ has is $\dim\mathcal{M}_k^0 = k-3$ for $k\le n-1$. Besides, $\mathcal{M}_k^I$ is the moduli space reproduced by vertices $\mathcal{V}_{v_1},\cdots,\mathcal{V}_{v_{I+1}}$ and I propagators. The vertex $\mathcal{V}_{v_m}$ has its dimension of moduli v_m-3 and the propagator has the dimension one, which is the parameter of the length of the evolution of the open string $\mathbb{R}_+$ (see Fig.5.(a)). Therefore, the dimension of $\mathcal{M}_k^I$ is $(v_1-3)+\cdots+(v_{I+1}-3)+I = (k+2I-3(I+1))+I = k-3$ because $k+2I=\sum_{m=1}^{I+1}v_m$. Note that the dimension of $\mathcal{M}_k^I$ is independent of the number of the propagators I. Thus, the moduli space is decomposed consistently as eq.(3.2) for $k\le n-1$. Here we want to determine the decomposition of the moduli space $\mathcal{M}_n$ as in eq.(3.2) in order to construct the vertex $\mathcal{V}_n$. $\mathcal{M}_n$ is of course given, because it is determined only from the Riemann surface. Alternatively, for $I\ge 1$ $\mathcal{M}_n^I$ are determined by the induction hypothesis, that is, they are determined by the vertices $\mathcal{V}_k$ with $k\le n-1$ and the propagators. Thus one gets $\mathcal{M}_n^0$ and consequently $\mathcal{V}_n$ of the form in eq.(3.4).

Next, it will be explained that the SFT action $S=S_0+\mathcal{V}$ of which vertices are constructed as above in eq.(3.4) satisfies the classical master equation (3.11). Let us consider the infinitesimal variation of the decomposition of Riemann surfaces. More precisely consider to take the boundary of each $\mathcal{M}_n^I$, and denote the operation as ∂, and write the integral of $\frac{1}{n}\langle\Omega||\Phi\rangle\cdots|\Phi\rangle$ over $\mathcal{M}_n$ as

$$\frac{1}{n}\int_{\mathcal{M}_n}\langle\Omega||\Phi\rangle\cdots|\Phi\rangle = \frac{1}{n}\int_{\mathcal{M}_k^0}\langle\Omega||\Phi\rangle\cdots|\Phi\rangle + \frac{1}{n}\int_{\mathcal{M}_k^1}\langle\Omega||\Phi\rangle\cdots|\Phi\rangle + \cdots\ . \tag{3.5}$$

Taking its boundary yields

$$0=\partial(\mathcal{V}_n)+\sum_{\substack{k_1+k_2=n+2\\ k_1,k_2\ge 3}}\frac{1}{2}\begin{pmatrix}\partial(\mathcal{V}_{k_1})\,\text{—}(\mathcal{V}_{k_2})\\ +(\mathcal{V}_{k_1})\,\text{—}\partial(\mathcal{V}_{k_2})\\ +(\mathcal{V}_{k_1})\partial(\text{—})(\mathcal{V}_{k_2})\end{pmatrix}+\sum_{\substack{k_1+k_2+k_3=n+4\\ k_1,k_2,k_3\ge 3}}(\cdots)+\cdots\ . \tag{3.6}$$

The first equality in the above equation follows from the fact that the left hand side of the equation (3.5) does not depend on the way of the decomposition of $\mathcal{M}_n$, *i.e.* $\partial\mathcal{M}_n=0$. This equation exists for $n\ge 3$, and the constraint for $n=3,4$ read

$$n=3\ :\ 0=\partial\mathcal{V}_3\ ,\qquad n=4\ :\ 0=\partial\mathcal{V}_4+\mathcal{V}_3\partial(\text{—})\mathcal{V}_3\ . \tag{3.7}$$

The first equation ($n=3$) means $\mathcal{M}_3$ has no moduli (a point). As will be clear later, the vertex $\mathcal{V}_{k+1}$ corresponds to the A_∞-structure m_k (eq.(4.2) or eq.(4.16)), and the first equation and the second equation corresponds to the second equation and the third equation in eq.(2.7), respectively. The equation (3.6) is, in fact, equivalent to

$$0=\partial(\mathcal{V}_n)+\sum_{\substack{k_1+k_2=n+2\\ k_1,k_2\ge 3}}\frac{1}{2}(\mathcal{V}_{k_1})\partial(\text{—})(\mathcal{V}_{k_2})\ , \tag{3.8}$$

which is the first term and one of the second term on the right hand side of the identity(3.6). The reason why these are equivalent is that the other parts of eq.(3.6) cancel by induction. For example, $\partial(\mathcal{V}_{k_1})$ —$(\mathcal{V}_{k_2})$ in the second term cancels with one of the third term $(\cdots)$ of the form $\sum_{\substack{k+l=k_1+2\\ k,l\geq 3}}((\mathcal{V}_k)\partial(—)(\mathcal{V}_l))$ —$(\mathcal{V}_{k_2})$. The recursion equation (3.8) is called *the string factorization equation*[13].

Finally we will rewrite the string factorization equation (3.8) in BV-formalism. The $(\mathcal{V}_{k_1})\partial(—)(\mathcal{V}_{k_2})$ means sewing up these vertices with the shortest propagator length, where the corresponding moduli is a subspace of the boundary of $\mathcal{M}_n^1$ which is common with the boundary of $\mathcal{M}_n^0$. This sewing $\partial(—)$ is given by the inverse reflection operator $|\omega\rangle$, and as will be clarified later in subsection 4.3, $|\omega\rangle$ is equivalent to the BV-bracket $\frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j} = (\ ,\ )$. This leads

$$(\mathcal{V}_k)\partial(—)(\mathcal{V}_l) = \mathcal{V}_k\frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j}\mathcal{V}_l = (\mathcal{V}_k,\mathcal{V}_l)\ . \tag{3.9}$$

On the other hands, after some calculation in conformal field theory one obtains[5, 7]

$$2\partial\mathcal{V}_n = 2\int_{\partial\mathcal{M}_n^0}\langle\Omega||\Phi\rangle\cdots|\Phi\rangle = 2(S_0,\mathcal{V}_n)\ . \tag{3.10}$$

Rewriting eq.(3.8) with (3.9) and (3.10) for $n\geq 3$ and summing up these, we obtain the classical BV-master equation

$$(S,S)=0\ . \tag{3.11}$$

The precise definitions for the notation used here are summarized in subsection 4.3. After preparing those and other identities the recursion relation (3.8) is derived again explicitly in Appendix B.1. One explicit example of constructing SFT in this procedure will be given in section 6.

4 A_∞-structure and BV-formalism

Continuing the argument in the previous section, the relation between an A_∞-structure and BV-formalism will be discussed for classical open SFT. In subsection 4.1, it is explained that the SFT constructed above, which satisfies the BV-master equation (3.11), has an A_∞-structure. In subsection 4.2, the BV-gauge transformations for the SFTs are identified with the gauge transformation for the Maurer-Cartan equations for A_∞-algebras. Finally in subsection 4.3, the notation and definitions, including those which are used implicitly in the previous section and this section, are summarized. The relation between the operator language of SFT, A_∞-language, and its dual representation is clarified and their graphical representation is also presented.

4.1 A_∞-structure in SFT

Represent the string state as $|\Phi\rangle = |\mathbf{e}_i\rangle\phi^i$ where $\{\mathbf{e}_i\}$ is the basis of the string Hilbert space $\mathcal{H}$ and ϕ^i are its coordinate whose degree is minus the degree of $\mathbf{e}_i$, and take a component representation of the above constructed SFT action as

$$\begin{aligned}\mathcal{V}_k &= \frac{1}{k}\langle V_k||\mathbf{e}_{i_1}\phi^{i_1}\rangle\cdots|\mathbf{e}_{i_k}\phi^{i_k}\rangle = \frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\\ \frac{1}{2}\omega(\Phi,Q\Phi) &= \frac{1}{2}\mathcal{V}_{i_1i_2}\phi^{i_2}\phi^{i_1}\ .\end{aligned} \tag{4.1}$$

Moreover we define for $k\geq 2$,

$$c^j_{i_1\cdots i_k} := (-1)^{\mathbf{e}_l}\omega^{jl}\mathcal{V}_{li_1\cdots i_k}\ . \tag{4.2}$$

On the other hand, the BV-BRST transformation is defined as

$$\delta = (\ ,S)\ . \tag{4.3}$$

With this δ, the classical master equation (3.11) is written as $\delta S=0$ and by using the Jacobi identity of the BV-bracket, $\delta^2 = (\ ,\frac{1}{2}(S,S))$ is satisfied. These facts read that the following three statements are equivalent : the action S satisfies the BV-master equation (3.11), the action S is invariant under the BV-BRST transformation (4.3), and the BV-BRST transformation δ is nilpotent.

When the action is written with $\{c^j_{i_1\cdots i_k}\}$ defined in eq.(4.2), the BV-BRST transformation becomes

$$\delta = (\ ,S) = \sum_{k=1}^{\infty}\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\ . \tag{4.4}$$

Because S satisfies the BV-master equation(3.11), this δ is nilpotent, and the fact means that $\{c^j_{i_1\cdots i_k}\}$ define an A_∞-algebra in the dual picture as explained in (Rem.2.1). Note however that the $\{c^j_{i_1\cdots i_k}\}$ does not only define an A_∞-structure but has the cyclic structure by lowering the upper index by the symplectic structure ω_{ij}. The algebraic structure of classical open SFT is the A_∞-structure with cyclic symmetry through an appropriate inner product, where the symplectic structure of BV-formalism plays the role of the inner product.

4.2 BV-gauge transformation

The BV-BRST transformation for Φ is $\delta\Phi = \sum_{k\geq 1}m_k(\Phi) = \mathfrak{m}_*(e^\Phi)$. The corresponding gauge transformation is then written as

$$\begin{aligned}\delta_\alpha\Phi &= \mathfrak{m}_*(e^\Phi\alpha e^\Phi)\\ &= Q\alpha + m_2(\alpha,\Phi)+m_2(\Phi,\alpha)+m_3(\alpha,\Phi,\Phi)+m_3(\Phi,\alpha,\Phi)+m_3(\Phi,\Phi,\alpha)+\cdots\ ,\end{aligned} \tag{4.5}$$

where $\alpha=\mathbf{e}_i\alpha^i$ is a gauge parameter of degree minus one, therefore the degree of α^i is minus the degree of $\mathbf{e}_i$ minus one. This is exactly the gauge transformation for Maurer-Cartan equations given by (Rem.2.5). Therefore if any two SFT action with A_∞-structures and an A_∞-morphism between them are given, the gauge transformations eq.(4.5) on both sides are compatible with the A_∞-morphism in the solution spaces of the Maurer-Cartan equations. The gauge transformation is written as $\delta_\alpha\Phi = m_*(e^\Phi)\frac{\overleftarrow{\partial}}{\partial\phi^i}\alpha^i$, and in the language of the component fields, it is

$$\delta_\alpha = \sum_{k=1}^{\infty}\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j_{i_1\cdots i_k}\left(\phi^{i_k}\cdots\phi^{i_1}\frac{\overleftarrow{\partial}}{\partial\phi^i}\alpha^i\right)\ .$$

The action is invariant under this δ_α because $0 = (S,S)\frac{\overleftarrow{\partial}}{\partial\phi^i}\alpha^i$ leads $\delta_\alpha S = 0$ [8]. Thus the gauge transformation for A_∞-algebras fit the standard argument of BV-formalism[38, 39]. Moreover,

$$0 = \omega_{kj}\frac{\overrightarrow{\partial}}{\partial\phi^j}(S,S)\frac{\overleftarrow{\partial}}{\partial\phi^i}\alpha^i|_{\frac{\partial S}{\partial\phi}=0} \tag{4.6}$$

indicates that the generator of the gauge transformation is degenerate and the rank of the Hessian for the quadratic part of the action S_0 is less than half of the number of the basis $\{\mathbf{e}_i\}$ on the space $\{\phi|\frac{\partial S}{\partial\phi}=0\}$, though the number of the basis is infinity. The origin $\phi = 0$ is also the solution for $\{\phi|\frac{\partial S}{\partial\phi}=0\}$, and in SFT case eq.(4.6) at the origin is nothing but the condition $(Q)^2 = 0$. When the ratio of the rank of the Hessian over the number of the basis is just half, the action is called *proper*. SFT is just the case. The Hessian at the origin is $\mathcal{V}_{i_1 i_2}$ in eq.(4.1), which is determined by Q. (The reducibility of the gauge group of SFT action then comes from the Virasoro symmetry of Q.) The above arguments lead that the rank of the Hessian is equal to the rank of unphysical states $\mathcal{H}^u$ which generate the gauge transformations. It is much larger than the rank of physical states $\mathcal{H}^p$, which is the cohomology class with respect to Q. Let $\mathcal{H}^t$ be Q-trivial states then $\mathcal{H} = \mathcal{H}^t \cup \mathcal{H}^u \cup \mathcal{H}^p$ and rank $\mathcal{H}^u =$ rank $\mathcal{H}^t$. From these it can be seen that actually rank $\{\mathcal{V}_{i_1 i_2}\}$/rank $\mathcal{H} = \frac{1}{2}$. and SFT actions are proper at the origin of $\mathcal{H}$.

Though SFT is treated in the context of the BV-formalism, the use is different from that in the original context of BV-formalism[37, 38, 39], where beginning with the gauge invariant action which does not include antifields, the terms including antifields are added to the original action so that the action satisfies the master equation and is proper. Restricting the antifields to zero recovers the original action, where the rank of the Hessian is less than the rank of the fields. We call it the trivial gauge. The gauge fixing is then performed by shifting the trivial gauge and restricting the antifields so that the rank of the Hessian is equal to the rank of the fields, *i.e.* half of the rank of the total space including antifields ($\mathcal{H}$ in SFT). In SFT, however, the antifields are originally included in the quadratic term S_0 and BV-master equation is used in order to determine the form of higher vertices. Therefore the trivial gauge fixing can be consistent gauge fixing in SFT. Actually the quadratic term S_0 reads that the rank of the Hessian of the gauge fixed quadratic part $\sim \frac{1}{2}$rank $\mathcal{H}$ at least as perturbation theory around the origin $\phi = 0$. This trivial gauge is called *Siegel gauge* in SFT and used in more explicit arguments in subsection 5.2 and section 6.

Coming back to the property of the gauge transformation, we mention two remarks about it. The gauge transformation makes Lie algebra on-shell. In the case of classical closed SFT (L_∞-algebra), the fact can be found in [5]. Moreover, if two classical open SFT and the A_∞-morphism between them are given, the gauge transformations on both SFTs are compatible with the A_∞-morphism on-shell, where on-shell means the solution of the Maurer-Cartan equations. This fact follows from (Rem.2.5).

[8] It holds even if the Poisson structure ω^{ij} is non-constant. It follows from the Jacobi identity of ω^{ij} and $0 = ((S,S), \phi^i\omega_{ij}\alpha^j)$ (See comments in *cyclic algebra with BV-Poisson structure* in the next subsection).

4.3 Operator language, A_∞-algebra, its dual, and their graphical representation

In this subsection some notation used before and later is summarized. We identify the operator language and the coalgebraic representation as

$$|\mathbf{e}_{i_1}\rangle_1\cdots|\mathbf{e}_{i_n}\rangle_n=\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_n}\ .$$

This leads, for instance, $|\Phi\rangle_1\cdots|\Phi\rangle_n=|\mathbf{e}_{i_1}\rangle_1\cdots|\mathbf{e}_{i_n}\rangle_n\phi^{i_n}\cdots\phi^{i_1}$.

- *Symplectic form and Poisson structure*

First, a (constant) symplectic structure $\omega:\mathcal{H}\otimes\mathcal{H}\to\mathbb{C}$ induced from the BPZ-inner product is defined. In the operator language, the symplectic structure is defined as

$$\omega(\mathbf{e}_i,\mathbf{e}_j):=\langle\omega|_{12}|\mathbf{e}_i\rangle_1|\mathbf{e}_j\rangle_2\ .$$

Here the $\langle\omega|_{12}$ is the reflection operator, which denotes the sewing of the state $|\ \rangle_2$ at the origin of the upper half plane with the state $|\ \rangle_1$ at infinity on it. Define the property of the exchanging the labels for kets as

$$\langle\omega|_{21}=-\langle\omega|_{12}\ ,$$

and then $\langle\omega|_{21}|\mathbf{e}_j\rangle_2|\mathbf{e}_i\rangle_1=(-1)^{\mathbf{e}_i\mathbf{e}_j}\langle\omega|_{21}|\mathbf{e}_i\rangle_1|\mathbf{e}_j\rangle_2=-(-1)^{\mathbf{e}_i\mathbf{e}_j}\langle\omega|_{12}|\mathbf{e}_i\rangle_1|\mathbf{e}_j\rangle_2$. The symplectic structure is translated into its component expression as

$$\langle\omega|_{12}|\mathbf{e}_i\rangle_1|\mathbf{e}_j\rangle_2=\omega_{ij}\ ,$$

and the above calculation reads

$$\omega_{ji}=-(-1)^{\mathbf{e}_i\mathbf{e}_j}\omega_{ij}\ .$$

Thus ω determines a graded symplectic structure. ω_{ij} is by definition constant on $\mathcal{H}$, *i.e.* it is independent of $\{\phi\}$. Now $\omega_{ij}\neq 0$ iff the degree of $\mathbf{e}_i$ plus the degree of $\mathbf{e}_j$ is equal to one. Therefore one always gets $(-1)^{\mathbf{e}_i\mathbf{e}_j}=1$ and then $\omega_{ji}=-\omega_{ij}$. Moreover in this reason the degree of $\langle\omega|_{12}$ is minus one so that the degree of $\omega_{ij}\in\mathbb{C}$ has degree zero.

The inverse reflector is defined as the inverse of $|\omega\rangle_{12}$

$$\langle\omega|_{12}|\omega\rangle_{23}={}_3\mathbf{1}_1\ , \tag{4.7}$$

where ${}_3\mathbf{1}_1$ denotes the identity operator which maps from $|\ \rangle_1$ to $|\ \rangle_3$. The degree of $|\omega\rangle_{12}$ is plus one. Expand it as

$$|\omega\rangle_{23}=|\mathbf{e}_j\rangle_2|\mathbf{e}_k\rangle_3\omega^{jk}(-1)^{\mathbf{e}_k} \tag{4.8}$$

and eq.(4.7) is rewritten as ${}_3\mathbf{1}_1=\langle\omega|_{12}|\mathbf{e}_j\rangle_2|\mathbf{e}_k\rangle_3\omega^{jk}(-1)^{\mathbf{e}_k}=|\mathbf{e}_k\rangle_3\langle\omega|_{12}|\mathbf{e}_j\rangle_2\omega^{jk}(-1)^{\mathbf{e}_j\mathbf{e}_k}$. We then define the dual state as

$$\langle\mathbf{e}^k|_1:=\langle\omega|_{12}|\mathbf{e}_j\rangle_2\omega^{jk}(-1)^{\mathbf{e}_j\mathbf{e}_k}\ . \tag{4.9}$$

By definition, the degree of $\langle\mathbf{e}^k|_1$ is minus the degree of $\mathbf{e}_k$. The equation (4.7) is then expressed as ${}_3\mathbf{1}_1=|\mathbf{e}_k\rangle_3\langle\mathbf{e}^k|_1$. The dual states must have the following inner product

$$\langle\mathbf{e}^k|_1|\mathbf{e}_i\rangle_1=\delta_i^k\ . \tag{4.10}$$

This gives a condition for ω^{jk} in eq.(4.9), which is

$$\omega_{ij}\omega^{jk} = \delta_i^k = \omega^{kj}\omega_{ji} \ . \tag{4.11}$$

Moreover this identity leads

$$\omega^{kj} = -(-1)^{\mathbf{e}_j\mathbf{e}_k}\omega^{jk} = -\omega^{jk} \quad (= -(-1)^{(\mathbf{e}_j+1)(\mathbf{e}_k+1)}\omega^{jk} \ ) \ . \tag{4.12}$$

Using this, one can see from (4.8) that the inverse reflector is symmetric with respect to the labels for bras

$$|\omega\rangle_{23} = |\omega\rangle_{32} \ ,$$

and the dual state (4.9) is rewritten as

$$\langle\mathbf{e}^k|_1 := \omega^{kj}\langle\omega|_{01}|\mathbf{e}_j\rangle_0 \ . \tag{4.13}$$

The complete set of $\mathbf{1}$ is then defined by this dual state as

$$\mathbf{1} = |\mathbf{e}_i\rangle\langle\mathbf{e}^i| \ . \tag{4.14}$$

The equations (4.11) and (4.12) indicate that ω^{ij}, the inverse of the symplectic structure ω_{ij}, gives a Poisson structure. Explicitly

$$\frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j} =: (\ , \ )$$

is the *BV-Poisson bracket*. This gives the identification of the inverse reflection operator $|\omega\rangle_{23}$ and the BV-bracket assumed in eq.(3.9). Its compatibility with the string vertices will be checked after introducing the vertices in *cyclic algebra with BV-Poisson structure*.

The symplectic structure is also expanded with the dual basis as

$$\langle\omega|_{12} = \omega_{ij}\langle\mathbf{e}^j|_2\langle\mathbf{e}^i|_1 \ .$$

Note that the dual state (4.13) and the inner product (4.10) can be regarded as those used in Appendix A.1.

- *String vertex and A_∞-structure*

The vertex with the operator representation and its component representation were related in eq.(4.1) as

$$\langle V_{k+1}||\Phi\rangle\cdots|\Phi\rangle = \langle V_{k+1}||\mathbf{e}_{i_1}\rangle\cdots|\mathbf{e}_{i_{k+1}}\rangle\phi^{i_{k+1}}\cdots\phi^{i_1} = \mathcal{V}_{i_1\cdots i_{k+1}}\phi^{i_{k+1}}\cdots\phi^{i_1} \ ,$$

where the indices which label the bras and kets are omitted. On the other hand, the A_∞-structures in both representations were related by eq.(2.8) as

$$m_k(\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_k}) = \mathbf{e}_j c^j_{i_1\cdots i_k} \ ,$$

and this $\{c^j_{i_1\cdots i_k}\}$ and $\{\mathcal{V}_{i_1\cdots i_{k+1}}\}$ were connected to the component representation by eq.(4.2) as

$$c^j_{i_1\cdots i_k} := (-1)^{\mathbf{e}_l}\omega^{jl}\mathcal{V}_{li_1\cdots i_k} \ . \tag{4.15}$$

The corresponding one in the operator representation is

$$\langle V_{k+1}||\Phi\rangle\cdots|\Phi\rangle=\langle\omega|_{12}|\Phi\rangle_1|m_k(\Phi,\cdots,\Phi)\rangle_2 \tag{4.16}$$

for $k\geq 1$ [9]. Actually, by using eq.(4.15), this identity can be checked

$$\begin{aligned}\omega(\Phi,m_k(\Phi))&=\omega(\mathbf{e}_{i_1}\phi^{i_1},m_k(\mathbf{e}_{i_2}\phi^{i_2},\cdots,\mathbf{e}_{i_{k+1}}\phi^{i_{k+1}}))\\&=\phi^{i_1}\omega(\mathbf{e}_{i_1},\mathbf{e}_j c^j_{i_2\cdots i_{k+1}})\phi^{i_{k+1}}\cdots\phi^{i_2}\\&=(-1)^{\mathbf{e}_{i_1}}\omega_{i_1 j}c^j_{i_2\cdots i_{k+1}}\phi^{i_{k+1}}\cdots\phi^{i_2}\cdot\phi^{i_1}\\&=\mathcal{V}_{i_1\cdots i_{k+1}}\phi^{i_{k+1}}\cdots\phi^{i_1}=\langle V_{k+1}||\Phi\rangle\cdots|\Phi\rangle\end{aligned}$$

where $m_k(\Phi):=m_k(\Phi,\cdots,\Phi)$. Thus the action can be rewritten in this form,

$$S=\frac{1}{2}\omega(\Phi,Q\Phi)+\sum_{k\geq 2}\frac{1}{k+1}\omega(\Phi,m_k(\Phi))\ . \tag{4.17}$$

On the other hand, one can define m_k by $\langle V_{k+1}|$ and the inverse reflector $|\omega\rangle$ as follows

$$|m_k(\Phi,\cdots,\Phi)\rangle_0=\langle V_{k+1}|_{1\cdots k+1}|\omega\rangle_{01}|\Phi\rangle_2\cdots|\Phi\rangle_{k+1}\ . \tag{4.18}$$

In fact, the coefficient for $\mathbf{e}_j\cdot\phi^{i_k}\cdots\phi^{i_1}$ reproduces eq.(4.15), and acting $\langle\omega|_{a0}|\Phi\rangle_a$ on both sides of the eq.(4.18) from left reproduces eq.(4.16) using the identity (4.7).

- *cyclic algebra with BV-Poisson structure*

The vertices $\frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}$ is cyclic, *i.e.*

$$\begin{aligned}\frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}&=(-1)^{(\mathbf{e}_{i_k}+\cdots+\mathbf{e}_{i_2})\mathbf{e}_{i_1}}\frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_1}\phi^{i_k}\cdots\phi^{i_2}\\&=\frac{1}{k}\mathcal{V}_{i_2\cdots i_k i_1}\phi^{i_1}\phi^{i_k}\cdots\phi^{i_2}\end{aligned}$$

This cyclic symmetry is the consequence of the property of trace in the terminology of [2]. The BV-bracket of two cyclic vertices are then rewritten as the sewing of the vertices

$$\begin{aligned}&\frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j}\frac{1}{l}\mathcal{V}_{j_1\cdots j_l}\phi^{j_l}\cdots\phi^{j_1}\\&=\frac{1}{k}\langle V_k|_{1\cdots k}|\Phi\rangle_1\cdots|\Phi\rangle_k\frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j}\frac{1}{l}\langle V_l|_{1'\cdots l'}|\Phi\rangle_{1'}\cdots|\Phi\rangle_{l'}\\&=\langle V_k|_{1\cdots k}|\Phi\rangle_1\cdots|\Phi\rangle_{k-1}|\mathbf{e}_i\rangle_k\omega^{ij}(-1)^{\mathbf{e}_j}\langle V_l|_{1'\cdots l'}|\mathbf{e}_j\rangle_{1'}|\Phi\rangle_{2'}\cdots|\Phi\rangle_{l'}\\&=\langle V_k|_{1\cdots k}\langle V_l|_{1'\cdots l'}|\omega\rangle_{k1'}(|\Phi\rangle)^n\ ,\end{aligned}$$

where $k+l=n+2$. This gives the explicit calculation in eq.(3.9). Here $(|\Phi\rangle)^n$ are inserted on the boundary of the disk S^1, so the last line of the above equation is also cyclic and the cyclic vertices close as a Lie algebra with respect to Lie bracket (,). In component language it is of the form

$$\langle V_k|_{1\cdots k}\langle V_l|_{1'\cdots l'}|\omega\rangle_{k1'}(|\Phi\rangle)^n=\frac{1}{n}\left(\langle V_k|_{1\cdots k}\langle V_l|_{1'\cdots l'}|\omega\rangle_{k1'}|\mathbf{e}_{i_1}\rangle\cdots|\mathbf{e}_{i_n}\rangle+cyclic\right)\phi^{i_n}\cdots\phi^{i_1}$$

[9] The coefficient for $\phi^{i_{k+1}}\cdots\phi^{i_1}$ reads $\langle V_{k+1}||\mathbf{e}_{i_1}\rangle\cdots|\mathbf{e}_{i_{k+1}}\rangle=(-1)^{\mathbf{e}_{i_1}}\langle\omega|_{12}|\mathbf{e}_{i_1}\rangle_1|m_k(\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_{k+1}})\rangle_2$.

where *cyclic* means $|\mathbf{e}_{i_1}\rangle\cdots|\mathbf{e}_{i_n}\rangle$ is moved cyclic with appropriate sign and each term is summed up. Using this we can also rewrite the recursion relation directly and arrive at the condition for the A_∞-structure(2.6). Furthermore if we define the free tensor algebra of this cyclic algebra, the BV-bracket (,) then defines a Poisson structure on the free tensor algebra. We can also extend the BV-Poisson structure ω^{ij} non-constant as the Poisson structure on a formal noncommutative supermanifold. However in the present paper we discuss mainly in the case that ω^{ij} is constant, so we want to report the issue separately elsewhere. Some related concepts are found in [46].

Note that $\langle V_k|$ is written as

$$\langle V_k| = \langle V_k||\mathbf{e}_{i_1}\rangle\langle\mathbf{e}^{i_1}|\cdots|\mathbf{e}_{i_k}\rangle\langle\mathbf{e}^{i_k}| = \mathcal{V}_{i_1\cdots i_k}\langle\mathbf{e}^{i_k}|\cdots\langle\mathbf{e}^{i_1}|\ ,$$

which is the same form as $\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}$. Thus the coordinates of the states and the dual states are identified as a vector space (Appendix A.1). Similarly $\frac{\partial}{\partial\phi^i}$ can be identified with $\mathbf{e}_i$.

- *other algebraic relations used later*

In addition to the BRST charge Q, in section 5 the propagator Q^+ which has degree minus one is introduced. Here the algebraic properties of these operators together with ω are summarized. We impose the propagator satisfies the following relation

$$(Q^+)^2 = 0\ , \qquad Q^+QQ^+ = Q^+\ . \tag{4.19}$$

From the properties of the BPZ-inner product, Q and Q^+ operate on $\langle\omega|$ and $|\omega\rangle$ as

$$\langle\omega|_{12}Q^{(2)} = -\langle\omega|_{12}Q^{(1)}\ , \quad \langle\omega|_{12}(Q^+)^{(2)} = \langle\omega|_{12}(Q^+)^{(1)}\ , \quad (Q^+)^{(2)}|\omega\rangle_{12} = (Q^+)^{(1)}|\omega\rangle_{12} \tag{4.20}$$

where the indices $^{(1)}$ or $^{(2)}$ denote the bras or kets where the operators act on. The justification for these relations are clarified in subsection 5.2.

By employing the property of Q acting on $\langle\omega|$ (4.20), here the orthogonal decomposition of the inner product ω is examined. Let us define the Hodge-Kodaira decomposition of string Hilbert space $\mathcal{H}$ as

$$QQ^u + Q^uQ + P^p = \mathbf{1}\ , \tag{4.21}$$

where degree minus one operator Q^u is defined so that QQ^u, Q^uQ and P^p are the projections onto Q-trivial states, unphysical states and physical states, respectively. Here define the space of physical states as

$$\mathcal{H}^p := P^p\mathcal{H}\ . \tag{4.22}$$

Similarly define the projections onto Q-trivial states and unphysical states as $QQ^u = P^t$ and $Q^uQ = P^u$ and express the decomposition of $\mathcal{H}$ as

$$\mathcal{H}^t := P^t\mathcal{H}\ , \quad \mathcal{H}^u := P^u\mathcal{H}\ , \qquad \mathcal{H} = \mathcal{H}^t\cup\mathcal{H}^u\cup\mathcal{H}^p\ .$$

There are ambiguities of the choice of Q^u. Q-trivial states $\mathcal{H}^t$ is unique but $\mathcal{H}^p$ is unique up to the Q-trivial states $\mathcal{H}^t$, and unphysical states $\mathcal{H}^u$ is unique up to Q-trivial and physical states $\mathcal{H}^t\cup\mathcal{H}^p$. Consider the inner product

$$\omega_{ij} = \omega(\mathbf{e}_i, \mathbf{e}_j)\ .$$

From the first equation in (4.20), we can see the following properties without fixing the ambiguities : when $\mathbf{e}_j \in \mathcal{H}^t$ then $\omega_{ij} = 0$ for $\mathbf{e}_i \in \mathcal{H}^t \cup \mathcal{H}^p$, when $\mathbf{e}_j \in \mathcal{H}^t \cup \mathcal{H}^p$ then $\omega_{ij} = 0$ for $\mathbf{e}_i \in \mathcal{H}^t$. If we denote the block element of matrix $\{\omega_{ij}\}$ where $\mathbf{e}_i \in \mathcal{H}^u$ and $\mathbf{e}_j \in \mathcal{H}^p$ as ω_{up} and similar for other eight block elements, matrix $\{\omega_{ij}\}$ is represented as the left hand side of eq.(4.23). This implies that by basis transformation corresponding to the ambiguity in Q^u the inner product ω is decomposed as the right hand side

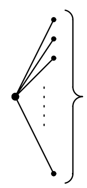
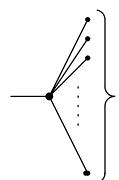
$$\{\omega_{ij}\} = \begin{pmatrix} \omega_{uu} & \omega_{up} & \omega_{ut} \\ \omega_{pu} & \omega_{pp} & \omega_{pt} \\ \omega_{tu} & \omega_{tp} & \omega_{tt} \end{pmatrix} = \begin{pmatrix} \omega_{uu} & \omega_{up} & \omega_{ut} \\ \omega_{pu} & \omega_{pp} & 0 \\ \omega_{tu} & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & \omega_{ut} \\ 0 & \omega_{pp} & 0 \\ \omega_{tu} & 0 & 0 \end{pmatrix} . \tag{4.23}$$

These orthogonal decomposition of the inner product will be used in subsection 5.3 when onshell reduction of SFT is discussed. In closed string case the explicit orthogonal decomposition for string states is given in Appendix of [15].

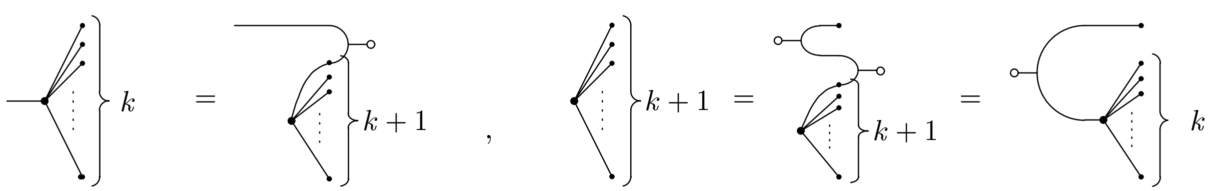
- *Graphical representation*

Here give some graphical representations for those defined above. It is helpful to realize intuitively, and used in subsection 5.3 to simplify the arguments when using the Feynman graphs of SFT.

Express $\langle\omega|$ and $|\omega\rangle$ as $\circ\!\!-\!\!\subset$ and $\supset\!\!-\!\!\circ$, respectively. Moreover, vertices and the A_∞-structure are expressed as follows

$\langle V_k| =$  k , $\qquad m_k =$  k .

In coalgebras or the operator representation, the operators act from left and accumulated on left. According to this order, we define the order of the operation for these graphs from right to left. The relations between the cyclic vertices and the A_∞-structure are then represented as

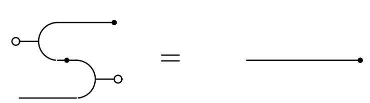


$$|m_k\rangle_1 = \langle V_{k+1}|_{a\cdots}|\omega\rangle_{1a} \qquad \langle V_{k+1}|_{1\cdots} = \langle V_{k+1}|_{b\cdots}\langle\omega|_{1a}|\omega\rangle_{ab} = \langle\omega|_{12}|m_k(\ ,\cdots,\ )\rangle_2$$

$$c^j_{i_1\cdots i_k} = (-1)^{\mathbf{e}_l}\omega^{jl}\mathcal{V}_{li_1\cdots i_k} \qquad \mathcal{V}_{i_1\cdots i_{k+1}} = \omega_{i_1 j}\omega^{jl}\mathcal{V}_{li_2\cdots i_{k+1}} = (-1)^{\mathbf{e}_{i_1}}\omega_{i_1 l}c^l_{i_2\cdots i_{k+1}}$$

where in the second equality in the right hand side, the identity (4.7) is written as

$$\begin{array}{c} \langle\omega|_{\cdot a}|\omega\rangle_{a\cdot} = .\mathbf{1}. \\ \omega_{ij}\omega^{jk} = \delta_i^k \end{array} \qquad \Longleftrightarrow$$



Note that $\langle\omega|$ and $|\omega\rangle$ correspond to lowering and raising indices with ω_{ij} and ω^{ij} in their component language, and which correspond to reversing the outgoing lines in these graphs.

5 A_∞-morphism and field transformation

For classical open SFT, an A_∞-morphism corresponds to a field transformation and the Maurer-Cartan equation is the equation of motion of SFT. It has been explained in section 2.3 that A_∞-morphisms preserve the Maurer-Cartan equations for A_∞-algebras. This means that if an A_∞-morphism is given, we get a field transformation which preserves the equation of motion. Generally when two A_∞-algebras are given, it is difficult to construct an A_∞-morphism between them. However it is known that any A_∞-algebras (with nonvanishing m_1) have a canonical A_∞-quasi-isomorphism[22, 23, 25] between the original A_∞-algebra and another canonical A_∞-algebra. The A_∞-quasi-isomorphism can be constructed in a canonical way in terms of Feynman graphs[25], and it fits for SFT very much. Here will explain the way of constructing the canonical A_∞-quasi-isomorphism in terms of the Feynman graphs in subsection 5.1, clarify the identification between the Feynman graphs and that in SFTs in subsection 5.2. The arguments in subsection 5.1 is then applied to SFT in subsection 5.3 and we show that the canonical A_∞-algebra is nothing but the on-shell S-matrix elements (Lem.5.1). From this result it is shown in (Thm.5.1) that every SFT constructed as explained in section 3 are quasi-isomorphic, and the quasi-isomorphism between those can be constructed in subsection 5.4. All the arguments are applicable also for classical closed SFT *i.e.* L_∞-algebras.

5.1 The minimal model theorem

Let $(\mathcal{H},\mathfrak{m})$ be an A_∞-algebra with $m_1\neq 0$. As mentioned above, for any $(\mathcal{H},\mathfrak{m})$, a canonical A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p$ from another canonical A_∞-algebra $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ to the A_∞-algebra $(\mathcal{H},\mathfrak{m})$ exists[25]. This is called *minimal model theorem* [10].

Here we construct the A_∞-morphism $\{\tilde{f}^p_k\}$ and A_∞-structure $\{\tilde{m}^p_k\}$ with $k\geq 2$ naturally as the problem of finding the solutions for the equation of motion (Maurer-Cartan equation) for SFT, and prove that the $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ and $\tilde{\mathcal{F}}$ are indeed an A_∞-algebra and a quasi-isomorphism, respectively [11]. The procedure of finding the solution is quite natural and standard, and so similar procedures can be found in various problems. The procedure also relates to the way of finding some classical solutions in closed SFT [47, 48] (see the next subsection) or constructing the tachyon potential[49] (see *tachyon condensation* in Discussions).

Consider solving the equation of motion for classical open SFT,

$$\sum_{k\geq 1} m_k(\Phi)=0 \tag{5.1}$$

with $m_1=Q$. For Q the coboundary operator of complex $(\mathcal{H},Q)$, we give the analogue of the Hodge-Kodaira decomposition

$$QQ^+ + Q^+Q + P = \mathbf{1}\ . \tag{5.2}$$

In this section for simplicity we assume this identity gives just the the Hodge-Kodaira decomposition of $\mathcal{H}$, that is, Q^+ is the adjoint of Q and P is the projection onto the harmonic form

[10]This naming has nothing to do with minimal model in the context of two-dimensional field theory directly.

[11]This explanation of the minimal model theorem from the problem of finding the solutions for the Maurer-Cartan equation is motivated in the lecture by K. Fukaya at Inst. of Tech. in Tokyo in December, 2000.

$(KerQ) \cap (KerQ^+)$. Q^+ can be regarded as the propagator for the SFT action and also plays the role of the gauge fixing. P is related to the projection onto the physical states in SFT, so we denote $\mathcal{H}^p := P\mathcal{H}$ in this subsection. These will be clarified in the next subsection, where the condition that QQ^+, Q^+Q, and P are projections must be relaxed [12]. Actually, the condition can be relaxed later in and after (Def.5.1) and only the identity (5.2) between Q, Q^+ and P will be required.

Since the solutions for eq.(5.1) are preserved under the gauge transformation $\delta_\alpha \Phi = Q(\alpha) + m_2(\alpha, \Phi) + m_2(\Phi, \alpha) + \cdots$, we will find the gauge fixed solutions $Q^+\Phi = 0$. Here we assume that Φ is sufficiently 'small', then the solution is almost the solution of $Q(\Phi) = 0$. Express Φ as $\Phi = \Phi^p + \Phi^u$ where $\Phi^p \in \mathcal{H}^p$ and $\Phi^u \in Q^+Q\mathcal{H}$. As will be explained below, Φ^u can be solved recursively with the power of Φ^p. Because here we regard that Φ^p is 'small', one can define a degree by the power of Φ^p. Substituting $\Phi = \Phi^p + \Phi^u$ in e.o.m (5.1) leads

$$Q(\Phi^u) + \sum_{k\geq 2} m_k(\Phi^p + \Phi^u) = 0 \ , \tag{5.3}$$

and acting Q^+ to both sides of this equation yields

$$\Phi^u = -\sum_{k\geq 2} Q^+ m_k(\Phi^p + \Phi^u) \ . \tag{5.4}$$

Here we get the Φ^u recursively by eq.(5.4). However not all $\Phi = \Phi^p + \Phi^u$ expressed in terms of Φ^p give the solution of eq.(5.1) because eq.(5.4) is derived from Q^+ *acting* eq.(5.1). In order to find Φ^u which is the solution of eq.(5.1), we substitute eq.(5.4) to e.o.m(5.1) once again,

$$\begin{aligned} 0 &= Q(\Phi^p + \Phi^u) + \sum_{k\geq 2} m_k(\Phi) \\ &= (Q^+Q + P - 1)\sum_{k\geq 2} m_k(\Phi) + \sum_{k\geq 2} m_k(\Phi) \\ &= Q^+Q\sum_{k\geq 2} m_k(\Phi) + \sum_{k\geq 2} Pm_k(\Phi) \end{aligned} \tag{5.5}$$

and we can get a condition (obstruction) for Φ^p. The first term in the third line of eq.(5.5) vanishes due to e.o.m(5.1) because $Q\sum_{k\geq 2} m_k(\Phi) = -QQ(\Phi) = 0$, and the condition for Φ^p is derived as

$$\sum_{k\geq 2} Pm_k(\Phi^p + \Phi^u) = 0 \ . \tag{5.6}$$

The above Φ^u can be represented recursively in terms of Φ^p by eq.(5.4). This equation can be regarded as the Maurer-Cartan equation on $\mathcal{H}^p$.

The equation (5.4) can be regarded as a nonlinear map from $\mathcal{H}^p$ to $\mathcal{H}$. Here we want to distinct the element of $\mathcal{H}^p$ with that of $\mathcal{H}$, so we rewrite $\Phi^p \in \mathcal{H}^p$ as $\tilde{\Phi}^p \in \mathcal{H}^p$. If we write the map defined by eq.(5.4) recursively as

$$\Phi := \tilde{f}^p_1(\tilde{\Phi}^p) + \tilde{f}^p_2(\tilde{\Phi}^p, \tilde{\Phi}^p) + \tilde{f}^p_3(\tilde{\Phi}^p, \tilde{\Phi}^p, \tilde{\Phi}^p) + \cdots \tag{5.7}$$

[12] In addition we change the definition of $\mathcal{H}^p$ after this section from $\mathcal{H}^p := P\mathcal{H}$ to $\mathcal{H}^p = P^p\mathcal{H}$ where P^p is defined as the precise projection onto physical states.

with $\tilde{f}_1^p$ the identity map (inclusion map), $\tilde{f}_l^p(\tilde{\Phi}^p)$ is given by connecting tree graphs corresponding to $\{-Q^+ m_k(\cdots)\}$ with all possible combination, summing up these, and picking up the term involving l powers of $\tilde{\Phi}^p$ (see (Def.5.1) below). Alternatively eq.(5.6) is also expressed as an equation for $\tilde{\Phi}^p$. It is obtained by substituting eq.(5.4) or eq.(5.7) into eq.(5.6). Let us define $\tilde{m}_l^p(\tilde{\Phi}^p), l \geq 2$ as the term involving l powers of $\tilde{\Phi}^p$ in eq.(5.6). In other words we define them so that eq.(5.6) is rewritten as

$$\sum_{k\geq 2} \tilde{m}_k^p(\tilde{\Phi}^p) = 0 \ . \tag{5.8}$$

Each $\tilde{m}_l^p(\tilde{\Phi}^p)$ for $l \geq 2$ is then given in the same way as $\tilde{f}_l^p$ but replacing $-Q^+$ on the last outgoing line by P. Here denote the structure $\{\tilde{m}_k^p\}_{k\geq 2}$ as $\tilde{\mathfrak{m}}^p$. The equation (5.8) is regarded as the Maurer-Cartan equation on $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$. Note that as will be proven, the $\tilde{\mathfrak{m}}^p$ defines an A_∞-structure on $\mathcal{H}^p$ and the nonlinear map (5.7) defines the A_∞-quasi-isomorphism from $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}, \mathfrak{m})$. Thus the canonical A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ and the A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p$ can be defined naturally in the problem of solving the Maurer-Cartan equations (the problem of constructing Kuranishi map in mathematical language, see (Rem.5.4)). From SFT point of view the above result means that if the expectation value of physical states which satisfies the Maurer-Cartan equation (5.8) is given, the solution of e.o.m for SFT (5.1) is obtained by the A_∞-quasi-isomorphism (5.7). This statement will be explained more precisely in the next subsection.

Mention that as can be seen from eq.(5.5) if we begin with $\tilde{\Phi} \in \mathcal{H}$ with $Q\tilde{\Phi} \neq 0$ instead of $\tilde{\Phi}^p$, the Maurer-Cartan equation (5.8) may be corrected by adding the term $\tilde{m}_1(\tilde{\Phi})$ with $\tilde{m}_1 := m_1 = Q$. This case is also considered later (see (Rem.5.1)).

Here summarize the definition of the A_∞-structure and the A_∞-morphism derived above.

Definition 5.1 Let $(\mathcal{H}, \mathfrak{m})$ be an A_∞-algebra and assume that we have degree minus one operator Q^+ and degree zero operator P which satisfies $QQ^+ + Q^+Q + P = \mathbf{1}$ on $\mathcal{H}$. Then another A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ with $\mathcal{H}^p := P\mathcal{H}$ and an A_∞-morphism from $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}, \mathfrak{m})$ are constructed. We define those with the following three equivalent expressions :

- $\tilde{\mathfrak{m}}^p = \{\tilde{m}_k^p\}_{k\geq 2}$ is given by eq.(5.6) and eq.(5.8) :

  $$(0 =) \sum_{k\geq 2} P m_k(\tilde{\Phi}^p + \tilde{\Phi}^u) = \sum_{k\geq 2} \tilde{m}_k^p(\tilde{\Phi}^p) \ ,$$

  together with eq.(5.4): $\tilde{\Phi}^u = -\sum_{k\geq 2} Q^+ m_k(\tilde{\Phi}^p + \tilde{\Phi}^u)$ and the A_∞-morphism $\tilde{\mathcal{F}}^p = \{\tilde{f}_k^p\}_{k\geq 2}$ is defined by eq.(5.4) :

  $$\begin{aligned}\Phi = \tilde{\Phi}^p + \tilde{\Phi}^u &= \tilde{\Phi}^p - \sum_{k\geq 2} Q^+ m_k(\tilde{\Phi}^p + \tilde{\Phi}^u) \\ &= \tilde{\Phi}^p + \tilde{f}_2^p(\tilde{\Phi}^p, \tilde{\Phi}^p) + \tilde{f}_3^p(\tilde{\Phi}^p, \tilde{\Phi}^p, \tilde{\Phi}^p) + \cdots \ .\end{aligned}$$

- $\{\tilde{f}_k^p\}$ are defined recursively as

  $$\tilde{f}_k^p(\tilde{\Phi}^p) = -Q^+ \sum_{1\leq k_1 < k_2 \cdots < k_i = k} m_i(\tilde{f}_{k_1}^p(\tilde{\Phi}^p), \tilde{f}_{k_2-k_1}^p(\tilde{\Phi}^p), \cdots, \tilde{f}_{k-k_{i-1}}^p(\tilde{\Phi}^p))$$

with $\tilde{f}_1^p(\tilde{\Phi}^p) = \tilde{\Phi}^p$. $\{\tilde{m}_k^p\}_{k\geq 2}$ are then defined as

$$\tilde{m}_k^p(\tilde{\Phi}^p) = \sum_{1\leq k_1<k_2\cdots<k_i=k} Pm_i(\tilde{f}_{k_1}^p(\tilde{\Phi}^p), \tilde{f}_{k_2-k_1}^p(\tilde{\Phi}^p), \cdots, \tilde{f}_{k-k_{i-1}}^p(\tilde{\Phi}^p)) \ .$$

- $\tilde{f}_l^p(\tilde{\Phi}^p)$ is given by connecting tree graphs $\{-Q^+m_k(\cdots)\}$ with all possible combination, summing up these, and picking up the term involving l powers of $\tilde{\Phi}^p$. $\tilde{m}_l^p(\tilde{\Phi}^p)$ is defined in the same way as $\tilde{m}_l^p$ but replacing $-Q^+$ on the last outgoing line by P.

  Let G_l be the set of the graphs with l incoming states and an element in it as $\Gamma_l \in G_l$. For each Γ_l associate the operator $\tilde{m}_{\Gamma_l}^p$, which is defined by attaching m_k to each vertex with k incoming legs and one outgoing legs, attaching $-Q^+$ to each internal edges, and connecting them (see (Fig.2)). The derived A_∞-structure and A_∞-quasi-isomorphism are then given as

$$\tilde{m}_l^p = P\sum_{\Gamma_l\in G_l} \tilde{m}_{\Gamma_l}^p \ , \qquad \tilde{f}_l^p = -Q^+\sum_{\Gamma_l\in G_l} \tilde{m}_{\Gamma_l}^p \ . \tag{5.9}$$

Note that once getting $\tilde{m}_k^p(\tilde{\Phi}^p)$ and $\tilde{f}_k^p(\tilde{\Phi}^p)$, then $\tilde{m}_k^p(\mathbf{e}_1^p, \cdots, \mathbf{e}_k^p)$ and $\tilde{f}_k^p(\mathbf{e}_1^p, \cdots, \mathbf{e}_k^p)$ are immediately obtained by reading the coefficient of $\phi^k\cdots\phi^1$, where $\mathbf{e}_i^p, i = 1, \cdots k$ are the basis of $\mathcal{H}^p$ and $\tilde{\Phi}^p = \mathbf{e}_i^p\phi^i$.

The explicit example is given in (Fig.2). In the order of the graphs in (Fig.2),

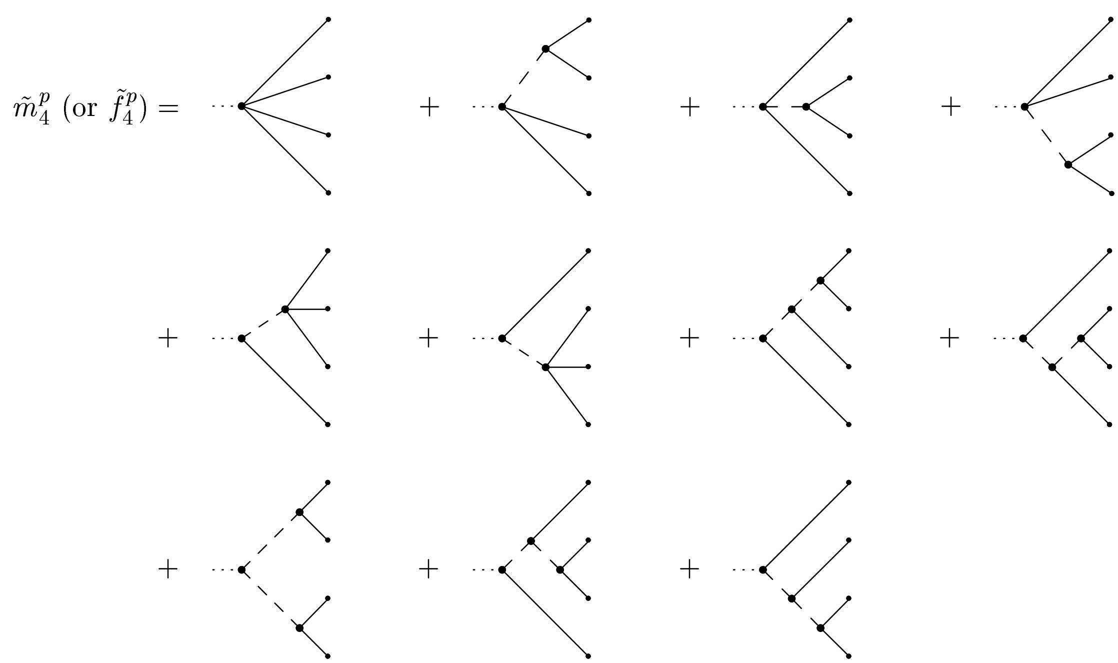


Figure 2: For example $\tilde{m}_4^p$ and $\tilde{f}_4^p$ are given. The large dots represent the vertices $\{m_k\}$. The dashed lines denote the propagators and we attach $-Q^+$ on them. The dotted line on each graph indicates the outgoing external line. We attach P for $\tilde{m}_k^p$ and $-Q^+$ for $\tilde{f}_k^p$. For $\tilde{m}_4^p$ and $\tilde{f}_4^p$, all such graphs with four incoming external states are summed up with weight $+1$.

$$
\begin{aligned}
\tilde{m}_4^p(\mathbf{e}_1^p,\mathbf{e}_2^p,\mathbf{e}_3^p,\mathbf{e}_4^p) &= Pm_4(\mathbf{e}_1^p,\mathbf{e}_2^p,\mathbf{e}_3^p,\mathbf{e}_4^p) + Pm_3(-Q^+m_2(\mathbf{e}_1^p,\mathbf{e}_2^p),\mathbf{e}_3^p,\mathbf{e}_4^p)\\
&+ Pm_3(\mathbf{e}_1^p,-Q^+m_2(\mathbf{e}_2^p,\mathbf{e}_3^p),\mathbf{e}_4^p) + Pm_3(\mathbf{e}_1^p,\mathbf{e}_2^p,-Q^+m_2(\mathbf{e}_3^p,\mathbf{e}_4^p))\\
&+ Pm_2(-Q^+m_3(\mathbf{e}_1^p,\mathbf{e}_2^p,\mathbf{e}_3^p),\mathbf{e}_4^p) + Pm_2(\mathbf{e}_1^p,-Q^+m_3(\mathbf{e}_2^p,\mathbf{e}_3^p,\mathbf{e}_4^p))\\
&+ Pm_2(-Q^+m_2(-Q^+m_2(\mathbf{e}_1^p,\mathbf{e}_2^p),\mathbf{e}_3^p),\mathbf{e}_4^p)\\
&+ Pm_2(\mathbf{e}_1^p,-Q^+m_2(-Q^+m_2(\mathbf{e}_2^p,\mathbf{e}_3^p),\mathbf{e}_4^p))\\
&+ Pm_2(-Q^+m_2(\mathbf{e}_1^p,\mathbf{e}_2^p),-Q^+m_2(\mathbf{e}_3^p,\mathbf{e}_4^p))\\
&+ Pm_2(-Q^+m_2(\mathbf{e}_1^p,-Q^+m_2(\mathbf{e}_2^p,\mathbf{e}_3^p)),\mathbf{e}_4^p)\\
&+ Pm_2(\mathbf{e}_1^p,-Q^+m_2(\mathbf{e}_2^p,-Q^+m_2(\mathbf{e}_3^p,\mathbf{e}_4^p)))\ ,
\end{aligned}
$$

and $\tilde{f}_4^p$ is obtained similarly but replaced each P on the outgoing line to $-Q^+$.

Remark 5.1 Though an A_∞-algebra $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ and an A_∞-quasi-isomorphism $\mathcal{F}$ have been given here, we can get another A_∞-algebra. It is obtained by replacing the $\mathcal{H}^p$ of $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ to $\mathcal{H}$. We denote this A_∞-algebra by $(\mathcal{H},\tilde{\mathfrak{m}})$ or simply $\tilde{\mathcal{H}}$. The A_∞-structure $\tilde{\mathfrak{m}}$ is that of $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$, but it has $\tilde{m}_1=m_1=Q$. (This $\tilde{m}_1$ vanishes trivially on $\mathcal{H}^p$ so the A_∞-structure of $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ is $\tilde{\mathfrak{m}}^p=\{\tilde{m}_k^p\}_{k\geq 2}$.) Thus the A_∞-algebra $(\mathcal{H},\tilde{\mathfrak{m}})$ is defined by (Def.5.1) but relaxing the restriction $\tilde{\Phi}^p\in\mathcal{H}^p$ as $\tilde{\Phi}\in\mathcal{H}$. $\tilde{\mathcal{F}}:=\tilde{\mathcal{F}}^p=\{\tilde{f}_1^p=\mathrm{Id},\tilde{f}_k^p(k\geq 2)\}$ then defines the A_∞-quasi-isomorphism from $(\mathcal{H},\tilde{\mathfrak{m}})$ to $(\mathcal{H},\mathfrak{m})$ [13]. Note that this is not only an A_∞-quasi-isomorphism but also an A_∞-isomorphism. The following diagram is obtained

$$
\begin{array}{ccc}
(\mathcal{H},\tilde{\mathfrak{m}}) & \xrightarrow{\quad\tilde{\mathcal{F}}\quad} & (\mathcal{H},\mathfrak{m})\\
{\scriptstyle P}\downarrow\uparrow{\scriptstyle \iota} & \nearrow_{\tilde{\mathcal{F}}^p} & \\
(\mathcal{H}^p,\tilde{\mathfrak{m}}^p) & &
\end{array}
$$

where $\iota:\mathcal{H}^p\to\mathcal{H}$ is the inclusion map. Explicitly $\tilde{\mathfrak{m}}^p$ and $\tilde{\mathcal{F}}^p$ are given by $\tilde{\mathfrak{m}}^p=P\circ\tilde{\mathfrak{m}}\circ\iota$ and $\tilde{\mathcal{F}}^p=\tilde{\mathcal{F}}\circ\iota$. We will consider these two A_∞-algebras $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ and $(\mathcal{H},\tilde{\mathfrak{m}})$ later when an A_∞-algebra $(\mathcal{H},\mathfrak{m})$ will be given. $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ is used when a SFT is reduced to its on-shell physics in subsection 5.3, and $(\mathcal{H},\tilde{\mathfrak{m}})$ is considered when the $\tilde{\mathcal{F}}$ is used as a field redefinition between two SFTs in subsection 5.4 and section 6.

Remark 5.2 Both A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p:(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)\to(\mathcal{H},\mathfrak{m})$ and $\tilde{\mathcal{F}}:(\mathcal{H},\tilde{\mathfrak{m}})\to(\mathcal{H},\mathfrak{m})$ have their inverse A_∞-quasi-isomorphisms. An quasi-inverse $(\tilde{\mathcal{F}}^p)^{-1}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ is given simply by the projection P

$$
\begin{array}{cccc}
(\tilde{\mathcal{F}}^p)_*^{-1} & : & \mathcal{H} & \longrightarrow & \mathcal{H}^p\\
 & & \Phi & \mapsto & \tilde{\Phi}^p=P\Phi\ .
\end{array}
$$

The inverse $(\tilde{\mathcal{F}})^{-1}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H},\tilde{\mathfrak{m}})$ is obtained by

$$
\begin{array}{cccc}
(\tilde{\mathcal{F}})_*^{-1} & : & \mathcal{H} & \longrightarrow & \mathcal{H}\\
 & & \Phi & \mapsto & \tilde{\Phi}=\Phi-\tilde{f}(\Phi)
\end{array}
$$

[13]Using them an explicit proof of (Rem.5.4) was presented by M. Akaho at topology seminar in Univ. of Tokyo in July, 2001.

where $\tilde{\Phi}=\Phi+\tilde{f}(\Phi)=\Phi-Q^{+}\sum_{k\geq 2}m_k(\Phi)$. This $(\tilde{\mathcal{F}})_*^{-1}$ restricted to the solutions for the Maurer-Cartan equations is nothing but the Kuranishi map.

Remark 5.3 (Geometric realization) The A_∞-structures $\mathfrak{m}$ and $\tilde{\mathfrak{m}}$ on $\mathcal{H}$ are expressed as $\delta=\frac{\overleftarrow{\partial}}{\partial\phi^j}c_i^j\phi^i+\sum_{k=2}^{\infty}\frac{\overleftarrow{\partial}}{\partial\phi^j}c_{i_1\cdots i_k}^j\phi^{i_k}\cdots\phi^{i_1}$ and $\tilde{\delta}=\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^j}\tilde{c}_i^j\tilde{\phi}^i+\sum_{k=2}^{\infty}\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^j}\tilde{c}_{i_1\cdots i_k}^j\tilde{\phi}^{i_k}\cdots\tilde{\phi}^{i_1}$ in the dual picture. The leading coefficients of both formal vector fields are identical $c_i^j=\tilde{c}_i^j$, because $m_1=\tilde{m}_1=Q$. The A_∞-quasi-isomorphism $\tilde{\mathcal{F}}$ gives the coordinate transformation $\tilde{\mathcal{F}}_*$ on $\mathcal{H}$. Now by the definition of $\tilde{\mathfrak{m}}$, $\mathbf{e}_j\tilde{c}_{i_1\cdots i_k}^j\in\mathcal{H}^p$ for $k\geq 2$ (and $\mathbf{e}_j\tilde{c}_i^j\in\mathcal{H}^t$). Thus the A_∞-quasi-isomorphism $\tilde{\mathcal{F}}$ is constructed so that the coefficients $\tilde{c}_{i_1\cdots i_k}^j$ for $k\geq 2$ satisfies $\mathbf{e}_j\tilde{c}_{i_1\cdots i_k}^j\in\mathcal{H}^p$. Then the A_∞-structure $\tilde{\mathfrak{m}}$ on $\mathcal{H}$ can be reduced to the one on $\mathcal{H}^p$. Namely, the minimal model theorem says geometrically that for any graded vector space $\mathcal{H}$ with a formal vector field δ which satisfies $\delta\cdot\delta=0$, there exists a formal coordinate transformation $\tilde{\mathcal{F}}_*$ so that the vector field $\sum_{k=2}^{\infty}\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^j}\tilde{c}_{i_1\cdots i_k}^j\tilde{\phi}^{i_k}\cdots\tilde{\phi}^{i_1}$ is along the $\mathcal{H}^p$ direction.

In the rest of this subsection, we shall give a proof that $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ and $(\mathcal{H},\tilde{\mathfrak{m}})$ are in fact A_∞-algebras and $\tilde{\mathcal{F}}^{(p)}$ (by $\tilde{\mathcal{F}}^{(p)}$ we denote $\tilde{\mathcal{F}}$ or $\tilde{\mathcal{F}}^p$) is an A_∞-morphism. The fact that $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ is an A_∞-algebra and $\tilde{\mathcal{F}}^p$ is an A_∞-quasi-isomorphism between $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ to $(\mathcal{H},\mathfrak{m})$ immediately follows from the fact that $(\mathcal{H},\tilde{\mathfrak{m}})$ is an A_∞-algebra and $\tilde{\mathcal{F}}$ is an A_∞-quasi-isomorphism between $(\mathcal{H},\tilde{\mathfrak{m}})$ to $(\mathcal{H},\mathfrak{m})$. The former is obtained by restricting the $\mathcal{H}$ of $(\mathcal{H},\tilde{\mathfrak{m}})$ to $\mathcal{H}^p$ in the latter case. Therefore we will prove the latter fact. In order to see this, it is enough to confirm the following two fact : $\mathfrak{m}\tilde{\mathcal{F}}=\tilde{\mathcal{F}}\tilde{\mathfrak{m}}$ and $(\tilde{\mathfrak{m}})^2=0$ on $\mathcal{H}$. We begin with a proof for the first statement. As has been seen in subsection 2.3, because $\tilde{\mathcal{F}}$ is the coalgebra homomorphism, $\tilde{\mathcal{F}}(e^{\tilde{\Phi}}-\mathbf{1})=e^{\Phi}-\mathbf{1}$ holds. Then $\mathfrak{m}\tilde{\mathcal{F}}(e^{\tilde{\Phi}})=\sum_{k\geq 1}m_k(\Phi)+\cdots$ for $\cdots\in\mathcal{H}^{\otimes n\geq 2}$. $\sum_{k\geq 1}m_k(\Phi)$ can be rewritten similarly as eq.(5.5), but now we consider generally when $\sum_{k\geq 1}m_k(\Phi)$ is not zero, and the rewriting leads the following equation,

$$\sum_{k\geq 1}m_k(\Phi)=\sum_{k\geq 1}\tilde{m}_k(\tilde{\Phi})+Q^{+}Q\sum_{k\geq 2}m_k(\Phi)\ . \tag{5.10}$$

We can show from now that this equation is in fact the $\mathcal{H}^{\otimes 1}$ part of the equation $\mathfrak{m}\tilde{\mathcal{F}}(e^{\tilde{\Phi}})=\tilde{\mathcal{F}}\tilde{\mathfrak{m}}(e^{\tilde{\Phi}})$. Note that it is sufficient for the condition $\mathfrak{m}\tilde{\mathcal{F}}=\tilde{\mathcal{F}}\tilde{\mathfrak{m}}$ (by an inductive argument). By utilizing the A_∞-condition for $\mathfrak{m}$, the second term on the right hand side of eq.(5.10) becomes

$$Q^{+}m_1\sum_{k\geq 2}m_k(\Phi)=-Q^{+}\sum_{l\geq 2}\left(\mathfrak{m}_l(e^{\Phi}\sum_{k\geq 1}m_k(\Phi)e^{\Phi})\Big|_{\mathcal{H}^{\otimes 1}}\right)\ .$$

where $|_{\mathcal{H}^{\otimes 1}}$ means picking up the $\mathcal{H}^{\otimes 1}$ part from $C(\mathcal{H})$. Thus one gets the following recursive formula,

$$\mathfrak{m}_*(e^{\Phi})=\tilde{\mathfrak{m}}_*(e^{\tilde{\Phi}})-Q^{+}\sum_{l\geq 2}\left(\mathfrak{m}_l(e^{\Phi}\mathfrak{m}_*(e^{\Phi})e^{\Phi})|_{\mathcal{H}^{\otimes 1}}\right)\ . \tag{5.11}$$

Since Φ is represented in the expansion of the power of $\tilde{\Phi}$, the above equation is satisfied separately in the homogeneous degree of $\tilde{\Phi}$. Here we argue inductively and suppose that $(\mathfrak{m}\tilde{\mathcal{F}}-\tilde{\mathcal{F}}\tilde{\mathfrak{m}})e^{\tilde{\Phi}}|_{\mathcal{H}^{\otimes 1}}=0$ is satisfied at degree $(\tilde{\Phi})^k$ with $1\leq k\leq n-1$. Then consider the $(\tilde{\Phi})^n$ parts of this equation. The left hand side $\mathfrak{m}_*(e^\Phi)$ involves n powers of $\tilde{\Phi}$, however the $\mathfrak{m}_*(e^\Phi)$ has $(\tilde{\Phi})^k$ with $k \leq n-1$ because the summention for l begins at $l=2$. By the induction hypothesis, $e^\Phi \mathfrak{m}_*(e^\Phi) e^\Phi$ in the second term on the right hand side of the equation (5.11) can be rewritten when restricted to $(\tilde{\Phi})^n$ part as

$$\begin{aligned} e^\Phi \mathfrak{m}_*(e^\Phi)e^\Phi|_{\mathcal{H}^{\otimes l\geq 2},(\tilde{\Phi})^n} &= \mathfrak{m}\tilde{\mathcal{F}}(e^{\tilde{\Phi}})|_{\mathcal{H}^{\otimes l\geq 2},(\tilde{\Phi})^n} = \tilde{\mathcal{F}}\tilde{\mathfrak{m}}(e^{\tilde{\Phi}})|_{\mathcal{H}^{\otimes l\geq 2},(\tilde{\Phi})^n} \\ &= \tilde{\mathcal{F}}\left(e^{\tilde{\Phi}}\tilde{\mathfrak{m}}_*(e^{\tilde{\Phi}})e^{\tilde{\Phi}}\right)\Big|_{\mathcal{H}^{\otimes l\geq 2},(\tilde{\Phi})^n}\ , \end{aligned}$$

where the induction hypothesis is used in the second equality. Now the $(\tilde{\Phi})^n$ parts of the right hand side of eq.(5.11) is

$$\tilde{\mathfrak{m}}_*(e^{\tilde{\Phi}})|_{(\tilde{\Phi})^n} - Q^+ \sum_{l\geq 2}\left(\mathfrak{m}_l \tilde{\mathcal{F}}\left(e^{\tilde{\Phi}}\tilde{\mathfrak{m}}_*(e^{\tilde{\Phi}})e^{\tilde{\Phi}}\right)\right)\Big|_{\mathcal{H}^{\otimes 1},(\tilde{\Phi})^n}\ ,$$

which is exactly equal to $\tilde{\mathcal{F}}\tilde{\mathfrak{m}}(e^{\tilde{\Phi}})|_{\mathcal{H}^{\otimes 1},(\tilde{\Phi})^n}$. This completes the proof by the induction that $(\mathfrak{m}\tilde{\mathcal{F}} - \tilde{\mathcal{F}}\tilde{\mathfrak{m}})(e^{\tilde{\Phi}})|_{(\mathcal{H})^{\otimes 1}} = 0$ [14].

Once getting $\mathfrak{m}\tilde{\mathcal{F}} = \tilde{\mathcal{F}}\tilde{\mathfrak{m}}$, it is easy to show that $\tilde{\mathfrak{m}}$ defines an A_∞-algebra. As was noted in (Rem.5.4), $\tilde{\mathcal{F}}$ has its inverse isomorphism $(\tilde{\mathcal{F}})^{-1}$. Acting the $(\tilde{\mathcal{F}})^{-1}$ on the both sides of $\mathfrak{m}\tilde{\mathcal{F}} = \tilde{\mathcal{F}}\tilde{\mathfrak{m}}$ from left and $\tilde{\mathfrak{m}}$ can be expressed as $\tilde{\mathfrak{m}} = \tilde{\mathcal{F}}^{-1}\mathfrak{m}\tilde{\mathcal{F}}$. We then get $(\tilde{\mathfrak{m}})^2 = 0$ immediately from $(\mathfrak{m})^2 = 0$. Thus we has concreted the proof of the statements that $\tilde{\mathfrak{m}}$ gives an A_∞-structure on $(\mathcal{H}, \tilde{\mathfrak{m}})$ and $\mathcal{F}$ is an A_∞-morphism between $(\mathcal{H}, \tilde{\mathfrak{m}})$ and $(\mathcal{H}, \mathfrak{m})$. ■

Remark 5.4 The fact proved above can be applied to show the following statement. Let $(\mathcal{H}, \mathfrak{m})$ and $(\mathcal{H}', \mathfrak{m}')$ be two A_∞-algebras, and let an A_∞-quasi-isomorphism $\mathcal{F}$ from $(\mathcal{H}, \mathfrak{m})$ to $(\mathcal{H}', \mathfrak{m}')$ is given. Then there exists an inverse A_∞-quasi-isomorphism $(\mathcal{F})^{-1} : (\mathcal{H}', \mathfrak{m}') \to (\mathcal{H}, \mathfrak{m})$. It can be now proved easily by applying the above results as follows. First, we can transform both A_∞-algebras $(\mathcal{H}, \mathfrak{m})$ and $(\mathcal{H}', \mathfrak{m}')$ to $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ and $(\mathcal{H}'^p, \tilde{\mathfrak{m}'}^p)$ by the canonical A_∞-quasi-isomorphisms $\tilde{\mathcal{F}}^p$ and $\tilde{\mathcal{F}'}^p$ in the above procedure. $\tilde{\mathcal{F}}^p$ and $\tilde{\mathcal{F}'}^p$ have their inverse quasi-isomorphisms, and the A_∞-quasi-isomorphism from $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}'^p, \tilde{\mathfrak{m}'}^p)$ $(\star)$ is then given by the composition $(\tilde{\mathcal{F}'}^p)^{-1} \circ \mathcal{F} \circ \tilde{\mathcal{F}}^p$

$$\begin{tikzcd} (\mathcal{H}, \mathfrak{m}) \arrow[r, "\mathcal{F}"] & (\mathcal{H}', \mathfrak{m}') \\ (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \arrow[u, "\tilde{\mathcal{F}}^p"] \arrow[r, "(\star)"] & (\mathcal{H}'^p, \tilde{\mathfrak{m}'}^p) \arrow[u, "\tilde{\mathcal{F}'}^p"] \end{tikzcd}$$

so that the diagram commutes. Because (the leading of) the quasi-isomorphism $(\star)$ is isomorphism, $(\star)$ has its inverse, and one can obtain an A_∞-quasi-isomorphism as $\tilde{\mathcal{F}}^p \circ (\star)^{-1} \circ (\tilde{\mathcal{F}'}^p)^{-1}$.

In subsection 5.4 and section 6 the canonical A_∞-quasi-isomorphism and the canonical A_∞-structure are applied to SFT in analogous way, though not precisely the same.

[14] The coefficient of $\phi^n \cdots \phi^1$ of the equation reads the identity (2.10).

5.2 Minimal model theorem in gauge fixed SFT

In the previous subsection, the analogue of the Hodge-Kodaira decomposition for the string Hilbert space $\mathcal{H}$ was given in eq.(5.2), and an A_∞-morphism $(\mathcal{H}, \tilde{\mathfrak{m}}^{(p)})$ is constructed using Q^+. We claim that this Q^+ is the propagator in SFT. In this subsection we will explain the statement. Note that this means that the diagrams defined in order to construct the A_∞-morphism $\tilde{\mathcal{F}}^{(p)}$ and A_∞-structure $\tilde{\mathfrak{m}}^{(p)}$ are actually the Feynman diagram in SFT.

In SFT, when one considers a propagator, first the action is gauge fixed. The propagator Q^+ is then defined as the inverse map of Q onto the cokernel of Q on the gauge. This propagator Q^+ actually gives the identity (5.2). Indeed define $P := \mathbf{1} - \{Q, Q^+\}$ and the identity is obtained. Here for simplicity, we will argue the properties of the propagator mainly in the Siegel gauge.

As was mentioned, the string field Φ includes both fields and antifields in the context of BV-formalism[37, 38, 39]. Let c_0 be a degree one operator. In order to express antifields explicitly, only in this subsection we represent the string field as $\Phi = \mathbf{e}_i\phi^i + \mathbf{e}_{\bar{i}}\phi^{\bar{i}}$ for $\mathbf{e}_{\bar{i}} := c_0\mathbf{e}_i$ and $\phi^{\bar{i}} := \bar{\phi}^i$. $\{\phi\}$ is the fields and $\{\bar{\phi}\}$ is the antifields. $\{\bar{\mathbf{e}}\} := \{\mathbf{e}_{\bar{i}}\}$ and $\{\mathbf{e}\} := \{\mathbf{e}_i\}$ are the basis of the string Hilbert space $\mathcal{H}$ which does and does not contain c_0, respectively. The degree of ϕ^i (resp. $\phi^{\bar{i}}$) is defined to be minus the degree of $\mathbf{e}_i$ (resp. $\mathbf{e}_{\bar{i}}$), so that the degree of Φ is zero. The degree of antifield $\phi^{\bar{i}}$ is defined to be minus the degree of the corresponding field ϕ^i minus one in the context of BV-formalism. In particular in the usual oscillator representation, let a_p be the matter oscillator of degree zero, b_q and c_r be the ghost oscillator of degree minus one and one, respectively. Then the 'antistate' $\mathbf{e}_{\bar{i}}$ corresponding to a state $\mathbf{e}_i \sim a_{-p_1}\cdots a_{-p_l}b_{-q_1}\cdots b_{-q_m}c_{-r_1}\cdots c_{-r_n}|0\rangle$ is taken to be $\mathbf{e}_{\bar{i}} \sim a_{-p_1}\cdots a_{-p_l}c_0c_{-q_1}\cdots c_{-q_m}b_{-r_1}\cdots b_{-r_n}|0\rangle$ with an appropriate normalization, where $p_k, \in \mathbb{Z}_{\geq 0}$, $q_k, r_k \in \mathbb{Z}_{>0}$ and $|0\rangle$ denotes the Fock space vacuum. The degree of the states are defined as the ghost number, that is, the number of c_{-r} (including c_0) minus the number of b_{-q} where the degree of the Fock vacuum $|0\rangle$ is counted here as zero. The degree of $\mathbf{e}_{\bar{i}}$ is then minus the degree of $\mathbf{e}_i$ plus one. Therefore the pair $\{\phi\}$ and $\{\bar{\phi}\}$ has consistent degree in BV-formalism [3] [15].

The Siegel gauge fixing is then $b_0\Phi = 0$, which restricts all the basis of $\mathcal{H}$ to the basis $\{\mathbf{e}\}$, in other words, restrict the space of fields to $\bar{\phi} = 0$. Express Q manifestly such as the c_0 including part and b_0 including part and the rest part,

$$Q = c_0L_0 + b_0M + \tilde{Q}\ .$$

The kinetic term $\frac{1}{2}\omega(\Phi, Q\Phi)|_{b_0\Phi=0}$ then reduces to $\frac{1}{2}\omega(\mathbf{e}_i\phi^i, c_0L_0\mathbf{e}_j\phi^j)$ and the propagator is defined as $Q^+ = b_0\frac{1}{L_0}$, which acts as the inverse of Q on the cokernel of L_0 in $\{\bar{\mathbf{e}}\}$. Here define the projection onto the kernel of L_0 in $\mathcal{H}$ as P. Q^+ is then extended to be the operator on $\mathcal{H}$, which is written as $Q^+ = b_0\frac{1}{L_0}(\mathbf{1} - P)$. Since Q commutes with Virasoro generators L_m, in particular with L_0, and L_0 does not include c_0 or b_0, L_0 commutes with c_0L_0, b_0M and $\tilde{Q}$ independently. Therefore Q^+ anticommutes with $\tilde{Q}$, and also does trivially with b_0M. Then $\{Q, Q^+\} = \{c_0L_0, b_0\frac{1}{L_0}(\mathbf{1} - P)\} = \mathbf{1} - P$, which is the desired form of the identity (5.2). Note

[15] In [3] the explicit correspondence between the operator description for SFT and BV-formalism for field theories is found.

that from this explicit form of Q^+ one can see that eq.(4.20) holds in the Siegel gauge (see for example [7]).

More generally, denote the propagator in Schwinger representation with cut-off Λ as

$$Q^+ = b_0 \int_0^\Lambda e^{-\tau L_0} d\tau \tag{5.12}$$

and we can define the identity (5.2). In this case P which satisfies the identity (5.2) is the boundary term of Q^+

$$P = e^{-\Lambda L_0} \ . \tag{5.13}$$

The cut-off Λ is IR cut-off in target space theory, but short distance cut-off in string world sheet[50, 7].

The $\tilde{\mathfrak{m}}$ in (Rem.5.1) with Q^+ such a SFT propagator in fact defines an A_∞-structure. No modification of the definition is needed. The proof that $\tilde{\mathfrak{m}}$ is an A_∞-structure and A_∞-quasi-isomorphic to $\mathfrak{m}$ requires only the identity (5.2) and the condition that QQ^+, Q^+Q or P is a projection is not necessary. The construction of the A_∞-algebra $(\mathcal{H}, \tilde{\mathfrak{m}})$ is thus considerably universal concept which is independent of some reguralization scheme. The fact holds true for $\tilde{\mathfrak{m}}^p$, but the story is a little different. Let us introduce Q^u of degree minus one and denote the Hodge-Kodaira decomposition of $\mathcal{H}$ as

$$QQ^u + Q^uQ + P^p = \mathbf{1} \ . \tag{5.14}$$

In this expression $P^t := QQ^u$, $P^u := Q^uQ$ and P^p are projections onto null states (Q-trivial states), unphysical states, physical states, respectively. Here define $\mathcal{H}^p$ as physical state space

$$\mathcal{H}^p := P^p\mathcal{H} \ . \tag{5.15}$$

$P\mathcal{H}$ is then different from $\mathcal{H}^p$. Especially $P\mathcal{H}$ which satisfies $QQ^+ + Q^+Q + P = \mathbf{1}$ includes unphysical states $\mathcal{H}^u$. This makes some trouble when we define the on-shell A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ because the image of $\tilde{m}^p_k$ on $(\mathcal{H}^p)^{\otimes k}$ is not guaranteed to belong to $\mathcal{H}^p$. However relating $\tilde{\mathfrak{m}}^p$ to string vertices provides us with the fact that actually $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ is an A_∞-algebra. This fact will be explained in the next subsection and Appendix B.3, where the relation between $\tilde{\mathfrak{m}}^p$ and on-shell string S-matrix elements is examined. Although the reduction to physical states $\mathcal{H}^p$ only has been mentioned above, one can also include the Q-trivial states and define the reduced A_∞-structure on $\mathcal{H}^t \cup \mathcal{H}^p$. It will also be explained there.

As was stated in eq.(4.19), we can also see from the above expression of Q^+ that $(Q^+)^2 = 0$ holds. $Q^+QQ^+ = Q^+$ holds only when P is a projection. P of the form in eq.(5.13) is not a projection and spoil the identity. However when some tree amplitudes are calculated with Feynman diagram in SFT, the P in $\{Q, Q^+\} = \mathbf{1} - P$ should only contribute to the poles. When the external states are put so that some propagators get poles, the amplitudes are not well-defined. Therefore we ignore the case and then one may define as $\{Q, Q^+\} = \mathbf{1}$ between the vertices. This leads $Q^+QQ^+ = Q^+$ even when P is not defined as a projection. This identity will not be used in (Lem.5.1),(Thm.5.1) and (Prop.5.1). It is used to rewrite SFT actions by field redefinitions in subsection 5.4 and section 6, but they are justified by (Prop.5.1). The origin of this problem can be found in Appendix B.3.

Thus one can obtain the A_∞-morphism $\tilde{\mathcal{F}}^{(p)}$ and A_∞-structure $\tilde{\mathfrak{m}}^{(p)}$, constructed in the previous subsection, in the context of SFT in BV-formalism. The A_∞-morphism constructed with $Q^+ = b_0\frac{1}{L_0}(1-P)$ is an A_∞-morphism from $(\mathcal{H}^{(p)}, \tilde{\mathfrak{m}}^{(p)})$ to $(\mathcal{H}, \mathfrak{m})$. Here on the Siegel gauge let us give more precise SFT interpretation of this A_∞-morphism $\tilde{\mathcal{F}}^{(p)}$ and A_∞-structure $\tilde{\mathfrak{m}}^{(p)}$ obtained here. It relates to the way of constructing some solutions of the equations of motions discussed for classical closed (non-polynomial) SFT in [47, 48]. The Maurer-Cartan equation (5.1) corresponding to the equation of motion for the original SFT restricted on the Siegel gauge $\bar{\phi} = 0$ is now written as

$$c_0 L_0 \mathbf{e}_j \phi^j + \mathbf{e}_{\bar{i}} \sum_{n\geq 2} m_n^{\bar{i}}(\mathbf{e}_{j_1}\phi^{j_1}, \cdots, \mathbf{e}_{j_n}\phi^{j_n}) = 0\ , \tag{5.16}$$

$$\tilde{Q}\mathbf{e}_j\phi^j + \mathbf{e}_i \sum_{n\geq 2} m_n^{i}(\mathbf{e}_{j_1}\phi^{j_1}, \cdots, \mathbf{e}_{j_n}\phi^{j_n}) = 0\ , \tag{5.17}$$

where $m_n^{\bar{i}}(\mathbf{e}_{j_1}\phi^{j_1}, \cdots, \mathbf{e}_{j_n}\phi^{j_n})$ means the coefficient of $\mathbf{e}_{\bar{i}}$ for $m_n(\mathbf{e}_{j_1}\phi^{j_1}, \cdots, \mathbf{e}_{j_n}\phi^{j_n})$ and similar for m_n^i. The first equation is the equation on $\{\bar{\mathbf{e}}\}$ and it is the equation of motion for the Siegel gauge fixed action. The second one is that on $\{\mathbf{e}\}$ and it means that the BRST transformation of the field $\{\phi\}$ which satisfies the equation of motion (the first equation) is zero on this Siegel gauge, where the gauge fixed BRST transformation acting on the fields $\{\phi\}$ is defined as [16]

$$\delta_{gf} = (\ , S)|_{\bar{\phi}=0} = \sum_i \frac{\overleftarrow{\partial}}{\partial\phi^i}\frac{\overrightarrow{\partial} S}{\partial\phi^{\bar{i}}}\Big|_{\bar{\phi}=0}\ . \tag{5.18}$$

One can easily see that eq.(5.17) is nothing but the statement that $\frac{\overrightarrow{\partial} S}{\partial\phi^{\bar{i}}}$ in the right hand side of the above equation is equal to zero. On the other hand, there exists the Maurer-Cartan equation on $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$. To represent the $\{\mathbf{e}\}$ part and $\{\bar{\mathbf{e}}\}$ part separately similarly as in eq.(5.16) and (5.17), these are of the form

$$\mathbf{e}^p_{\bar{i}} \sum_{n\geq 2} \tilde{m}_n^{p,\bar{i}}(\mathbf{e}^p_{j_1}\tilde{\phi}^{j_1}, \cdots, \mathbf{e}^p_{j_n}\tilde{\phi}^{j_n}) = 0\ , \tag{5.19}$$

$$\mathbf{e}^p_{i} \sum_{n\geq 2} \tilde{m}_n^{p,i}(\mathbf{e}^p_{j_1}\tilde{\phi}^{j_1}, \cdots, \mathbf{e}^p_{j_n}\tilde{\phi}^{j_n}) = 0 \tag{5.20}$$

on the Siegel gauge. As was explained in subsection 2.3, A_∞-morphisms preserve the solutions of Maurer-Cartan equations. Eq.(5.4) reads that the A_∞-morphism $\tilde{\mathcal{F}}^p$ is given by $\Phi|_{b_0\Phi=0} = \Phi^p|_{b_0\Phi=0} + \Phi^u|_{b_0\Phi=0}$ and

$$\begin{aligned}\Phi^u|_{b_0\Phi=0} &= -\sum_{k\geq 2} b_0\frac{1}{L_0}(\mathbf{1}-P)m_k(\mathbf{e}^p_j\tilde{\phi}^j + \mathbf{e}^u_l\phi^l)\\ &= -\sum_{k\geq 2}\frac{1}{L_0}(\mathbf{1}-P)\mathbf{e}_i m_k^{\bar{i}}(\mathbf{e}^p_j\tilde{\phi}^j + \mathbf{e}^u_l\phi^l)\end{aligned} \tag{5.21}$$

[16]By definition (5.18), the gauge fixed BRST-transformation is nilpotent and keeps the action invariant only up to the (gauge fixed) equation of motion, which are the standard facts in BV-formalism.

on the Siegel gauge, where $\mathbf{e}^p_j \in P^p\{\mathbf{e}\}$ and $\mathbf{e}^u_l \in P^u\{\mathbf{e}\}$. Substituting $\Phi^u|_{b_0\Phi=0} = \mathbf{e}^u_l\phi^l$ on the left hand side of the above equation into the right hand side recursively, we obtain the A_∞-morphism $\tilde{\mathcal{F}}^p = \{\tilde{f}^p_k\}_{k\geq 2}$ with $\tilde{f}^p_k : (P^p\{\mathbf{e}\})^{\otimes k} \to \{\mathbf{e}\}$. The fact that $\tilde{\mathcal{F}}^p$ preserves the Maurer-Cartan equations is then easily seen because substituting eq.(5.21) into eq.(5.16) and (5.17) leads eq.(5.19) and (5.20), respectively. Note that the A_∞-morphism $\Phi|_{b_0\Phi=0} = \Phi^p|_{b_0\Phi=0} + \tilde{f}(\Phi^p|_{b_0\Phi=0})$ given by eq.(5.21) is rewritten as

$$
\begin{aligned}
\Phi|_{b_0\Phi=0} &= \Phi^p|_{b_0\Phi=0} - \sum_{k\geq 2}\sum_i \frac{1}{L_0}(\mathbf{1}-P)\mathbf{e}_i\omega(\mathbf{e}_i, m_k(\mathbf{e}^p_j\tilde{\phi}^j + \mathbf{e}^u_l\phi^l)) \\
&= \Phi^p|_{b_0\Phi=0} - \sum_{k\geq 2}\sum_i \frac{1}{L_0}(\mathbf{1}-P)\mathbf{e}_i\tilde{\mathcal{V}}_{k+1}(\mathbf{e}_i, \mathbf{e}^p_{j_1}\tilde{\phi}^{j_1}, \cdots, \mathbf{e}^p_{j_k}\tilde{\phi}^{j_k})
\end{aligned}
\tag{5.22}
$$

where $\tilde{\mathcal{V}}_{k+1}$ denotes the tree $k+1$ point (off-shell) correlation function. The second equality is justified in the next subsection. When $\Phi^p|_{b_0\Phi=0}$ satisfies the equation of motion (5.19), eq.(5.22) gives the solution of eq.(5.16). In fact the A_∞-morphism restricted on the Siegel gauge (5.21) is derived by regarding only eq.(5.16) as a Maurer-Cartan equation and applying the arguments in the previous subsection. The solutions derived in [47, 48] for closed SFT are exactly this $\Phi|_{b_0\Phi=0}$ in eq.(5.22). $\Phi^p|_{b_0\Phi=0} = \mathbf{e}^p_i\tilde{\phi}^i$ express the condensation of marginal operators. In [47, 48] the zero momentum dilaton condensation is discussed in closed SFT. The condensation of the zero momentum states corresponding to background g and B is also considered[48]. In both case the obstruction eq.(5.19) is expected to vanish and all $\Phi^p|_{b_0\Phi=0}$ given by eq.(5.22) are the solutions, but generally there exists the obstruction (5.19). Furthermore, even if in the neighborhood of the origin the Siegel gauge is consistent in the sense in subsection 4.2, it is not necessarily true apart the origin. In order for the solutions to be the ones on which the space of the Siegel gauge condition $\bar{\phi} = 0$ is transversal to the gauge orbit, one must confirm that the solution actually satisfies eq.(5.20). This is equivalent to the condition that on the solution of eq.(5.16) the gauge fixed BRST-transformation is zero. In *tachyon condensation* in Discussions, a few comments about the solution describing the tachyonic vacuum[35] are presented from these viewpoints.

Though in the above argument the Siegel gauge is considered for simplicity, one can take another gauge for constructing Q^+. Generally in the context of BV-formalism the antifields are gauge fixed as $\phi^{\bar{i}} = \frac{\partial\Psi(\phi)}{\partial\phi^i}$, where $\Psi(\phi)$ has degree minus one in order for $\phi^{\bar{i}}$ to have consistent degree and in this reason is called *gauge fixing fermion*. In order to keep the gauge fixed kinetic term quadratic, here we consider the quadratic gauge fixing fermion. It is generally of the form

$$\Psi(\phi) = \phi^i M_{ij}\phi^j$$

where M_{ij} is an appropriate $\mathbb{C}$ valued matrix. By this gauge fixing, the antifield is restricted to

$$\phi^{\bar{i}} = M_{ij}\phi^j \ .$$

The string field Φ is then restricted to $\Phi|_{gf} = \mathbf{e}_i\phi^i + \mathbf{e}_{\bar{i}}M_{ij}\phi^j = (\mathbf{e}_i + \mathbf{e}_{\bar{j}}M_{ji})\phi^i$. Let $S_0|_{gf} := \frac{1}{2}\omega(\Phi|_{gf}, Q\Phi|_{gf})$ be the gauge fixed kinetic term and the propagator Q^+ is then defined as this inverse. The most important thing here is that the propagator Q^+ with the identity (5.2) $QQ^+ + Q^+Q + P = \mathbf{1}$ always satisfies

$$P^pP = P^p \tag{5.23}$$

independent of the choice of the gauge. Note that for (Thm.5.1), only this identity is needed for Q^+. Indeed the P in Siegel gauge (5.13) satisfies this identity, and it is clear that in any other gauge QQ^+ or Q^+Q can not detect the physical states similarly as $P^t = QQ^u$ and $P^u = Q^uQ$. In addition $(Q^+)^2 = 0$ holds, and $Q^+QQ^+ = Q^+$ is assumed as argued above on the Siegel gauge.

We will see in the next subsection that in any gauge as far as the identity (5.23) holds $(\mathcal{H}, \tilde{\mathfrak{m}}^p)$ defines an A_∞-algebra. In addition we will propose there that for general configuration of fields of $\Phi|_{gf}$ one may choose the appropriate gauge fixing and construct the A_∞-morphism.

5.3 On-shell reduction of classical SFT

In subsection 5.1, the A_∞-structure $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ naturally induced from the A_∞-algebra $(\mathcal{H}, \mathfrak{m})$ was given. At the same time, an A_∞-morphism $\tilde{\mathcal{F}}^p$ from A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}, \mathfrak{m})$ was constructed. Here let $(\mathcal{H}, \mathfrak{m})$ be the A_∞-algebra which defines a classical open SFT action $S(\Phi)$. Then by comparing the construction of SFT in section 3 and the construction of the A_∞-structure $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ in subsection 5.1, we can see that the n-point vertex defined by A_∞-structure $\tilde{\mathfrak{m}}^p$ is nothing but the tree level n-point correlation function of open string(Lem.5.1). (Thm.5.1) in the next subsection immediately follows from this fact.

Here will explain the fact. Because the construction of SFT as in section 3 guarantees that the scattering amplitudes of the SFT with A_∞-structure $(\mathcal{H}, \mathfrak{m})$ computed by the Feynman rule reproduce the correlation function of open string on-shell, what should be shown is that the scattering amplitudes of the SFT computed by the Feynman rule coincides with $-\frac{1}{n}\omega(\mathbf{e}^p_{i_1}, \tilde{m}^p_{n-1}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n}))$ where $\mathbf{e}^p_{i_1}, \cdots, \mathbf{e}^p_{i_n} \in \mathcal{H}^p$ are the external states of the amplitude.

The n-point amplitude is computed with the Feynman rules as follows. Formally, when a SFT action $S = S_0 + \mathcal{V}$ is given, the scattering amplitudes are computed with the partition function of SFT,

$$Z = \int \mathcal{D}\Phi e^{-S} = \int \mathcal{D}\Phi \left(\sum_{k=1}^{\infty} \frac{1}{k!} (-\mathcal{V}_3 - \mathcal{V}_4 - \cdots)^k \right) e^{-S_0} \ . \tag{5.24}$$

In the second equality, $e^{-\mathcal{V}} = e^{-(\mathcal{V}_3 + \mathcal{V}_4 + \cdots)}$ is expanded as a perturbation. Represent the vertex $\mathcal{V}_k$ as $\mathcal{V}_k = \frac{1}{k}\langle V_k||\Phi\rangle_1 \cdots |\Phi\rangle_k$. Fixing a gauge, constructing the propagator in the gauge, and its contraction between any two vertices $\langle V_{v_1}|$ and $\langle V_{v_2}|$ are described as

$$\langle V_{v_1}|\langle V_{v_2}|Q^+|\omega\rangle_{ab} \tag{5.25}$$

where $|\omega\rangle_{ab}$ is the inverse reflection operator (4.7), and the indices a, b denote that the propagator connects the a-th legs of the vertices $\langle V_{v_1}|$ with the b-th legs of the vertices for $1 \le a \le v_1$ and $1 \le b \le v_2$. Since the value of eq.(5.25) does not rely on whether Q^+ operates on the ket $|\ \rangle_a$ or on the ket $|\ \rangle_b$, the index for Q^+ is omitted. The Feynman rule is then defined by the usual Wick contraction using eq.(5.25) [17]. We are interested in tree amputated amplitudes. When the tree

[17] Concerning the relation between this Feynman rule of the world sheet picture and the one of component field theory picture, the reference [3] also provides us with useful information.

n-point amplitude with the external propagators is defined in the path integral form, the Wick contraction with the propagators can be divided into two processes : the contraction between the n external states and the vertices, and the contraction between the vertices. Performing the latter process leads some function of n powers of Φ. Here define it as

$$-\frac{1}{n}\langle\tilde{V_n}||\Phi\rangle\cdots|\Phi\rangle = -\frac{1}{n}\tilde{\mathcal{V}}_n(\Phi,\cdots,\Phi)\ . \tag{5.26}$$

The former process, contracting the Φ in $\tilde{\mathcal{V}}_n(\Phi,\cdots,\Phi)$ with n external fields, finishes the calculation of the amplitude. Instead, the coefficient of $\phi^{i_n}\cdots\phi^{i_1}$ for eq.(5.26) reads

$$-\frac{1}{n}\langle\tilde{V_n}||\mathbf{e}_{i_1}\rangle\cdots|\mathbf{e}_{i_n}\rangle = -\frac{1}{n}\tilde{\mathcal{V}}_{i_1\cdots i_n}\ , \tag{5.27}$$

the amputated n-point amplitude with external states $\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_n}$. Restricting the external states to physical states leads the on-shell n-point amplitude and express it as

$$-\frac{1}{n}\langle\tilde{V_n}||\mathbf{e}^p_{i_1}\rangle\cdots|\mathbf{e}^p_{i_n}\rangle = -\frac{1}{n}\tilde{\mathcal{V}}^p_{i_1\cdots i_n}\ . \tag{5.28}$$

We can also include the Q-trivial states for the external states but the on-shell amplitudes vanish even if one of the external states is Q-trivial. To avoid to increase the notation, here we argue in the case of physical state space $\mathcal{H}^p$ only.

More precisely, when computing the Feynman diagram we should comment about the gauge fixing . Generally let $\mathcal{O}(\Phi^n)$ be any operators of $(\Phi)^n$ powers, then its expectation value is calculated as

$$\langle\mathcal{O}(\Phi^n)\rangle \sim \int\mathcal{D}\Phi\ \mathcal{O}(\Phi^n)e^{-S}\Big|_{gf} = \int\mathcal{D}\Phi_{gf}\ \mathcal{O}(\Phi^n_{gf})e^{-\sum_{k\geq 3}\frac{1}{k}\mathcal{V}_k(\Phi_{gf},\cdots,\Phi_{gf})}e^{-S_0|_{gf}} \tag{5.29}$$

where $|_{gf}$ denotes a gauge fixing discussed in the previous subsection, Φ_{gf} denotes the gauge fixed Φ, and $S_0|_{gf}$ means S_0 but Φ in S_0 is replaced by Φ_{gf}. The propagator Q^+ is derived from this gauge fixed kinetic term $S_0|_{gf}$. The expectation value $\langle\mathcal{O}(\Phi^n)\rangle$ does not depend on the gauge fixing only when $\mathcal{O}(\Phi^n)$ is a gauge invariant operator. The amputated n point amplitudes $-\frac{1}{n}\tilde{\mathcal{V}}_{i_1\cdots i_n}$ generally depend on the gauge. It is calculated by using the propagator Q^+, which depends on the choice of the gauge. As will be seen, the vertex $\tilde{\mathcal{V}}_{i_1\cdots i_n}$ relates to the A_∞-structure $\tilde{\mathfrak{m}}$, which implies that the set of the vertices satisfies BV-master equation with an appropriate symplectic structure $\tilde{\omega}$. Here in any gauge we can define an A_∞-structure. Thus there are the ambiguities of the gauge choice when constructing the A_∞-structure $\tilde{\mathfrak{m}}$. Mathematically the propagator is related to a homotopy operator. It is interesting that the ambiguities of homotopy operators are physically those of the propagators through the choice of the gauge, and are those of constructing higher vertices $\tilde{\mathcal{V}}$ which satisfies BV-master equation. However from the expression in eq.(5.29), it is natural that the vertices $\tilde{\mathcal{V}}_k(\Phi,\cdots,\Phi)$ is defined such that its value on $\Phi|_{gf}$ is $\tilde{\mathcal{V}}_k(\Phi_{gf},\cdots,\Phi_{gf})$ calculated by the propagator with the gauge $|_{gf}$. We then propose this definition for the A_∞-structure $\tilde{\mathfrak{m}}$ (and an A_∞-quasi-isomorphism $\tilde{\mathcal{F}}$ which induces the transformation from the original A_∞-structure $\mathfrak{m}$ to $\tilde{\mathfrak{m}}$), though later arguments do not depend on this choice. Any way, the on-shell amplitudes $-\frac{1}{n}\tilde{\mathcal{V}}^p_{i_1\cdots i_n}$ is gauge invariant and is independent of the choice of the gauge. In this subsection the arguments are restricted to the on-shell physics and it is shown below that $\tilde{\mathcal{V}}^p_{i_1\cdots i_n}$ defines the A_∞-structure $\tilde{\mathfrak{m}}^p$. Hereafter in this paper $|_{gh}$ is omitted.

By construction, eq.(5.26), (5.27) or (5.28) are defined as the sum of Feynman diagrams which is related to $\Gamma_{n-1} \in G_{n-1}$ in (Def.5.1). Here will show that the on-shell n-point amplitudes coincide with the amplitudes defined by $\tilde{m}^p_{n-1}$. In order to see that, the following two statements must be confirmed : each Feynman graph in eq.(5.28) coincides with some $\tilde{m}_{\Gamma_{n-1}}$, and all the weight of these Feynman graphs are one because $\tilde{m}_{n-1}$ is constructed by summing over each $\tilde{m}_{\Gamma_{n-1}}$ with weight one. We shall confirm these two statements at the same time below.

As argued in section 3, when the number of the propagators in one of these Feynman graphs is I, the number of the vertices is $I+1$, and we have an identity $n = \sum_{m=1}^{I+1} v_m - 2I$, where v_m is the number of the legs of the vertex. Because the tree diagrams are considered, there are not more than one propagator between any two vertices. Each term of eq.(5.26) corresponding to each Feynman graph is then represented as

$$(sym.\ fac.)\ \frac{-1}{v_1}\langle V_{v_1}| \cdots \frac{-1}{v_{I+1}}\langle V_{v_{I+1}}|\ (Q^+|\omega\rangle)^I(|\Phi\rangle)^n \tag{5.30}$$

where $(sym.\ fac.)$ means the symmetric factor appearing if $v_i = v_j$ for any $i \neq j$. Such factor comes from the coefficient of the Taylor expansion of $e^{-\mathcal{V}_{v_m}}$ in eq.(5.24). Both $Q^+|\omega\rangle$ and Φ have degree zero, so eq.(5.30) does not depend on the order of them. Each $Q^+|\omega\rangle$ connects any two external states of any two different vertices to each other. The simplest term is that of $I=0$, which is $-\frac{1}{n}\langle V_n|(|\Phi\rangle)^n$. Therefore it is clear that $\langle \tilde{V}^p_n| = \langle V_n| + \cdots$ where $\cdots$ are the terms of $I \geq 1$. Note that $\langle \tilde{V}_n|$ is cyclic-symmetrized by construction. Therefore for each term choose $\mathbf{e}^p_{i_1}$ from $(\Phi)^n$, assign $\mathbf{e}^p_{i_2}\cdots\mathbf{e}^p_{i_n}$ to the rest $(\Phi)^{n-1}$ with cyclic order, and the on-shell amplitude eq.(5.28) with external states $\mathbf{e}^p_{i_1},\cdots,\mathbf{e}^p_{i_n}$ is obtained by summing over all these graphs and dividing by n. This n is represented explicitly in the expression (5.26),(5.27) and (5.28). Let us concentrate on one of such n point tree Feynman graph. We choose $\mathbf{e}^p_{i_1}$ as the end points of the graph and introduce an orientation on the edge of the graphs as follows. Let the vertex one of whose external states is $\mathbf{e}^p_{i_1}$ be $\langle V_{v_1}|$. On the edge between vertex $\langle V_{v_1}|$ and $\mathbf{e}^p_{i_1}$, we introduce the orientation from $\langle V_{v_1}|$ to $\mathbf{e}^p_{i_1}$ [18]. Other edges connected to $\langle V_{v_1}|$ are ordered so that the orientations on them flow into $\langle V_{v_1}|$. We write the edge connected to $\mathbf{e}^p_{i_1}$ on the left hand side and the others on the right hand side of the vertex $\langle V_{v_1}|$ keeping its cyclic order (see the second step of an example in (Fig.3)). Some of the edges written on the right hand side connect to other vertices. We write the edges of those vertices, except the edge connecting to $\langle V_{v_1}|$, on the right hand side of those vertices keeping its cyclic order, and the orientations are ordered so that the flow of each edge is from right to left. Repeating this, we get a tree graph. An example of the Feynman graph in (Fig.1) in section 3 is figured in (Fig.3). The above procedure gives the one-to-one correspondence between the n-point tree graphs of with external states fixed and the tree graphs $\Gamma_{n-1} \in G_{n-1}$ defined in (Def.5.1). The value corresponding to the Feynman graph

[18] An edge corresponds to a propagator $Q^+|\omega\rangle_{ab}$. The $|\omega\rangle_{ab}$ is symmetric with respect to the label a, b and eq.(5.25) does not rely on whether Q^+ operates on $|\ \rangle_a$ or on $|\ \rangle_b$. Therefore this orientation of the edges does not have any physical meaning.

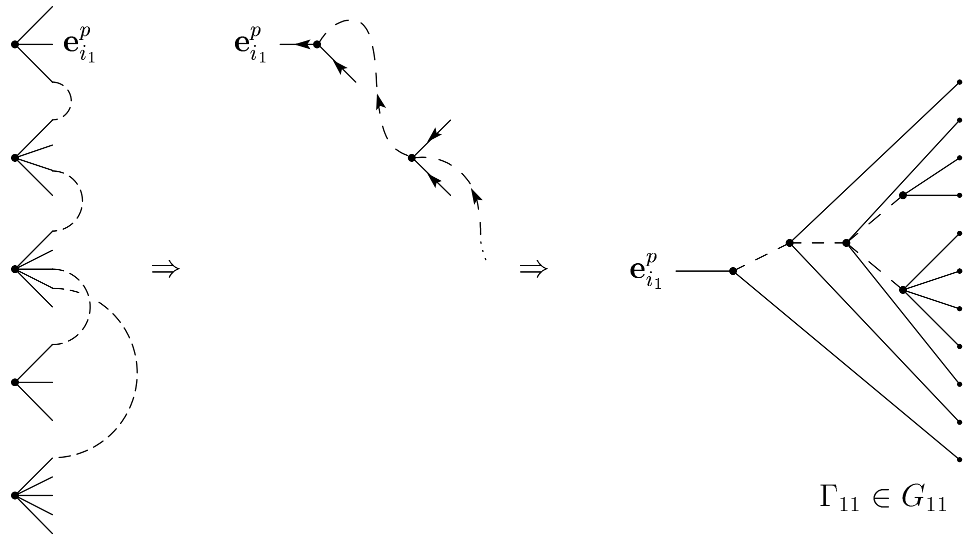


Figure 3: The graph on the left hand side is rewritten as the one on the right hand side through introducing the orientation on each edges and enjoying the graphical representations in subsection 4.3. Here on the left hand side, we adjust the outgoing legs of vertices to the top by employing the cyclic symmetry of the vertices. In this process for each vertex v_m number of graphs are identified, and the factor $\frac{1}{v_m}$ in front of $\langle V_{v_m}|$ cancels. $\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_{12}}$ is assigned to the rest edges in cyclic order.

is expressed in terms of $\langle V|$, Q^+ and $|\omega\rangle$ as in eq.(5.30), but we can rather express in terms of m_k, Q^+ and $|\omega\rangle$. It can be done using the relations in subsection 4.3 such as

$$\langle V_{k+1}|\ |\omega\rangle = m_k\ . \tag{5.31}$$

The contribution of a graph Γ_{n-1} to $\tilde{\mathcal{V}}^p_{i_1 \cdots i_n}$ in eq.(5.28) is then evaluated as

$$-(sym.fac.) \cdot \frac{1}{v_1 \cdots v_{I+1}} (-1)^I (-1)^{I+1} \omega(\mathbf{e}^p_{i_1}, \tilde{m}^p_{\Gamma_{n-1}}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n}))\ , \tag{5.32}$$

where $(-1)^{I+1}$ comes from the vertices and $(-1)^I$ appears because Q^+ in eq.(5.30) is replaced to $-Q^+$ and the above equation includes I number of propagators Q^+. In practice, let us evaluate the value corresponding to the graph Γ_{11} in (Fig.3). It is

$$\begin{aligned}
&-\left(\frac{1}{2}\frac{1}{2}\right) \frac{(-1)^4(-1)^5}{3 \cdot 4 \cdot 5 \cdot 3 \cdot 5} \langle V_3|\langle V_4|\langle V_5|\langle V_3|\langle V_5|Q^+|\omega\rangle_{(12)} Q^+|\omega\rangle_{(23)} Q^+|\omega\rangle_{(34)} Q^+|\omega\rangle_{(35)} |\mathbf{e}^p_{i_1}\rangle \cdots |\mathbf{e}^p_{i_{12}}\rangle \\
&= \left(\frac{1}{2}\frac{1}{2}\right) \frac{1}{3 \cdot 4 \cdot 5 \cdot 3 \cdot 5} \cdot \\
&\omega(\mathbf{e}_{i_1}, m_2(-Q^+ m_3(\mathbf{e}^p_{i_2}, -Q^+ m_4(\mathbf{e}^p_{i_3}, -Q^+ m_2(\mathbf{e}^p_{i_4}, \mathbf{e}^p_{i_5}), -Q^+ m_4(\mathbf{e}^p_{i_6}, \mathbf{e}^p_{i_7}, \mathbf{e}^p_{i_8}, \mathbf{e}^p_{i_9}), \mathbf{e}^p_{i_{10}}), \mathbf{e}^p_{i_{11}}), \mathbf{e}^p_{i_{12}}) \\
&=: \left(\frac{1}{2}\frac{1}{2}\right) \frac{1}{3 \cdot 4 \cdot 5 \cdot 3 \cdot 5} \omega(\mathbf{e}^p_{i_1}, \tilde{m}^p_{\Gamma_{11}}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_{12}}))\ .
\end{aligned}$$

On the first line of the above equation the vertices are labeled as $1 \cdots 5$ in the order, and the indices (ab) for $a, b = 1 \cdots 5$ denote that the propagator contracts vertices a with b.

On the other hand, the n point vertex given by $\tilde{\mathfrak{m}}^p$ is

$$\omega(\mathbf{e}^p_{i_1}, \tilde{m}^p_{n-1}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})) = \sum_{\Gamma_{n-1}\in G_{n-1}} \omega(\mathbf{e}^p_{i_1}, P\tilde{m}^p_{\Gamma_{n-1}}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})) \ .$$

The rest of the proof is then to confirm that the Feynman rule gives the contribution of a graph Γ_{n-1} to $\tilde{\mathcal{V}}^p_{i_1\cdots i_n}$ is $\omega(\mathbf{e}^p_{i_1}, \tilde{m}^p_{\Gamma_{n-1}}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n}))$ with weight $+1$ for each $\Gamma_{n-1} \in G_{n-1}$. First, the factor $\frac{1}{v_m}$ in front of $\langle V_{v_m}|$ cancels because choosing one outgoing edges from v_m legs creates the factor v_m as stated in (Fig.3). Next, if the graph includes k vertices with the same v_m, the exchanging of the vertices creates the factor $k!$, which cancels with the symmetric factor $(sym.fac.)$. Thus for each graph Γ_{n-1} the same weight $+1$ is obtained and one can get

$$\tilde{\mathcal{V}}^p_{i_1\cdots i_n} = \omega(\mathbf{e}^p_{i_1}, \sum_{\Gamma_{n-1}\in G_{n-1}} \tilde{m}^p_{\Gamma_{n-1}}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})) \ . \tag{5.33}$$

Here we claim that the above relation can also be written as

$$\tilde{\mathcal{V}}^p_{i_1\cdots i_n} = \omega(\mathbf{e}^p_{i_1}, \tilde{m}^p_{n-1}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})) \ . \tag{5.34}$$

It is guaranteed if P would be the projection onto physical states, but as was seen in the previous subsection, it fails. However the claim still holds. It follows from the existence of the orthogonal decomposition (4.23) and $P^pP = P^p$ (eq.(5.23)). ■

Lemma 5.1 $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ defines the A_∞-algebra of string S-matrix elements on $\mathcal{H}^p$, and the co-homomorphism $\tilde{\mathcal{F}}^p$ defined in (Def.5.1) is the A_∞-quasi-isomorphism between the A_∞-algebras $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ and $(\mathcal{H}, \mathfrak{m})$.

proof. The relation between $\tilde{\mathfrak{m}}^p$ and the string S-matrix elements was given in (5.34). However it is necessary to show that $\tilde{\mathfrak{m}}^p$ actually defines an A_∞-structure on physical states $\mathcal{H}^p$. Because P and P^p are different, if we use $\tilde{\mathfrak{m}}^p$ naively as an A_∞-structure, the image of $\tilde{m}^p_k$ does not possibly belong to $\mathcal{H}^p$, and if we define the A_∞-structure on $\mathcal{H}^p$ as the image of $\tilde{m}^p_k$ is projected onto $\mathcal{H}^p$, it is not guaranteed that it defines an A_∞-condition (2.6). However fortunately it can be shown that

$$\tilde{m}^p_{n-1}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n}) \ \in \mathcal{H}^t \cup \mathcal{H}^p$$

for any $\mathbf{e}^p_i \in \mathcal{H}^p$. The proof is given in Appendix B.3. As was mentioned, in addition to $\mathbf{e}^p_i \in \mathcal{H}^p$, any Q-trivial states can be included and the same result is obtained. The structure $\tilde{\mathfrak{m}}^p$ defines the A_∞-structure on on-shell states $\mathcal{H}^t \cup \mathcal{H}^p$. Furthermore as is shown in Appendix B.3, the A_∞-structure can be reduced on $\mathcal{H}^p$. We denote the A_∞-algebra by $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$. The following diagram is then obtained as the modified version in (Rem.5.1)

$$\begin{tikzcd}
(\mathcal{H},\tilde{\mathfrak{m}}) \arrow[r, "\tilde{\mathcal{F}}"] \arrow[d, shift right, "P^t+P^p"'] & (\mathcal{H},\mathfrak{m}) \\
(\mathcal{H}^t\cup\mathcal{H}^p,\tilde{\mathfrak{m}}^p) \arrow[u, shift right, "\iota"'] \arrow[ur, "\tilde{\mathcal{F}}^p"] \arrow[d, shift right, "P^p"'] & \\
(\mathcal{H}^p,\tilde{\mathfrak{m}}^p) \arrow[u, shift right, "\iota"'] \arrow[uur, "\tilde{\mathcal{F}}^p"] &
\end{tikzcd}$$

where $\tilde{\mathfrak{m}}$ and $\tilde{\mathcal{F}}$ are strictly the same ones in (Rem.5.1), and $\iota:\mathcal{H}^p\to\mathcal{H}^t\cup\mathcal{H}^p$, $\iota:\mathcal{H}^t\cup\mathcal{H}^p\to\mathcal{H}$ are the inclusion maps. $\iota:\mathcal{H}^p\to\mathcal{H}^t\cup\mathcal{H}^p$ is extended to an A_∞-quasi-isomorphism $\mathcal{F}^\iota=\{f_k^\iota\}_{k\geq 1}$ as $f_1^\iota=\iota$ and $f_2^\iota=f_3^\iota=\cdots=0$. We represent $\mathcal{F}^\iota$ simply as ι. Similarly $\iota:\mathcal{H}^t\cup\mathcal{H}^p\to\mathcal{H}$, P^p and P^t+P^p are regarded as A_∞-quasi-isomorphisms. The A_∞-structure $\tilde{\mathfrak{m}}^p$ on $\mathcal{H}^t\cup\mathcal{H}^p$ and $\mathcal{H}^p$ are then given by the composition $(P^t+P^p)\circ\tilde{\mathfrak{m}}\circ\iota$ and $P^p\circ\tilde{\mathfrak{m}}\circ\iota\circ\iota$, respectively. The A_∞-quasi-isomorphisms $\tilde{\mathcal{F}}^p:(\mathcal{H}^t\cup\mathcal{H}^p,\tilde{\mathfrak{m}}^p)\to(\mathcal{H},\mathfrak{m})$ and $\tilde{\mathcal{F}}^p:(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)\to(\mathcal{H},\mathfrak{m})$ are also given as $\tilde{\mathcal{F}}^p=\tilde{\mathcal{F}}\circ\iota$ and $\tilde{\mathcal{F}}^p=\tilde{\mathcal{F}}\circ\iota\circ\iota$, respectively. $\tilde{\mathcal{F}}$ and these two $\tilde{\mathcal{F}}^p$ also have the inverse A_∞-quasi-isomorphism similarly as (Rem.5.4).

As will be seen, when the A_∞-structure is applied together with the symplectic structure, the symplectic structure on the A_∞-algebra $(\mathcal{H}^t\cup\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ is degenerate and the A_∞-algebra $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ is more convenient. ■

We have been seen that there exists an A_∞-structure $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ on physical states $\mathcal{H}^p$ (or on on-shell Hilbert space $\mathcal{H}^t\cup\mathcal{H}^p$) and the n-point amplitudes defined by $\tilde{\mathfrak{m}}^p$ coincides with the on-shell n-point tree amplitudes of open strings. The fact that the n point correlation functions in two-dimensional theory possess the A_∞-structure are essentially already known. It is described in [32] that the S^2 tree amplitudes for closed strings has a L_∞-structure, where the external states are restricted to physical states and therefore it has vanishing Q. This implies that the tree level closed string free energy satisfies the classical BV-master equation. The result is extended to quantum closed string, and it is shown that the free energy which consists of the closed string loop amplitudes satisfies the quantum BV-master equation[33] [19]. Moreover, in [5], for classical closed SFT the $\mathcal{M}_k^0\to\mathcal{M}_k$ limits are considered, and it is shown that restricting the external states to physical states yields the L_∞-structure found in [32]. The open string version of this L_∞-structure is nothing but the A_∞-structure $\tilde{\mathfrak{m}}^p$ considered in this paper (and reviewed in Appendix B.2). What is obtained newly from the above result is that the A_∞-algebras associated with SFTs and the A_∞-algebra of two-dimensional theory are connected by an A_∞-morphism, which preserves the equation of motions, and the explicit form of the A_∞-morphism are realized with such a familiar language of the Feynman graph.

5.4 Field transformation between family of classical SFTs

In section 3 we construct SFTs and which are defined more precisely in subsection 4.3. As their propagator we consider the one which satisfies eq.(5.23). For the family of those well-

[19] In [32, 33] these structures are derived in the context of 2D-string theory, *i.e.* the dimension of the target space is two. However they are in fact the general structures of the string world sheet.

defined SFTs, (Lem.5.1) leads the main claim of the present paper (Thm.5.1) described below. In this subsection after proving it, we show that they preserve the value of the actions in a certain subspace of $\mathcal{H}$ using (Prop.5.1) presented later. The field transformations induce one-to-one correspondence of moduli spaces of classical solutions between such SFTs in the context of deformation theory. We will explain that the classical solutions are regarded as those corresponding to marginal deformations. Finally in this subsection a boundary SFT like action corresponding to the A_∞-algebra $(\mathcal{H}, \tilde{\mathfrak{m}})$ defined in (Rem.5.1) is proposed.

Theorem 5.1 All the well-defined classical SFTs which are constructed on a fixed conformal field theory are quasi-isomorphic to each other.

proof. As was explained in section 3, when the decomposition of the moduli space of Riemann surfaces is given, then the vertices of SFT are determined, *i.e.* SFT action is determined. Different decompositions of moduli space lead different SFTs. By construction the Feynman graphs for those SFTs should reproduce the string correlation functions on-shell. Here let $S(\Phi)$, $S'(\Phi')$ be such two SFT actions and $(\mathcal{H}, \mathfrak{m})$, $(\mathcal{H}, \mathfrak{m}')$ be the corresponding A_∞-algebras. (Lem.5.1) states that the set of the on-shell string correlation functions defines an A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ and the A_∞-algebras of SFTs $(\mathcal{H}, \mathfrak{m})$, $(\mathcal{H}, \mathfrak{m}')$ are A_∞-quasi-isomorphic to $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$. Thus the A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p$ from $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}, \mathfrak{m})$ and $\tilde{\mathcal{F}'}^p$ from $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}, \mathfrak{m}')$ exist. Note that on a fixed conformal background any such A_∞-algebras of SFTs are A_∞-quasi-isomorphic to the same A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$. The composition $\tilde{\mathcal{F}'}^p \circ (\tilde{\mathcal{F}}^p)^{-1}$ then defines the A_∞-quasi-isomorphism from $(\mathcal{H}, \mathfrak{m}')$ to $(\mathcal{H}, \mathfrak{m})$. This map is in fact a quasi-isomorphism because the inverse of quasi-isomorphism is a quasi-isomorphism and the composition of quasi-isomorphisms is a quasi-isomorphism. ■

This result indicates that the equations of motions for any classical SFTs constructed from the same on-shell S-matrix elements are transformed to each other by the above quasi-isomorphism $\tilde{\mathcal{F}'}^p \circ (\tilde{\mathcal{F}}^p)^{-1}$. However the quasi-isomorphism is not an isomorphism, it does not guarantee that there exists a field redefinition between them which preserves the value of the actions. In contrast, any diffeomorphisms $\mathcal{F}$ on $\mathcal{H}$ which preserve the value of the actions of the form

$$\Phi' = \mathcal{F}_*(\Phi) = f_1(\Phi) + f_2(\Phi, \Phi) + \cdots \tag{5.35}$$

are A_∞-isomorphisms, by defining the symplectic structures on $S(\Phi)$ and $S'(\Phi')$ so that the diffeomorphism $\mathcal{F}^*$ preserves these symplectic structures[44]. The statement that $\mathcal{F}$ preserves the values of the actions is, in other words, that $\mathcal{F}$ satisfies $S(\Phi) = \mathcal{F}^* S'(\Phi') := S'(\mathcal{F}_*(\Phi))$. The SFTs, which are characterized by the pair $(S(\Phi), \omega)$, are called equivalent when the SFTs are connected by such a diffeomorphism preserving the action and the symplectic structures[41]. The fact that the diffeomorphism which connects equivalent SFTs is an A_∞-morphism is clearly understood in the dual (component field) picture as follows. Express two string fields as $\Phi = \mathbf{e}_i \phi^i$ and $\Phi' = \mathbf{e}'_i {\phi'}^i$, and acting $\frac{\overleftarrow{\partial}}{\partial \phi^i} \omega^{ij} \frac{\overrightarrow{\partial}}{\partial \phi^j}$ on the identity $S(\Phi) = S'(\mathcal{F}_*(\Phi))$ leads

$$\frac{\overleftarrow{\partial}}{\partial \phi^i} \omega^{ij} \frac{\overrightarrow{\partial}}{\partial \phi^j} S(\Phi) = \frac{\overleftarrow{\partial}}{\partial \phi^i} \omega^{ij} \frac{\overrightarrow{\partial}}{\partial \phi^j} S'(\mathcal{F}_*(\Phi)) \ .$$

The left hand sides is exactly δ : the dual description of the A_∞-coderivative in eq.(2.9). The right hand side is rewritten as

$$\frac{\overleftarrow{\partial}}{\partial\phi'^k}\frac{\phi'^k\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}\phi'^l}{\partial\phi^j}\frac{\overrightarrow{\partial}}{\partial\phi'^l}S'(\Phi')\ .$$

Thus, if $\mathcal{F}$ preserves the symplectic structure or equivalently the Poisson structure is preserved

$$\mathcal{F}^*\omega'^{kl}=\frac{\phi'^k\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}\phi'^l}{\partial\phi^j}\ , \tag{5.36}$$

it is clear that the right hand side gives $\delta'=\frac{\overleftarrow{\partial}}{\partial\phi'^k}\omega'^{kl}\frac{\overrightarrow{\partial}}{\partial\phi'^l}S'(\Phi')=(\quad,S'(\Phi'))$. To summarize and extend the above arguments, one can obtain the following fact :

Proposition 5.1 When a cohomomorphism $\mathcal{F}$ of the form in eq.(5.35) between two actions $S(\Phi)$ and $S'(\Phi')$ and two symplectic structures which are preserved by the cohomomorphism $\mathcal{F}$ are given, then the following two statements are equivalent :

- $\mathcal{F}$ is an A_∞-morphism.
- $\mathcal{F}$ preserves the value of the action, that is, $S(\Phi)=\mathcal{F}^*S'(\Phi')$.

This equivalence follows from the fact that the symplectic structures on both sides are non-degenerate. Note that here f_1 may not be an isomorphism. (Prop.5.1) is well-defined and holds for general symplectic structures on $\mathcal{H}$ which depend on $\{\phi\}$. Because in this paper mainly we deal with the constant symplectic structures associated with the BPZ-inner product, we avoid the explanation of the issue in this paper (see *cyclic algebra with BV-Poisson structure* in subsection 4.3). Of course when considering the graded commutative fields such as $U(1)$ gauge fields, the symplectic structure is that on supermanifold[51] and is known to be well-defined.

Here we come back to the physical consequence of (Thm.5.1). We define the action $\tilde{S}(\tilde{\Phi}^p)$ on the A_∞-algebra $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ by summing up $\frac{1}{n}\tilde{\mathcal{V}}^p_n(\tilde{\Phi}^p,\cdots,\tilde{\Phi}^p)$ in eq.(5.26) as

$$\tilde{S}(\tilde{\Phi}^p)=\sum_{k\geq 2}\frac{1}{k+1}\omega(\tilde{\Phi}^p,\tilde{m}^p_k(\tilde{\Phi}^p))\ . \tag{5.37}$$

Now we have an A_∞-quasi-isomorphism $\tilde{\mathcal{F}}'^p\circ(\tilde{\mathcal{F}}^p)^{-1}$ between $(\mathcal{H},\mathfrak{m})$ and $(\mathcal{H},\mathfrak{m}')$. Though it has not assumed in the proof of (Thm.5.1), the $\tilde{\mathcal{F}}^p$ and $\tilde{\mathcal{F}}'^p$ preserve the value of the actions, that is, the A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p$ satisfies

$$\tilde{S}(\tilde{\Phi}^p)=(\tilde{\mathcal{F}}^p)^*S(\Phi)=S((\tilde{\mathcal{F}}^p)_*(\tilde{\Phi}^p)) \tag{5.38}$$

and similar for $\tilde{\mathcal{F}}'^p$. This fact yields that, on the subspace of $\mathcal{H}$ which is the image of $\tilde{\mathcal{F}}^p$ from $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$, any actions $S(\Phi)$ are preserved. Eq.(5.38) follows from (Prop.5.1) and the fact that the $\tilde{\mathcal{F}}^p$ preserves the symplectic structures. Let ω_{ij} be the symplectic structures on $(\mathcal{H},\mathfrak{m})$. Each term in $\tilde{S}(\tilde{\Phi}^p)$ was made of the A_∞-structure $\tilde{m}^p_k$ as $\frac{1}{k+1}\omega(\tilde{\Phi}^p,\tilde{m}_k(\tilde{\Phi}^p))$, so the symplectic structure on $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ is ω_{ij} restricted on $\mathcal{H}^p \subset \mathcal{H}$. On the other hand, define $\tilde{\omega}^p_{ij}$ as a symplectic structure on $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ which is preserved under the transformation $\tilde{\mathcal{F}}^p$. $\tilde{\omega}^p = (\tilde{\mathcal{F}}^p)^*\omega$ is written as [20]

$$\tilde{\omega}^p_{ij} = \frac{\overrightarrow{\partial}\phi^k}{\partial\tilde{\phi}^{p,i}}\omega_{kl}\frac{\phi^l\overleftarrow{\partial}}{\partial\tilde{\phi}^{p,j}} = (-1)^{\mathbf{e}^p_i}\omega\left(\frac{\overrightarrow{\partial}}{\partial\tilde{\phi}^{p,i}}\Phi, \Phi\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^{p,j}}\right) .$$

Here we choose the basis so that the inner product ω is decomposed orthogonally as in eq.(4.23). Since $\Phi = \tilde{\Phi}^p - Q^+\sum_{k\geq 2} m_k(\Phi)$ and the image of Q^+ vanishes in the symplectic inner product in the right hand side of the above equation by using eq.(4.20), the right hand side becomes $(-1)^{\mathbf{e}^p_i}\omega(\frac{\overrightarrow{\partial}}{\partial\tilde{\phi}^{p,i}}\tilde{\Phi}^p, \tilde{\Phi}^p\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^{p,j}}) = \omega_{ij}$. Thus the $\tilde{\omega}^p_{ij}$ coincides with the ω_{ij} restricted on $\mathcal{H}^p$, the map $\tilde{\mathcal{F}}^p$ from $\tilde{S}(\tilde{\Phi}^p)$ to $S(\Phi)$ preserves the symplectic structures, and (Prop.5.1) leads that eq.(5.38) holds [21].

Although the proof of eq.(5.38) has been concreted, it is interesting to observe $S((\tilde{\mathcal{F}}^p)_*(\Phi))$ directly by substituting $\Phi = \tilde{\mathcal{F}}^p_*(\tilde{\Phi}^p)$ in $S(\Phi)$

$$S((\tilde{\mathcal{F}}^p)_*(\Phi)) = \frac{1}{2}\omega\left(\tilde{f}^p(\tilde{\Phi}^p), Q\tilde{f}^p(\tilde{\Phi}^p)\right) + \sum_{k\geq 2}\frac{1}{k+1}\omega\left(\tilde{\Phi}^p + \tilde{f}^p(\tilde{\Phi}^p), m_k(\tilde{\Phi}^p + \tilde{f}^p(\tilde{\Phi}^p))\right) \quad (5.39)$$

and check that $S((\tilde{\mathcal{F}}^p)_*(\Phi))$ in fact coincides with $\tilde{S}(\tilde{\Phi}^p)$. The equation (5.39) is written as the power series of $\tilde{\Phi}^p$ and let us observe the $(\tilde{\Phi}^p)^n$ parts of eq.(5.39) for $n \geq 3$. For $n = 3$ the first term in the right hand side of this equation drops out and it can be seen clearly that $S((\tilde{\mathcal{F}}^p)_*(\Phi))|_{(\tilde{\Phi}^p)^{\otimes 3}} = \frac{1}{3}\omega(\tilde{\Phi}^p, \tilde{m}^p_2(\tilde{\Phi}^p, \tilde{\Phi}^p))$. Generally both first and second term contribute. Using $\tilde{m}^p_{\Gamma_k}$ in (Def.5.1), the contributions of the first and second terms to the terms of n powers of $\tilde{\Phi}^p$ are

$$\sum_{\Gamma_{k_1\geq 2}, \Gamma_{k_2\geq 2}, k_1+k_2=n} -\frac{1}{2}\omega(\tilde{m}^p_{\Gamma_{k_1}}(\tilde{\Phi}^p), -Q^+\tilde{m}^p_{\Gamma_{k_2}}(\tilde{\Phi}^p)) \quad (5.40)$$

and

$$\sum_{l\geq 3}^{n}\frac{1}{l}\sum_{\substack{\Gamma_{k_1\geq 1},\cdots,\Gamma_{k_l\geq 1},\\ k_1+\cdots+k_l=n}}\omega\left(-Q^+\tilde{m}^p_{\Gamma_{k_1}}(\tilde{\Phi}^p), m_{l-1}(-Q^+\tilde{m}^p_{\Gamma_{k_2}}(\tilde{\Phi}^p), \cdots, -Q^+\tilde{m}^p_{\Gamma_{k_l}}(\tilde{\Phi}^p))\right) , \quad (5.41)$$

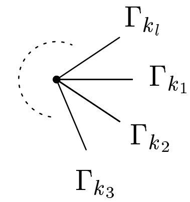
respectively. In eq.(5.40) $Q^+QQ^+ = Q^+$ is used. Here in eq.(5.41) we denoted $\tilde{\Phi}^p$ by $-Q^+\tilde{m}^p_{\Gamma_1}(\tilde{\Phi}^p)$. In the expression where the cyclicity of vertices $\{m_{l-1}\}$ are emphasized, eq.(5.40) and (5.41) are

[20] The equation below is actually equivalent to eq.(5.36). The equivalence follows from $\omega_{ij}\omega^{jk} = \delta^k_i$ and $\tilde{\omega}^p_{ij}\tilde{\omega}^{p,jk} = \delta^k_i$. The argument is well-defined even for the non-constant symplectic structure.

[21] In the proof the appropriate basis is chosen, but it can also be seen that the result is independent of the choice because ω_{ij} restricted on $\mathcal{H}^p$ is independent of it.

graphically pictured as

$$-\sum_\Gamma (sym.fac.) \quad \Gamma_{k_1} \;—\bullet—\; \Gamma_{k_2} \qquad + \qquad \sum_{l\geq 3}^{n} \sum_\Gamma (sym.fac.)$$



,

where Γ_k denotes that $\tilde{m}^p_{\Gamma_k}(\tilde{\Phi}^p)$ is in this place. The summations $\sum_\Gamma$ in the first and second term are the summations for Γ_{k_i}'s in eq.(5.40) and (5.41), respectively. The $(sym.fac.)$ in the first and second term are the symmetric factors with respect to the Γ_{k_i}'s. When $\Gamma_{k_i} \neq \Gamma_{k_j}$ for any $i \neq j$, then $(sym.fac.) = 1$. In the first term when $\Gamma_{k_1} = \Gamma_{k_2}$ then $(sym.fac.) = \frac{1}{2}$, and in the second term when $\Gamma_{k_i} = \Gamma_{k_j}$ for all $1 \leq i, j \leq l$ then $(sym.fac.) = \frac{1}{l}$. These factor comes from the coefficients in eq.(5.40) and (5.41). Each term contributes to the term $\omega(\tilde{\Phi}^p, \tilde{m}^p_{\Gamma_{n-1}}(\tilde{\Phi}^p, \cdots, \tilde{\Phi}^p))$ for some Γ_{n-1}. Here fix the tree graph Γ_{n-1} and treat it as a cyclic graph, *i.e.* a outgoing leg and $n-1$ incoming legs are not distinguished and the graphs which coincides with each other by moving cyclic are identified. Denote it by Γ^{cyc}_{n-1} and let us observe the coefficient of $\omega(\tilde{\Phi}^p, \tilde{m}^p_{\Gamma^{cyc}_{n-1}}(\tilde{\Phi}^p, \cdots, \tilde{\Phi}^p))$ from the above two contributions. As was seen in

(a) (b)

Figure 4: Suppose that Γ^{cyc}_{n-1} is the graph (a). For each vertex $\bullet$ in (a) eq.(5.41) contributes and for each propagator (dashed line) in (a) eq.(5.41) contributes. The term corresponding to the vertex surrounded by the circle of dotted line has $(sym.fac.) = \frac{1}{3}$. On the other hand, consider the case when Γ^{cyc}_{n-1} is the graph in (b). This is the exceptional case. The graph has $(sym.fac.) = \frac{1}{2}$ with respect to the propagator marked by the circle.

section 3 and the previous subsection, if Γ^{cyc}_{n-1} has I propagators, it contains $I+1$ vertices. We label the vertices in Γ^{cyc}_{n-1} as v_i, $i = 1, \cdots, I+1$ and the propagators as $j = 1, \cdots, I$. Without an exceptional case explained later, the first terms contribute for each propagators j in Γ^{cyc}_{n-1} and the second terms contribute for each vertex v_i in Γ^{cyc}_{n-1}. The coefficient for $\omega(\tilde{\Phi}^p, \tilde{m}^p_{\Gamma^{cyc}_{n-1}}(\tilde{\Phi}^p, \cdots, \tilde{\Phi}^p))$ is then

$$-\sum_{j=1}^{I}(sym.fac.)_j + \sum_{i=1}^{I+1}(sym.fac.)_{v_i} \ . \tag{5.42}$$

Note that if $(sym.fac.)_{v_i} \neq 1$ for certain i, then $(sym.fac.)_{v_{i'}} = 1$ for all $i' \neq i$ (this fact can be read easily from the graph). In the same way if $(sym.fac.)_j = \frac{1}{2}$ for certain j, then the other $(sym.fac.)_{j'}$ is equal to one. Thus in the case when $(sym.fac.)_j = 1$ for all propagators

j, eq.(5.42) becomes $-I + \sum_{i=1}^{I+1}(sym.fac.)_{v_i} = (sym.fac.)_{\Gamma^{cyc}_{n-1}}$ where $(sym.fac.)_{\Gamma^{cyc}_{n-1}}$ denotes the cyclic symmetric factor for Γ^{cyc}_{n-1} which, if not equal to one, comes from $(sym.fac.)_{v_i}$ for certain i. The case when $(sym.fac.)_j = \frac{1}{2}$ for certain j is the exceptional case mentioned above (see for example the graph (b) in (Fig.4)), and the coefficient for $\omega(\tilde{\Phi}^p, \tilde{m}^p_{\Gamma^{cyc}_{n-1}}(\tilde{\Phi}^p, \cdots, \tilde{\Phi}^p))$ is not given by eq.(5.42). In this case $I+1$ is even and the graph Γ^{cyc}_{n-1} is symmetric with respect to the propagator j, so the overcounting must be divided in eq.(5.42). The coefficient is then $-\frac{1}{2}(I-1) - \frac{1}{2} + \frac{1}{2}(I+1)$ because in this case $(sym.fac.)_{v_i} = 1$ for all i. This is equal to $\frac{1}{2}$, which is exactly the cyclic symmetric factor for Γ^{cyc}_{n-1} also in this case. From the above result, eq.(5.39) is rewritten as

$$S((\tilde{\mathcal{F}}^p)_*(\Phi)) = \sum_{n\geq 3, \Gamma^{cyc}_{n-1}} (sym.fac.)_{\Gamma^{cyc}_{n-1}} \omega(\tilde{\Phi}^p, \tilde{m}^p_{\Gamma^{cyc}_{n-1}}(\tilde{\Phi}^p, \cdots, \tilde{\Phi}^p)) \tag{5.43}$$

where the summation for Γ^{cyc}_{n-1} runs over the Γ_{n-1}'s which are identified by the cyclic symmetry. Therefore re-symmetrizing the sum reproduces the desired form of the action (5.37). It is interesting and worth emphasizing that for each edge of Γ^{cyc}_{n-1} the first term (5.40) contributes, for each vertex of Γ^{cyc}_{n-1} the second term (5.41) contributes, and the overcounting of the graphs are just canceled. ■

The action $\tilde{S}(\tilde{\Phi}^p)$ is an effective action in the following sense. $\tilde{S}(\tilde{\Phi}^p)$ is obtained by substituting $\Phi = \tilde{\mathcal{F}}^p(\tilde{\Phi}^p)$ into $S(\Phi)$ as explained above. When we express $\Phi = \Phi^p + \Phi^u$ where Φ^p and Φ^u denotes the physical and unphysical modes of Φ, respectively, the substituting means $\Phi^p = \tilde{\Phi}^p$ and $\Phi^u = f(\tilde{\Phi}^p)$. As is seen from eq.(5.4), the latter is nothing but the equation of motion for Φ^u. Moreover $\tilde{S}(\tilde{\Phi}^p)$ is related to $S(\Phi)$ by integrating out Φ^u at tree level through a gauge fixing as

$$\int \mathcal{D}\Phi^u e^{-S(\Phi)} = e^{-\tilde{S}(\tilde{\Phi}^p)} \ . \tag{5.44}$$

In this sense the action $\tilde{S}(\tilde{\Phi}^p)$ is an effective action.

Here we clarify the properties of the solutions of the equations of motions. As was explained in subsection 2.3, because the A_∞-morphism preserves the solutions of Maurer-Cartan equations, there is one-to-one correspondence between the set of the equations of motions for different SFTs on the same conformal background. However in the context of deformation theory, the solution for the Maurer-Cartan equation are assumed to be of the form $\Phi = \epsilon\tilde{\Phi}^p + \mathcal{O}(\epsilon^2)$ for ϵ a 'small' formal deformation parameter. The argument in subsection 5.1 is just the case where the small parameter is thought to be included in $\tilde{\Phi}^p$. Such solutions are expressed as $\Phi = \tilde{\mathcal{F}}^p_*(\tilde{\Phi}^p)$ where $\tilde{\Phi}^p$ is a solution for the Maurer-Cartan equation $\tilde{\mathfrak{m}}^p_*(e^{\tilde{\Phi}^p}) = 0$. The solutions can be regarded as those corresponding to the marginal deformation as will be explained below. Infinitesimally around the origin of $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ the Maurer-Cartan equation is the quadratic form $\tilde{m}^p_2(\tilde{\Phi}, \tilde{\Phi}) \sim 0$ and generally the path of the solutions which flows from the origin exists [22]. Consider the continuous deformation of the solutions on this path. Near the origin $\tilde{\Phi}^p = 0$ *i.e.* $\Phi = 0$, the equation of motion is $Q\Phi = 0$ so the solution is the one corresponding to the marginal deformation. Note that at the origin the value of the action $S(\Phi)$ is zero. Since the

[22]Such path exists if the Hessian has the eigenvalues of opposite sign with respect to the states $\{\mathbf{e}^p_i\}$.

continuous family of the solutions which connects to the origin is now considered, $S(\Phi)$ is kept to be zero. Next, apart from the origin, we consider the physics around a solution Φ_{bg} on the path. Expanding the action $S(\Phi)$ around Φ_{bg}, another action $S'(\Phi') := S(\Phi_{bg} + \Phi')$ is obtained where $\Phi = \Phi_{bg} + \Phi'$ and $\Phi' \in \mathcal{H}'$: the string Hilbert space on another conformal background. As will be explained later in Discussions about the background independence, $S'(\Phi')$ also has an A_∞-structure. Here represent the kinetic term of $S'(\Phi')$ as

$$S'(\Phi') = \frac{1}{2}\omega'(\Phi', Q'\Phi') + \cdots \ .$$

The infinitesimal deformation along the path from $\Phi_{bg} \in \mathcal{H}$ then corresponds to the solution of $Q'\Phi' = 0$ on $\mathcal{H}'$ because $S'(\Phi') \sim \frac{1}{2}\omega'(\Phi', Q'\Phi') = 0$. Assuming the background independence of the action $S(\Phi)$, Q' is regarded as the BRST operator on another conformal background (up to the isomorphism of the vector space $\mathcal{H}'$), and therefore the infinitesimal deformation on the path can be regarded as the marginal deformation even if finitely apart from the origin.

In the above arguments we obtained a quasi-isomorphism $\tilde{\mathcal{F}}^p$ and discussed various meaning it has. However some additional input (from world sheet picture) may derive more strong results. For instance in [17] it is shown that for closed SFT the infinitesimal variation of the way of the decomposition of the moduli space leads the infinitesimal field redefinition preserving the value of the actions and the BV-symplectic structures. Similarly for an one parameter family of classical open SFTs, the infinitesimal field redefinition preserving the actions is found[7], which is discussed in the next section. If one wishes to find a field redefinition preserving the value of the action, we must consider an isomorphism instead of the quasi-isomorphism $\tilde{\mathcal{F}}^p$. From the general arguments in this paper, there exists only one candidate for the isomorphism, which is the A_∞-isomorphism $\tilde{\mathcal{F}} : (\mathcal{H}, \tilde{\mathfrak{m}}) \to (\mathcal{H}, \mathfrak{m})$. However when any two actions $S(\Phi)$ and $S'(\Phi')$ on the same conformal background are given, generally $\tilde{\mathcal{F}}^* S(\Phi)$ and $\tilde{\mathcal{F}'}^* S'(\Phi')$, both of which are the functional of $\tilde{\Phi}$, do not coincide off-shell. Therefore we cannot apply the isomorphism $\tilde{\mathcal{F}}$ in order to construct a field redefinition preserving the value of the action. Only on-shell $\tilde{\mathcal{F}}^* S(\Phi)$ and $\tilde{\mathcal{F}'}^* S'(\Phi')$ coincide and the above argument has held.

Finally we comment about the SFT action $\tilde{\mathcal{F}}^* S(\Phi)$, which is the one obtained by substituting the field redefinition $\Phi = \tilde{\mathcal{F}}(\tilde{\Phi})$ into the original SFT action $S(\Phi)$. The form of the action is derived directly in the same way as $\tilde{S}(\tilde{\Phi}^p) = (\tilde{\mathcal{F}}^p)^* S(\Phi)$

$$\begin{aligned} \tilde{S}(\tilde{\Phi}) := \tilde{\mathcal{F}}^* S(\Phi) = \frac{1}{2}\omega(\tilde{\Phi}, Q\tilde{\Phi}) + \sum_{k\geq 2} \frac{1}{k+1}\omega(\tilde{\Phi}, \tilde{m}_k^{cyc}(\tilde{\Phi})) \\ - \omega(Q^+ Q\tilde{\Phi}, \sum_{k\geq 2} \tilde{m}_k^{cyc}(\tilde{\Phi})) \ . \end{aligned} \tag{5.45}$$

Note that $Q^+ Q\tilde{\Phi}$ is almost equal to $P^u\tilde{\Phi}$. $\tilde{m}_k^{cyc}$ denotes the one which is obtained by removing P on the outgoing line of $\tilde{m}_k$ defined in (Def.5.1) and (Rem.5.1). $\tilde{m}_k^{cyc} = \sum_{\Gamma_k \in G_k} \tilde{m}_{\Gamma_k}$ and $P\tilde{m}_k^{cyc} = \tilde{m}_k$ holds. Note that these $\{\tilde{m}_k^{cyc}\}_{k\geq 2}$ with $\tilde{m}_1^{cyc} := Q$ do not define an A_∞-structure. Instead, $\tilde{\mathcal{V}}(\ ,\cdots,\ ) = \omega(\ ,\tilde{m}_k^{cyc}(\ ,\cdots,\ ))$ has the cyclic symmetry similarly as m_k or $\tilde{m}_k^p$ does. The second term is derived in the same way as the on-shell action $\tilde{S}(\tilde{\Phi}^p)$ in eq.(5.37). Here in addition the kinetic term and the third term appear. These vanish when $\tilde{\Phi}$ is restricted to

$\tilde{\Phi}^p \in \mathcal{H}^p$, so it can be seen that $\tilde{\mathcal{F}}^* S(\Phi)$ reduces to $\tilde{S}(\tilde{\Phi}^p)$ on-shell. Both the kinetic term and the third term in the right hand side of eq.(5.45) come from the kinetic term of the action $S(\Phi)$, which vanish in eq.(5.39) because the fields $\tilde{\Phi}^p$ is restricted on-shell.

Thus we get an off-shell action which coincides with the string S-matrix elements on-shell. On this action, we define the symplectic structure $\tilde{\omega}$, which is different from ω, so that the A_∞-isomorphism $\tilde{\mathcal{F}}$ from $(\mathcal{H}, \tilde{\mathfrak{m}}, \tilde{\omega})$ to $(\mathcal{H}, \mathfrak{m}, \omega)$ preserves the symplectic structures. This $\tilde{\omega}$ is written as

$$\tilde{\omega}_{ij} = \frac{\overrightarrow{\partial}\phi^k}{\partial\tilde{\phi}^i}\omega_{kl}\frac{\phi^l\overleftarrow{\partial}}{\partial\tilde{\phi}^j} = (-1)^{\mathbf{e}_i}\omega\left(\frac{\overrightarrow{\partial}}{\partial\tilde{\phi}^i}\Phi, \Phi\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^j}\right) , \tag{5.46}$$

which coincides with ω_{ij} when $\mathbf{e}_i, \mathbf{e}_j$ are restricted on-shell, but generally different from ω_{ij} off-shell. Thus $\tilde{\omega}$ is a field dependent symplectic form. By (Prop.5.1),

$$\tilde{\delta} = \frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^i}\tilde{\omega}^{ij}\frac{\overrightarrow{\partial}}{\partial\tilde{\phi}^j}\tilde{S}(\tilde{\Phi}))$$

coincides with the dual of the A_∞-structure $\tilde{\mathfrak{m}}$. Moreover the fact that this $\tilde{\delta}$ defines an A_∞-structure on $(\mathcal{H}, \tilde{\mathfrak{m}})$ implies that the action $\tilde{S}(\tilde{\Phi})$ satisfies the BV-master equation with respect to the symplectic structure $\tilde{\omega}$. Consequently, an action $\tilde{S}(\tilde{\Phi})$, which coincides with the string correlation functions on-shell and satisfies the BV-master equation, is obtained. In this sense, this action $\tilde{S}(\tilde{\Phi})$ can be regarded as one definition of boundary SFT[43] on the neighborhood of the origin of two-dimensional theory space $\mathcal{H}$. It is interesting that, although rather formally, the action $\tilde{S}(\tilde{\Phi})$ is related to the original open classical SFT action by the field redefinition $\tilde{\mathcal{F}}$. This $\tilde{\mathcal{F}}$ is nothing but the coordinate transformation on a formal noncommutative supermanifold in (Rem.5.3). See also *boundary string field theory* in Discussions.

6 RG-flow and Field redefinition

In this section we discuss the arguments in subsection 5.4 on a more explicit description of SFT : the classical open SFT discussed in [7].

The most explicit way of creating a SFT action is based on the variation of the cut-off length of the propagator as in [13, 52, 50] for closed SFT, and in [6, 7] for open SFT. The cut-off of the propagator can be regarded as an UV-reguralization on target space, and the variation of the cut-off length has been discussed in these literatures in the context of Polchinski's renormalization group[40]. One can consider the one parameter family of SFTs in this procedure. Let ζ be the cut-off length of the propagator and $S(\Phi^\zeta;\zeta)$ be the SFT action in this scale. This ζ parameterizes an one parameter family of SFT $S(\Phi^\zeta;\zeta)$. The action $S(\Phi^\zeta;\zeta)$ is ζ-dependent in two means : explicit ζ-dependence of $\mathcal{V}_{i_1\cdots i_n}$ or equivalently $\mathfrak{m}$ (and ω^{ij}), and ζ-dependence through the ζ-dependence of Φ^ζ. In classical SFT case, the renormalization group flow is then defined so that the total ζ-dependence of $S(\Phi^\zeta;\zeta)$ cancels

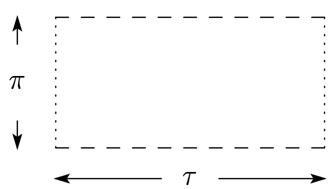
$$0 = \frac{d}{d\zeta}S(\Phi^\zeta;\zeta) = \frac{\partial S(\Phi^\zeta;\zeta)}{\partial\zeta} + \frac{\partial\Phi^\zeta}{\partial\zeta}\frac{\partial S(\Phi^\zeta;\zeta)}{\partial\Phi^\zeta} . \tag{6.1}$$

Here we concentrate on the classical open SFT which possesses an A_∞-structure[7]. In [7] the infinitesimal field redefinition $\frac{\partial\Phi^\zeta}{\partial\zeta}$ which satisfies the above equation is derived (eq.(6.4)). We restrict the arguments on the Siegel gauge in this section. First in (Def.6.1) we define the A_∞-structure explicitly. In (Prop.6.1) it is shown that this field redefinition is an A_∞-isomorphism on the Siegel gauge. Next the action $S(\Phi^\zeta;\zeta)$ are transformed to another action $(\tilde{\mathcal{F}}^\zeta)^*S(\Phi^\zeta;\zeta)$ by the A_∞-isomorphism $\tilde{\mathcal{F}}^\zeta$ and its properties are observed. The action $(\tilde{\mathcal{F}}^\zeta)^*S(\Phi^\zeta;\zeta)$ is the one which coincides with the string S-matrix on-shell (5.45). Using this, it is shown that on the subspace of $\mathcal{H}$ which is the image of $(\tilde{\mathcal{F}}^\zeta)^p$ from $(\mathcal{H}^p, \tilde{\mathfrak{m}}^{\zeta,p})$, the finite field redefinition $\tilde{\mathcal{F}}^{\zeta',p}\circ(\tilde{\mathcal{F}}^{\zeta,p})^{-1}$ from $S(\Phi^\zeta;\zeta)$ to $S(\Phi^{\zeta'};\zeta')$ reduces to the above A_∞-isomorphism when its infinitesimal limit is taken. Finally various pictures are summarized on this explicit model.

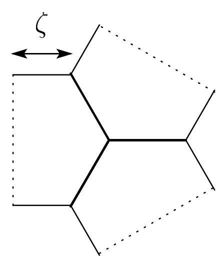
We begin with a brief review of the construction of one parameter family of classical open SFT[7, 6]. It is argued in [7] very clear. The main idea was explained in section 3, but this procedure relies on the fact that all moduli space of disks with n punctures can be reproduced by connecting Witten's type trivalent vertex[2] with propagators. We simply represent the propagator Q^+ in the Siegel gauge : $b_0\Phi = 0$ as $b_0\frac{1}{L_0}$ since the arguments in this section do not depend on the detail. In the Schwinger representation it is

$$b_0\frac{1}{L_0} = b_0\int_0^\infty e^{-\tau L_0}\ . \tag{6.2}$$

One can interpret $e^{-\tau L_0}$ as the evolution operator for the open string. The width of the propagator is set to be π (Fig.5.(a).). When cutting-off the propagator with length 2ζ, the length of the propagator τ runs from $\tau = 2\zeta$ to $\tau = \infty$. Therefore the subspace of moduli space, which has been reproduced by connecting trivalent vertices with the propagator with length $0 \le \tau \le 2\zeta$ in $2\zeta = 0$ (no-cut-off) theory, can not be reproduced by the trivalent vertices in the theory of cut-off length 2ζ. Such diagram must be add to the SFT action as higher vertices. Vertices in the 2ζ cut-off theory are constructed recursively in this way. By construction, the width of the external legs of n point vertex with $n \ge 3$ are of course π.



(a) propagator.



(b) trivalent vertex with strip.

Figure 5: (a). τ runs between $0 \le \tau \le \infty$ in Witten's cubic open SFT. When Cutting-off the propagator with length 2ζ, τ runs from 2ζ to ∞. The moduli which is not reproduced by connecting Witten's trivalent vertices with such propagators equal to the moduli which is not reproduced by connecting the modified trivalent vertices (b) with the usual propagator ($0 \le \tau \le \infty$).

Cutting-off the propagator with length 2ζ can be replaced by sewing the strip with width π

and length ζ to all external legs of the vertices. Such trivalent vertex is pictured in (Fig.5.(b).) for example. It is carried out by attaching the evolution operator $e^{-\zeta L_0}$ to each external legs. Then the length of the propagator τ runs from $\tau = 0$ to $\tau = \infty$ and the modification for the propagator does not need.

Let $\mathfrak{m}^\zeta$ be the A_∞-structure corresponding to the vertices in the 2ζ-cut-off theory. The A_∞-structure for Witten's cubic SFT is described as $\mathfrak{m}^0 = \{m_1^0 = Q, m_2^0, m_3^0 = m_4^0 = \cdots = 0\}$. Recalling the arguments in subsection 5.2 and 5.3 yields that $\mathfrak{m}^\zeta$ is given essentially as $\tilde{\mathfrak{m}}$ by replacing Q^+ and P in the definition of $\tilde{\mathfrak{m}}^{(p)}$ in (Def.5.1) to $Q^{\zeta,+}$ and P^ζ defined below in the context of the present paper. Here explicitly present it as follows.

Definition 6.1 (A_∞-structure $\mathfrak{m}^\zeta$) Define $Q^{\zeta,+} := b_0 \int_0^{2\zeta} e^{-\tau L_0} d\tau$ and $P^\zeta := e^{-2\zeta L_0}$. These satisfy the following identity

$$\{Q, Q^{\zeta,+}\} + P^\zeta = \mathbf{1} \ .$$

Let us define as an intermediate step $\{f_k^{\tilde{\zeta}}\}$ and $\{m_k^{\tilde{\zeta}}\}_{k\geq 2}$ recursively by

$$f_k^{\tilde{\zeta}}(\Phi^{\tilde{\zeta}}) := -Q^{\zeta,+} \sum_{1\leq k_1<k_2=k} m_2^0(f_{k_1}^{\tilde{\zeta}}(\Phi^{\tilde{\zeta}}), f_{k_2-k_1}^{\tilde{\zeta}}(\Phi^{\tilde{\zeta}}))$$

with $f_1^{\tilde{\zeta}}(\Phi^{\tilde{\zeta}}) = \Phi^{\tilde{\zeta}}$ and

$$m_k^{\tilde{\zeta}}(\Phi^{\tilde{\zeta}}) := \sum_{1\leq k_1<k_2=k} P^\zeta m_2^0(f_{k_1}^{\tilde{\zeta}}(\Phi^{\tilde{\zeta}}), f_{k_2-k_1}^{\tilde{\zeta}}(\Phi^{\tilde{\zeta}})) \ .$$

for $k \geq 2$. By shifting the field as $\Phi^{\tilde{\zeta}} = e^{-\zeta L_0}\Phi^\zeta$, the A_∞-structure $\mathfrak{m}^\zeta$ is defined as

$$\begin{aligned} m_k^\zeta(\Phi^\zeta) &:= e^{\zeta L_0} m_k^{\tilde{\zeta}}(e^{-\zeta L_0}\Phi^\zeta) \\ &= e^{-\zeta L_0} \sum_{1\leq k_1<k_2=k} m_2^0(f_{k_1}^{\tilde{\zeta}}(e^{-\zeta L_0}\Phi^\zeta), f_{k_2-k_1}^{\tilde{\zeta}}(e^{-\zeta L_0}\Phi^\zeta)) \ . \end{aligned} \tag{6.3}$$

The fact that actually $\mathfrak{m}^\zeta$ defines an A_∞-structure follows from this construction and the fact that $\mathfrak{m}^0$ defines an A_∞-structure. The fact that the vertices have cyclic symmetry is also clear by construction.

For example when $k = 2$, the three point vertex m_2^ζ is $e^{-\zeta L_0} m_2^0(e^{-\zeta L_0} \quad , e^{-\zeta L_0} \quad)$, which is just the one in (Fig.5.(b)). For any $n \geq 3$, one can see that m_n^ζ includes $n-3$ propagators all of which have length $0 \leq \tau \leq 2\zeta$. They cannot be reproduced by connecting lower vertices m_k^ζ $(k \leq n-1)$ with the propagators.

In this situation, the flow of Φ^ζ which satisfies eq.(6.1) is defined in [7] as

$$\frac{\partial \Phi^\zeta}{\partial \zeta} = b_0 \sum_{k\geq 2} m_k^\zeta(\Phi^\zeta) \tag{6.4}$$

in the Siegel gauge. The fact that this infinitesimal field redefinition preserves the value of the actions can be checked directly by substituting this into eq.(6.1) because now the variation of m_k^ζ with respect to ζ is derived directly by the explicit construction of $\mathfrak{m}^\zeta$ in (Def.6.1).

Proposition 6.1 This field redefinition (6.4) is an A_∞-isomorphism on the Siegel gauge.

The field redefinition (6.4) is defined only on the Siegel gauge $b_0\Phi^\zeta = 0$. Therefore this proposition claims, in other words, that this field redefinition can be extended to the infinitesimal neighborhood of the submanifold $b_0\Phi^\zeta = 0$ so that the derivative with respect to $\bar{\phi}$ can be defined.

proof. Now the symplectic structure for $S(\Phi^\zeta;\zeta)$ is ω and is independent of ζ. Because this field redefinition $\frac{\partial\Phi^\zeta}{\partial\zeta}$ preserves the value of the action, by (Prop.5.1) it is sufficient for the proof to show that the field redefinition preserves the symplectic form ω. Let us consider the infinitesimal field transformation $\mathcal{F}^{\delta_\epsilon}$ defined by ϵ as follows

$$
\begin{aligned}
(\mathcal{F}^{\delta_\epsilon})^* a(\phi^{\zeta'}) &:= a(\phi^\zeta) + \delta_\epsilon a(\phi^\zeta) \\
&:= a(\phi^\zeta) + (a(\phi^\zeta), \epsilon(\phi^\zeta)) \ ,
\end{aligned}
\tag{6.5}
$$

where $a(\phi^\zeta)$ is a functions of ϕ^ζ and $\epsilon(\phi^\zeta)$ is a infinitesimal function of ϕ^ζ which determines the infinitesimal transformation. $(\ ,\ )$ is the BV-Poisson structure with respect to ω. Such infinitesimal transformation is called *canonical transformation* of (BV-)symplectic structure and is applied to the infinitesimal field redefinition in closed SFT in [17]. This transformation preserves the Poisson structure, that is,

$$
(a(\phi^\zeta), b(\phi^\zeta)) + \delta_\epsilon(a(\phi^\zeta), b(\phi^\zeta)) = (a(\phi^\zeta) + \delta_\epsilon a(\phi^\zeta), b(\phi^\zeta) + \delta_\epsilon b(\phi^\zeta)) \tag{6.6}
$$

up to $(\epsilon)^2$. It immediately follows from the Jacobi identity of $(\ ,\ )$. Now $(\ ,\ )$ is defined on the theory with parameter ζ. On the other hand, if the symplectic structure $(\ ,\ )_{\zeta'}$ on the theory with parameter ζ' is defined so that $\mathcal{F}^{\delta_\epsilon}$ preserves the symplectic structures, we have the following identity

$$
(\mathcal{F}^{\delta_\epsilon})^*(a(\phi^{\zeta'}), b(\phi^{\zeta'}))_{\zeta'} = ((\mathcal{F}^{\delta_\epsilon})^* a(\phi^{\zeta'}), (\mathcal{F}^{\delta_\epsilon})^* b(\phi^{\zeta'})) \ . \tag{6.7}
$$

The right hand side exactly coincides with the right hand side in eq.(6.6), therefore the symplectic structure $(\ ,\ )_{\zeta'}$ induced by the transformation $\mathcal{F}^{\delta_\epsilon}$ is determined by the equality between the left hand sides in eq.(6.6) and eq.(6.7). When we set $a(\phi^{\zeta'}) = \phi^{\zeta',i}$ and $b(\phi^{\zeta'}) = \phi^{\zeta',j}$, the equality becomes

$$
(\mathcal{F}^{\delta_\epsilon})^* \omega^{\zeta',ij} = \omega^{ij} + \delta_\epsilon \omega^{ij}
$$

where $\omega^{\zeta',ij}$ denotes the Poisson tensor of $(\ ,\ )_{\zeta'}$. This equality implies that if ω^{ij} is constant, $\omega^{\zeta',ij}$ is equal to ω^{ij}. Thus it is shown that the constant symplectic structure ω^{ij} is preserved under the infinitesimal field redefinition of the form in eq.(6.5). Here the field redefinition $\frac{\partial\Phi^\zeta}{\partial\zeta}$ can be rewritten in the form (6.5) as $\epsilon(\phi^\zeta) = \omega(\Phi^\zeta, b_0 \sum_{k\ge 2} m_k^\zeta(\Phi^\zeta))$. In fact on the Siegel gauge,

$$
(\Phi^\zeta, \epsilon)|_{b_0\Phi^\zeta=0} = b_0 \sum_{k\ge 2} m_k^\zeta(\Phi^\zeta)|_{b_0\Phi^\zeta=0}
$$

holds. This completes the proof of (Prop.6.1). ∎

Next let us observe the action $(\tilde{\mathcal{F}}^\zeta)^* S(\Phi)$. As was seen in eq.(5.45), it is of the form

$$\begin{aligned}\tilde{S}(\tilde{\Phi}^\zeta) := (\tilde{\mathcal{F}}^\zeta)^* S(\Phi^\zeta;\zeta) = \frac{1}{2}\omega(\tilde{\Phi}^\zeta, Q\tilde{\Phi}^\zeta) + \sum_{k\geq 2}\frac{1}{k+1}\omega(\tilde{\Phi}^\zeta, \tilde{m}_k^{\zeta,cyc}(\tilde{\Phi}^\zeta)) \\ - \omega(Q^+Q\tilde{\Phi}^\zeta, \sum_{k\geq 2}\tilde{m}_k^{\zeta,cyc}(\tilde{\Phi}^\zeta)) \ ,\end{aligned} \tag{6.8}$$

where $\tilde{m}_k^{\zeta,cyc}$ is the one related to m_k^ζ as in eq.(5.45). At the same time we have the field redefinition $\Phi^\zeta = \tilde{\mathcal{F}}_*^\zeta(\tilde{\Phi}^\zeta) = \tilde{\Phi}^\zeta + \tilde{f}^\zeta(\tilde{\Phi})$. By construction, comparing with $\tilde{m}^{0,cyc}$, $\tilde{m}_k^{\zeta,cyc}$ has its external legs of length ζ, which indicates

$$\tilde{m}_k^{\zeta,cyc}(\ ,\cdots,\ ) = e^{-\zeta L_0}\tilde{m}_k^{0,cyc}(e^{-\zeta L_0}\ ,\cdots, e^{-\zeta L_0}\ )\ .$$

In the same way the following relation between $\tilde{f}_k^\zeta$ with different k holds,

$$\tilde{f}_k^\zeta(\ ,\cdots,\ ) = e^{-\zeta L_0}\tilde{f}_k^0(e^{-\zeta L_0}\ ,\cdots, e^{-\zeta L_0}\ )\ ,$$

where we used the $Q^+ = b_0\frac{1}{L_0}$ on the outgoing states of $\tilde{f}_k^\zeta$ and $e^{-\zeta L_0}$ commute.

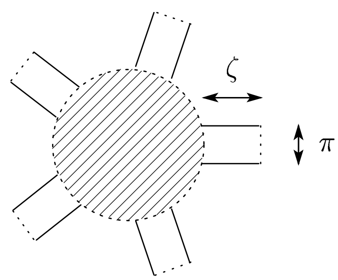


Figure 6: $\tilde{m}_4^{\zeta,cyc}$ is figured. The interior of the circle denotes that all tree five point Feynman graphs are summed up and which means that the integral runs the whole moduli space $\mathcal{M}_5$. Comparing to $\tilde{m}_4^{0,cyc}$ the additional strips with length ζ are attached.

Let us restrict the external states $\tilde{\Phi}$ to $\tilde{\Phi}^p \in \mathcal{H}^p$. In this case $\tilde{m}_k^{\zeta,cyc}$ and $\tilde{f}_k^\zeta$ are replaced by $\tilde{m}_k^{\zeta,p}$ and $\tilde{f}_k^{\zeta,p}$. Then $\tilde{m}_k^{\zeta,p}$ coincides with $\tilde{m}_k^{0,p}$ because $e^{-\zeta L_0} = 1$ on $\tilde{\Phi}^p$. Thus the on-shell effective action $\tilde{S}(\tilde{\Phi}^{\zeta,p}) := (\tilde{\mathcal{F}}^{\zeta,p})^* S(\Phi^\zeta)$ is independent of ζ. On the other hand, the situation is not the same for $\tilde{f}_k^\zeta$, because the outgoing states of f_k^ζ do not belong to $\mathcal{H}^p$ but the image of Q^+. The outgoing legs has its length ζ and then the propagator acts on it. The facts leads

$$\tilde{f}_k^{\zeta,p} = e^{-\zeta L_0}\tilde{f}_k^{0,p}\ . \tag{6.9}$$

Note that $\tilde{f}_k^{0,p}$ has no ζ-dependence. We can then consider the infinitesimal variation of the field redefinition $\Phi^\zeta = \tilde{\Phi}^{\zeta,p} + \tilde{f}^{\zeta,p}(\tilde{\Phi}^p)$,

$$\begin{aligned}\frac{\partial\Phi^\zeta}{\partial\zeta} &= -L_0 e^{-\zeta L_0}\tilde{f}^{0,p}(\tilde{\Phi}^{0,p}) \\ &= -L_0\tilde{f}^{\zeta,p}(\tilde{\Phi}^{\zeta,p})\end{aligned}$$

where $\tilde f^{\zeta,p}(\tilde\Phi^{\zeta,p}) = \sum_{k\geq 2}\tilde f_k^{\zeta,p}(\tilde\Phi^{\zeta,p})$. By definition, one can rewrite $\tilde f_k^{\zeta,p}(\tilde\Phi^{\zeta,p})$ as

$$\tilde f^{\zeta,p}(\tilde\Phi^{\zeta,p}) = -Q^+\sum_{k\geq 2} m_k^\zeta(\Phi^\zeta)$$

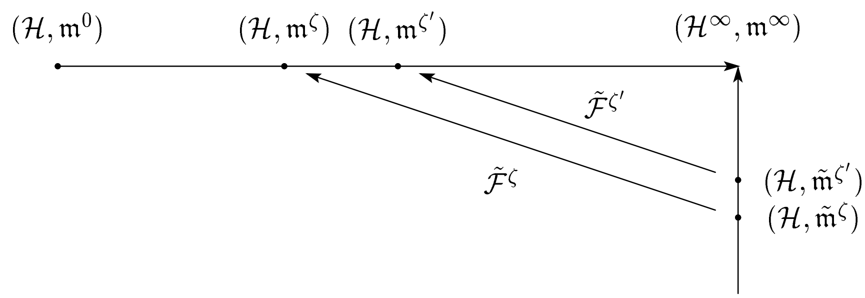
where Φ^ζ in the right hand side is the image of $\tilde{\mathcal{F}}_*^{\zeta,p}$ from $\tilde\Phi^p \in \mathcal{H}^p$. Recalling $Q^+ = b_0\frac{1}{L_0}$, the infinitesimal field redefinition is derived as

$$\frac{\partial\Phi^\zeta}{\partial\zeta} = b_0\sum_{k\geq 2} m_k^\zeta(\Phi^\zeta)\ ,$$

which exactly coincides with the renormalization group flow (6.4). The finite field redefinition from $S(\Phi^\zeta;\zeta)$ to $S(\Phi^{\zeta'};\zeta')$ on this subspace is given by

$$\begin{aligned}\Phi^{\zeta'} &= \Phi^\zeta + e^{-\zeta' L_0} f^{\zeta'}(\Phi^{0,\zeta'}) - e^{-\zeta L_0} f^{\zeta'}(\Phi^{0,\zeta}) \\ &= \Phi^\zeta + b_0\int_0^{\zeta'-\zeta} e^{-\tau L_0} d\tau \sum_{k\geq 2} m_k^\zeta(\Phi^\zeta)\ .\end{aligned}$$

Finally the various SFT action obtained here and their relation between each other are summarized. $(\mathcal{H},\mathfrak{m}^0)$ is the cubic open SFT. On the horizontal line the one parameter family of



SFT $(\mathcal{H},\mathfrak{m}^\zeta)$ is defined and there exists the infinitesimal field redefinition on it. This field redefinition preserves the A_∞-structure *i.e.* the BRST-symmetry on the Siegel gauge, and formally by integrating it there exists a field redefinition between any two SFTs on this one parameter family. Alternatively, for each $(\mathcal{H},\mathfrak{m}^\zeta)$ there exists an equivalent SFT $(\mathcal{H},\tilde{\mathfrak{m}}^\zeta)$. These are related to each other by $\tilde{\mathcal{F}}^\zeta$. The field transformations on the horizontal line and $\tilde{\mathcal{F}}^\zeta$ are compatible. When $(\mathcal{H},\tilde{\mathfrak{m}}^\zeta)$ is restricted to physical states, the reduced theory does not depend on ζ and coincides with $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$. In this subspace the composition $\tilde{\mathcal{F}}^{\zeta',p}\circ(\tilde{\mathcal{F}}^\zeta)^{-1}$ defines the finite field transformation between $(\mathcal{H},\mathfrak{m}^\zeta)$ and $(\mathcal{H},\mathfrak{m}^{\zeta'})$. Its infinitesimal version actually coincides with the field redefinition given in [7].

Note that in the limit $\zeta\to\infty$, each vertex of the action on $(\mathcal{H},\mathfrak{m}^\infty)$ has the whole moduli and it coincides with the correlation function itself. $(\mathcal{H},\mathfrak{m}^\infty)$ then coincides with $(\mathcal{H},\tilde{\mathfrak{m}}^\infty)$ and its restriction onto the physical state space $\mathcal{H}^p$ is just $(\mathcal{H},\tilde{\mathfrak{m}}^p)$. This is the open string version of the argument given in [5], where the L_∞-structure of closed SFT is reduced to the L_∞-structure in string world sheet theory[32, 33]. This argument is reviewed in Appendix B.2.

Thus we have two one parameter families of SFT $(\mathcal{H},\mathfrak{m}^\zeta)$ and $(\mathcal{H},\tilde{\mathfrak{m}}^\zeta)$. The flow on $(\mathcal{H},\mathfrak{m}^\zeta)$ is discussed from the viewpoint of the renormalization group in [7]. The flow on $(\mathcal{H},\tilde{\mathfrak{m}}^\zeta)$ might also be interpreted as the renormalization flow of boundary SFT.

7 Conclusions and Discussions

We discussed classical open SFTs with cyclic vertices and argued that what extent their structure is governed by their general properties. These SFTs have the structure of A_∞-algebras, and it is shown that the A_∞-algebras of them are A_∞-quasi-isomorphic to the A_∞-algebra of on-shell S-matrix elements. Moreover applying this, it is shown that any such SFTs which differ in the decomposition of moduli space are A_∞-quasi-isomorphic to each other. This implies that between any such SFTs there is one-to-one correspondence of the solutions of the equations of motions which describe marginal deformations. In subspaces which relate to physical state space $\mathcal{H}^p$ the finite field transformation between two SFTs are described in terms of the Feynman graph.

We discuss the above arguments explicitly on the one parameter family of classical open SFTs on the Siegel gauge. On this one parameter family, there exists a field redefinition preserving the value of the actions. We showed that the infinitesimal field redefinition is an A_∞-isomorphism. It was observed that the infinitesimal version of the above finite field transformation in the subspace coincides with the A_∞-isomorphism preserving the actions.

Through the explanation of the above statements, various expressions for A_∞-algebras of SFTs are used and their relations were summarized. The sign attending on the degree (ghost number) is uniformed self-consistently. The relation to the conventional definition of the sign is not denoted explicitly in this paper, but one can easily obtain the relation by comparing the two definitions of A_∞-algebras the elements of which have their degree differ by one (see (Rem.2.2) and, for example, [45]. Essentially the relation can be read from the relation between the convention in first half and the latter half of [7].).

The problem of taking dual of coalgebras has some subtlety when the graded vector space is infinite dimensional. SFT is just the case. However SFTs are field theories. Therefore as far as assuming that the SFT is well-defined as field theory the dual of the coalgebras should be able to be taken. Moreover the dual language is introduced in the present paper only for intuitive and geometric understanding. All the arguments on the dual are rearranged in coalgebra language and then hold even in the model where the decomposition of fields and basis is difficult.

The convergence was not discussed. The finite field redefinitions or the solutions for the Maurer-Cartan equations for SFTs, which are formally preserved under the field redefinitions, are defined by polynomials of infinite powers. Of course many of the arguments in this paper make sense as formal power series. For instance each coefficient of the Maurer-Cartan equations for the canonical A_∞-algebra defines each on-shell S-matrix element. However, it should be checked that when the solutions converge, etc. . Their seems no ways to confirm the convergences instead of doing some numerical analysis explicitly. However, the finite field redefinitions and the equations of motions are defined by the Feynman graph of SFT. Therefore the problem of the convergences relates to the problem of the original SFT itself. Looking for some 'good' model on a conformal background might be a good issue. Alternatively, one can also argue these on an appropriate subspace, due to, for instance, the momentum conservation of the vertices. Therefore we think that some well-defined field redefinitions or the solutions of the equations of motions are obtained in the subspace. One such example will be commented below in *tachyon* *condensation* in Discussions.

We ends with presenting the following related topics or future directions.

- *the background independence.*

SFTs have mainly two directions of deformations : changing the decomposition of moduli space of Riemann surfaces as discussed in this paper, and transferring to other backgrounds.

The issue of the background independence can be treated in the category of weak A_∞-algebras as follows. Consider two points x and y on CFT theory space. x and y denote two conformal backgrounds. Let $\mathcal{H}_x$ and $\mathcal{H}_y$ be two string Hilbert spaces on the conformal backgrounds and let $\Phi_x \in \mathcal{H}_x$ and $\Phi_y \in \mathcal{H}_y$ be the string fields. Generally the field transformation $\mathcal{F}_y$ is of the form

$$\Phi_y := \mathcal{F}_{y,*}(\Phi_x) = \Phi_{bg} + \mathcal{F}_*(\Phi_x) = \Phi_{bg} + f_1(\Phi_x) + f_2(\Phi_x, \Phi_x) + \cdots \ ,$$

where $\Phi_{bg} = \mathbf{e}_i \Phi^i_{bg}$ denotes a background in $\mathcal{H}_y$. Only the degree zero part of Φ^i_{bg} can be nonzero, because the vacuum expectation value of the action should belong to $\mathbb{R}$. The case $\Phi_{bg} = 0$ reduces to the problem on the same conformal background.

If a SFT gives a background independent formulation of string theory, each solution Φ_{bg} of the equation of motion for the SFT $S_y(\Phi_y)$ on $\mathcal{H}_y$ describes a conformal background. Moreover $S(\Phi)$ re-expanded around the equation of motion Φ_{bg} should define a SFT action on the conformal background of $\mathcal{H}_x$.

Let $S_x(\Phi_x)$ be a SFT action on $\mathcal{H}_x$. Suppose that these two actions $S_y(\Phi_y)$ and $S_x(\Phi_x)$ satisfy the BV-master equations on their conformal backgrounds. Then if there exists $\mathcal{F}_y$ which preserves the symplectic structures and satisfies $\mathcal{F}^*_y S_y(\Phi_y) = S_y(\mathcal{F}_{y,*}(\Phi_x)) = S_x(\Phi_x)$, the action $S_y(\Phi_y)$ is certainly background independent.

Here let us express this $\mathcal{F}_y$ as the composition $\mathcal{F}^*_y = \mathcal{F}^* \circ \mathcal{F}^*_{bg}$ where

$$\Phi_y = \mathcal{F}_{bg,*}(\Phi') = \Phi_{bg} + \Phi' \ , \qquad \Phi' = \mathcal{F}_*(\Phi_x) \ .$$

By $\mathcal{F}_{bg}$ the SFT $S_y(\Phi_y)$ on $\mathcal{H}_y$ is transformed to a SFT on $\mathcal{H}_x$ which is regarded as a SFT on the conformal background corresponds to Φ_{bg}. It is known that when the Φ_{bg} denotes an equation of motion for $S_y(\Phi_y)$ the action expanded around Φ_{bg}, $\mathcal{F}^*_{bg} S_y(\Phi_y) = S_y(\Phi_{bg} + \Phi')$, also has an A_∞-structure[53] with symplectic structure unchanged. It is explained in *weak* A_∞ in Appendix A.2 in the dual picture. Denote by $\mathfrak{m}_y$ an A_∞-structure on $\Phi_y \in \mathcal{H}_y$, define $\mathcal{F}_{bg}$ as a cohomomorphism, and the induced A_∞-structure $\mathfrak{m}'$ on $\mathcal{H}_x$ is given by (A.23) [23]

$$\mathfrak{m}' = \mathfrak{m}_y \circ \mathcal{F}_{bg} \ . \tag{7.1}$$

Generally this induced A_∞-structure $\mathfrak{m}'$ is not quasi-isomorphic to the original one, since $m'_1 =: Q'$, and especially its cohomology class, is changed.

Next we consider the field redefinition

$$\mathcal{F}_*(\Phi_x) = f_1(\Phi_x) + f_2(\Phi_x, \Phi_x) + \cdots \ . \tag{7.2}$$

[23] These arguments are treated on the category of weak A_∞-algebras. Actually when Φ_{bg} does not satisfy the equation of motion, $\mathfrak{m}'$ in eq.(7.1) defines a weak A_∞-structure on $\mathcal{H}_x$.

In order for $\Phi_{bg} \in \mathcal{H}_y$ describes the conformal background x, f_1 should be an isomorphism. We use this f in order for the kinetic term of $S_y(\Phi_{bg} + \Phi')$ to coincides with that of $S_x(\Phi_x)$. Note that by (Prop.5.1) as far as $\mathcal{F}$ preserves the symplectic structures, $\mathcal{F}$ is an A_∞-isomorphism between two A_∞-algebras.

Locally $\mathcal{H}_y$ can be regarded as a fiber on y, and the total space can be viewed as a vector bundle. In [41] for classical closed SFT the infinitesimal background independence is proved by utilizing the CFT theory space connection [54] (and the argument is extended for quantum closed SFT in [42]). Here the 'infinitesimal' means that the two conformal backgrounds which relate to each other by infinitesimal marginal deformation are considered. In this case $\Phi_y - \Phi_x$ is infinitesimal and the infinitesimal deformation of f_1 corresponds to the connection on the vector bundle. $f_2, f_3, \cdots$ preserving the symplectic structures are constructed in [41].

Generally it is difficult to construct $\{f_k\}_{k\geq 2}$. Then giving up constructing $\{f_k\}_{k\geq 2}$ and one can consider the reduction of the shifted action $S_y(\Phi_{bg} + f_1(\Phi'')) - S_y(\Phi_{bg})$ to "minimal" as in section 5. By construction the shifted action satisfies the BV-master equation. Therefore if the corresponding "minimal" action coincides with the on-shell S-matrix elements, the shifted action might be regarded as the SFT action on x, and the action $S_y(\Phi_y)$ is background independent. This argument is an reformulation of the arguments in [15], where infinitesimal marginal deformation Φ_{bg} is discussed.

- *tachyon condensation.* The solution describing the tachyonic nonperturbative vacuum in cubic open SFT[35] is one of the solution of the e.o.m (5.16) with the condition (5.17). The issue can be viewed as a toy model for applying the argument in the present paper, but a modification is needed.

Consider the tachyonic solution in the Siegel gauge. Since we are interested in the Lorentz invariant solution with twist symmetry, the solution is non-zero only for even level scalar fields which is constant in space-time. Each corresponding state is then not physical state because for the basis corresponding to constant (zero-momentum) fields, the eigenvalues of L_0 for the base of level 0, 2, 4 $\cdots$ are -1, 1, 3, $\cdots$, respectively. The field corresponding to the level zero state is the constant tachyon field, and we denote it by t. In order to obtain the tachyonic solution for t and the value of the action at the solution, the tachyon effective potential $V(t)$ has been needed. Its equation of motion is $\frac{\partial}{\partial t}V(t) = 0$. We want to relate $V(t)$ with $\tilde{S}(\tilde{\Phi}^p)$ in eq.(5.37) and $\frac{\partial}{\partial t}V(t) = 0$ with $\sum_{k\geq 2}\tilde{m}^p(\tilde{\Phi}^p) = 0$ (5.8) : the Maurer-Cartan equation on $\mathcal{H}^p$. Other fields corresponding to the states of level $2, 4, \cdots$ have been expressed as the power series of the tachyon field t. It is regarded as the field redefinition $\tilde{\mathcal{F}}^p$ from $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}, \mathfrak{m})$, that is, we want to regard the tachyon field as $\tilde{\Phi}^p \in \mathcal{H}^p$ and other fields as $\tilde{\Phi}^u = f(\tilde{\Phi}^p) \in \mathcal{H}^u$. However a modification is needed because the tachyon is not physical and the tachyonic solution can not be obtained by marginal deformation. We then modify the definition of P, which was essentially the projection onto the physical state. Let the new P be the projection onto the tachyon t. The P in the propagator $Q^+ = b_0\frac{1}{L_0}(1-P)$ is also replaced to this P. Then beginning with the equations (5.16) and (5.17) with $m_3 = m_4 = \cdots = 0$, the field redefinition $\tilde{\mathcal{F}}^p$ and the Maurer-Cartan equations (5.19) and (5.20) with P replaced are obtained. Then the solution of eq.(5.19) is the one we are looking for. The corresponding effective potential is exactly what is mentioned in [55] and presented explicitly using Feynman graphs in [49]. However in order that the solution is well-defined on the Siegel gauge, it should be BRST-invariant. In [56] it is checked in level $(2,6)$ approximation with good agreement. The condition is nothing but eq.(5.20) with replaced P. By discussing the issue from this viewpoints, some symmetry around this tachyon effective potential can be seen and it might give some insight also for the problem of the exact solution and the physics around it.

- *boundary string field theory.*

In subsection 5.4 and section 6 a boundary SFT like action is obtained in eq.(5.45) and (6.8). Each vertex in the action coincides with the string correlation function on-shell, and is extended off-shell in the similar notion as [57]. The action relates to the original SFT action by a finite field redefinition constructed by the Feynman graphs of SFT and satisfies the classical BV-master equation. The field redefinition can be realized as the coordinate transformation on a formal noncommutative supermanifold in (Rem.5.3). However no relation to other literature has been clarified. The property of the BV-BRST transformation $\tilde{\delta}$ and the symplectic form $\tilde{\omega}$ should be investigated further. This argument is rather formal but the definition of boundary SFT[43] is also formal, so relating it to the ordinary SFT might help us to realize the structure of boundary SFT.

Note that action $\tilde{S}(\tilde{\Phi})$ reproduces the string S-matrix on-shell even if it is treated as the fundamental action of field theory. Since in eq.(5.45) $Q^+Q\tilde{\Phi}$ is almost $P^u\tilde{\Phi}$, let us identify it with unphysical fields. The action $\tilde{S}(\tilde{\Phi})$ then does not contain the terms which is linear for unphysical fields. Therefore when computing the on-shell amplitudes the exchanging diagrams do not appear and they indeed coincide with the on-shell string S-matrices.

- *other SFT.* In this paper we deal with the classical open SFT and its A_∞-structure. The argument holds true also for the classical closed SFT, because by commutative-symmetrizing the arguments on A_∞-structures reduce to L_∞-algebras and it is known that the classical closed SFT is described by L_∞-algebras[5]. (The definition of the symplectic structure etc. in subsection 4.3 is necessarily modified.) In other case, like quantum closed, classical and quantum open closed case, the algebraic structures are governed by the BV-algebra in any case. However in order to extend the use of the quasi-isomorphism in the minimal model theorem in these case, it is necessary to refine the algebraic structure in more detail. Such study might makes clear the general structure of these SFTs. Some study for the algebraic structure of quantum closed SFT are found in [42, 58, 59].

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A Dual description of homotopy algebras

We shall give the dual description of A_∞-algebras. In this picture, A_∞-algebras are understood more geometrically. The definition of the dual of a coalgebra in the present paper is given in subsection A.1, and its geometrical point of view is explained in subsection A.2, where we deal with a formal noncommutative supermanifold. These arguments hold similarly for L_∞-algebras.

A.1 The definition of the dual of a coalgebra

Let $\mathcal{H}$ be a graded vector space, and $C(\mathcal{H}) := \oplus_{n=1}^{\infty}(\mathcal{H}^{\otimes n})$ be its tensor algebra. The basis of $\mathcal{H}$ is denoted by $\{\mathbf{e}_i\}$, and here we define the dual basis of $\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k}\in\mathcal{H}^{\otimes k}$ with an inner product as follows. At first, denote the dual basis of $\{\mathbf{e}_i\}$ by $\{\mathbf{e}^i\}$, and define an inner product between $\mathcal{H}$ and $\mathcal{H}^*$ as

$$\langle\mathbf{e}^i|\mathbf{e}_j\rangle=\delta^i_j\ . \tag{A.1}$$

We represent an elements of $C(\mathcal{H})$ as $g=\sum_{k=1}^{\infty}g^{i_k\cdots i_1}\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k}$, and an element of $C(\mathcal{H})^*$, the dual of $C(\mathcal{H})$ as $a=\sum_{k=1}^{\infty}a_{i_1\cdots i_k}\mathbf{e}^{i_k}\cdots\mathbf{e}^{i_1}$. Generalizing the above inner product between $\mathcal{H}$ and $\mathcal{H}^*$ (A.1), here the inner product between $C(\mathcal{H})$ and $C(\mathcal{H})^*$ is defined as

$$\langle\mathbf{e}^{i_k}\cdots\mathbf{e}^{i_1}|\mathbf{e}_{j_1}\cdots\mathbf{e}_{j_l}\rangle=\epsilon^{i_1\cdots i_k}_{j_1\cdots j_l}, \tag{A.2}$$

where $\epsilon^{i_1\cdots i_k}_{j_1\cdots j_l}$ equal zero for $k\neq l$, and if $k=l$, $\epsilon^{i_1\cdots i_k}_{j_1\cdots j_k}=\delta^{i_1}_{j_1}\cdots\delta^{i_k}_{j_k}$. Moreover, for $a_1,\cdots,a_n\in C(\mathcal{H})^*$ and $g_1,\cdots,g_n\in C(\mathcal{H})$, the inner product of n-tensor is given by

$$\langle a_1\otimes\cdots\otimes a_n|g_1\otimes\cdots\otimes g_n\rangle=\langle a_1|g_1\rangle\cdots\langle a_n|g_n\rangle\ .$$

Now we have obtained the inner product between $C(\mathcal{H})$ and its dual $C(\mathcal{H})^*$, we will translate operations on $C(\mathcal{H})$ into those on $C(\mathcal{H})^*$. For the coproducts $\triangle$ on $C(\mathcal{H})$, the product m on $C(\mathcal{H})^*$ is defined as

$$\langle m(a\otimes b)|g\rangle=\langle a\otimes b|\triangle g\rangle\ , \tag{A.3}$$

the derivation δ corresponding to the coderivation $\mathfrak{m}$ is defined as

$$\langle\delta(a)|g\rangle=\langle a|\mathfrak{m}(g)\rangle\ , \tag{A.4}$$

and homomorphism $\mathbf{f}$ corresponds to the cohomomorphism $\mathcal{F}$ from $C(\mathcal{H})$ to another tensor algebra $C(\mathcal{H}')$ is determined as

$$\langle\mathbf{f}(a)|g\rangle=\langle a|\mathcal{F}(g)\rangle\ . \tag{A.5}$$

Because $g\in C(\mathcal{H})$ and $a\in C(\mathcal{H}')^*$, the homomorphism $\mathbf{f}$ is a map from $C(\mathcal{H}')^*$ to $C(\mathcal{H})^*$. Therefore $\mathbf{f}$ can be regarded as $\mathcal{F}^*$: the pullback of $\mathcal{F}$. Here we write the elements of $C(\mathcal{H})$ on the left hand side and the elements of $C(\mathcal{H})^*$ on the right hand side. The operations on $C(\mathcal{H}')$ or $C(\mathcal{H}')^*$ are distinguished by attaching $'$ to them. The above definitions of the operations on $C(\mathcal{H})^*$ translate various conditions for the operations on $C(\mathcal{H})$ into those on $C(\mathcal{H})^*$ as follows. The coassociativity of $\triangle$ is equivalent to the associativity of m :

$$\begin{array}{rcl}\langle m(m(a\otimes b)\otimes c)|g\rangle&=&\langle a\otimes b\otimes c|(\triangle\otimes\mathbf{1})\triangle(g)\rangle\\ &&\qquad\qquad\|\\ \langle m(a\otimes m(b\otimes c))|g\rangle&=&\langle a\otimes b\otimes c|(\mathbf{1}\otimes\triangle)\triangle g\rangle\end{array}\ . \tag{A.6}$$

The condition that $\mathfrak{m}$ is the coderivation is translated into the Leibniz rule for δ :

$$\begin{array}{ccc} \langle \delta \cdot m(a\otimes b)|g\rangle & = & \langle a\otimes b|\triangle \cdot \mathfrak{m}(g)\rangle \\ & & \| \\ \langle m(\delta\otimes \mathbf{1}+\mathbf{1}\otimes\delta)(a\otimes b)|g\rangle & = & \langle a\otimes b|(\mathfrak{m}\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{m})\triangle g\rangle \end{array} . \tag{A.7}$$

The condition that $\mathcal{F} : C(\mathcal{H}) \to C(\mathcal{H}')$ is a cohomomorphism is rewritten as the one that $\mathbf{f} : C(\mathcal{H}')^* \to C(\mathcal{H})^*$ is a homomorphism :

$$\begin{array}{ccc} \langle \mathbf{f} \cdot m'(a\otimes b)|g\rangle & = & \langle a\otimes b|\triangle' \cdot \mathcal{F}(g)\rangle \\ & & \| \\ \langle m(\mathbf{f}(a)\otimes \mathbf{f}(b))|g\rangle & = & \langle a\otimes b|(\mathcal{F}\otimes\mathcal{F})\triangle g\rangle \end{array} . \tag{A.8}$$

$(\mathcal{H},\mathfrak{m})$ is an A_∞-algebra means that $(C(\mathcal{H})^*,\delta)$ is a complex on the dual :

$$0 = \langle \delta\cdot\delta(a)|g\rangle = \langle a|\mathfrak{m}\cdot\mathfrak{m}(g)\rangle = 0 \ . \tag{A.9}$$

Finally the condition that $\mathcal{F}$ is an A_∞-morphism is tranlated into the equivariance of $\mathbf{f}$:

$$\begin{array}{ccc} \langle \delta \cdot \mathbf{f}(a)|g\rangle & = & \langle a|\mathcal{F}\cdot\mathfrak{m}(g)\rangle \\ & & \| \\ \langle \mathbf{f}\cdot\delta'(a)|g\rangle & = & \langle a|\mathfrak{m}'\cdot\mathcal{F}(g)\rangle \end{array} . \tag{A.10}$$

The above statement will be realized with some graphs [24]. In the above explanation, the elements of $C(\mathcal{H})$ are written in the left hand side of the inner products (ket), and the elements of the dual algebra $C(\mathcal{H})^*$ are in the right hand side (bra). Here, for the algebra on the left hand side, we represent the product m, the derivation δ, and the homomorphism $\mathbf{f}$ as $m = \supset\!-$, $\delta = -\boxed{\delta}-$, $\mathbf{f} = -\boxed{\mathbf{f}}-$. According to the operations of the algebra from left, the lines of the graphs are connected to the right direction. In other words, the operations on the algebra $C(\mathcal{H}^*)$ in the left hand side from left yields the flow from the left to the right on the lines of the graphs. Next, for the coalgebra $C(\mathcal{H})$ in the right hand side, we represent the coproduct $\triangle$, the coderivation $\mathfrak{m}$, and the cohomomorphism $\mathcal{F}$ as $\triangle = \supset\!-$, $\mathfrak{m} = -\boxed{\mathfrak{m}}-$, $\mathcal{F} = -\boxed{\mathcal{F}}-$, and define the orientation of the operation from the right to the left on the lines of the graphs. Lastly, in order to distinguish the left and right in the inner products, we introduce $\langle\ |\ \rangle$ between the algebra $C(\mathcal{H})$ and the coalgebra $C(\mathcal{H})^*$.

The definition of the algebra $C(\mathcal{H})^*$ dual to the coalgebra $C(\mathcal{H})$ (A.3)(A.4)(A.5) are written graphically as follows : The graphs in both sides of the equations represent the $\mathbb{C}$ valued inner products. The arrow on the dashed line in (Fig.7) denotes the orientation of the operations in both sides. The m is defined so that the inner product is invariant when the $|$ on the right hand side of (Fig.7) is moved to the left. Then the $\supset\!-$ is m on the left of $|$, and it becomes $\triangle$ on the right of $|$. Similarly, in (Fig.8), the $-\boxed{\delta}-$ on the left of $|$ becomes $-\boxed{\mathfrak{m}}-$ on the right and the $-\boxed{\mathfrak{m}}-$ is $-\boxed{\delta}-$ when is transferred to the left. The situation is similar for $-\boxed{\mathbf{f}}-$ and $-\boxed{\mathcal{F}}-$ (Fig.9).

[24]The graphs used below is different from that in the body of this paper. In fact, a line denotes the flow of an element of $\mathcal{H}$ in the body of this paper, but the line used below denotes an element of $C(\mathcal{H})$.

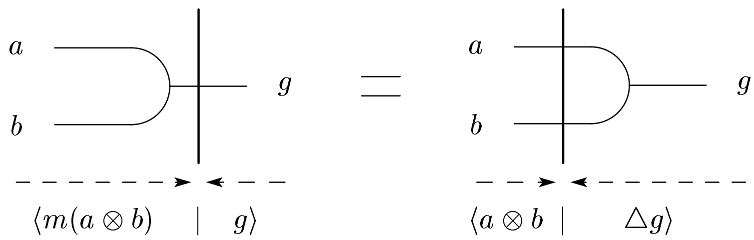

Figure 7: $\langle m(a \otimes b) | g \rangle = \langle a \otimes b | \triangle g \rangle$ eq.(A.3)

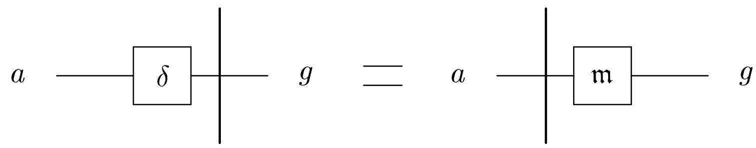

Figure 8: $\langle \delta(a) | g \rangle = \langle a | \mathfrak{m}(g) \rangle$ eq.(A.4)

By the benefit of the above rewriting, the following facts can be understood naturally with these graphs : the equivalence between that $\mathfrak{m}$ is a coderivative and that the δ is a derivative (A.7), the one between that the $\mathcal{F}$ is a cohomomorphism and that the $\mathbf{f}$ is a homomorphism (A.8), the one between that the $(\mathcal{H}, \mathfrak{m})$ is an A_∞-algebra and that the $(C(\mathcal{H})^*, \delta)$ is a complex(A.9), and the one between the $\mathcal{F}$ is an A_∞-morphism and that the $\mathbf{f}$ is δ-equivariant (A.10). For instance, eq.(A.8) is shown as (Fig.10).

A.2 The geometry on $C(\mathcal{H})^*$: formal noncommutative supermanifold

In this subsection, we represent explicitly m, δ and $\mathbf{f}$, which correspond to $\triangle$, $\mathfrak{m}$ and $\mathcal{F}$, respectively, and realize them geometrically on the algebra $C(\mathcal{H})^*$ dual to the $C(\mathcal{H})$. For the coassociative coproduct

$$\triangle(\mathbf{e}_1 \cdots \mathbf{e}_n) = \sum_{k=1}^{n-1} (\mathbf{e}_1 \cdots \mathbf{e}_k) \otimes (\mathbf{e}_{k+1} \cdots \mathbf{e}_n) \ ,$$

the corresponding associative product m defined in eq.(A.3) are written as

$$m((\mathbf{e}^{i_k} \cdots \mathbf{e}^{i_1}) \otimes (\mathbf{e}^{j_l} \cdots \mathbf{e}^{j_1})) = \mathbf{e}^{j_l} \cdots \mathbf{e}^{j_1} \mathbf{e}^{i_k} \cdots \mathbf{e}^{i_1} \ . \tag{A.11}$$

For $a = \sum_{k=1}^{\infty} a_{i_1 \cdots i_k} \mathbf{e}^{i_k} \cdots \mathbf{e}^{i_1}$ and $b = \sum_{l=1}^{\infty} b_{j_1 \cdots j_l} \mathbf{e}^{j_l} \cdots \mathbf{e}^{j_1}$, $m(a \otimes b)$ becomes

$$m((\sum_{k=1}^{\infty} a_{i_1 \cdots i_k} \mathbf{e}^{i_k} \cdots \mathbf{e}^{i_1}) \otimes (\sum_{l=1}^{\infty} b_{j_1 \cdots j_l} \mathbf{e}^{j_l} \cdots \mathbf{e}^{j_1})) = \sum_n (a \cdot b)_{m_1 \cdots m_n} \mathbf{e}^{m_n} \cdots \mathbf{e}^{m_1}$$

$$(a \cdot b)_{m_1 \cdots m_n} = \sum_{p=1}^{n-1} \epsilon_{m_1 \cdots m_n}^{i_1 \cdots i_p j_1 \cdots j_{n-p}} a_{i_1 \cdots i_p} b_{j_1 \cdots j_{n-p}} \ .$$

It is easily seen that by the above definition of m, $(a \cdot b)_{m_1 \cdots m_n} = \langle m(a \otimes b) | \mathbf{e}_{m_1} \cdots \mathbf{e}_{m_n} \rangle = \langle a \otimes b | \triangle(\mathbf{e}_{m_1} \cdots \mathbf{e}_{m_n}) \rangle$ holds.

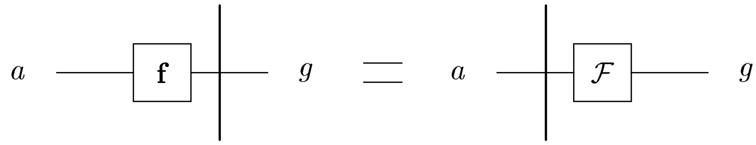

Figure 9: $\langle \mathbf{f}(a)|g\rangle = \langle a|\mathcal{F}(g)\rangle$ eq.(A.5)

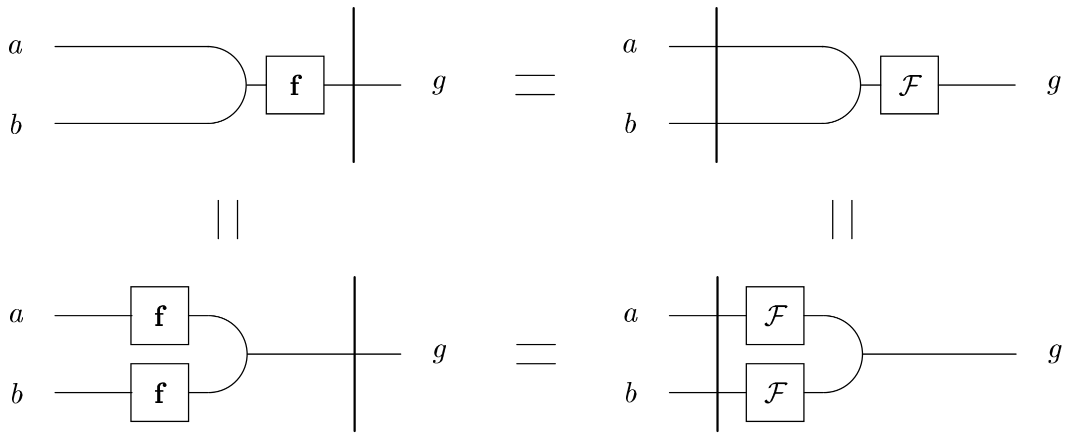

Figure 10: $\langle \mathbf{f}\cdot m(a\otimes b)|g\rangle = \langle m(\mathbf{f}(a)\otimes \mathbf{f}(b))|g\rangle$ eq.(A.8)

$a, b \in C(\mathcal{H})^*$ can be regarded as the polynomial functions on the graded vector space $\mathcal{H}$. Consider $\Phi = \mathbf{e}_i\phi^i \in \mathcal{H}$. ϕ^i is a coordinate of $\mathcal{H}$, and its degree is set to be minus the degree of $\mathbf{e}_i$ in order for Φ to have its degree zero. $\{\phi^i\}$ is isomorphic to $\mathcal{H}^*$, and so it is identified with $\mathcal{H}^*$. Actually, introduce a natural pairing ( ) between $\mathcal{H}$ and $\mathcal{H}^*$ [25], and one can define $a(\Phi)$ as

$$a(\Phi) := \sum_{k=1}^{\infty} a_{i_1\cdots i_k}(\mathbf{e}^{i_k}(\Phi))\cdots(\mathbf{e}^{i_1}(\Phi)) = \sum_{k=1}^{\infty} a_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\ .$$

This $a(\Phi) \in C(\mathcal{H})^*$ is nothing but a polynomial function on the graded vector space $\mathcal{H}$ because $\{\phi^i\}$ is the coordinate on $\mathcal{H}$. The pair of the graded vector space and the algebra of formal power series of the coordinates on the graded vector space is called as *formal supermanifold*[8, 22]. In this case $a(\Phi = 0) = 0$ for any $a \in C(\mathcal{H})^*$, and the coordinates are associative but noncommutative, so this is a formal noncommutative pointed supermanifold. We can translate L_∞-algebras in the language of the formal supermanifold, too. In this situation the coordinates are graded commutative which reflects the cocommutativity of the L_∞-algebra, and we get a formal (commutative) pointed supermanifold.

[25]The inner product is the same type that is defined in eq.(A.1) and different from that in eq.(A.2).

Remark A.1 Note that the product (A.11) is defined so as to satisfy the following compatibility

$$\begin{array}{ccc} \triangle(\Phi^{\otimes n}) & = & \sum_{k=1}^{n}(\Phi^{\otimes k})\otimes(\Phi^{\otimes n-k}) \\ \| & & \| \\ \triangle(\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_n})\phi^{i_n}\cdots\phi^{i_1} & & \sum_{k=1}^{n-1}(\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k})\phi^{i_k}\cdots\phi^{i_1}\otimes(\mathbf{e}_{i_{k+1}}\cdots\mathbf{e}_{i_n})\phi^{i_n}\cdots\phi^{i_{k+1}}\ . \end{array} \tag{A.12}$$

Recall that $\triangle(\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_n})=\sum_{k=1}^{n}(\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k})\otimes(\mathbf{e}_{i_{k+1}}\cdots\mathbf{e}_{i_n})$. Here we identify ϕ^i and $\mathbf{e}^i$. Then the equality between the two terms in the second line in eq.(A.12) means that the operation of ϕ^i' in the tensor coalgebra $C(\mathcal{H})$ are defined by eq.(A.11). Thus the coalgebra $C(\mathcal{H})$ can be defined as a coalgebra of $C(\mathcal{H})^*$-module.

- *coderivation*

Next, for a coderivation $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2+\cdots$,

$$\mathfrak{m}_k(\mathbf{e}_1\cdots\mathbf{e}_n)=\sum_{p=1}^{n-k}(-1)^{\mathbf{e}_1+\cdots+\mathbf{e}_p}\mathbf{e}_1\cdots\mathbf{e}_{p-1}m_k(\mathbf{e}_p\cdots\mathbf{e}_{p+k-1})\mathbf{e}_{p+k}\cdots\mathbf{e}_n\ ,\quad \mathbf{e}_i\in\mathcal{H}\ , \tag{A.13}$$

we construct δ which corresponds to $\mathfrak{m}$. By the definition of δ (A.4), one sees that a derivation corresponding to the coderivative may be constructed separately for k. Express $m_k:\mathcal{H}^k\to\mathcal{H}$ as

$$m_k(\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k})=\mathbf{e}_j c^j_{i_1\cdots i_k} \tag{A.14}$$

and $\delta_k: C(\mathcal{H})^*\to C(\mathcal{H})^*$,

$$\delta_k=\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}$$

is a derivation. Here we identify the coordinate $\{\phi^i\}$ with $\mathcal{H}^*$ and replace $\mathbf{e}^i$ to ϕ^i. The derivation δ is constructed as $\delta=\delta_1+\delta_2+\cdots$. It is regarded as an (odd) *formal vector field* on the formal noncommutative pointed supermanifold. Note that the condition that $\mathfrak{m}_k$ is a coderivation is replaced to that δ_k satisfies the Leibniz rule on the polynomials of ϕ^i's. Moreover, as will be seen explicitly, $\delta^2=0$ holds iff $\mathfrak{m}$ define an A_∞-algebra. The formal manifold with such δ is called *Q-manifold* in [8] [26].

Remark A.2 The operation of $\mathfrak{m}_k$ is compatible with the decomposition of the supercoordinates in the following sense. Here compute $\mathfrak{m}_k(\Phi^{\otimes n})$ in two ways. One way is acting $\mathfrak{m}_k$ after rewriting $\Phi^{\otimes n}=(\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_n})\phi^{i_n}\cdots\phi^{i_1}$ and we get the result in eq.(A.13) as the coefficient of $\phi^{i_n}\cdots\phi^{i_1}$. Another way is computing $\mathfrak{m}_k(\Phi^{\otimes n})$ as

$$\begin{aligned}\mathfrak{m}_k(\Phi^{\otimes n})&=\sum_{p=1}^{n-k}\Phi^{\otimes p}m_k(\Phi)\Phi^{\otimes n-k-p}\\ &=\sum_{p=1}^{n-k}(-1)^{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_{p_1}}}(\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_{p-1}})m_k(\mathbf{e}_{i_p}\cdots\mathbf{e}_{i_{p+k-1}})(\mathbf{e}_{i_{p+k}}\cdots\mathbf{e}_{i_n})\phi^{i_n}\cdots\phi^{i_1}\ ,\end{aligned} \tag{A.15}$$

[26] This Q does not correspond to the BRST operator Q in the body of this paper but δ : the BRST-generator in gauge theory. δ in this paper is written as Q in [8].

and picking up the coefficient of $\phi^{i_n}\cdots\phi^{i_1}$. One can see that this leads the same results as in (A.13) and these arguments are compatible. In the second equality of eq.(A.15), one gets the sign $(-1)^{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_{p_1}}}$ because $\phi^{i_1},\cdots,\phi^{i_{p-1}}$ pass through m_k which has degree one.

Remark A.3 When $\mathfrak{m}$ satisfies $\mathfrak{m}\cdot\mathfrak{m}=0$, we have relations between $\mathfrak{m}_k$, Rewriting m_k using eq.(A.14) yields relations between $c^j_{i_1\cdots i_k}$. On the other hand, in the dual language, the condition $\mathfrak{m}\cdot\mathfrak{m}=0$ is $\delta\cdot\delta=0$. Calculating $\delta\cdot\delta$ and concentrating on the term of n powers of ϕ^i leads

$$\begin{aligned}\sum_{k+l=n+1}\delta_k\cdot\delta_l &= \left(\frac{\overleftarrow{\partial}}{\partial\phi^i}c^i_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\right)\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j_{j_1\cdots j_l}\phi^{j_l}\cdots\phi^{j_1}\\ &= \frac{\overleftarrow{\partial}}{\partial\phi^i}\sum_{k+l=n+1}\sum_{m=1}^{k}(-1)^{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_{m-1}}}c^i_{i_1\cdots i_k}c^{i_m}_{j_1\cdots j_l}\phi^{i_k}\cdots\phi^{i_{m+1}}\left(\phi^{j_l}\cdots\phi^{j_1}\right)\phi^{i_{m-1}}\cdots\phi^{i_1}\ .\end{aligned}$$

The coefficient of $\phi^n\cdots\phi^1$ then reads

$$0=\sum_{\substack{k+l=n+1\\ m=0,\cdots,k-1}}(-1)^{\mathbf{e}_1+\cdots+\mathbf{e}_m}c^i_{1\cdots m,i_m,m+l+1\cdots n}c^{i_m}_{m+1\cdots m+l}\ . \tag{A.16}$$

This is exactly the relation $\mathfrak{m}\cdot\mathfrak{m}=0$ (or eq.(2.6)) rewritten with $\{c^i_{i_1\cdots i_k}\}$.

- *cohomomorphism*

In the terminology of the formal supermanifold, a homomorphism corresponding to a co-homomorphism $\mathcal{F}$ are constructed as follows. Let $\mathcal{H},\mathcal{H}'$ be two graded vector space and $\mathcal{F}_n\ :\mathcal{H}^{\otimes n}\longrightarrow\mathcal{H}'$. A cohomomorphism $\mathcal{F}$ from $C(\mathcal{H})$ to $C(\mathcal{H}')$ is now given by

$$\begin{aligned}&\mathcal{F}=\mathcal{F}^1+\mathcal{F}^2+\mathcal{F}^3+\cdots\ ,\qquad \mathcal{F}^l\ :\ C(\mathcal{H})\longrightarrow{\mathcal{H}'}^{\otimes l}\\ &\mathcal{F}^l(\mathbf{e}_1\cdots\mathbf{e}_n)=\sum_{\substack{n_1,\cdots,n_l\geq 1\\ n_1+\cdots+n_l=n}}f_{n_1}(\mathbf{e}_1\cdots\mathbf{e}_{n_1})\otimes\cdots\otimes f_{n_l}(\mathbf{e}_{n-n_l+1}\cdots\mathbf{e}_n)\ .\end{aligned}\tag{A.17}$$

Now we express f_n as

$$f_n(\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_n})=\mathbf{e}_{j'}f^{j'}_{i_1\cdots i_n}\ .$$

The homomorphism $\mathbf{f}$ gives the pullback from $C(\mathcal{H}')^*$, the formal power series ring on $\mathcal{H}'$, $C(\mathcal{H})^*$. In this reason we can write as $\mathbf{f}=\mathcal{F}^*$. Let $\{\phi^i\}$ and $\{\phi^{i'}\}$ be the coordinates on $\mathcal{H}$ and $\mathcal{H}'$, respectively, and take an element of $C(\mathcal{H}')^*$:$a(\phi'):=\sum_{k=1}^{\infty}a_{i_1\cdots i_k}\phi^{i'_k}\cdots\phi^{i'_1}$. Then $\mathbf{f}:C(\mathcal{H}')^*\to C(\mathcal{H})^*$ is induced from $\mathcal{F}_*$:

$$\begin{array}{cccc}\mathcal{F}_*: & \mathcal{H} & \to & \mathcal{H}'\\ & \phi & \mapsto & \phi'=\mathcal{F}_*(\phi)\end{array}\ ,\qquad \phi^{j'}=\mathcal{F}^{j'}_*(\phi)=f^{j'}_i\phi^i+f^{j'}_{i_1i_2}\phi^{i_2}\phi^{i_1}+\cdots+f^{j'}_{i_1\cdots i_n}\phi^{i_n}\cdots\phi^{i_1}+\cdots \tag{A.18}$$

as $\mathbf{f}(a(\phi'))=a(\mathcal{F}_*(\phi))$. One can see that the cohomomorphism $\mathcal{F}$ is, in the dual geometric picture, a nonlinear map $\mathcal{F}_*$ from a formal supermanifold $\mathcal{H}$ to $\mathcal{H}'$ preserving the origin.

- *A_∞-morphism*

The condition that this $\mathcal{F}$ is an A_∞-morphism is equivalent to the statement that this map $\mathcal{F}_*$ between two formal supermanifolds is compatible with the action of δ and δ' on both sides, *i.e.* $\mathcal{F}_*$ is a morphism between Q-manifolds. For any $a(\phi') \in C(\mathcal{H}')^*$, the condition is

$$\mathbf{f}\delta'(a(\phi')) = \delta\mathbf{f}a(\phi') \ , \tag{A.19}$$

and is written explicitly as

$$\mathbf{f}\left(a(\phi')\frac{\overleftarrow{\partial}}{\partial\phi^{j'}}c^{j'}(\phi')\right) = a(\mathcal{F}_*(\phi))\frac{\overleftarrow{\partial}}{\partial\phi^{j}}c^{j}(\phi)$$

where we expressed $\delta = \frac{\overleftarrow{\partial}}{\partial\phi^j}c^j(\phi)$. Because $a(\mathcal{F}_*(\phi))\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j(\phi) = \mathbf{f}\left(a(\phi')\frac{\overleftarrow{\partial}}{\partial\phi^{j'}}\right)\frac{\phi^{j'}\overleftarrow{\partial}}{\partial\phi^j}c^j(\phi)$ in the right hand side, we get

$$\mathbf{f}\left(c^{j'}(\phi')\right) = \frac{\phi^{j'}\overleftarrow{\partial}}{\partial\phi^j}c^j(\phi) \ . \tag{A.20}$$

We can see that when δ and $\mathbf{f}$ are given and $\mathbf{f}$ has its inverse, then δ' is induced as $c^{j'}(\phi') = \mathbf{f}^{-1}\left(\frac{\phi^{j'}\overleftarrow{\partial}}{\partial\phi^j}c^j(\phi)\right)$.

- *weak* A_∞

Let us add the term $\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j$ to δ where c^j is a constant and write it as δ_w. Explicitly δ_w is of the form

$$\delta_w = \frac{\overleftarrow{\partial}}{\partial\phi^j}c^j + \frac{\overleftarrow{\partial}}{\partial\phi^j}c^j(\phi) = \sum_{k=0}^{\infty}\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} \ .$$

Acting it on $C(\mathcal{H})^*$ yields constant term generally. Thus let us enlarge $C(\mathcal{H})^*$ as $C(\mathcal{H})^*_w$, which denotes the space of $C(\mathcal{H})^*$ plus constant, *i.e.* $C(\mathcal{H})^*_w = \mathbb{C}\oplus C(\mathcal{H})^*$. $C(\mathcal{H})^*_w$ is regarded as the space of functions on $\mathcal{H}$ which do not vanish at the origin generally. Similarly consider the map $\mathbf{f}_w : C(\mathcal{H}')^*_w \to C(\mathcal{H})^*_w$ induced from $\mathcal{F}_{w,*}$ defined as

$$\phi^{j'} = \mathcal{F}^{j'}_{w,*}(\phi) = f^{j'} + f^{j'}_i\phi^i + f^{j'}_{i_1i_2}\phi^{i_2}\phi^{i_1} + \cdots + f^{j'}_{i_1\cdots i_n}\phi^{i_n}\cdots\phi^{i_1} + \cdots$$

where $f^{j'} \in \mathbb{C}$. It can be seen that this map does not preserve the origin. In this extended situation, we can again consider the following conditions

$$\delta_w \cdot \delta_w = 0 \tag{A.21}$$

$$\mathbf{f}_w\delta'_w = \delta_w\mathbf{f}_w \ . \tag{A.22}$$

The condition (A.21) and (A.22) are dual version of the definition of weak A_∞-algebras and weak A_∞-morphisms, respectively.

Let us consider a weak A_∞-isomorphism of the form

$$\phi^{j'} = \mathcal{F}^{j'}_{w,*}(\phi) = f^{j'} + f^{j'}_i\phi^i \ ,$$

where $f^{j'} \in \mathbb{C}$ and $f^{j'}_i$ has its inverse. For simplicity let $f^{j'}_i = \delta^{j'}_i$. The weak A_∞ version of eq.(A.20) then becomes

$$c^j_w(\phi^i) = c'_w(f^i + \phi^i) \ .$$

The weak A_∞-structure δ_w on $C(\mathcal{H})^*$ is naturally induced from δ'_w on $C(\mathcal{H}')^*$ by $\mathbf{f}_w$. Note that when δ'_w defines strictly an A_∞-structure, that is, the constant part c'_w vanishes, and f^i is the solution of the Maurer-Cartan equation on $C(\mathcal{H}')^*$, then $c^j_w \in \mathbb{C}$ vanishes and the induced δ_w is also an A_∞-algebra. Express the coalgebra representation corresponding to δ_w and δ'_w as $\mathfrak{m}_w$ and $\mathfrak{m}'_w$, respectively, and $\mathfrak{m}_w$ is given by

$$\mathfrak{m}_w(e^\Phi) = \mathfrak{m}'_w(e^{\Phi_{bg}+\Phi}) \tag{A.23}$$

where $\Phi_{bg} = \mathbf{e}_i f^i$. Note that since $f^{j'} \in \mathbb{C}$, the corresponding $\mathbf{e}_i$ has degree zero.

B Some relations on vertices in SFT

In this section some properties of vertices in SFT are derived using identities and notations in the body of this paper.

B.1 The recursion relation

In section 3 it was explained that in order for the Feynman rule to reproduce the single covered moduli spaces of Riemann surfaces, the vertices in SFT must satisfy the string factorization equations (3.8),

$$0 = \partial(\mathcal{V}_n) + \sum_{\substack{k_1+k_2=n+2 \\ k_1,k_2\geq 3}} \frac{1}{2}(\mathcal{V}_{k_1})\partial(-)(\mathcal{V}_{k_2}) \ . \tag{B.1}$$

Here let us define the (off-shell) n point tree amplitude using that defined in eq.(5.26) as

$$\frac{1}{n}\int_{\mathcal{M}_n} \langle\Omega|(|\tilde{\Phi}\rangle)^n := \frac{1}{n}\tilde{\mathcal{V}}_n(\tilde{\Phi}) = \frac{1}{n}\langle\tilde{V}_n|(|\Phi\rangle)^n \ . \tag{B.2}$$

In this subsection we shall derive the recursion relation (B.1) or equivalently the classical master equation by employing only the following two relations :

$$0 = \int_{\partial\mathcal{M}_n} \langle\Omega|(|\tilde{\Phi}\rangle)^n = \partial(\langle\tilde{V}_n|)(\tilde{\Phi})^n \ , \tag{B.3}$$

where $\partial(\langle\tilde{V}_n|_{1\cdots n}) = \langle\tilde{V}_n|_{1\cdots k\cdots n}\sum_{k=1}^n Q^{(k)}$ and

$$\{Q, Q^+\} = \mathbf{1} - P \sim \mathbf{1} \ . \tag{B.4}$$

The first equality in eq.(B.3) follows from $\partial\mathcal{M}_n = 0$. As argued in subsection 5.2, the P in eq.(B.4) just contribute to the poles in eq.(B.2). Thus when the external states $\tilde{\Phi}$ are set so that the propagators in $\langle\tilde{V}_n|$ have the pole, eq.(B.3) itself is not well defined. Therefore including this case we define $\{Q, Q^+\} = \mathbf{1}$ between vertices $\{\langle V_k|\}$. Originally the recursion relations (B.1) are defined as the relations between (the subspaces of) the moduli spaces. The problem here arises from attaching the value $\langle\tilde{V}_n| \in (\mathcal{H}^*)^{\otimes n}$ to each the moduli space $\mathcal{M}_n$.

Expanding eq.(B.3) with respect to the number of the propagators Q^+ leads

$$\partial(\langle V_n|) + \sum_{\substack{k_1+k_2=n+2\\k_1,k_2\geq 3}} \frac{1}{2}\partial(\langle V_{k_1}|\langle V_{k_2}| \ -Q^+|\omega\rangle)|Q\Phi\rangle(|\Phi\rangle)^{n-1} + \cdots \ . \tag{B.5}$$

The origin of the minus in front of Q^+ can be found by recalling the calculation using Feynman diagram in subsection 5.3. Each of the second term is rewritten as

$$\begin{aligned}\partial(\langle V_{k_1}|\langle V_{k_2}| \ -Q^+|\omega\rangle) &= (\partial\langle V_{k_1}|)\langle V_{k_2}| \ -Q^+|\omega\rangle \\ &+ \langle V_{k_1}|(\partial\langle V_{k_2}|) \ -Q^+|\omega\rangle \\ &- \langle V_{k_1}|\langle V_{k_2}| \ \{Q,-Q^+\}|\omega\rangle\end{aligned}$$

and the identity (B.4) leads the third term is $\langle V_{k_1}|\langle V_{k_2}| \ |\omega\rangle$. As was explained in section 3, eq.(B.5) is equivalent to

$$\partial(\langle V_n|) + \sum_{\substack{k_1+k_2=n+2\\k_1,k_2\geq 3}} \frac{1}{2}\langle V_{k_1}|\langle V_{k_2}| \ |\omega\rangle = 0 \ .$$

Thus the recursion relation (3.8) is derived. This identity can be rewritten as

$$(S_0, \mathcal{V}_n) + \sum_{\substack{k_1+k_2=n+2\\k_1,k_2\geq 3}} \frac{1}{2}(\mathcal{V}_k, \mathcal{V}_l) = 0$$

and summing up this equation for $n \geq 3$ leads the classical BV-master equation (3.11).

B.2 On-shell reduction of the vertices I

Let us consider the $\mathcal{M}_n^0 \to \mathcal{M}_n$ limit for each vertex as was mentioned in subsection 5.3 and restrict their external states on-shell. This should give the on-shell string correlation functions. In [5] such arguments are given in order to derive the L_∞-structure for string world sheet theory found in [32, 33]. Here it is reviewed in open string case and derive the on-shell A_∞-structure from the A_∞-structure of string field vertices.

In order for $\langle V_n|$ to cover the whole moduli $\mathcal{M}_n$ of $\dim\mathcal{M}_n = n-3$,

$$\sum_{\substack{k_1+k_2=n+2\\k_1,k_2\geq 3}} \frac{1}{2}\langle V_{k_1}|\langle V_{k_2}| \ -Q^+|\omega\rangle$$

must cover the subspace of $\mathcal{M}_n$ whose dimension is less than $n-3$. Because the sum of the dimensions of the moduli spaces corresponding to $\langle V_{k_1}|$ and $\langle V_{k_2}|$ is $(k_1-3)+(k_2-3) = n-4$, the propagator Q^+ must not create one more dimension. Consequently, each vertex has infinite length strips for their external states. A_∞-structure $\mathfrak{m}^{\zeta\to\infty}$ in section 6 is just the case and the vertex $\mathcal{V}_n = \omega(\Phi, m_{n-1}^\infty(\Phi,\cdots,\Phi))$ is the string correlation function when the external states are strictly restricted on-shell (or on physical states).

Now $\mathcal{V}_n$ is of the form $\mathcal{V}_n = \int_{\mathcal{M}_n} \langle\Omega|(|\Phi\rangle)^n$ and $\partial\mathcal{M}_n = 0$, the recursion relation (B.1) is saturated separately as

$$\partial(\langle V_n|) = 0 \ , \qquad \sum_{\substack{k_1+k_2=n+2 \\ k_1,k_2\geq 3}} \frac{1}{2}\langle V_{k_1}|\langle V_{k_2}||\omega\rangle = 0 \ . \tag{B.6}$$

The second identity is just the condition of A_∞-structures with $Q=0$. By employing the first identity, it is reduced to on-shell A_∞-structure as follows.

From eq.(4.8) and (4.18) the relation between A_∞-structure m_{n-1} and $\mathcal{V}_n$ can be read as

$$m_{n-1}(\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}) = (-1)^{\mathbf{e}_k}\mathbf{e}_j\omega^{jk}\mathcal{V}_n(\mathbf{e}_k,\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}) \ . \tag{B.7}$$

Here we restrict the external states as $\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}\in\mathcal{H}^p\cup\mathcal{H}^t$. Choosing the orthogonal basis as in eq.(4.23), m_{n-1} is then decomposed as

$$\begin{aligned} m_{n-1}(\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}) = &(-1)^{\mathbf{e}_k}P^p\mathbf{e}_j\omega^{jk}\mathcal{V}_n(P^p\mathbf{e}_k,\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}) \\ &+(-1)^{\mathbf{e}_k}P^t\mathbf{e}_j\omega^{jk}\mathcal{V}_n(P^u\mathbf{e}_k,\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}) \\ &+(-1)^{\mathbf{e}_k}P^u\mathbf{e}_j\omega^{jk}\mathcal{V}_n(P^t\mathbf{e}_k,\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}) \ . \end{aligned} \tag{B.8}$$

The term on the third line vanishes due to the first identity in eq.(B.6). Because $P^t\mathbf{e}_k$ is Q-exact, write this as $P^t\mathbf{e}_k = Q(Q^u\mathbf{e}_k)$, and

$$\mathcal{V}_n(Q(Q^u\mathbf{e}_k),\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}) = 0 \tag{B.9}$$

follows from $(\partial\mathcal{V}_n)(Q^u\mathbf{e}_k,\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}) = 0$. The fact that eq.(B.9) is hold is an expected result since the string correlation function vanishes even if one Q-trivial external state is included.

Thus it is shown that

$$m_{n-1}(\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n})\in\mathcal{H}^p\cup\mathcal{H}^t \tag{B.10}$$

for any $\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n}\in\mathcal{H}^p\cup\mathcal{H}^t$ and the A_∞-structure m_{n-1} can be reduced on-shell $\mathcal{H}^p\cup\mathcal{H}^t$.

Furthermore one can see that even if one of the external states $\mathbf{e}_{i_k}$ for $2\leq k\leq n$ belongs to $\mathcal{H}^t$, $\mathcal{V}_n(P^p\mathbf{e}_k,\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_n})$ vanishes in the same reason as above and only the term on the first line in eq.(B.8) survives.

On the other hand, the A_∞-condition corresponding to the second identity in eq.(B.6) is eq.(2.6) with $Q=0$:

$$\sum_{\substack{k+l=n+1,\ k,l\geq 2 \\ j=0,\cdots,k-1}} (-1)^{\mathbf{e}_1+\cdots+\mathbf{e}_j} m_k(\mathbf{e}_1,\cdots,\mathbf{e}_j,m_l(\mathbf{e}_{j+1},\cdots,\mathbf{e}_{j+l}),\mathbf{e}_{j+l+1},\cdots,\mathbf{e}_n) = 0 \tag{B.11}$$

with $k=k_1-1$ and $l=k_2-1$. Acting P^p on left and restricting the external states $\mathbf{e}_1,\cdots,\mathbf{e}_n$ on physical states $\mathcal{H}^p$ then leads

$$\sum_{\substack{k+l=n+1,\ k,l\geq 2 \\ j=0,\cdots,k-1}} (-1)^{\mathbf{e}^p_1+\cdots+\mathbf{e}^p_j} P^p m_k(\mathbf{e}^p_1,\cdots,\mathbf{e}^p_j,P^p m_l(\mathbf{e}^p_{j+1},\cdots,\mathbf{e}^p_{j+l}),\mathbf{e}^p_{j+l+1},\cdots,\mathbf{e}^p_n) = 0 \tag{B.12}$$

The reason why m_l can be replaced by $P^p m_l$ is that the contribution of $P^t m_l(\mathbf{e}^p_{j+1}, \cdots, \mathbf{e}^p_{j+l})$ to $m_k(\ , \cdots, \ )$ necessarily belongs to $\mathcal{H}^t$ as was stated above and is projected out by P^p acting in front of m_k.

This concretes the proof that the A_∞-structure of string vertices can be reduced to the A_∞-structure $\{P^p m_k\}_{k\geq 2}$ in string world sheet theory.

B.3 On-shell reduction of the vertices II

In this subsection it will be shown that $\tilde{\mathfrak{m}}^p$ defined in (Def.5.1) indeed define an A_∞-structure on-shell and it can be reduced to an A_∞-structure on physical state space $\mathcal{H}^p$, which was postponed in (Lem.5.1).

The $\tilde{\mathfrak{m}}^p$ defines the on-shell S-matrix elements and the corresponding vertices $\tilde{\mathcal{V}}^p_n$ satisfies $\partial(\langle \tilde{V}^p_n|) = 0$ because of $\partial\mathcal{M}_n = 0$ similarly as the first identity in eq.(B.6). Thus the proof is almost the same as that in the above subsection. Replacing m_{n-1} in the previous subsection by $\tilde{m}^p_{n-1}$ in subsection 5.3 and repeating the argument from eq.(B.7) to eq.(B.12) give us the proof. Only one different point is that $\tilde{\mathfrak{m}}^p_n$ in (Def.5.1) contains P and the identity corresponding to eq.(B.7) is

$$\tilde{m}^p_{n-1}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n}) = (-1)^{\mathbf{e}_k} P\mathbf{e}_j\omega^{jk}\tilde{\mathcal{V}}_n(\mathbf{e}_k, \mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})$$

where the external states $\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n}$ are now restricted on physical state space $\mathcal{H}^p$. In the orthogonal basis, this decompose as

$$\begin{aligned}\tilde{m}^p_{n-1}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n}) &= (-1)^{\mathbf{e}_k} PP^p\mathbf{e}_j\omega^{jk}\tilde{\mathcal{V}}_n(P^p\mathbf{e}_k, \mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})\\ &+ (-1)^{\mathbf{e}_k} PP^t\mathbf{e}_j\omega^{jk}\tilde{\mathcal{V}}_n(P^u\mathbf{e}_k, \mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})\\ &+ (-1)^{\mathbf{e}_k} PP^u\mathbf{e}_j\omega^{jk}\tilde{\mathcal{V}}_n(P^t\mathbf{e}_k, \mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})\end{aligned}$$

and the term in the third line of the above equation vanishes for the same reason as in eq.(B.8). Here recall that $PP^p = P^p$ and note that the identity $QQ^+ + Q^+Q + P = \mathbf{1}$ leads Q and P commute to each another. The above equation is then rewritten as

$$\begin{aligned}\tilde{m}^p_{n-1}(\mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n}) &= (-1)^{\mathbf{e}_k} P^p\mathbf{e}_j\omega^{jk}\tilde{\mathcal{V}}_n(P^p\mathbf{e}_k, \mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})\\ &+ (-1)^{\mathbf{e}_k} Q(PQ^u\mathbf{e}_j)\omega^{jk}\tilde{\mathcal{V}}_n(P^u\mathbf{e}_k, \mathbf{e}^p_{i_2}, \cdots, \mathbf{e}^p_{i_n})\end{aligned} \tag{B.13}$$

where in the second line $PP^t = PQQ^u = QPQ^u$ is used. Thus it has been shown that the image of $\tilde{\mathfrak{m}}^p$ indeed belongs to on-shell $\mathcal{H}^p \cup \mathcal{H}^t$ similarly as the previous subsection. It is easily seen that this result does not change when the elements in $\mathcal{H}^t$ are included as the external states. Furthermore, because the term in the second line in eq.(B.13) belong to $\mathcal{H}^t$, the above $\tilde{m}^p_{n-1}$ can be reduced to the A_∞-structure on $\mathcal{H}^p$ similarly as in eq.(B.12). Let $\iota : \mathcal{H}^p \to \mathcal{H}^p \cup \mathcal{H}^t$ be the inclusion map and the reduced A_∞-structure is given as

$$P^p \circ \tilde{\mathfrak{m}}^p \circ \iota \ ,$$

where P^p and ι is extended naturally as A_∞-morphisms. This is equal to $P^p\mathfrak{m}$ derived in eq.(B.12). The reduced A_∞-algebra is denoted as $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ again to avoid increasing notations.

Thus we complete the proof that $\tilde{\mathfrak{m}}^p$ defines an A_∞-structure on physical states $\mathcal{H}^p$.

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