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The category of cosheaves and Laplace transforms

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Abstract

In this paper, we develop the theory of cosheaves which is based on J.P. Schneiders' work. We define a complex of cosheaves for the Whitney holomorphic functions which is an analogy of tempered holomorphic functions by Kashiwara-Schapira. As one of applications, we obtain a representation of the Laplace transforms on the category of cosheaves.

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1 Introduction

In algebraic analysis, we use various algebraic tools such as the sheaf theory and the localization of categories. J. P. Schneiders [10] introduced the theory of cosheaves by using the concept of a pro-object which is based on Grothendieck's work [9] and applied to Borel-Moore homology. However, it seems that there are no examples of applications to algebraic analysis. In this paper, we develop his theory of cosheaves and apply it to the Laplace transforms by Kashiwara-Schapira [5].

The Laplace transforms is as follows. Let E be an n-dimensional $\mathbb{C}$-vector space, and let j be the inclusion map from E to its projective compactification P. E^* denotes the dual space of E. If F is a $\mathbb{R}$-constructible and $\mathbb{R}^+$-conic sheaf on E, then they set for short :

$$\begin{aligned} \mathrm{THom}\,(F, \mathcal{O}_E) &:= \mathrm{R}\Gamma(P; T\mathcal{H}om\,(j_! F, \mathcal{O}_P)), \\ F \overset{\mathrm{W}}{\otimes} \mathcal{O}_E &:= \mathrm{R}\Gamma(P; j_! F \overset{\mathrm{w}}{\otimes} \mathcal{O}_P) \end{aligned}$$

and gave the Laplace isomorphisms

$$L : F \overset{\mathrm{W}}{\otimes} \mathcal{O}_E \xrightarrow{\sim} F^\wedge[n] \overset{\mathrm{W}}{\otimes} \mathcal{O}_{E^*}. \tag{1.1}$$

$${}^tL : \mathrm{THom}\,(F, \mathcal{O}_E) \xleftarrow{\sim} \mathrm{THom}\,(F^\wedge[n], \mathcal{O}_{E^*}). \tag{1.2}$$

A conic sheaf $\mathcal{O}^t_E$ is defined as the associated sheaf of the presheaf

$$U \mapsto \mathrm{THom}\,(\mathbb{C}_U, \mathcal{O}_E)$$

for any open subanalytic cone $U \subset E$. The morphism (1.2) induces

$$(\mathcal{O}^t_E)^\wedge[n] \simeq \mathcal{O}^t_{E^*}. \tag{1.3}$$

We consider an analogy of (1.3) for Whitney holomorphic functions. We define a complex $\mathcal{O}^{\mathrm{cw}}_E$ of cosheaves which is the dual of $\mathcal{O}^t_E$, and we prove that the Laplace isomorphism (1.1) induces an isomorphism

$$(\mathcal{O}^{\mathrm{cw}}_E)^\wedge[n] \simeq \mathcal{O}^{\mathrm{cw}}_{E^*}. \tag{1.4}$$

We remark the definition of cosheaves. Let X be a topological space, and let $\mathrm{Op}(X)$ be the set of open subsets of X. Suppose that k is a commutative ring with unit. We denote by $\mathrm{Mod}(k)$ the category of k-modules. Since a sheaf is a certain functor from $\mathrm{Op}(X)^{\mathrm{op}}$ to $\mathrm{Mod}(k)$, it seems naturally that we define a cosheaf as a certain functor from $\mathrm{Op}(X)^{\mathrm{op}}$ to $\mathrm{Mod}(k)^{\mathrm{op}}$. However, if we select this definition, then we cannot expect good properties, because projective limits on $\mathrm{Mod}(k)$ is not always exact. So, we introduce the category of promodule, denoted by $\mathrm{Pro}(k)$, and we regard a cosheaf as a certain functor from $\mathrm{Op}(X)^{\mathrm{op}}$ to $\mathrm{Pro}(k)^{\mathrm{op}}$. $\mathrm{Mod}(k)$ is a full subcategory of $\mathrm{Pro}(k)$, and projective limits on $\mathrm{Pro}(k)$ are always exact.

Now let us briefly explain the content of each section.

Section 2 is a preparation for defining the category of cosheaves. This preparation consists of four parts. First, we introduce the category of promodules and its properties.

Secondly, we recall the definition of sheaves with values in a k-abelian category $\mathcal{C}$, denoted by $\mathrm{Sh}(k_X, \mathcal{C})$. Thirdly, we mention several functors such as the direct image and the inverse image. Finally, we study two new functors $\otimes$ and $\mathcal{C}hom$. Although we do not assume that $\mathrm{Sh}(k_X, \mathcal{C})$ is Tannakian, we have few difficulty because of these functors.

In Section 3, we define precosheaves and cosheaves by using the preceding preparation. A cosheaf is a sheaf with value in $\mathrm{Pro}(k)^{\mathrm{op}}$. We remark that this definition is the same as Schneiders' one essentially. After that, we study the proper direct image and its properties on the category of cosheaves.

In Section 4, we first introduce the new concept of "c-injective", and we prove that the category of cosheaves has enough c-injective objects. Next, we mention the derived category of cosheaves, and we define several derived functors. Particularly, we show that the functor $\mathcal{C}hom$ is right derivable by using "c-injective". Finally, we give an analogy of Poincaré-Verdier duality theorem and sevral formulas.

In Section 5, we first discuss the cosheafication. After that, we define a new functor c. If B is a sheaf, then

$$U \mapsto \Gamma_c(U; B)$$

is a precosheaf. We write the associated cosheaf by $\mathrm{c}(B)$. Moreover, we prove that this functor c is left derivable under a certain condition.

In final section, we give the isomorphism (1.4) as the main theorem after we review the Laplace transforms by Kashiwara-Schapira [5]. In order to get (1.4), we first mention the cosheaf version of Fourier-Sato transformations. Next, we define the conic cosheaf $\mathcal{C}_E^{\infty\mathrm{cw}}$ and study its behavior. Finally, we define the complex $\mathcal{O}_E^{\mathrm{cw}}$ of cosheaves by using the conic cosheaf $\mathcal{C}_E^{\infty\mathrm{cw}}$ and prove that the Laplace isomorphism (1.1) induces (1.4)

2 Preliminaries

2.1 Promodules

Let k be a commutative ring with unit. $\mathrm{Mod}(k)$ will denote the category of k-modules. We first establish the concept of a k-promodule. Let $\mathcal{C}$ be a k-abelian category and let $\mathcal{C}^\vee$ denote the category of k-additive functors from $\mathcal{C}$ to $\mathrm{Mod}(k)$. If I is a filtrant set and $\alpha : I^{\mathrm{op}} \to \mathcal{C}$ is a functor, then an object "$\varprojlim_{i\in I}$"$\alpha(i)$ of $\mathcal{C}^\vee$ is defined by

$$\begin{array}{rcccl} \text{“}\varprojlim_{i\in I}\text{”}\alpha(i) & : & \mathcal{C} & \to & \mathrm{Mod}(k), \\ & & A & \mapsto & \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(\alpha(i), A). \end{array}$$

We say that $X \in \mathcal{C}^\vee$ is a pro-object of $\mathcal{C}$ if there exists a filtrant set I and a functor $\alpha : I^{\mathrm{op}} \to \mathcal{C}$ such that X is isomorphic to "$\varprojlim_{i\in I}$"$\alpha(i)$. We denote by $\mathrm{Pro}(\mathcal{C})$ the full subcategory of $\mathcal{C}^\vee$ consisting of pro-objects. Note that morphisms of $\mathrm{Pro}(\mathcal{C})$ are represented as follows :

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\text{“}\varprojlim_{i\in I}\text{”}\alpha(i), \text{“}\varprojlim_{j\in J}\text{”}\beta(j)) \simeq \varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(\alpha(i), \beta(j)).$$

Since $\mathrm{Mod}(k)$ is a k-abelian category, by putting $\mathcal{C} := \mathrm{Mod}(k)$, we get a category $\mathrm{Pro}(\mathrm{Mod}(k))$. We write $\mathrm{Pro}(k)$ instead of $\mathrm{Pro}(\mathrm{Mod}(k))$.

Definition 2.1.1. We call each object of $\mathrm{Pro}(k)$ a k-promodule, and call each morphism of $\mathrm{Pro}(k)$ a k-promodule map.

In this section, we simply write $\mathrm{Hom}_k(\,\cdot\,,\,\cdot\,)$ instead of $\mathrm{Hom}_{\mathrm{Pro}(k)}(\,\cdot\,,\,\cdot\,)$. The basic properties of $\mathrm{Pro}(k)$ are as follows.

Theorem 2.1.2 (Grothendieck [9]). (i) $\mathrm{Pro}(k)$ *is a k-abelian category.*

(ii) *The natural functor* $\mathrm{Mod}(k) \to \mathrm{Pro}(k)$ *is fully faithful and exact.*

(iii) $\mathrm{Pro}(k)$ *admits exact small filtrant projective limits.*

(iv) $\mathrm{Pro}(k)$ *admits direct sums and inductive limits.*

"$\varprojlim$" denotes the usual projective limit in $\mathrm{Pro}(k)$.

Next, we shall consider ring action on k-promodules. Let R be a k-algebra with unit and let $\mathrm{Mod}(R)$ denote the category of left R-modules. For any k-abelian category $\mathcal{C}$, Kashiwara-Schapira [3] defined a category $\mathrm{Mod}(R, \mathcal{C})$ as follows :

$$\mathrm{Ob}(\mathrm{Mod}(R,\mathcal{C})) := \{(X, \xi_X); X \in \mathcal{C}, \xi_X : R \to \mathrm{End}_{\mathcal{C}}(X) \text{ is a morphism of } k\text{-algebras}\},$$
$$\mathrm{Hom}_{\mathrm{Mod}(R,\mathcal{C})}((X, \xi_X), (Y, \xi_Y)) := \{f : X \to Y; f \circ \xi_X(r) = \xi_Y(r) \circ f \text{ for all } r \in R\}.$$

By putting $\mathcal{C} := \mathrm{Pro}(k)$, we obtain a category $\mathrm{Mod}(R, \mathrm{Pro}(k))$.

Definition 2.1.3. We call each object of $\mathrm{Mod}(R, \mathrm{Pro}(k))$ a left R-promodule, and call each morphism of $\mathrm{Mod}(R, \mathrm{Pro}(k))$ a left R-promodule map.

We shall mention the relation between $\mathrm{Mod}(R)$ and $\mathrm{Mod}(R, \mathrm{Pro}(k))$. Take a R-promodule $(X, \xi_X) \in \mathrm{Mod}(R, \mathrm{Pro}(k))$. If $X \in \mathrm{Mod}(k)$, then ξ_X defines the structure of a left R-modules in X. Conversely, if X is a left R-module, then the R-module structure induces the map $\xi_X : R \to \mathrm{End}_{\mathrm{Mod}(k)}(X)$. Moreover, a k-module map f is a morphism in $\mathrm{Mod}(R, \mathrm{Pro}(k))$ if and only if f is a morphism of R-modules. Hence $\mathrm{Mod}(R)$ is a full subcategory of $\mathrm{Mod}(R, \mathrm{Pro}(k))$.

Theorem 2.1.4 (Kashiwara-Schapira [3]).

(i) $\mathrm{Mod}(R, \mathrm{Pro}(k))$ *is a k-abelian category.*

(ii) *The forgetful functor* $\mathrm{Mod}(R, \mathrm{Pro}(k)) \to \mathrm{Pro}(k)$ *is faithful and exact.*

(iii) *The natural functor* $\mathrm{Mod}(R) \to \mathrm{Mod}(R, \mathrm{Pro}(k))$ *is fully faithful and exact.*

Remark 2.1.5. We have another approach of ring action on prolinear spaces; that is to say, we can consider $\mathrm{Pro}(\mathrm{Mod}(R))$. However, $\mathrm{Pro}(\mathrm{Mod}(R))$ is not always equivalent to $\mathrm{Mod}(R, \mathrm{Pro}(k))$. We will not use $\mathrm{Pro}(\mathrm{Mod}(R))$ in this paper since we do not need it later.

Finally, we discuss some results to prove Theorem 4.1.7.

Definition 2.1.6. $P \in \mathrm{Pro}(k)$ is called a quasi-projective module if the functor

$$\mathrm{Hom}_k(P, \ \cdot\ ) : \mathrm{Mod}(k) \to \mathrm{Mod}(k)$$

is exact.

Theorem 2.1.7 (Kashiwara-Schapira [3]). *The category* $\mathrm{Pro}(k)$ *has enough quasi-projective objects.*

Theorem 2.1.8 (Schneiders [10]). *There exists a unique functor*

$$\cdot \ \otimes_k \ \cdot \ : \mathrm{Pro}(k) \times \mathrm{Mod}(k) \to \mathrm{Pro}(k) \tag{2.1.1}$$

with the following property. For any $X,\ Y \in \mathrm{Pro}(k)$ *and* $A \in \mathrm{Mod}(k)$*, we have an isomorphism*

$$\mathrm{Hom}_k(A, \mathrm{Hom}_k(X,Y)) \simeq \mathrm{Hom}\,(X \otimes_k A, Y).$$

Proof. We first assume that A is a free k-module. Then, by taking a basis of A, there is an isomorphism $A \simeq \bigoplus_{i\in I} k_{(i)}$, where I is an index set and each $k_{(i)}$ is a copy of k. So, we have

$$\begin{aligned}\mathrm{Hom}_k(A, \mathrm{Hom}_k(X,Y)) &\simeq \mathrm{Hom}_k(\bigoplus_{i\in I} k_{(i)}, \mathrm{Hom}_k(X,Y))\\ &\simeq \prod_{i\in I} \mathrm{Hom}_k(X,Y)_{(i)}\\ &\simeq \mathrm{Hom}_k(\bigoplus_{i\in I} X_{(i)}, Y).\end{aligned}$$

Therefore $X \otimes_k A$ is represented as $\bigoplus_{i\in I} X_{(i)}$.

Next, we assume that A is a general k-module. Take a resolution $B_2 \to B_1 \to A \to 0$ such that both B_1 and B_2 are free k-modules. Then, $X \otimes_k A$ is defined by the following exact sequence

$$X \otimes_k B_2 \to X \otimes_k B_1 \to X \otimes_k A \to 0.$$

q.e.d.

Remark 2.1.9. The functor (2.1.1) is compatible with the usual tenser product of k-module.

Definition 2.1.10. We say that k satisfies the condition A if for any quasi-projective object $P \in \mathrm{Pro}(k)$ and for any injective object $I \in \mathrm{Mod}(k)$, $\mathrm{Hom}_k(P, I)$ is also injective in $\mathrm{Mod}(k)$.

For example, when k is a Dedekind domain, k satisfies the condition A.

Conjecture 2.1.11. *This condition A seems to be satisfied even if* k *is more general commutative ring. However, I do not know whether the conjecture is true for the present.*

Theorem 2.1.12. *Suppose that* k *satisfies the condition A. If* $P \in \mathrm{Pro}(k)$ *is a quasi-projective object, then the functor* $P \otimes_k \ \cdot$ *is exact.*

Proof. Let $A' \to A \to A''$ be an exact sequence in $\mathrm{Mod}(k)$. Take an injective object $I \in \mathrm{Mod}(k)$. Then, $\mathrm{Hom}_k(P, I)$ is also injective by the hypothesis, so the sequence

$$\mathrm{Hom}_k(A'', \mathrm{Hom}_k(P, I)) \to \mathrm{Hom}_k(A, \mathrm{Hom}_k(P, I)) \to \mathrm{Hom}_k(A', \mathrm{Hom}_k(P, I))$$

is exact. This sequence can be rewritten as

$$\mathrm{Hom}_k(P \otimes_k A'', I) \to \mathrm{Hom}_k(P \otimes_k A, I) \to \mathrm{Hom}_k(P \otimes_k A', I).$$

Since this sequence is exact for any injective object $I \in \mathrm{Mod}(k)$, the following sequence is also exact :

$$P \otimes_k A' \to P \otimes_k A \to P \otimes_k A''.$$

Therefore, $P \otimes_k \cdot$ is an exact functor. q.e.d.

2.2 Generalized sheaves

Let k be a commutative ring with unit and let $\mathcal{C}$ be a k-abelian category. Suppose that X is a topological space. We denote by $\mathrm{Op}(X)$ the set of open subsets of X. We regard $\mathrm{Op}(X)$ as the directed set. In this section, we recall the definition of $\mathcal{C}$-valued presheaves and sheaves without proof. This theory is well known for the specialists. Detailed proof can be found on Kashiwara-Schapira [3].

Remark 2.2.1. We denote by $\mathrm{PSh}(k_X)$ (resp. $\mathrm{Sh}(k_X)$) the category of ordinary presheaves (resp. sheaves) on X.

Remark 2.2.2. We assume that $\mathcal{C}$ satisfies the following conditions.

(i) $\mathcal{C}$ admits small direct sums and small direct products.

(ii) Small filtrant inductive limits in $\mathcal{C}$ are exact.

(iii) If $\{I(j)\}_{j \in J}$ is a family of small filtrant categories indexed by a set J, and if $\{X_{i,j}\}_{i \in I(j)}$ is an inductive system in $\mathcal{C}$, then the natural morphism

$$\varinjlim_{\varphi \in A} \prod_{j \in J} X_{\varphi(j),j} \to \prod_{j \in J} \varinjlim_{i \in I(j)} X_{i,j}$$

is an isomorphism, where $A := \{\varphi : J \to \bigsqcup_{j \in J} I(j) | \varphi(j) \in I(j)\}$.

Definition 2.2.3. A $\mathcal{C}$-valued presheaf F on X is a functor $F : \mathrm{Op}(X)^{\mathrm{op}} \to \mathcal{C}$, that is, the data of

(i) An assignment to each open set $U \subset X$ of a object $F(U) \in \mathcal{C}$,

(ii) A collection of morphisms in $\mathcal{C}$,

$$\rho_{VU} : F(U) \to F(V) \tag{2.2.1}$$

for each pair of open sets U and V with $V \subset U$, these data satisfying

(i) $\rho_{UU} = \mathrm{id}_{F(U)}$,

(ii) For any open inclusions $W \subset V \subset U$, we have $\rho_{WV} \circ \rho_{VU} = \rho_{WU}$.

Definition 2.2.4. Let F and G be two $\mathcal{C}$-valued presheaves on X. A morphism $\varphi : F \to G$ of presheaves is a morphism of functors. In other words, for each open subset U of X, we have a morphism

$$\varphi_U : F(U) \to G(U), \tag{2.2.2}$$

and the collection $\{\varphi_U\}_{U \subset X}$ of the morphisms satisfy the following commutative diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi_V} & G(U) \\ {\scriptstyle \rho_{UV}}\downarrow & & {\scriptstyle \rho_{UV}}\downarrow \\ F(V) & \xrightarrow{\varphi_U} & G(V), \end{array} \qquad V \subset U \subset X. \tag{2.2.3}$$

We denote by $\mathrm{PSh}(k_X, \mathcal{C})$ the category of $\mathcal{C}$-valued presheaves on X.

Remark 2.2.5. If $\mathcal{C}$ is the category of k-modules, then $\mathrm{PSh}(k_X, \mathcal{C})$ is the category consisting of ordinary presheaves.

Remark 2.2.6. We sometimes write $\Gamma(U; F)$ instead of $F(U)$. So, we have a functor

$$\Gamma(U;\ \cdot\ ) : \mathrm{PSh}(k_X, \mathcal{C}) \to \mathcal{C},$$

and a morphism of functors

$$\rho_{VU} : \Gamma(U;\ \cdot\ ) \to \Gamma(V;\ \cdot\ )$$

for any open sets U, V with $V \subset U$.

The category $\mathrm{PSh}(k_X, \mathcal{C})$ admits finite direct sums. If F and G are two $\mathcal{C}$-valued presheaf on X, then the direct sum of F and G is represented as

$$(F \oplus G)(U) = F(U) \oplus G(U). \tag{2.2.4}$$

Moreover, $\mathrm{PSh}(k_X, \mathcal{C})$ admits kernels and cokernels. The kernel and cokernel of a morphism $\varphi : F \to G$ are represented as follows :

$$(\ker \varphi)(U) = \ker(\varphi_U : F(U) \to G(U)), \tag{2.2.5}$$

$$(\operatorname{coker} \varphi)(U) = \operatorname{coker}(\varphi_U : F(U) \to G(U)). \tag{2.2.6}$$

As a result, we know that $\mathrm{PSh}(k_X, \mathcal{C})$ is a k-abelian category.

Let F be a presheaf on X. Suppose that U is an open subset of X and that $\{U_i\}_{i \in I}$ is a family of open sets with $U = \cup_{i \in I} U_i$. A collection of morphisms $\{\rho_{U_i U}\}_{i \in I}$ gives rise to the following morphism

$$d : F(U) \to \prod_{i \in I} F(U_i).$$

Next, consider the natural projections $\varepsilon_i : \prod_{i\in I} F(U_i) \to F(U_i)$ and put $d_{jk} := \rho_{(U_j\cap U_k)\,U_j} \circ \varepsilon_j - \rho_{(U_j\cap U_k)\,U_k} \circ \varepsilon_k$. the collection of morphisms $\{d_{jk}\}_{jk\in I}$ induces the morphism

$$d' : \prod_{i\in I} F(U_i) \to \prod_{j,k\in I} F(U_j \cap U_k).$$

As a result, we obtain the following complex :

$$0 \to F(U) \xrightarrow{d} \prod_{i\in I} F(U_i) \xrightarrow{d'} \prod_{j,k\in I} F(U_j \cap U_k). \tag{2.2.7}$$

Definition 2.2.7. A presheaf $F \in \mathrm{PSh}(k_X, \mathcal{C})$ is called a sheaf if (2.2.7) is exact for any open set $U \subset X$ and any open covering $\{U_i\}_{i\in I}$ of U.

Morphisms of sheaves are defined simply as morphisms of the underlying presheaves. So, the category of $\mathcal{C}$-valued sheaves on X, denoted by $\mathrm{Sh}(k_X, \mathcal{C})$, is clearly the full subcategory of $\mathrm{PSh}(k_X, \mathcal{C})$. Moreover, $\mathrm{Sh}(k_X, \mathcal{C})$ is a k-additive category because the direct sum (2.2.4) of two sheaves F and G is a sheaf.

Theorem 2.2.8. *The forgetful functor*

$$\iota : \mathrm{Sh}(k_X, \mathcal{C}) \to \mathrm{PSh}(k_X, \mathcal{C})$$

has a left adjoint functor.

We write the left adjoint functor of the forgetful functor ι as $(\,\cdot\,)^a$. Then, we have

$$\mathrm{Hom}_{\mathrm{PSh}(k_X,\mathcal{C})}(F, \iota G) \simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(F^a, G).$$

F^a is called the sheafication of F or is called the associated sheaf of F.

As a result of Theorem 2.2.8, we obtain the following corollary.

Corollary 2.2.9. $\mathrm{Sh}(k_X, \mathcal{C})$ *is a k-abelian category.*

Remark 2.2.10. If $\mathcal{C}$ is the category of k-modules, then $\mathrm{Sh}(k_X, \mathcal{C})$ is the category of ordinary sheaves.

Remark 2.2.11. Let $\varphi : F \to G$ be a morphism in $\mathrm{Sh}(k_X, \mathcal{C})$. Then, the kernel of φ on $\mathrm{Sh}(k_X, \mathcal{C})$ is represented as (2.2.5). However, (2.2.6) is not always a sheaf. The cokernel of φ on $\mathrm{Sh}(k_X, \mathcal{C})$ is the sheafication of (2.2.6). Therefore, we know that the functor

$$\Gamma(U;\,\cdot\,) : \mathrm{Sh}(k_X, \mathcal{C}) \to \mathcal{C} \tag{2.2.8}$$

is left exact for any open set $U \subset X$.

Definition 2.2.12. Let F be a $\mathcal{C}$-valued sheaf on X and let U be an open set of X. Then, a sheaf $F|_U$ on U is obtained by putting $F|_U(V) := F(V)$ for each open subset $V \subset U$, and we call $F|_U$ the restriction of F on U.

Definition 2.2.13. Let F and $G \in \mathrm{Sh}(k_X, \mathcal{C})$. Then, the presheaf

$$U \mapsto \mathrm{Hom}_{\mathrm{Sh}(k_U,\mathcal{C})}(F|_U, G|_U) \tag{2.2.9}$$

is a sheaf on X. We denote this sheaf by $\mathcal{H}om_{k_X}(F, G)$.

As a result, we obtain a left exact bifunctor

$$\mathcal{H}om_{k_X}(\,\cdot\,,\,\cdot\,) : \mathrm{Sh}(k_X, \mathcal{C})^{\mathrm{op}} \times \mathrm{Sh}(k_X, \mathcal{C}) \to \mathrm{Sh}(k_X).$$

Let $\mathcal{R}$ be a sheaf of (not necessarily commutative) k-algebra on X.

Definition 2.2.14. Let $M \in \mathrm{Sh}(k_X, \mathcal{C})$ and let ν_M be a morphism of k_X-algebra from $\mathcal{R}$ to $\mathcal{H}om_{k_X}(M, M)$. Then, we call (M, ν_M) a $\mathcal{C}$-valued sheaf with left $\mathcal{R}$-action.

Definition 2.2.15. Let (M, ν_M) and (N, ν_N) be a $\mathcal{C}$-valued sheaves with left $\mathcal{R}$-action. Then, a morphism from (M, ν_M) to (N, ν_N) is a morphism $\varphi \in \mathrm{Hom}_{k_X}(M, N)$ which satisfies the following commutative diagram

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "\nu_M"] \arrow[d, "\nu_N"'] & \mathcal{H}om_{k_X}(M, M) \arrow[d, "\mathcal{H}om\,(M,\varphi)"'] \\
\mathcal{H}om_{k_X}(N, N) \arrow[r, "\mathcal{H}om\,(\varphi,N)"] & \mathcal{H}om_{k_X}(M, N).
\end{tikzcd}$$

We denote by $\mathrm{Sh}(\mathcal{R}, \mathcal{C})$ the category of $\mathcal{C}$-valued $\mathcal{R}$-sheaves with left $\mathcal{R}$-action. Note that $\mathrm{Sh}(\mathcal{R}, \mathcal{C})$ is a k-abelian category and that the forgetful functor $\mathrm{Sh}(\mathcal{R}, \mathcal{C}) \to \mathrm{Sh}(k_X, \mathcal{C})$ is exact and faithful.

We have a left exact bifunctor

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{R},\mathcal{C})}(\,\cdot\,,\,\cdot\,) : \mathrm{Sh}(\mathcal{R}, \mathcal{C})^{\mathrm{op}} \times \mathrm{Sh}(\mathcal{R}, \mathcal{C}) \to \mathrm{Mod}(k),$$

and an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathrm{Sh}(\mathcal{R},\mathcal{C})}(M, N) &\to \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(M, N) \\ &\xrightarrow{e} \mathrm{Hom}_{\mathrm{Sh}(k_X)}(\mathcal{R}, \mathcal{H}om_{k_X}(M, N)), \end{aligned} \tag{2.2.10}$$

where $e(\varphi) := \mathcal{H}om_{k_X}(M, \varphi) \circ \nu_M - \mathcal{H}om_{k_X}(\varphi, N) \circ \nu_N$.

Definition 2.2.16. Let M and $N \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$. Then, we define a sheaf $\mathcal{H}om_{\mathcal{R}}(M, N)$ as

$$U \mapsto \mathrm{Hom}_{\mathrm{Sh}(\mathcal{R}|_U,\mathcal{C})}(M|_U, N|_U).$$

By the definition above, we have a left exact bifunctor

$$\mathcal{H}om_{\mathcal{R}}(\,\cdot\,,\,\cdot\,) : \mathrm{Sh}(\mathcal{R}, \mathcal{C})^{\mathrm{op}} \times \mathrm{Sh}(\mathcal{R}, \mathcal{C}) \to \mathrm{Sh}(k_X),$$

and an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{R}}(M, N) \to \mathcal{H}om_{k_X}(M, N) \xrightarrow{e} \mathcal{H}om_{k_X}(\mathcal{R}, \mathcal{H}om_{k_X}(M, N)).$$

2.3 Direct and inverse image

Let X and Y be two topological spaces and let f be a continuous map from Y to X.

If G is a $\mathcal{C}$-valued sheaf on Y, then the presheaf

$$U \mapsto G(f^{-1}(U)) \tag{2.3.1}$$

is a $\mathcal{C}$-valued sheaf on X.

Definition 2.3.1. Let G be a $\mathcal{C}$-valued sheaf on Y. Then, we denote the sheaf (2.3.1) on X by f_*G, and we call it the direct image of G by f.

By the definition above, there is a left exact functor

$$f_* : \mathrm{Sh}(k_Y, \mathcal{C}) \to \mathrm{Sh}(k_X, \mathcal{C}).$$

Let F be a $\mathcal{C}$-valued sheaf on X. Then, we define a presheaf on Y as

$$U \mapsto \varinjlim_{f(U)\subset V} F(V). \tag{2.3.2}$$

However, this presheaf is not always a sheaf.

Definition 2.3.2. Let F be a $\mathcal{C}$-valued sheaf on X. Then, we denote the associated sheaf of (2.3.2) by $f^{-1}F$, and we call it the inverse image of F by f.

There is an exact functor

$$f^{-1} : \mathrm{Sh}(k_X, \mathcal{C}) \to \mathrm{Sh}(k_Y, \mathcal{C}).$$

Remark 2.3.3. Let $F \in \mathrm{Sh}(k_X, \mathcal{C})$. Then, we write the presheaf (2.3.2) as $f^\wedge F$. Since $f^{-1}F$ is the associated sheaf of $f^\wedge F$, there exists a natural morphism

$$f^\wedge F \to f^{-1}F,$$

and this morphism is a monomorphism.

Theorem 2.3.4. *f_* is a right adjoint to f^{-1}. More precisely, if $F \in \mathrm{Sh}(k_X, \mathcal{C})$ and $G \in \mathrm{Sh}(k_Y, \mathcal{C})$, then we have the following isomorphisms :*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(k_Y,\mathcal{C})}(f^{-1}F, G) &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(F, f_*G). \\ f_*\mathcal{H}om_{k_Y}(f^{-1}F, G) &\simeq \mathcal{H}om_{k_X}(F, f_*G). \end{aligned}$$

Proposition 2.3.5. *Let X and Y be two topological spaces and let $f : Y \to X$ be a continuous map. Then,*

(i) *We have the following functors :*

$$\begin{aligned} f_* &: \mathrm{Sh}(f^{-1}\mathcal{R}, \mathcal{C}) \to \mathrm{Sh}(\mathcal{R}, \mathcal{C}), \\ f^{-1} &: \mathrm{Sh}(\mathcal{R}, \mathcal{C}) \to \mathrm{Sh}(f^{-1}\mathcal{R}, \mathcal{C}). \end{aligned}$$

(ii) *f_* is a right adjoint to f^{-1}. In other words, for any $M \in \mathrm{Sh}(\mathcal{R})$ and $N \in \mathrm{Sh}(f^{-1}\mathcal{R})$, we have the following isomorphisms*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(f^{-1}\mathcal{R},\mathcal{C})}(f^{-1}M, N) &\simeq \mathrm{Hom}_{\mathrm{Sh}(\mathcal{R},\mathcal{C})}(M, f_*N), \\ f_*\mathcal{H}om_{f^{-1}\mathcal{R}}(f^{-1}M, N) &\simeq \mathcal{H}om_{\mathcal{R}}(M, f_*N). \end{aligned}$$

Let $G \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$. Then, for any locally closed set Z of X, we shall define a sheaf $\Gamma_Z F$ as follows. First, we assume that Z is an open set U and that $i : U \to X$ is the inclusion map, then we put $\Gamma_U G := i_* i^{-1} G$. Next, we suppose that Z is a closed set S. Then, there is a natural morphism $G \to \Gamma_{X-S} G$, so we put $\Gamma_S G := \ker(G \to \Gamma_{X-S} G)$. Finally, we consider the case that Z is a general locally closed. Select an open set U and a closed set S with $Z = U \cap S$. Then, we define $\Gamma_Z G$ as $\Gamma_U \circ \Gamma_S G$. Remark that $\Gamma_Z G$ is depend only upon Z. Consequently, there is a left exact functor

$$\Gamma_Z(\,\cdot\,) : \mathrm{Sh}(\mathcal{R}, \mathcal{C}) \to \mathrm{Sh}(\mathcal{R}, \mathcal{C}).$$

Moreover, when $F \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$, we also define a sheaf F_Z as follows. If Z is a closed set and if $i : S \to X$ is the inclusion map, then we put $F_S := i_{S*} i_S^{-1} F$. Next, if U is a open set, then there is a natural morphism $F \to F_{X-U}$, so we put $F_U := \ker(F \to F_{X-U})$. Finally, assume Z is a locally closed set. Then, we chose an open set U and a closed set S, and we put $F_Z := (G_U)_S$. Consequently, we have an exact functor

$$(\,\cdot\,)_Z : \mathrm{Sh}(\mathcal{R}, \mathcal{C}) \to \mathrm{Sh}(\mathcal{R}, \mathcal{C}).$$

The functor $\Gamma_Z(\,\cdot\,)$ is right adjoint to the functor $(\,\cdot\,)_Z$, and we have

$$\mathcal{H}om_{\mathcal{R}}(F_Z, G) \simeq \mathcal{H}om_{\mathcal{R}}(F, \Gamma_Z G) \simeq \Gamma_Z \mathcal{H}om_{\mathcal{R}}(F, G). \tag{2.3.3}$$

2.4 $\mathcal{C}hom$ and $\otimes$

We define two new bifunctors $\mathcal{C}hom$ and $\otimes$, and we study the properties. We shall recall the classical result below.

Proposition 2.4.1. *Let A be a sheaf on X. Then, there exists a family $\{U_i\}_{i \in I}$ of open sets of X and an epimorphism $\bigoplus_{i \in I} k_{U_i} \to A$ of sheaves.*

Theorem 2.4.2. *Let $A \in \mathrm{Sh}(k_X)$ and let F, $G \in \mathrm{Sh}(k_X, \mathcal{C})$. Then, there exists two $\mathcal{C}$-valued sheaves $\mathcal{C}hom_{k_X}(A, F)$ and $A \otimes_{k_X} G$ which satisfy the following isomorphisms*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G, F)) &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \mathcal{C}hom_{k_X}(A, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(A \otimes_{k_X} G, F). \end{aligned}$$

Proof. (a) First, we assume that U is an open set and that $A = k_U$. Then, by (2.3.3), we get

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G, F)) &= \mathrm{Hom}_{\mathrm{Sh}(k_X)}(k_U, \mathcal{H}om_{k_X}(G, F)) \\ &\simeq \Gamma(X; \Gamma_U \mathcal{H}om_{k_X}(G, F)) \\ &\simeq \Gamma(X; \mathcal{H}om_{k_X}(G, \Gamma_U F)) \\ &= \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \Gamma_U F)). \end{aligned}$$

Hence, $\mathcal{C}hom_{k_X}(A,F)$ is represented as $\Gamma_U F$. Similarly,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G,F)) &\simeq \Gamma(X; \Gamma_U \mathcal{H}om_{k_X}(G,F)) \\
&\simeq \Gamma(X; \mathcal{H}om_{k_X}(G_U,F)) \\
&= \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G_U,F)),
\end{aligned}$$

so $A \otimes_{k_X} G$ is represented as G_U.

(b) Next, we suppose that $A = \bigoplus_{i\in I} A_i$ and that $\mathcal{C}hom_{k_X}(A_i,F)$ and $A_i \otimes_{k_X} G$ are well-defined for $i \in I$. Then, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G,F)) &= \mathrm{Hom}_{\mathrm{Sh}(k_X)}(\bigoplus_{i\in I} A_i, \mathcal{H}om_{k_X}(G,F)) \\
&= \prod_{i\in I} \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A_i, \mathcal{H}om_{k_X}(G,F)) \\
&\simeq \prod_{i\in I} \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \mathcal{C}hom_{k_X}(A_i,F)) \\
&= \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \prod_{i\in I} \mathcal{C}hom_{k_X}(A_i,F))
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G,F)) &= \prod_{i\in I} \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A_i, \mathcal{H}om_{k_X}(G,F)) \\
&\simeq \prod_{i\in I} \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(A_i \otimes_{k_X} G, F) \\
&= \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(\bigoplus_{i\in I}(A_i \otimes_{k_X} G), F).
\end{aligned}$$

Therefore, we get $\mathcal{C}hom_{k_X}(A,F) \simeq \prod_{i\in I} \mathcal{C}hom_{k_X}(A_i,F)$ and $A \otimes_{k_X} G \simeq \bigoplus_{i\in I}(A_i \otimes_{k_X} G)$.

(c) Finally, we assume that A is a general sheaf on X. By applying Proposition 2.4.1, there exists an exact sequence $A_2 \to A_1 \to A \to 0$ such that $\mathcal{C}hom_{k_X}(A_i,F)$ and $A_i \otimes_{k_X} G$ are well-defined for $i = 1,2$. Hence, we define $\mathcal{C}hom_{k_X}(A,F)$ and $A \otimes_{k_X} G$ by the exact sequence

$$\begin{gathered}
0 \to \mathcal{C}hom_{k_X}(A,F) \to \mathcal{C}hom_{k_X}(A_1,F) \to \mathcal{C}hom_{k_X}(A_2,F), \\
A_2 \otimes_{k_X} G \to A_1 \otimes_{k_X} G \to A \otimes_{k_X} G \to 0.
\end{gathered}$$

It is easy to prove that $\mathcal{C}hom_{k_X}(A,F)$ and $A \otimes_{k_X} G$ depend only upon A.

q.e.d.

We obtain a left exact bifunctor

$$\mathcal{C}hom_{k_X}(\,\cdot\,,\,\cdot\,) : \mathrm{Sh}(k_X)^{\mathrm{op}} \times \mathrm{Sh}(k_X,\mathcal{C}) \to \mathrm{Sh}(k_X,\mathcal{C}),$$

and a right exact bifunctor

$$\cdot \otimes_{k_X} \cdot : \mathrm{Sh}(k_X) \times \mathrm{Sh}(k_X,\mathcal{C}) \to \mathrm{Sh}(k_X,\mathcal{C}).$$

Theorem 2.4.3. *Let Z be a locally closed subset of X. If $A \in \mathrm{Sh}(k_X)$ and $F \in \mathrm{Sh}(k_X, \mathcal{C})$, then we have*

$$\Gamma_Z \mathcal{C}hom_{k_X}(A, F) \simeq \mathcal{C}hom_{k_X}(A_Z, F) \simeq \mathcal{C}hom_{k_X}(A, \Gamma_Z(F)).$$

Proof. Let $G \in \mathrm{Sh}(k_X, \mathcal{C})$. By (2.3.3), we get

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \Gamma_Z \mathcal{C}hom_{k_X}(A, F)) &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G_Z, \mathcal{C}hom_{k_X}(A, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G_Z, F)), \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G_Z, F)) &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \Gamma_Z \mathcal{H}om_{k_X}(G, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A_Z, \mathcal{H}om_{k_X}(G, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \mathcal{C}hom_{k_X}(A_Z, F)), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G_Z, F)) &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G, \Gamma_Z F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \mathcal{C}hom_{k_X}(A, \Gamma_Z F)). \end{aligned}$$

Since these are true for any G, the proof follows. q.e.d.

Remark 2.4.4. Let Z be a locally closed subset of X. If $A \in \mathrm{Sh}(k_X)$ and $G \in \mathrm{Sh}(k_X, \mathcal{C})$, then we have

$$(A \otimes_{k_X} G)_Z \simeq A_Z \otimes_{k_X} G \simeq A \otimes_{k_X} G_Z.$$

This proof is similar to the proof of Theorem 2.4.3.

Theorem 2.4.5. *Under the hypothesis of Theorem 2.4.2, we have an isomorphism*

$$\begin{aligned} \mathcal{H}om_{k_X}(A, \mathcal{H}om_{k_X}(G, F)) &\simeq \mathcal{H}om\,(G, \mathcal{C}hom_{k_X}(A, F)) \\ &\simeq \mathcal{H}om_{k_X}(A \otimes_{k_X} G, F). \end{aligned}$$

Proof. By Theorem 2.4.3, for any open set $U \subset X$, we have

$$\begin{aligned} \Gamma(U; \mathcal{H}om_{k_X}(A, \mathcal{H}om_{k_X}(G, F))) &= \mathrm{Hom}_{\mathrm{Sh}(k_U)}(A|_U, \mathcal{H}om_{k_X}(G, F)|_U) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A_U, \mathcal{H}om_{k_X}(G, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \mathcal{C}hom_{k_X}(A_U, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \Gamma_U \mathcal{C}hom_{k_X}(A, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_U,\mathcal{C})}(G|_U, \mathcal{C}hom_{k_X}(A, F)|_U) \\ &= \Gamma(U; \mathcal{H}om_{k_X}(G, \mathcal{C}hom_{k_X}(A, F))). \end{aligned}$$

Similarly, by Remark 2.4.4, we also have

$$\begin{aligned} \Gamma(U; \mathcal{H}om_{k_X}(A, \mathcal{H}om_{k_X}(G, F)) &= \mathrm{Hom}_{\mathrm{Sh}(k_U)}(A|_U, \mathcal{H}om_{k_X}(G, F)|_U) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \Gamma_U \mathcal{H}om_{k_X}(G, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G, \Gamma_U F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(A \otimes_{k_X} G, \Gamma_U F) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_U,\mathcal{C})}((A \otimes_{k_X} G)|_U, F|_U) \\ &= \Gamma(U; \mathcal{H}om_{k_X}(A \otimes_{k_X} G, F)). \end{aligned}$$

q.e.d.

Theorem 2.4.6. *Let* A,$B \in \mathrm{Sh}(k_X)$ *and let* $F \in \mathrm{Sh}(k_X, \mathcal{C})$. *Then, we have*

$$\begin{aligned}\mathcal{C}hom_{k_X}(B, \mathcal{C}hom_{k_X}(A, F)) &\simeq \mathcal{C}hom_{k_X}(A \otimes_{k_X} B, F) \\ &\simeq \mathcal{C}hom_{k_X}(A, \mathcal{C}hom_{k_X}(B, F)).\end{aligned}$$

Proof. We prove the first isomorphism. Let $G \in \mathrm{Sh}(k_X, \mathcal{C})$. By Theorem 2.4.5, we have

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \mathcal{C}hom_{k_X}(B, \mathcal{C}hom_{k_X}(A, F))) \\ \simeq\ &\mathrm{Hom}_{\mathrm{Sh}(k_X)}(B, \mathcal{H}om_{k_X}(G, \mathcal{C}hom_{k_X}(A, F))) \\ \simeq\ &\mathrm{Hom}_{\mathrm{Sh}(k_X)}(B, \mathcal{H}om_{k_X}(A, \mathcal{H}om_{k_X}(G, F))) \\ \simeq\ &\mathrm{Hom}_{\mathrm{Sh}(k_X)}(A \otimes_{k_X} B, \mathcal{H}om_{k_X}(G, F)) \\ \simeq\ &\mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \mathcal{C}hom_{k_X}(A \otimes_{k_X} B, F)).\end{aligned}$$

Since this is true for any G, the first isomorphism follows. The second isomorphism is similar. q.e.d.

Remark 2.4.7. If A,$B \in \mathrm{Sh}(k_X)$ and $G \in \mathrm{Sh}(k_X, \mathcal{C})$, then we have an isomorphism

$$(A \otimes_{k_X} B) \otimes_{k_X} G \simeq A \otimes_{k_X} (B \otimes_{k_X} G).$$

This proof is similar to the proof of Theorem 2.4.6.

Theorem 2.4.8. *Let* X *and* Y *be two topological spaces and let* $f : Y \to X$ *be a continuous map. If* $A \in \mathrm{Sh}(k_X)$ *and* $F \in \mathrm{Sh}(k_Y, \mathcal{C})$, *then we have*

$$f_*\mathcal{C}hom_{k_Y}(f^{-1}A, F) \simeq \mathcal{C}hom_{k_X}(A, f_*F).$$

Proof. Let $G \in \mathrm{Sh}(k_X, \mathcal{C})$. By Theorem 2.3.4, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, f_*\mathcal{C}hom_{k_Y}(f^{-1}A, F)) &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_Y,\mathcal{C})}(f^{-1}G, \mathcal{C}hom_{k_Y}(f^{-1}A, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_Y)}(f^{-1}A, \mathcal{H}om_{k_Y}(f^{-1}G, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, f_*\mathcal{H}om_{k_Y}(f^{-1}G, F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(A, \mathcal{H}om_{k_X}(G, f_*F)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(G, \mathcal{C}hom_{k_X}(A, f_*F)).\end{aligned}$$

Since this is true for any G, the proof follows. q.e.d.

Remark 2.4.9. Let X and Y be two topological spaces and let $f : Y \to X$ be a continuous map. If $A \in \mathrm{Sh}(k_X)$ and $G \in \mathrm{Sh}(k_X, \mathcal{C})$, then we have

$$f^{-1}A \otimes_{k_Y} f^{-1}G \simeq f^{-1}(A \otimes_{k_Y} G).$$

This proof is similar to the proof of Theorem 2.4.8.

Remark 2.4.10. Let $M \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$. Then, by Theorem 2.4.2, the ring action $\nu_M : \mathcal{R} \to \mathcal{H}om_{k_X}(M, M)$ induces two morphisms

$$\lambda_M : M \to \mathcal{C}hom_{k_X}(\mathcal{R}, M),$$
$$\mu_M : \mathcal{R} \otimes M \to M.$$

If $M \in \mathrm{Sh}(\mathcal{R}^{\mathrm{op}}, \mathcal{C})$, then we use the notations λ_M^{op} and μ_M^{op} similarly.

Definition 2.4.11. Let $A \in \mathrm{Sh}(\mathcal{R})$ and $N \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$. Then we put

$$\mathcal{C}hom_{\mathcal{R}}(A, N) := \ker(\mathcal{C}hom_{k_X}(A, N) \xrightarrow{d} \mathcal{C}hom_{k_X}(\mathcal{R} \otimes A, N)),$$

where $d = \mathcal{C}hom_{k_X}(\mu_A, N) - \mathcal{C}hom_{k_X}(A, \lambda_N)$.

There exists a left exact bifunctor

$$\mathcal{C}hom_{\mathcal{R}}(\,\cdot\,,\,\cdot\,) : \mathrm{Sh}(\mathcal{R})^{\mathrm{op}} \times \mathrm{Sh}(\mathcal{R}, \mathcal{C}) \to \mathrm{Sh}(k_X, \mathcal{C}).$$

Remark 2.4.12. Suppose that $F \in \mathrm{Sh}(k_X, \mathcal{C})$, $A \in \mathrm{Sh}(\mathcal{R})$ and $N \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$. Then, by Theorem 2.4.2 and Definition 2.4.11, we have the following isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(\mathcal{R})}(A, \mathcal{H}om_{k_X}(F, N)) &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X, \mathcal{C})}(F, \mathcal{C}hom_{\mathcal{R}}(A, N)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(\mathcal{R}, \mathcal{C})}(A \otimes_{k_X} F, N). \end{aligned}$$

Theorem 2.4.13. *Suppose that $A \in \mathrm{Sh}(k_X)$, $B \in \mathrm{Sh}(\mathcal{R})$ and $N \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$. Then, we have isomorphisms*

$$\begin{aligned} \mathcal{C}hom_{\mathcal{R}}(B, \mathcal{C}hom_{k_X}(A, N)) &\simeq \mathcal{C}hom_{\mathcal{R}}(A \otimes_{k_X} B, N) \\ &\simeq \mathcal{C}hom_{k_X}(A, \mathcal{C}hom_{\mathcal{R}}(B, N)). \end{aligned}$$

Proof. We prove the first isomorphism. Let $F \in \mathrm{Sh}(k_X, \mathcal{C})$. By Remark 2.4.12, we get

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Sh}(k_X, \mathcal{C})}(F, \mathcal{C}hom_{\mathcal{R}}(B, \mathcal{C}hom_{k_X}(A, N))) \\ \simeq\ &\mathrm{Hom}_{\mathrm{Sh}(\mathcal{R})}(B, \mathcal{H}om_{k_X}(F, \mathcal{C}hom_{k_X}(A, N))) \\ \simeq\ &\mathrm{Hom}_{\mathrm{Sh}(\mathcal{R})}(B, \mathcal{H}om_{k_X}(A, \mathcal{H}om_{k_X}(F, N))) \\ \simeq\ &\mathrm{Hom}_{\mathrm{Sh}(\mathcal{R})}(A \otimes_{k_X} B, \mathcal{H}om_{k_X}(F, N)) \\ \simeq\ &\mathrm{Hom}_{\mathrm{Sh}(k_X, \mathcal{C})}(F, \mathcal{C}hom_{\mathcal{R}}(A \otimes_{k_X} B, N)). \end{aligned}$$

Since this is true for any F, the first isomorphism follows. The second isomorphism is also similar. q.e.d.

Theorem 2.4.14. *Let X and Y be two topological spaces and let $f : Y \to X$ be a continuous map. If $A \in \mathrm{Sh}(\mathcal{R})$ and $N \in \mathrm{Sh}(f^{-1}\mathcal{R}, \mathcal{C})$, then we have*

$$f_* \mathcal{C}hom_{f^{-1}\mathcal{R}}(f^{-1}A, N) \simeq \mathcal{C}hom_{\mathcal{R}}(A, f_* N).$$

Proof. Let $F \in \mathrm{Sh}(k_X, \mathcal{C})$. Then, by Remark 2.4.12, we get

$$\begin{aligned}
& \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(F, f_*\mathcal{C}hom_{f^{-1}\mathcal{R}}(f^{-1}A, N)) \\
\simeq\ & \mathrm{Hom}_{\mathrm{Sh}(k_Y,\mathcal{C})}(f^{-1}F, \mathcal{C}hom_{f^{-1}\mathcal{R}}(f^{-1}A, N)) \\
\simeq\ & \mathrm{Hom}_{\mathrm{Sh}(f^{-1}\mathcal{R})}(f^{-1}A, \mathcal{H}om_{k_Y}(f^{-1}F, N)) \\
\simeq\ & \mathrm{Hom}_{\mathrm{Sh}(\mathcal{R})}(A, f_*\mathcal{H}om_{k_Y}(f^{-1}F, N)) \\
\simeq\ & \mathrm{Hom}_{\mathrm{Sh}(\mathcal{R})}(A, \mathcal{H}om_{k_X}(F, f_*N)) \\
\simeq\ & \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(F, \mathcal{C}hom_{\mathcal{R}}(A, f_*N)).
\end{aligned}$$

Since this is true for any F, the proof follows. q.e.d.

Remark 2.4.15. Let Z be a locally closed set of X. If $A \in \mathrm{Sh}(\mathcal{R})$ and $N \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$, then we have isomorphisms

$$\Gamma_Z \mathcal{C}hom_{\mathcal{R}}(A, N) \simeq \mathcal{C}hom_{\mathcal{R}}(A_Z, N) \simeq \mathcal{C}hom_{\mathcal{R}}(A, \Gamma_Z N).$$

Definition 2.4.16. Let $A \in \mathrm{Sh}(\mathcal{R}^{\mathrm{op}})$ and $M \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$. Then we put

$$A \otimes_{\mathcal{R}} M := \mathrm{coker}(A \otimes_{k_X} \mathcal{R} \otimes_{k_X} M \xrightarrow{d} A \otimes_{k_X} M),$$

where $d = \mu_A^{\mathrm{op}} \otimes_{k_X} M - A \otimes_{k_X} \mu_M$.

There is a right exact bifunctor

$$\cdot \otimes_{\mathcal{R}} \cdot : \mathrm{Sh}(\mathcal{R}^{\mathrm{op}}) \times \mathrm{Sh}(\mathcal{R}, \mathcal{C}) \to \mathrm{Sh}(k_X, \mathcal{C}).$$

Remark 2.4.17. Suppose that $M \in \mathrm{Sh}(\mathcal{R}, \mathcal{C})$, $A \in \mathrm{Sh}(\mathcal{R}^{\mathrm{op}})$ and $G \in \mathrm{Sh}(k_X, \mathcal{C})$. Then, by Theorem 2.4.2 and Definition 2.4.16, we have the following isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}(\mathcal{R}^{\mathrm{op}})}(A, \mathcal{H}om_{k_X}(M, G)) &\simeq \mathrm{Hom}_{\mathrm{Sh}(\mathcal{R},\mathcal{C})}(M, \mathcal{C}hom_{k_X}(A, G)) \\
&\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X,\mathcal{C})}(A \otimes_{\mathcal{R}} M, G).
\end{aligned}$$

If $B \in \mathrm{Sh}(X)$, then

$$B \otimes_{k_X} (A \otimes_{\mathcal{R}} M) \simeq (B \otimes_{k_X} A) \otimes_{\mathcal{R}} M.$$

Suppose that $f : Y \to X$ be a continuous map. Then, we have

$$f^{-1}A \otimes_{f^{-1}\mathcal{R}} f^{-1}M \simeq f^{-1}(A \otimes_{\mathcal{R}} M).$$

If Z is a locally closed set of X, then we have

$$(A \otimes_{\mathcal{R}} M)_Z \simeq A_Z \otimes_{\mathcal{R}} M \simeq A \otimes_{\mathcal{R}} M_Z.$$

Remark 2.4.18. We have another tensor product

$$\cdot \otimes_{\mathcal{R}} \cdot : \mathrm{Sh}(\mathcal{R}^{\mathrm{op}}, \mathcal{C}) \times \mathrm{Sh}(\mathcal{R}) \to \mathrm{Sh}(k_X, \mathcal{C})$$

which satisfies

$$\mathrm{Hom}_{\mathrm{Csh}(k_X)}(N \otimes_{\mathcal{R}} B, G) \simeq \mathrm{Hom}_{\mathrm{Sh}(\mathcal{R})}(B, \mathcal{H}om(N, G))$$

for any $N \in \mathrm{Sh}(\mathcal{R}^{\mathrm{op}}, \mathcal{C})$, $B \in \mathrm{Sh}(\mathcal{R})$ and $G \in \mathrm{Sh}(k_X, \mathcal{C})$. Although we do not explain this functor in detail, we will use it later.

3 Category of cosheaves

3.1 Precosheaves and cosheaves

Let X be a topological space and let k be a commutative ring with unit. We redefine the category of cosheaves by Schneiders [10]. Although his notation is different from our one, his definition is the same as ours essentially. If we put $\mathcal{C} := \mathrm{Pro}(k)^{\mathrm{op}}$, then $\mathcal{C}$ satisfies the conditions of Remark 2.2.2.

Definition 3.1.1. We set

$$\begin{aligned} \mathrm{PCsh}(k_X) &:= \mathrm{PSh}(k_X, \mathrm{Pro}(k)^{\mathrm{op}}), \\ \mathrm{Csh}(k_X) &:= \mathrm{Sh}(k_X, \mathrm{Pro}(k)^{\mathrm{op}}). \end{aligned}$$

We call an object of $\mathrm{PCsh}(k_X)$ a precosheaf on X. Similarly, we call an object of $\mathrm{Csh}(k_X)$ a cosheaf on X.

By the previous section, we get several functors; f_*, f^{-1}, Γ_Z, $(\ \cdot\ )_Z$, $\mathcal{C}hom$ and $\otimes$. Note that Schneiders already gave these functors in [10].

Definition 3.1.2. Let F be a precosheaf on X. For any $T \in \mathrm{Mod}(k)$, we define a presheaf $\langle F, T\rangle$ on X by

$$U \mapsto \mathrm{Hom}_{\mathrm{Pro}(k)}(F(U)^{\mathrm{op}}, T).$$

We obtain a left exact bifunctor

$$\langle\ \cdot\ ,\ \cdot\ \rangle : \mathrm{PCsh}(k_X) \times \mathrm{Mod}(k) \to \mathrm{PSh}(k_X).$$

The next proposition is a convenient tool to construct morphisms of precosheaves.

Proposition 3.1.3 (See [10]). *Let F and G be two precosheaf on X. We suppose that a presheaf morphism $\gamma_T : \langle F, T\rangle \to \langle G, T\rangle$ is assigned to each $T \in \mathrm{Mod}(k)$ and that the collection $\{\gamma_T\}_{T\in\mathrm{Mod}(k)}$ satisfies the following condition. For any k-module map $a : S \to T$, the diagram below commutes :*

$$\begin{array}{ccc} \langle F, S\rangle & \xrightarrow{\gamma_S} & \langle G, S\rangle \\ {\scriptstyle\langle F,a\rangle}\downarrow & & \downarrow{\scriptstyle\langle G,a\rangle} \\ \langle F, T\rangle & \xrightarrow{\gamma_T} & \langle G, T\rangle\,. \end{array}$$

Then, there exists a unique morphism $\varphi : F \to G$ of precosheaf such that for any $T \in \mathrm{Mod}(k)$, we have $\langle \varphi, T\rangle = \gamma_T$.

Proof. Let U be an open subset of X. Then, for any $T \in \mathrm{Mod}(k)$, we have a morphism

$$\gamma_T(U) : \mathrm{Hom}_{\mathrm{Pro}(k)}(F(U)^{\mathrm{op}}, T) \to \mathrm{Hom}_{\mathrm{Pro}(k)}(G(U)^{\mathrm{op}}, T).$$

By the definition of k-promodules, there is a unique morphism

$$\varphi_U : F(U) \to G(U)$$

such that $\mathrm{Hom}_{\mathrm{Pro}(k)}(\varphi_U^{\mathrm{op}}, T) = \gamma_T(U)$. This collection $\{\varphi_U\}_{U\subset X}$ defines a morphism $\varphi : F \to G$ of precosheaves. q.e.d.

Remark 3.1.4. Let F be a precosheaf on X. F is a cosheaf if and only if $\langle F, T\rangle$ is a sheaf for any $T \in \mathrm{Mod}(k)$.

Remark 3.1.5. Let $F' \to F \to F''$ be a sequence of cosheaves on X. This sequence is exact if and only if

$$\langle F', T\rangle \to \langle F, T\rangle \to \langle F'', T\rangle$$

is exact for any injective k-module T.

Let $\mathcal{R}$ be a sheaf of (not necessarily commutative) k-algebra. Then, by putting

$$\mathrm{Csh}(\mathcal{R}) := \mathrm{Sh}(\mathcal{R}, \mathrm{Pro}(k)^{\mathrm{op}}),$$

we get the category of cosheaves with $\mathcal{R}$-action. If $M \in \mathrm{Csh}(\mathcal{R})$ and $T \in \mathrm{Mod}(k)$, then $\langle M, T\rangle$ is a $\mathcal{R}$-module, namely, a sheaf with $\mathcal{R}$-action.

Remark 3.1.6. There is a left exact functor

$$\langle\ \cdot\ ,\ \cdot\ \rangle : \mathrm{Csh}(\mathcal{R}) \times \mathrm{Mod}(k) \to \mathrm{Sh}(\mathcal{R}).$$

Let $F \in \mathrm{Csh}(\mathcal{R})$ and let Z be a locally closed subset of X. Then, we have isomorphisms

$$\begin{aligned} \langle \Gamma_Z F, T\rangle &\simeq \Gamma_Z \langle F, T\rangle, \\ \langle F_Z, T\rangle &\simeq \langle F, T\rangle_Z . \end{aligned}$$

For any $A \in \mathrm{Sh}(\mathcal{R})$ and $B \in \mathrm{Sh}(\mathcal{R}^{\mathrm{op}})$, there are the following isomorphisms

$$\begin{aligned} \langle \mathcal{C}hom_{\mathcal{R}}(A, F), T\rangle &\simeq \mathcal{H}om_{\mathcal{R}}(A, \langle F, T\rangle), \\ \langle B \otimes_{\mathcal{R}} F, T\rangle &\simeq B \otimes_{\mathcal{R}} \langle F, T\rangle . \end{aligned}$$

If $f : Y \to X$ is a continuous map and $G \in \mathrm{Csh}(f^{-1}\mathcal{R})$, then we have

$$\begin{aligned} \langle f_* G, T\rangle &\simeq f_* \langle G, T\rangle, \\ \left\langle f^{-1} F, T\right\rangle &\simeq f^{-1} \langle F, T\rangle . \end{aligned}$$

The above isomorphisms can also be found in [10].

Remark 3.1.7. The category $\mathrm{Csh}(\mathcal{R})$ is depend on the choice of k. More precisely, if k' is another commutative ring with unit and we have a ring morphism $k' \to k$, then the natural functor $\mathrm{Sh}(\mathcal{R}, \mathrm{Pro}(k)^{\mathrm{op}}) \to \mathrm{Sh}(\mathcal{R}, \mathrm{Pro}(k')^{\mathrm{op}})$ is not always an equivalence of categories (See Remark 2.1.5). Since we fix a commutative ring k from now on, there will be no risk of confusion.

3.2 Proper direct image

We will consider the proper direct image on cosheaves. Let X and Y be two locally compact and Housdorff spaces and let $f : Y \to X$ be a continuous map.

Definition 3.2.1. Suppose that G is a cosheaf on Y. Then, we define a cosheaf $f_! G$ on X as follows :

$$\Gamma(U; f_! G) := \varinjlim_{L} \Gamma(f^{-1}(U); \Gamma_L G),$$

where L ranges through the family of sets satisfying the following conditions :

(i) L is a closed set of $f^{-1}(U)$,

(ii) $f : L \to U$ is a proper map.

We have a left exact functor

$$f_! : \mathrm{Csh}(k_Y) \to \mathrm{Csh}(k_X).$$

Now let $\mathcal{R}$ be a sheaf of k-algebra on X. If $N \in \mathrm{Csh}(f^{-1}\mathcal{R})$, then $f_! N$ is a cosheaf with left $\mathcal{R}$-action. This action is defined by using the composition of the following morphisms

$$\mathcal{R} \to f_* \mathcal{H}om_{k_Y}(N, N) \to \mathcal{H}om_{k_X}(f_! N, f_! N).$$

Consequently, there is a left exact functor

$$f_! : \mathrm{Csh}(f^{-1}\mathcal{R}) \to \mathrm{Csh}(\mathcal{R}).$$

For any $T \in \mathrm{Mod}(k)$, we have the following isomorphism

$$\langle f_! N, T \rangle \;\simeq\; f_! \langle N, T \rangle .$$

Remark 3.2.2. There is a monomorphism of functors

$$f_! \to f_*.$$

Proposition 3.2.3. *Let $N \in \mathrm{Csh}(f^{-1}\mathcal{R}^{\mathrm{op}})$ and $A \in \mathrm{Sh}(\mathcal{R})$. Then, we have a natural morphism*

$$f_! N \otimes_{\mathcal{R}} A \to f_!(N \otimes_{f^{-1}\mathcal{R}} f^{-1}A). \tag{3.2.1}$$

Moreover, if A is flat , then (3.2.1) is an isomorphism.

Proof. There is a sequence of natural morphisms

$$f_! N \otimes_{\mathcal{R}} A \to f_* N \otimes_{\mathcal{R}} A \to f_*(N \otimes_{f^{-1}\mathcal{R}} f^{-1}A).$$

For any $T \in \mathrm{Mod}(k)$, we have a unique morphism γ_T with the following commutative diagram

$$\begin{array}{ccc} f_! \langle N, T \rangle \otimes_{\mathcal{R}} A & \xrightarrow{\gamma_T} & f_!(\langle N, T \rangle \otimes_{f^{-1}\mathcal{R}} f^{-1}A) \\ \downarrow & & \downarrow \\ f_* \langle N, T \rangle \otimes_{\mathcal{R}} A & \longrightarrow & f_*(\langle N, T \rangle \otimes_{f^{-1}\mathcal{R}} f^{-1}A). \end{array}$$

The family $\{\gamma_T\}_{T \in \mathrm{Mod}(k)}$ induces a morphism (3.2.1) by Proposition 3.1.3.

If A is $\mathcal{R}$-flat, it is well-known that γ_T is an isomorphism for any $T \in \mathrm{Mod}(k)$. Therefore, (3.2.1) must be an isomorphism. q.e.d.

Definition 3.2.4. $M \in \mathrm{Csh}(\mathcal{R})$ is $\mathcal{R}$-flat if the functor $\cdot \otimes_{\mathcal{R}} M$ is exact.

Remark 3.2.5. $M \in \mathrm{Csh}(\mathcal{R})$ is $\mathcal{R}$-flat if and only if $\langle M, T \rangle$ is $\mathcal{R}$-flat for any injective k-module T.

Remark 3.2.6. Let $B \in \mathrm{Sh}(f^{-1}\mathcal{R}^{\mathrm{op}})$ and $M \in \mathrm{Csh}(\mathcal{R})$. Then, we have a natural morphism

$$f_! B \otimes_{\mathcal{R}} M \to f_!(B \otimes_{f^{-1}\mathcal{R}} f^{-1}M).$$

Moreover, if M is $\mathcal{R}$-flat, then this morphism is an isomorphism. We can prove this fact by the same way as Proposition 3.2.3.

4 Cohomological algebra of cosheaves

4.1 c-injective objects

Let k be a commutative ring with the condition A (See Definition 2.1.10). We assume that X is a topological space, but X is not necessarily locally compact and Housdorff.

We have the notion of an injective cosheaf. We say that a cosheaf $I \in \mathrm{Csh}(k_X)$ is injective if the functor $\mathrm{Hom}_{\mathrm{Csh}(k_X)}(\,\cdot\,, I)$ is exact. However, we use another notion of "c-injective".

Remark 4.1.1. Let $A \in \mathrm{Sh}(k_X)$ and $F \in \mathrm{Csh}(k_X)$. Then, we put

$$\mathrm{CHom}_{k_X}(A, F) := \Gamma(X; \mathcal{C}hom_{k_X}(A, F)).$$

This gives a left exact functor

$$\mathrm{CHom}_{k_X}(\,\cdot\,,\,\cdot\,) : \mathrm{Sh}(k_X)^{\mathrm{op}} \times \mathrm{Csh}(k_X) \to \mathrm{Pro}(k)^{\mathrm{op}}.$$

$\mathrm{CHom}_{\mathcal{R}}$ is also similar.

Remark 4.1.2. We sometimes omit k_X, for example, we write $\mathcal{C}hom$ instead of $\mathcal{C}hom_{k_X}$.

Definition 4.1.3. We say that $I \in \mathrm{Csh}(k_X)$ is c-injective if $\mathrm{CHom}(\,\cdot\,, I)$ is an exact functor.

we denote by $\mathrm{CInj}(k_X)$ the full subcategory of $\mathrm{Csh}(k_X)$ consisting of c-injective objects.

Remark 4.1.4. $I \in \mathrm{Csh}(k_X)$ is c-injective if and only if $\langle I, T \rangle$ is an injective sheaf for any injective object $T \in \mathrm{Mod}(k)$. As a result, for any $A \in \mathrm{Sh}(k_X)$, the category $\mathrm{CInj}(k_X)$ is $\mathrm{CHom}(A,\,\cdot\,)$-injective.

Proposition 4.1.5. *Assume that $I \in \mathrm{Csh}(k_X)$ is c-injective.*

(i) *If $U \subset X$ is an open set, then $I|_U \in \mathrm{Csh}(k_U)$ is c-injective.*

(ii) *If W is a topological space and $g : X \to W$ is a continuous map, then $g_* I$ is a c-injective cosheaf on W.*

(iii) *For any locally closed set $Z \subset X$, a cosheaf $\Gamma_Z I$ is c-injective.*

Proof. We prove (i). We write by $i : U \to X$ the embedding map. If $A \in \mathrm{Sh}(U)$, then we have

$$\begin{aligned} \mathrm{CHom}\,(A, I|_U)) &\simeq \Gamma(U; \mathcal{C}hom\,((i_*A)|_U, I|_U)) \\ &\simeq \Gamma(X; \mathcal{C}hom\,((i_*A)_U, I)) \\ &= \mathrm{CHom}\,((i_*A)_U, I). \end{aligned}$$

Since the functor $A \mapsto (i_*A)_U$ is exact, $\mathrm{CHom}\,(\,\cdot\,, I|_U)$ is also exact, thus $I|_U$ is c-injective.
(ii) and (iii) follow from Theorem 2.4.8 and Theorem 2.4.3 respectively. q.e.d.

Corollary 4.1.6. *If $I \in \mathrm{Csh}(k_X)$ is c-injective, then $\mathcal{C}hom\,(\,\cdot\,, I)$ is exact.*

Theorem 4.1.7. *The category $\mathrm{CInj}(k_X)$ is cogenerating in $\mathrm{Csh}(k_X)$. Namely, for every $G \in \mathrm{Csh}(k_X)$, there exists a c-injective object I and a monomorphism $G \to I$.*

Proof. (a) We first assume that X is one point set $\{\mathrm{pt}\}$. Let $G \in \mathrm{Csh}(\{\mathrm{pt}\})$. Applying Theorem 2.1.7, there is a quasi-project $P \in \mathrm{Pro}(k)$ and an epimorphism $P \twoheadrightarrow G(\{\mathrm{pt}\})^{\mathrm{op}}$ in $\mathrm{Pro}(k)$. Defining $I \in \mathrm{Csh}(\{\mathrm{pt}\})$ by $I(\{\mathrm{pt}\}) := P^{\mathrm{op}}$, we get a monomorphism $G \rightarrowtail I$ of cosheaves, so it is sufficient to show that I is c-injective. By Theorem 2.1.8 and the definition of $\mathcal{C}hom$, we have an isomorphism

$$\mathrm{CHom}\,(\,\cdot\,, I) \simeq (I(\{\mathrm{pt}\})^{\mathrm{op}} \otimes_k \,\cdot\,(\{\mathrm{pt}\}))^{\mathrm{op}}.$$

in $\mathrm{Pro}(k)^{\mathrm{op}}$. Hence, the exactness of $\mathrm{CHom}\,(\,\cdot\,, I)$ follows from Theorem 2.1.12.

(b) Next, we suppose that X is a general topological space. We write by $\hat{X}$ the space X endowed with the discrete topology and by h the natural continuous map from $\hat{X}$ to X. Let $G \in \mathrm{Csh}(k_X)$. Then, applying Remark 2.3.3, we obtain a monomorphism

$$G \simeq h_* h^{\wedge} G \rightarrowtail h_* h^{-1} G.$$

Since $\hat{X}$ has the discrete topology, by (a) we can take a c-injective object $I \in \mathrm{Csh}(k_{\hat{X}})$ and a monomorphism $h^{-1}G \rightarrowtail I$. Therefore, the composition

$$G \rightarrowtail h_* h^{-1} G \rightarrowtail h_* I$$

is also a monomorphism, and $h_* I$ must be c-injective by Proposition 4.1.5.

q.e.d.

Definition 4.1.8. We say that $L \in \mathrm{Csh}(k_X)$ is flabby if $\rho_{UX} : L(X) \to L(U)$ is an epimorphism for any open set $U \subset X$.

Remark 4.1.9. $L \in \mathrm{Csh}(k_X)$ is flabby if and only if $\langle L, T \rangle$ is a flabby sheaf for any injective object $T \in \mathrm{Mod}(k)$. So, the category of flabby cosheaves is $\Gamma(X;\,\cdot\,)$-injective.

Proposition 4.1.10. *Let I be a c-injective cosheaf on X. Then, for any $A \in \mathrm{Sh}(k_X)$, $\mathcal{C}hom\,(A, I)$ is flabby. Moreover, if A is flat, then $\mathcal{C}hom\,(A, I)$ is c-injective.*

Proof. First assume that A is a general sheaf. Consider the exact sequence $0 \to A_U \to A$. Then, by applying the functor $\mathrm{CHom}\,(\,\cdot\, I)$, we get the exact sequence

$$\Gamma(X; \mathcal{C}hom\,(A, I)) \to \Gamma(U; \mathcal{C}hom\,(A, I)) \to 0.$$

Hence, $\mathcal{C}hom\,(A, I)$ is flabby. Next, we assume that A is flat. Then, by Theorem 2.4.6, $\mathrm{CHom}\,(\,\cdot\,, \mathcal{C}hom\,(A, I))$ is an exact functor. Therefore, $\mathcal{C}hom\,(A, I)$ is c-injective. q.e.d.

Let $\mathcal{R}$ be a k-algebra on X. We denote by $\mathrm{CInj}(\mathcal{R})$ the full subcategory of $\mathrm{Csh}(\mathcal{R})$ consisting of the objects isomorphic to $\mathcal{C}hom_{k_X}(\mathcal{R}, I)$ for some $I \in \mathrm{CInj}(k_X)$.

Theorem 4.1.11. (i) *The category* $\mathrm{CInj}(\mathcal{R})$ *is cogenerating in* $\mathrm{Csh}(\mathcal{R})$*.*

(ii) *For a monomorphism* $\mathcal{C}hom\,(\mathcal{R}, I') \rightarrowtail M$ *of left* $\mathcal{R}$*-modules with* $M \in \mathrm{Csh}(\mathcal{R})$ *and* $I' \in \mathrm{CInj}(k_X)$*, there exists an exact sequence* $0 \to I' \to I \to I'' \to 0$ *in* $\mathrm{Csh}(k_X)$ *such that the morphism* $\mathcal{C}hom\,(\mathcal{R}, I') \to \mathcal{C}hom\,(\mathcal{R}, I)$ *factors through* M *in* $\mathrm{Csh}(\mathcal{R})$*.*

Proof. (i) Let $M \in \mathrm{Csh}(\mathcal{R})$. Applying Theorem 4.1.7, we select a c-injective object $I \in \mathrm{CInj}(k_X)$ and a monomorphism $\varphi : M \rightarrowtail I$. Then, we obtain a sequence of morphisms

$$M \xrightarrow{\lambda_M} \mathcal{C}hom\,(\mathcal{R}, M) \xrightarrow{\mathcal{C}hom\,(\mathcal{R},\varphi)} \mathcal{C}hom\,(\mathcal{R}, I),$$

and this composition is a monomorphism.

(ii) Since $\mathcal{R}$ is k-algebra, the natural morphism $k_X \to \mathcal{R}$ induce a morphism

$$\mathcal{C}hom\,(\mathcal{R}, I') \to I'.$$

Let N be the fiber coproduct of I' and M over $\mathcal{C}hom\,(\mathcal{R}, I')$;

$$\begin{tikzcd} \mathcal{C}hom\,(\mathcal{R}, I') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & M \arrow[d] \\ I' \arrow[r] & N. \end{tikzcd}$$

Then, $I' \to N$ is a monomorphism in $\mathrm{Csh}(k_X)$. By Theorem 4.1.7, we can take a c-injective cosheaf I and a monomorphism $N \rightarrowtail I$. The cokernel I'' of the monomorphism $I' \to I$ belongs to $\mathrm{CInj}(k_X)$. The morphism $\mathcal{C}hom\,(\mathcal{R}, I') \to \mathcal{C}hom\,(\mathcal{R}, I)$ factors through M in $\mathrm{Csh}(\mathcal{R})$ as follows :

$$\mathcal{C}hom\,(\mathcal{R}, I') \rightarrowtail M \xrightarrow{\lambda_M} \mathcal{C}hom\,(\mathcal{R}, M) \to \mathcal{C}hom\,(\mathcal{R}, N) \to \mathcal{C}hom\,(\mathcal{R}, I).$$

q.e.d.

Remark 4.1.12. Assume that X is a locally compact and Housdorff space. We can define c-soft cosheaves by the same way as sheaf theory. $F \in \mathrm{Csh}(k_X)$ is c-soft if $\Gamma(X; F) \to \Gamma(K; F|_K)$ is surjective for any compact set $K \subset X$. Note that $F \in \mathrm{Csh}(k_X)$ is a c-soft cosheaf if and only if $\langle F, T\rangle$ is a c-soft sheaf for any injective object $T \in \mathrm{Mod}(k)$. Hence, if X is countable at infinity, then c-soft cosheaves are $\Gamma(X;\,\cdot\,)$-acyclic.

4.2 Derived category of cosheaves

Let X be a topological space and let k be a commutative ring with the condition A. Suppose that $\mathcal{R}$ is a (not necessarily commutative) k_X-algebra. If $\mathcal{C}$ is an abelian category, then we denote by $\mathbf{C}(\mathcal{C})$ the abelian category of complexes in $\mathcal{C}$. By regarding morphisms in $\mathbf{C}(\mathcal{C})$ which are homotopic to 0 as the zero morphism, we obtain the triangulated category $\mathbf{K}(\mathcal{C})$. The derived category $\mathbf{D}(\mathcal{C})$ is the localization of $\mathbf{K}(\mathcal{C})$ by the multiplicative system of quasi-isomorphisms. We denotes as usual by $\mathbf{D}^*(\mathcal{C})$ ($* = +,\ -$ or b) the full triangulated category of $\mathbf{D}(\mathcal{C})$ consisting of complexes bounded from below, from above, or bounded.

By Remark 3.1.5, we obtain a right derived functor

$$R\langle\ \cdot\ ,\ \cdot\ \rangle : \mathbf{D}^+(\mathrm{Csh}(\mathcal{R})) \times \mathbf{D}^+(\mathrm{Mod}(k)) \to \mathbf{D}^+(\mathrm{Sh}(\mathcal{R})).$$

Although the following proposition is not difficult, it is very important.

Proposition 4.2.1. *Let $F \in \mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))$ and let T be an injective k-module. Then, $H^k(R\langle F, T\rangle) \simeq \big\langle H^k(F), T\big\rangle$.*

Proof. Since the functor $\langle\ \cdot\ , T\rangle$ is exact, the proof is obvious. q.e.d.

Corollary 4.2.2. *Let $F \in \mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))$. Then, F is quasi-isomorphic to 0 if and only if $R\langle F, T\rangle \simeq 0$ for any injective k-module T.*

Since $\mathrm{CInj}(k_X)$ is $\Gamma(X;\ \cdot\ )$-injective, we obtain a right derived functor

$$\mathrm{R}\Gamma(X;\ \cdot\ ) : \mathbf{D}^+(\mathrm{Csh}(k_X)) \to \mathbf{D}^+(\mathrm{Pro}(k)^{\mathrm{op}}).$$

Suppose that Z is a locally closed set of X. Since $\mathrm{CInj}(\mathcal{R})$ is Γ_Z-injective, we get a functor

$$\mathrm{R}\Gamma_Z : \mathbf{D}^+(\mathrm{Csh}(\mathcal{R})) \to \mathbf{D}^+(\mathrm{Csh}(\mathcal{R})).$$

Moreover, since $(\ \cdot\ )_Z$ is an exact functor, we have a functor

$$(\ \cdot\ )_Z : \mathbf{D}^*(\mathrm{Csh}(\mathcal{R})) \to \mathbf{D}^*(\mathrm{Csh}(\mathcal{R})).$$

Theorem 4.2.3. *If M and $N \in \mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))$, then there exists an isomorphism*

$$\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M_Z, N) \simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, \mathrm{R}\Gamma_Z N).$$

Proof. By (2.3.3), we get

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M_Z, N) &\simeq \varinjlim_{\substack{\mathrm{qis} \\ M' \to M}} \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}({M'}_Z, N) \\
&\simeq \varinjlim_{\substack{\mathrm{qis} \\ M' \to M}} \varinjlim_{\substack{\mathrm{qis} \\ N \to N'}} \mathrm{Hom}_{\mathbf{K}^+(\mathrm{Csh}(\mathcal{R}))}({M'}_Z, N') \\
&\simeq \varinjlim_{\substack{\mathrm{qis} \\ M' \to M}} \varinjlim_{\substack{\mathrm{qis} \\ N \to N'}} \mathrm{Hom}_{\mathbf{K}^+(\mathrm{Csh}(\mathcal{R}))}(M', \Gamma_Z N') \\
&\simeq \varinjlim_{\substack{\mathrm{qis} \\ N \to N'}} \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, \Gamma_Z N') \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, \mathrm{R}\Gamma_Z N).
\end{aligned}$$

q.e.d.

Let Y be another topological space and let $f : Y \to X$ be a continuous map. Since f^{-1} is an exact functor, we have a functor

$$f^{-1} : \mathbf{D}^*(\mathrm{Csh}(\mathcal{R})) \to \mathbf{D}^*(\mathrm{Csh}(f^{-1}\mathcal{R})).$$

Similarly, the right derived functor

$$Rf_* : \mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R})) \to \mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))$$

is well-defined.

Theorem 4.2.4. *Let $M \in \mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))$ and let $N \in \mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R}))$. Then we have an isomorphism*

$$\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R}))}(f^{-1}M, N) \simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, Rf_*N).$$

Proof. By Proposition 2.3.5, we get

$$\begin{aligned}
& \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R}))}(f^{-1}M, N) \\
\simeq\ & \varinjlim_{M' \xrightarrow{\mathrm{qis}} M} \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R}))}(f^{-1}M', N) \\
\simeq\ & \varinjlim_{M' \xrightarrow{\mathrm{qis}} M} \varinjlim_{N \xrightarrow{\mathrm{qis}} N'} \mathrm{Hom}_{\mathbf{K}^+(\mathrm{Csh}(f^{-1}\mathcal{R}))}(f^{-1}M', N') \\
\simeq\ & \varinjlim_{M' \xrightarrow{\mathrm{qis}} M} \varinjlim_{N \xrightarrow{\mathrm{qis}} N'} \mathrm{Hom}_{\mathbf{K}^+(\mathrm{Csh}(\mathcal{R}))}(M', f_*N') \\
\simeq\ & \varinjlim_{N \xrightarrow{\mathrm{qis}} N'} \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, f_*N') \\
\simeq\ & \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, Rf_*N).
\end{aligned}$$

q.e.d.

By Theorem 4.1.11, the functor $\mathcal{C}hom_{\mathcal{R}}$ is right derivable, so we obtain a bifunctor

$$R\mathcal{C}hom_{\mathcal{R}}(\,\cdot\,,\,\cdot\,) : \mathbf{D}^-(\mathrm{Sh}(\mathcal{R}))^{\mathrm{op}} \times \mathbf{D}^+(\mathrm{Csh}(\mathcal{R})) \to \mathbf{D}^+(\mathrm{Csh}(k_X)).$$

Similarly, the functor $\mathrm{RCHom}_{\mathcal{R}}$ is defined.

Remark 4.2.5. Recall that the category $\mathrm{Csh}(\mathcal{R})$ depends on the choice of the commutative ring k (see Remark 3.1.7). If k satisfies the condition A and if $\mathcal{R}$ is k_X-algebra, then we can define $R\mathcal{C}hom_{\mathcal{R}}$ on the derived category of $\mathrm{Sh}(\mathcal{R}, \mathrm{Pro}(k)^{\mathrm{op}})$. Now we put k as the integer ring $\mathbb{Z}$, which satisfies the condition A. Then, for any ring sheaf $\mathcal{R}$, we can define $R\mathcal{C}hom_{\mathcal{R}}$ on the derived category of $\mathrm{Sh}(\mathcal{R}, \mathrm{Pro}(\mathbb{Z})^{\mathrm{op}})$, because $\mathcal{R}$ is always $\mathbb{Z}_X$-algebra.

If $A \in \mathrm{Sh}(\mathcal{R}^{\mathrm{op}})$ is flat over $\mathcal{R}$, then the functor

$$A \otimes_{\mathcal{R}} \,\cdot\, : \mathrm{Csh}(\mathcal{R}) \to \mathrm{Csh}(k_X)$$

is exact, so we get the derived functor

$$\cdot \otimes^L_{\mathcal{R}} \cdot\ : \mathbf{D}^-(\mathrm{Sh}(\mathcal{R}^{\mathrm{op}})) \times \mathbf{D}^-(\mathrm{Csh}(\mathcal{R})) \to \mathbf{D}^-(\mathrm{Csh}(k_X)).$$

In particular, if $\mathcal{R}$ has finite weak global dimension, then we also get

$$\cdot \otimes^L_{\mathcal{R}} \cdot\ : \mathbf{D}^*(\mathrm{Sh}(\mathcal{R}^{\mathrm{op}})) \times \mathbf{D}^*(\mathrm{Csh}(\mathcal{R})) \to \mathbf{D}^*(\mathrm{Csh}(k_X)).$$

From now on, we suppose that $\mathcal{R}$ has finite weak global dimension.

Theorem 4.2.6. *Let $A \in \mathbf{D}^-(\mathrm{Sh}(\mathcal{R}^{\mathrm{op}}))$, $M \in \mathbf{D}^-(\mathrm{Csh}(\mathcal{R}))$ and $G \in \mathbf{D}^+(\mathrm{Csh}(k_X))$. Then, we have*

$$\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(A \otimes^L_{\mathcal{R}} M, G) \simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, R\mathcal{C}hom\,(A, G)).$$

Proof. By Remark 2.4.17, we get

$$\begin{aligned}
& \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(A \otimes^L_{\mathcal{R}} M, G) \\
\simeq\ & \varinjlim_{A' \xrightarrow{\mathrm{qis}} A}\ \varinjlim_{M' \xrightarrow{\mathrm{qis}} M} \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(A' \otimes_{\mathcal{R}} M', G) \\
\simeq\ & \varinjlim_{A' \xrightarrow{\mathrm{qis}} A}\ \varinjlim_{M' \xrightarrow{\mathrm{qis}} M}\ \varinjlim_{G \xrightarrow{\mathrm{qis}} G'} \mathrm{Hom}_{\mathbf{K}^+(\mathrm{Csh}(k_X))}(A' \otimes_{\mathcal{R}} M', G') \\
\simeq\ & \varinjlim_{A' \xrightarrow{\mathrm{qis}} A}\ \varinjlim_{M' \xrightarrow{\mathrm{qis}} M}\ \varinjlim_{G \xrightarrow{\mathrm{qis}} G'} \mathrm{Hom}_{\mathbf{K}^+(\mathrm{Csh}(\mathcal{R}))}(M', \mathcal{C}hom\,(A', G')) \\
\simeq\ & \varinjlim_{A' \xrightarrow{\mathrm{qis}} A}\ \varinjlim_{G \xrightarrow{\mathrm{qis}} G'} \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, \mathcal{C}hom\,(A', G')) \\
\simeq\ & \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, R\mathcal{C}hom\,(A, G)).
\end{aligned}$$

q.e.d.

Remark 4.2.7. Let $A \in \mathbf{D}^-(\mathrm{Sh}(\mathcal{R}))$, $F \in \mathbf{D}^-(\mathrm{Csh}(k_X))$ and $N \in \mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))$. Then, we have

$$\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(A \otimes^L_{k_X} F, N) \simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(F, R\mathcal{C}hom_{\mathcal{R}}(A, N)).$$

This follows from Remark 2.4.12.

Theorem 4.2.8. *If $A \in \mathbf{D}^-(\mathrm{Sh}(k_X))$, $B \in \mathbf{D}^-(\mathrm{Sh}(\mathcal{R}))$ and $M \in \mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))$, then we have isomorphisms*

$$\begin{aligned}
R\mathcal{C}hom_{\mathcal{R}}(B, R\mathcal{C}hom\,(A, M)) &\simeq R\mathcal{C}hom_{\mathcal{R}}(A \otimes^L_{k_X} B, M) \\
&\simeq R\mathcal{C}hom\,(A, R\mathcal{C}hom_{\mathcal{R}}(B, M)).
\end{aligned}$$

Proof. By Proposition 4.1.10, if A is flat and $M \in \mathrm{CInj}(\mathcal{R})$, then $\mathcal{C}hom\,(A, M) \in \mathrm{CInj}(\mathcal{R})$. Hence, $R\mathcal{C}hom_{\mathcal{R}}(B, R\mathcal{C}hom\,(A, M)) \simeq \mathcal{C}hom_{\mathcal{R}}(B, \mathcal{C}hom\,(A, M))$ and $R\mathcal{C}hom_{\mathcal{R}}(A \otimes^L B, M) \simeq \mathcal{C}hom_{\mathcal{R}}(A \otimes B, M)$. Therefore, the first isomorphism follows from Theorem 2.4.13. In order to prove the second isomorphism, we also assume that the complex A

satisfies the following condition; each A^j is isomorphic to a direct sum of k_U (See Proposition 2.4.1). By Proposition 4.1.10, $\mathcal{C}hom_{\mathcal{R}}(B, M)$ is flabby. Consequently, we have

$$\begin{aligned} R\mathcal{C}hom\,(A, R\mathcal{C}hom_{\mathcal{R}}(B, M)) &\simeq \mathcal{C}hom\,(A, \mathcal{C}hom_{\mathcal{R}}(B, M)) \\ &\simeq \mathcal{C}hom_{\mathcal{R}}(A \otimes B, M) \\ &\simeq R\mathcal{C}hom_{\mathcal{R}}(A \otimes^L_{k_X} B, M). \end{aligned}$$

q.e.d.

Theorem 4.2.9. *Let $A \in \mathbf{D}^-(\mathrm{Sh}(\mathcal{R}))$ and let $M \in \mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R}))$. Then we have an isomorphism*

$$Rf_* R\mathcal{C}hom_{f^{-1}\mathcal{R}}(f^{-1}A, M) \simeq R\mathcal{C}hom_{\mathcal{R}}(A, Rf_*M).$$

Proof. Suppose that $M \in \mathrm{CInj}(f^{-1}\mathcal{R})$. Then, $f_*M \in \mathrm{CInj}(\mathcal{R})$. By Proposition 4.1.10, $\mathcal{C}hom_{f^{-1}\mathcal{R}}(f^{-1}A, M)$ is flabby. Hence, we have

$$\begin{aligned} Rf_* R\mathcal{C}hom_{f^{-1}\mathcal{R}}(f^{-1}A, M) &\simeq f_*\mathcal{C}hom_{f^{-1}\mathcal{R}}(f^{-1}A, M) \\ &\simeq \mathcal{C}hom_{\mathcal{R}}(A, f_*M) \\ &\simeq R\mathcal{C}hom_{\mathcal{R}}(A, Rf_*M). \end{aligned}$$

q.e.d.

Remark 4.2.10. If $A \in \mathbf{D}^-(\mathrm{Sh}(\mathcal{R}))$ and $M \in \mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))$, then we have isomorphisms

$$\mathrm{R}\Gamma_Z R\mathcal{C}hom_{\mathcal{R}}(A, M) \simeq R\mathcal{C}hom_{\mathcal{R}}(A_Z, M) \simeq R\mathcal{C}hom_{\mathcal{R}}(A, \mathrm{R}\Gamma_Z(M)).$$

This proof follows from Remark 2.4.15.

We assume that X and Y are locally compact and Hausdorff. Then, we get

$$Rf_! : \mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R})) \to \mathbf{D}^+(\mathrm{Csh}(\mathcal{R})).$$

Theorem 4.2.11. *Let $N \in \mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R}^{\mathrm{op}}))$ and let $A \in \mathbf{D}^+(\mathrm{Sh}(\mathcal{R}))$. Then, we have*

$$Rf_!N \otimes^L_{\mathcal{R}} A \xrightarrow{\sim} Rf_!(N \otimes^L_{f^{-1}\mathcal{R}} f^{-1}A). \tag{4.2.1}$$

Proof. Since $\mathcal{R}$ has finite weak global dimension, we may assume that A is a complex bounded from below consisting of $\mathcal{R}$-flat sheaves. If N is c-soft, then we have $Rf_!N \otimes^L_{\mathcal{R}} A \simeq f_!N \otimes_{\mathcal{R}} A$ and $Rf_!(N \otimes^L_{f^{-1}\mathcal{R}} f^{-1}A) \simeq f_!(N \otimes_{f^{-1}\mathcal{R}} f^{-1}A)$. Therefore, (4.2.1) follows from Proposition 3.2.3. q.e.d.

Remark 4.2.12. Let $F \in \mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))$ and let Z be a locally closed subset of X. Then, we have

$$\begin{aligned} R\langle \mathrm{R}\Gamma_Z F, T\rangle &\simeq \mathrm{R}\Gamma_Z R\langle F, T\rangle\,, \\ R\langle F_Z, T\rangle &\simeq R\langle F, T\rangle_Z\,. \end{aligned}$$

For any $A \in \mathbf{D}^+(\mathrm{Sh}(\mathcal{R}))$ and $B \in \mathbf{D}^+(\mathrm{Sh}(\mathcal{R}^{\mathrm{op}}))$, there are two isomorphisms

$$\begin{aligned} R\langle R\mathcal{C}hom_{\mathcal{R}}(A, F), T\rangle &\simeq R\mathcal{H}om_{\mathcal{R}}(A, R\langle F, T\rangle), \\ R\left\langle B \otimes^L_{\mathcal{R}} F, T\right\rangle &\simeq B \otimes^L_{\mathcal{R}} R\langle F, T\rangle \,. \end{aligned}$$

If $f : Y \to X$ is a continuous map and $G \in \mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R}))$, then we have

$$\begin{aligned} R\langle Rf_* G, T\rangle &\simeq Rf_* R\langle G, T\rangle \,, \\ R\left\langle f^{-1}F, T\right\rangle &\simeq f^{-1}R\langle F, T\rangle \,, \\ R\langle Rf_! G, T\rangle &\simeq Rf_! R\langle G, T\rangle \,. \end{aligned}$$

These proofs are easy.

4.3 Poincaré-Verdier duality

We prove the Poincaré-Verdier duality on the category of cosheaves. We note that Schneider did not give this Poincaré-Verdier duality in [10]. We assume that k is a field. Let X and Y be two locally compact and Hausdorff topological spaces and let $f : Y \to X$ be a continuous map. Moreover, we suppose that Y has finite c-soft dimension.

If S is a c-soft sheaf on Y and F is a cosheaf on X, then we define a precosheaf $f^!_S F$ on Y as

$$U \mapsto \Gamma(U; f^!_S F) := \mathrm{CHom}\,(f_!(S_U), F).$$

Lemma 4.3.1. *$f^!_S F$ is a cosheaf on Y.*

Proof. Let U be an open subset of X and let $\{U_i\}_{i\in I}$ be an open covering of U. Then, we have an exact sequence

$$\bigoplus_{i,j\in I} S_{U_i\cap U_j} \to \bigoplus_{i\in I} S_{U_i} \to S_U \to 0.$$

Since S is c-soft, by applying the functor $\mathrm{CHom}\,(f_!(\,\cdot\,), F)$, we get the exact sequence

$$0 \to \mathrm{CHom}\,(f_!(S_U), F) \to \prod_{i\in I} \mathrm{CHom}\,(f_!(S_{U_i}), F) \to \prod_{i,j\in I} \mathrm{CHom}\,(f_!(S_{U_i\cap U_j}), F).$$

q.e.d.

Remark 4.3.2. For any $T \in \mathrm{Mod}(k)$, we have an isomorphism

$$\left\langle f^!_S F, T\right\rangle \simeq f^!_S \left\langle F, T\right\rangle,$$

where $f^!_S$ of the right hand side is defined in Kashiwara-Schapira [2], p.141. Hence, if F is c-injective, then $f^!_S F$ is also c-injective.

Lemma 4.3.3. *Let $G \in \mathrm{Csh}(k_Y)$. Then, we have the following isomorphism*

$$\mathrm{Hom}_{\mathrm{Csh}(k_X)}(f_!(S \otimes G), F) \simeq \mathrm{Hom}_{\mathrm{Csh}(k_Y)}(G, f^!_S F).$$

Proof. Let $T \in \mathrm{Mod}(k)$. Then, by Lemma 3.1.3 of [2], we have

$$\mathrm{Hom}_{\mathrm{Sh}(k_X)}(f_!(S \otimes \langle G, T\rangle), \langle F, T\rangle) \simeq \mathrm{Hom}_{\mathrm{Sh}(k_Y)}(\langle G, T\rangle, f_S^! \langle F, T\rangle).$$

Since this is true for any $T \in \mathrm{Mod}(k)$, our lemma follows. q.e.d.

Theorem 4.3.4. *There exists a functor*

$$f^! : \mathbf{D}^+(\mathrm{Csh}(k_X)) \to \mathbf{D}^+(\mathrm{Csh}(k_Y))$$

such that for any $G \in \mathbf{D}^{\mathrm{b}}(\mathrm{Csh}(k_Y))$ *and* $F \in \mathbf{D}^+(\mathrm{Csh}(k_X))$*, we have an isomorphism*

$$\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(Rf_!G, F) \simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_Y))}(G, f^! F).$$

Proof. There exists a complex of c-soft sheaves $S \in \mathbf{K}^{\mathrm{b}}(\mathrm{Sh}(k_Y))$ quasi-isomorphic to the constant sheaf k_Y. So, we get

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(Rf_!G, F) &\simeq \varinjlim_{\substack{\mathrm{qis} \\ G' \to G}} \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(f_!(S \otimes G'), F) \\
&\simeq \varinjlim_{\substack{\mathrm{qis} \\ G' \to G}} \varinjlim_{\substack{\mathrm{qis} \\ F \to F'}} \mathrm{Hom}_{\mathbf{K}^+(\mathrm{Csh}(k_X))}(f_!(S \otimes G'), F') \\
&\simeq \varinjlim_{\substack{\mathrm{qis} \\ G' \to G}} \varinjlim_{\substack{\mathrm{qis} \\ F \to F'}} \mathrm{Hom}_{\mathbf{K}^+(\mathrm{Csh}(k_Y))}(G', f_S^! F') \\
&\simeq \varinjlim_{\substack{\mathrm{qis} \\ F \to F'}} \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_Y))}(G, f_S^! F') \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_Y))}(G, \text{“}\varinjlim_{\substack{\mathrm{qis} \\ F \to F'}}\text{”} f_S^! F')
\end{aligned}$$

Since $\mathrm{CInj}(k_X)$ is $f_S^!$-injective, $\text{“}\varinjlim_{\substack{\mathrm{qis} \\ F \to F'}}\text{”} f_S^! F'$ belongs to $\mathbf{D}^+(\mathrm{Csh}(k_Y))$. q.e.d.

Lemma 4.3.5. *For any* $K \in \mathrm{Csh}(k_X)$*, we have an isomorphism*

$$f_S^! \mathcal{H}om\,(K, F) \simeq \mathcal{H}om\,(f^{-1}K, f_S^! F),$$

Proof. For any open set $U \subset Y$, we get

$$\begin{aligned}
\Gamma(U; f_S^! \mathcal{H}om\,(K, F)) &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(f_!(S_U), \mathcal{H}om\,(K, F)) \\
&\simeq \mathrm{Hom}_{\mathrm{Csh}(k_X)}(K, \mathcal{C}hom\,(f_!(S_U), F)),
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(U; \mathcal{H}om\,(f^{-1}K, f_S^! F)) &= \mathrm{Hom}_{\mathrm{Csh}(k_U)}((f^{-1}K)|_U, (f_S^! F)|_U) \\
&\simeq \mathrm{Hom}_{\mathrm{Csh}(k_X)}(K, f_* \Gamma_U(f_S^! F))
\end{aligned}$$

Hence, it is sufficient to show $\mathcal{C}hom\,(f_!(S_U),F) \simeq f_*\Gamma_U(f^!_S F)$. This follows from

$$\begin{aligned}
\Gamma(V; f_*\Gamma_U(f^!_S F)) &\simeq \Gamma(f^{-1}(V)\cap U; f^!_S F)\\
&= \Gamma(X; \mathcal{C}hom\,(f_!(S_{f^{-1}(V)\cap U}),F))\\
&\simeq \Gamma(X; \mathcal{C}hom\,((f_!(S_U))_V,F))\\
&\simeq \Gamma(V; \mathcal{C}hom\,(f_!(S_U),F)).
\end{aligned}$$

q.e.d.

Lemma 4.3.6. *Let $A \in \mathrm{Sh}(k_Y)$. Then, we have an isomorphism*

$$\mathcal{C}hom\,(f_!(A\otimes S),F) \simeq f_*\mathcal{C}hom\,(A, f^!_S F). \tag{4.3.1}$$

Proof. For any $K \in \mathrm{Csh}(k_X)$, we get

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Csh}(k_X)}(K, \mathcal{C}hom\,(f_!(A\otimes S),F)) &\simeq \mathrm{Hom}_{\mathrm{Sh}(k_X)}(f_!(A\otimes S), \mathcal{H}om\,(K,F))\\
&\simeq \mathrm{Hom}_{\mathrm{Sh}(k_Y)}(A, f^!_S\mathcal{H}om\,(K,F))\\
&\simeq \mathrm{Hom}_{\mathrm{Sh}(k_Y)}(A, \mathcal{H}om\,(f^{-1}K, f^!_S F))\\
&\simeq \mathrm{Hom}_{\mathrm{Csh}(k_Y)}(f^{-1}K, \mathcal{C}hom\,(A, f^!_S F))\\
&\simeq \mathrm{Hom}_{\mathrm{Csh}(k_X)}(K, f_*\mathcal{C}hom\,(A, f^!_S F)).
\end{aligned}$$

Since this is true for any K, the proof follows. q.e.d.

Theorem 4.3.7. *Let $A \in \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(k_Y))$ and $F \in \mathbf{D}^{+}(\mathrm{Csh}(k_X))$. Then, we have an isomorphism*

$$R\mathcal{C}hom\,(Rf_!A, F) \simeq Rf_*R\mathcal{C}hom\,(A, f^!F).$$

Proof. We may assume that F is injective. If we take a complex of c-soft sheaves $S \in \mathbf{K}^{\mathrm{b}}(\mathrm{Sh}(k_Y))$ quasi-isomorphic to the constant sheaf k_Y, then $Rf_!A$ is represented as $f_!(A\otimes S)$. Hence, by Lemma 4.3.6, we obtain

$$\begin{aligned}
R\mathcal{C}hom\,(Rf_!A, F) &\simeq \mathcal{C}hom\,(f_!(A\otimes S),F)\\
&\simeq f_*\mathcal{C}hom\,(A, f^!_S F)\\
&\simeq Rf_*R\mathcal{C}hom\,(A, f^!F).
\end{aligned}$$

q.e.d.

Theorem 4.3.8. *Let $\mathcal{R}$ be a k_X-algebra. If $B \in \mathbf{D}^{+}(\mathrm{Sh}(f^{-1}\mathcal{R}^{\mathrm{op}}))$ and $M \in \mathbf{D}^{+}(\mathrm{Sh}(\mathcal{R}))$, then there is an isomorphism*

$$Rf_!B \otimes^L_{\mathcal{R}} M \simeq Rf_!(B \otimes^L_{f^{-1}\mathcal{R}} f^{-1}M). \tag{4.3.2}$$

Proof. Let $G \in \mathbf{D}^+(\mathrm{Csh}(k_X))$. Then, by Theorem 4.2.4, Theorem 4.2.6 and Theorem 4.3.7, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(Rf_!B \otimes^L_{\mathcal{R}} M, G) &\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, R\mathcal{C}hom\,(Rf_!B, G)) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(\mathcal{R}))}(M, Rf_*R\mathcal{C}hom\,(B, f^!G)) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(f^{-1}\mathcal{R}))}(f^{-1}M, R\mathcal{C}hom\,(B, f^!G)) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_Y))}(B \otimes^L_{f^{-1}\mathcal{R}} f^{-1}M, f^!G) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(Rf_!(B \otimes^L_{f^{-1}\mathcal{R}} f^{-1}M), G).
\end{aligned}$$

q.e.d.

Theorem 4.3.9. *Let $A \in \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(k_X))$ and let $F \in \mathbf{D}^+(\mathrm{Csh}(k_X))$. Then, we have*

$$f^!R\mathcal{C}hom\,(A, F) \simeq R\mathcal{C}hom\,(f^{-1}A, f^!F).$$

Proof. Let $G \in \mathbf{D}^+(\mathrm{Csh}(k_Y))$. Then, Theorem 4.2.11 and Theorem 4.3.7 we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_Y))}(G, f^!R\mathcal{C}hom\,(A, F)) &\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(Rf_!G, R\mathcal{C}hom\,(A, F)) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(A \otimes Rf_!G, F) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_X))}(Rf_!(f^{-1}A \otimes G), F) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_Y))}(f^{-1}A \otimes G, f^!F) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Csh}(k_Y))}(G, R\mathcal{C}hom\,(f^{-1}A, f^!F))
\end{aligned}$$

q.e.d.

Remark 4.3.10. Let $F \in \mathbf{D}^{\mathrm{b}}(\mathrm{Csh}(k_X))$ and let $T \in \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}(k))$. Then, we get a morphism $R\langle f^!F, T\rangle \simeq f^!R\langle F, T\rangle$. This follows from Remark 4.3.2.

Theorem 4.3.11. *Let $A \in \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(k_X))$ and let $F \in \mathbf{D}^{\mathrm{b}}(\mathrm{Csh}(k_X))$. Then, the morphism $Rf_!f^!A \otimes F \to A \otimes F$ induces a morphism*

$$f^!A \otimes f^{-1}F \to f^!(A \otimes F). \tag{4.3.3}$$

Moreover, if $A \simeq k_X$ and f is a topological submersion with finite fiber dimension, then this morphism is an isomorphism.

Proof. We show that (4.3.3) is an isomorphism. Let T be an injective k-module. Then, by Proposition 4.2.1, we have

$$\begin{aligned}
\left\langle H^k(f^!k_X \otimes f^{-1}F), T\right\rangle &\simeq H^k(R\langle f^!k_X \otimes f^{-1}F, T\rangle) \\
&\simeq H^k(f^!k_X \otimes f^{-1}R\langle F, T\rangle) \\
&\simeq H^k(f^!(k_X \otimes R\langle F, T\rangle)) \\
&\simeq H^k(R\langle f^!(k_X \otimes F), T\rangle) \\
&\simeq \left\langle H^k(f^!F), T\right\rangle,
\end{aligned}$$

where the third isomorphism follows from Proposition 3.3.2 of [2]. Finally, applying Corollary 4.2.2, we have

$$H^k(f^!k_X \otimes f^{-1}F) \simeq H^k(f^!F).$$

q.e.d.

5 Construction of cosheaves

5.1 Associated cosheaf

Let X be a locally compact and Hausdorff space. Assume that k is a commutative ring with the condition A. (See Definition 2.1.10). We consider an open base τ of X, and we suppose that τ satisfies the following condition. For any open set $U \subset X$ and any element $x \in U$, there exists $U' \in \tau$ such that $x \in U' \subset\subset U$. We may regard τ as a directed set by the natural inclusion relation of τ. We call a functor $A : \tau^{\mathrm{op}} \to \mathrm{Mod}(k)$ a presheaf on τ. We denote by $\mathrm{PSh}(\tau)$ the category of presheaves on τ. Then, we have the forgetful functor $\mathrm{Sh}(k_X) \to \mathrm{PSh}(\tau)$. We shall recall the following basic result.

Proposition 5.1.1. *The forgetful functor* $\mathrm{Sh}(k_X) \to \mathrm{PSh}(\tau)$ *has a left adjoint functor.*

Since this proposition is important for us, we will give the proof.

Let A be a presheaf on τ. We define a presheaf A' on X as

$$A'(U) := \varprojlim_{U' \in I(U)} A(U'),$$

where $I(U) := \{U' \in \tau : U' \subset\subset U\}$. Moreover, we have a natural morphism $a : A \to A'$ on $\mathrm{PSh}(\tau)$. We need the following lemma.

Lemma 5.1.2. *For any* $B \in \mathrm{Sh}(k_X)$ *and any morphism* $b : A \to B$ *on* $\mathrm{PSh}(\tau)$*, there exists a unique morphism* $c : A' \to B$ *such that* $c \circ a = b$.

Proof. We prove the uniqueness of c. Let U be an open set of X. A element $s \in A'(U)$ is a collection $\{s_{U'}\}_{U' \in I(U)}$ with $s_{U'} \in A(U')$ and the collection satisfies the following condition. For any U_1', $U_2' \in I(U)$, $U_1' \subset U_2'$ implies $s_{U_2'}|_{U_1'} = s_{U_1'}$. Take an element $U' \in I(U)$. Then, we get $s|_{U'} = \{s_{U'}|_{U''}\}_{U'' \in I(U')}$, and

$$c_U(s)|_{U'} = c_{U'}(s|_{U'}) = c_{U'} \circ a_{U'}(s_{U'}) = b_{U'}(s_{U'}).$$

Since B is a sheaf, there is an element $t \in B(U)$ such that $t|_{U'} = b_{U'}(s_{U'})$ for any $U' \in I(U)$. Hence, we obtain $c_U(s) = t$, and this means the uniqueness of c.

Conversely, the existence of c follows from putting $c_U(s) = t$. q.e.d.

The proof of Proposition 5.1.1 is as follows. Let A'' be the associated sheaf of A'. Then, by the previous lemma, the functor $A \mapsto A''$ is just a left adjoint functor of the forgetful functor $\mathrm{Sh}(k_X) \to \mathrm{PSh}(\tau)$.

Proposition 5.1.3. *We suppose that* τ *and* A *satisfy the following conditions.*

(i) U', $U'' \in \tau$ *implies* $U' \cup U''$, $U' \cap U''$, $U' \setminus \overline{U''} \in \tau$.

(ii) for any U', $U'' \in \tau$, *the sequence*

$$0 \to A(U' \cup U'') \to A(U') \oplus A(U'') \to A(U' \cap U'')$$

is exact.

Then, A' is a sheaf on X. Moreover, we also assume that

(iii) for any U' and $U'' \in \tau$ with $U'' \subset U'$, the sequence $A(U') \to A(U'') \to 0$ is exact.

Then, A' is a c-soft sheaf on X.

Proof. First, we assume (i) and (ii). Let $U \in \mathrm{Op}(X)$ and let $\{U_j\}_{j\in J}$ be an open covering of U. By the assumption (i), for any $V \in I(U)$, there exists a collection $\{V_q\}_{q\in Q}$ with the three conditions

(a) Q is a finite set,

(b) $\bigcup_{q\in Q} V_q = V$,

(c) $\forall q \in Q$, $\exists j \in J$; $V_q \in I(U_j)$.

Moreover, by the condition (ii), we get the following exact sequence

$$0 \to A(V) \to \prod_{q\in Q} A(V_q) \to \prod_{r,s\in Q} A(V_r \cap V_s).$$

The sequence induces the fact that A' is a sheaf.

Next, we also assume (iii). Take a compact set $K \subset X$. Then, for any $V \in I(X)$ with $K \subset V$, there is an set $V_1 \in I(V)$ such that $K \subset V_1$. By (iii), we have an epimorphism

$$A(V) \twoheadrightarrow A(V \setminus \overline{V_1}).$$

Hence, we know that A' is a c-soft sheaf. q.e.d.

Definition 5.1.4. We call a functor $F : \tau^{\mathrm{op}} \to \mathrm{Pro}(k)^{\mathrm{op}}$ a precosheaf on τ. We denote by $\mathrm{PCsh}(\tau)$ the category of precosheaf on τ.

Theorem 5.1.5. *The forgetful functor*

$$\mathrm{Csh}(k_X) \to \mathrm{PCsh}(\tau)$$

has a left adjoint functor.

Proof. Let $F \in \mathrm{PCsh}(\tau)$. We define a precosheaf on X as

$$F'(U) := \varprojlim_{U'\in I(U)} F(U), \tag{5.1.1}$$

and there is a natural morphism $a : F \to F'$. Now we take a cosheaf G on X and a morphism $b : F \to G$. Applying Lemma 5.1.2, for any $T \in \mathrm{Mod}(k)$, there exists a unique morphism $\langle F', T\rangle \to \langle G, T\rangle$ such that the diagram below commutes :

$$\begin{tikzcd} \langle F,T\rangle \arrow[r, "\langle b,T\rangle"] \arrow[d, "\langle a,T\rangle"'] & \langle G,T\rangle \arrow[d, "\mathrm{id}"] \\ \langle F',T\rangle \arrow[r] & \langle G,T\rangle . \end{tikzcd}$$

Hence, we also have a morphism $F' \to G$. Finally, apply Theorem 2.2.8. q.e.d.

Remark 5.1.6. Let $F \in \mathrm{PCsh}(\tau)$. We suppose the following conditions :

(i) U', $U'' \in \tau$ implies $U' \cup U''$, $U' \cap U''$, $U' \setminus \overline{U''} \in \tau$.

(ii) for any U', $U'' \in \tau$, the sequence

$$0 \to F(U' \cup U'') \to F(U') \oplus F(U'') \to F(U' \cap U'')$$

is exact.

Then, F' is a cosheaf on X. Moreover, if we also assume that

(iii) for any U', $U'' \in \tau$ with $U'' \subset U'$, the sequence $F(U') \to F(U'') \to 0$ is exact,

Then, F' is a c-soft cosheaf on X

This proof follows from Proposition 5.1.3.

5.2 The functor c

We suppose that k is a field and that X is a locally compact and Hausdorff topological space. We study the functor c and its properties. We remark that this functor can also be found in Schneiders [10], and he gave an application to the Borel-Moore homology. We will give another application later.

Definition 5.2.1. Let B be a sheaf on X. Consider $\mathrm{Mod}(k)^{\mathrm{op}} \hookrightarrow \mathrm{Pro}(k)^{\mathrm{op}}$. Then,

$$U \mapsto \Gamma_c(U; B)^{\mathrm{op}}$$

is a precosheaf on X. We denote the associated cosheaf by $\mathrm{c}B$.

We obtain a right exact functor

$$\mathrm{c} : \mathrm{Sh}(\mathcal{R}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Csh}(\mathcal{R}).$$

Proposition 5.2.2. *Let $B \in \mathrm{Sh}(\mathcal{R}^{\mathrm{op}})$. If Z is a locally closed subset of X, then we have an isomorphism*

$$\mathrm{c}(B_Z) \simeq \Gamma_Z \mathrm{c}B.$$

Proof. If U is an open subset of X, then we have an isomorphism $\Gamma_c(V; B_U) \simeq \Gamma_c(V \cap U; B)$ for each open subset $V \subset X$. Hence, we obtain $\mathrm{c}(B_U) \simeq \Gamma_U \mathrm{c}B$. If S is a closed subset of X, by using the exact sequence $B_{X-S} \to B \to B_S \to 0$, we get $\mathrm{c}(B_S) \simeq \Gamma_S \mathrm{c}B$. In case $Z = S \cap U$, we have

$$\mathrm{c}(B_Z) = \mathrm{c}((B_U)_S) \simeq \Gamma_U \Gamma_S \mathrm{c}B = \Gamma_Z \mathrm{c}B.$$

q.e.d.

Theorem 5.2.3. *Let $A \in \mathrm{Sh}(\mathcal{R})$ and $B \in \mathrm{Sh}(\mathcal{R}^{\mathrm{op}})$. Then, we have a natural isomorphism*

$$\mathcal{C}hom_{\mathcal{R}}(A, \mathrm{c}B) \simeq \mathrm{c}(B \otimes_{\mathcal{R}} A).$$

We use the following basic result to prove the proposition above.

Proposition 5.2.4. *For any $A \in \mathrm{Sh}(\mathcal{R})$, there exists a family $\{U_i\}_{i \in I}$ of open sets and an epimorphism $\bigoplus_{i \in I} \mathcal{R}_{U_i} \twoheadrightarrow A$.*

We begin the proof of Theorem 5.2.3.

Proof. By Proposition 5.2.4, we may assume that A is isomorphic to $\mathcal{R}_U$, where U is an open set of X. Then, we get

$$\begin{aligned} \mathcal{C}hom_{\mathcal{R}}(A, \mathrm{c}B) &= \mathcal{C}hom_{\mathcal{R}}(\mathcal{R}_U, \mathrm{c}B) \\ &\simeq \Gamma_U \mathrm{c}B \\ &\simeq \mathrm{c}(B_U) \\ &\simeq \mathrm{c}(B \otimes_{\mathcal{R}} \mathcal{R}_U) \\ &= \mathrm{c}(B \otimes_{\mathcal{R}} A). \end{aligned}$$

q.e.d.

Proposition 5.2.5. *Let $f : Y \to X$ be a continuous map. If $B \in \mathrm{Sh}(f^{-1}\mathcal{R}^{\mathrm{op}})$, then we have an isomorphism*

$$\mathrm{c}f_! B \simeq f_* \mathrm{c}B.$$

Proof. The proof follows from an isomorphism $\Gamma_c(U; f_! B) \simeq \Gamma_c(f^{-1}(U); B)$ for each open set $U \subset X$. q.e.d.

Theorem 5.2.6. *Let S be a c-soft sheaf on X. Then the precosheaf*

$$U \mapsto \Gamma_c(U; S)^{\mathrm{op}} \tag{5.2.1}$$

is a flabby cosheaf on X.(See Definition 4.1.8).

Proof. (i) We will check that (5.2.1) is a sheaf. Let $\{U_i\}_{i \in I}$ be an open covering of U. We first prove that

$$\bigoplus_{i \in I} \Gamma_c(U_i; S) \xrightarrow{d} \Gamma_c(U; S) \tag{5.2.2}$$

is surjective. Take $s \in \Gamma_c(U; S)$ and set $K := \operatorname{supp} s$. Since K is compact, we may assume that I is a finite set. Moreover, by induction, we may assume that $I = \{1, 2\}$. Since S is c-soft, there exists $t \in \Gamma_c(U_2; S)$ and open set W such that $U_2 \setminus U_1 \subset W \subset U_2$ and $t|_W = s|_W$. Set $L := \operatorname{supp} t$. Then $\operatorname{supp}(s - t) \subset (K \cup L) \setminus W \subset U_1$.

Next, we prove that the following sequence :

$$\bigoplus_{i,j \in I} \Gamma_c(U_i \cap U_j; S) \xrightarrow{d'} \bigoplus_{i \in I} \Gamma_c(U_i; S) \xrightarrow{d} \Gamma_c(U; S) \tag{5.2.3}$$

is exact. We may assume that $I := \{1, \cdots, n\}$. Let us show (5.2.3) by induction on n. Take a collection $\{s_1, \cdots, s_n : s_i \in \Gamma_c(U_i; S)\}$ with the condition $\sum_{i=1}^{n} s_i = 0$. If we put $s' := \sum_{i=1}^{n-1} s_i$, then $s' + s_n = 0$, so we get $\operatorname{supp} s_n \subset U_n \cap (U_1 \cup \cdots \cup U_{n-1})$.

Applying (5.2.2), we can select a collection $\{t_1, \cdots, t_{n-1} : t_i \in \Gamma_c(U_i \cap U_n; F)\}$ which satisfies $s_n = \sum_{i=1}^{n-1} t_i$. Then, for any $i = 1, \cdots, n-1$, we have $\mathrm{supp}(s_i + t_i) \subset U_i$. We also have $\sum_{i=1}^{n-1}(s_i + t_i) = 0$. By the hypothesis of induction, there exists a collection $\{s_{ij} : s_{ij} \in \Gamma_c(U_i \cap U_j; S)\}_{1 \le i,j \le n-1}$ such that $\sum_{j=1}^{n-1} s_{ij} - \sum_{j=1}^{n-1} s_{ji} = s_i + t_i$ for $1 \le i \le n-1$. Moreover, we set $s_{ni} := t_i$, $s_{in} = 0$ and $s_{nn} = 0$, where $1 \le i \le n-1$. Then, we have $\sum_{j=1}^{n} s_{ij} - \sum_{j=1}^{n} s_{ji} = s_i$ for $1 \le i \le n$.

(ii) The flabbiness is obvious.

q.e.d.

Proposition 5.2.7. *Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence of sheaves on X. If B'' is c-soft, then the sequence*

$$0 \to \mathrm{c}B'' \to \mathrm{c}B \to \mathrm{c}B' \to 0$$

is also exact.

Proof. Since B'' is c-soft, for any open set $U \subset X$, the sequence

$$0 \to \Gamma_c(U; B') \to \Gamma_c(U; B) \to \Gamma_c(U; B'') \to 0,$$

is exact. Hence, the proof follows. q.e.d.

By Proposition 5.2.7, we get the following corollary.

Corollary 5.2.8. *There exists a derived functor of the functor* c

$$\mathrm{Lc} : \mathbf{D}^{+}(\mathrm{Sh}(\mathcal{R}^{\mathrm{op}}))^{\mathrm{op}} \to \mathbf{D}^{-}(\mathrm{Csh}(\mathcal{R})).$$

In particular, if X has finite c-soft dimension, then we have

$$\mathrm{Lc} : \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(\mathcal{R}^{\mathrm{op}}))^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathrm{Csh}(\mathcal{R})).$$

From now on, we assume that X has finite c-soft dimension.

Theorem 5.2.9. *Let Y be a locally compact and Hausdorff space with finite c-soft dimension, and let $f : Y \to X$ be a continuous map. If $B \in \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(f^{-1}\mathcal{R}^{\mathrm{op}}))$, then we have*

$$\mathrm{Lc}(Rf_!B) \simeq Rf_*\mathrm{Lc}(B).$$

Proof. We may assume that B is c-soft. Then, applying Proposition 5.2.5, we get

$$\begin{aligned} \mathrm{Lc}(Rf_!B) &\simeq \mathrm{c}(f_!B) \\ &\simeq f_*\mathrm{c}(B) \\ &\simeq Rf_*\mathrm{Lc}(B). \end{aligned}$$

q.e.d.

Remark 5.2.10. Let $B \in \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(\mathcal{R}^{\mathrm{op}}))$. If Z is a locally closed set, then

$$\mathrm{Lc}(B_Z) \simeq \mathrm{R}\Gamma_Z \mathrm{Lc}(B).$$

If $\mathcal{R}$ has finite weak global dimension and $A \in \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(\mathcal{R}))$, then

$$R\mathcal{C}hom_{\mathcal{R}}(A, \mathrm{Lc}(B)) \simeq \mathrm{Lc}(B \otimes^{L}_{\mathcal{R}} A).$$

This proof is similar to the proof of Theorem 5.2.9. Apply Proposition 5.2.2 and Theorem 5.2.3.

Theorem 5.2.11. *Let $B \in \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(k_X))$. If $f : Y \to X$ is a continuous map, then*

$$f^! \mathrm{Lc}(B) \simeq \mathrm{Lc}(f^{-1}B).$$

Proof. Let $A \in \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(k_X))$. Then, by applying Theorem 4.2.11, Theorem 4.3.7, Theorem 5.2.9 and Remark 5.2.10, we have

$$\begin{aligned}
\mathrm{RCHom}\,(A, f^! \mathrm{Lc}(B)) &\simeq \mathrm{RCHom}\,(f_! A, \mathrm{Lc}(B)) \\
&\simeq \mathrm{R}\Gamma(X; \mathrm{Lc}(B \otimes f_! A)) \\
&\simeq \mathrm{R}\Gamma(X; \mathrm{Lc}(f_!(f^{-1}B \otimes A))) \\
&\simeq \mathrm{R}\Gamma(X; f_* \mathrm{Lc}(f^{-1}B \otimes A)) \\
&\simeq \mathrm{R}\Gamma(Y; \mathrm{Lc}(f^{-1}B \otimes A)) \\
&\simeq \mathrm{RCHom}\,(A, \mathrm{Lc}(f^{-1}B)).
\end{aligned}$$

Since this is true for any A, the proof follows. q.e.d.

Let us define conic cosheaves. Let $\mathbb{R}^+$ denote the multiplicative group of strictly positive numbers, and suppose that X has an action of $\mathbb{R}^+$. In other words we have a continuous map :

$$\mu : X \times \mathbb{R}^+ \to X,$$

which satisfies for each $x \in X$, t_1, $t_2 \in \mathbb{R}^+$:

$$\mu(x, t_1, t_2) = \mu(\mu(x, t_1), t_2), \quad \mu(x, 1) = x.$$

Consider the maps:

$$X \xrightarrow{j} X \times \mathbb{R}^+ \xrightarrow{p} X,$$

where $j(x) = (x, 1)$, and p is the projection. We have

$$\mu^{-1}F \leftarrow p^{-1}p_*\mu^{-1}F \to p^{-1}p_*j_*j^{-1}\mu^{-1}F \simeq p^{-1}F.$$

Definition 5.2.12. (i) We denote by $\mathrm{Csh}_{\mathbb{R}^+}(k_X)$ the full subcategory of $\mathrm{Csh}(k_X)$ consisting of cosheaves F such that $\mu^{-1}F \xleftarrow{\sim} p^{-1}p_*\mu^{-1}F \xrightarrow{\sim} p^{-1}F$.

(ii) We denote by $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Csh}(k_X))$ the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{Csh}(k_X))$ consisting of objects F such that for all $j \in \mathbb{Z}$, $H^j(F) \in \mathrm{Csh}_{\mathbb{R}^+}(k_X)$.

(iii) We call an object of $\mathrm{Csh}_{\mathbb{R}^+}(k_X)$ a conic cosheaf.

Remark 5.2.13. Let $F \in \mathbf{D}^{\mathrm{b}}(\mathrm{Csh}(k_X))$. Then, the following statements are equivalent.

(i) $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Csh}(k_X))$.

(ii) $\mu^{-1}F \simeq p^{-1}F$.

(iii) $\mu^! F \simeq p^! F$.

The proof follows from Proposition 3.7.2 of [2] and Corollary 4.2.2.

Theorem 5.2.14. *The following functor is well-defined.*

$$\mathrm{Lc} : \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Sh}(k_X))^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Csh}(k_X)).$$

Proof. Let $B \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Sh}(k_X))$. Then, by Theorem 5.2.11, we get

$$\mu^! \mathrm{Lc}(B) \simeq \mathrm{Lc}(\mu^{-1}B) \simeq \mathrm{Lc}(p^{-1}B) \simeq p^! \mathrm{Lc}(B).$$

Applying Remark 5.2.13, the proof follows. q.e.d.

6 Applications

6.1 Review of Laplace transforms

From now on, the base field k is $\mathbb{C}$. Let X be a real analytic manifold and let $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(X))$ be the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(X))$ consisting of objects whose cohomology groups are $\mathbb{R}$-constructible sheaves. We denote by $\mathcal{D}b_X$ (resp. $\mathcal{C}^\infty_X$) the sheaf of Schwartz's distributions (resp. C^∞-class functions), and by $\mathcal{D}_X$ the sheaf of finite-order differential operators with coefficients in analytic functions. Recall the functor

$$T\mathcal{H}om\,(\,\cdot\,, \mathcal{D}b_X) : \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(X))^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(\mathcal{D}_X))$$

by Kashiwara [6] and the functor

$$\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X : \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(X)) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(\mathcal{D}_X)).$$

by Kashiwara-Schapira [4].

Let X be a complex manifold, $X^{\mathbb{R}}$ the real analytic underlying manifold and $\overline{X}$ the complex conjugate manifold. If there is no risk of confusion, we write X instead of $X^{\mathbb{R}}$. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions and by $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients. The functors

$$\begin{array}{rcll} T\mathcal{H}om\,(\,\cdot\,, \mathcal{O}_X) & : & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(X))^{\mathrm{op}} & \to \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(\mathcal{D}_X)), \\ \cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X & : & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(X)) & \to \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(\mathcal{D}_X)). \end{array}$$

are defined as

$$\begin{aligned} T\mathcal{H}om\,(F,\mathcal{O}_X) &:= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, T\mathcal{H}om\,(F,\mathcal{D}b_{X^{\mathbb{R}}})),\\ F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X &:= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{X^{\mathbb{R}}}). \end{aligned}$$

An $\mathcal{O}_X$-module $\mathcal{F}$ is called quasi-good if any compact subset of X has a neighborhood U such that $\mathcal{F}|_U$ is a union of an increasing countable family of coherent $\mathcal{O}_X|_U$-submodule. A $\mathcal{D}_X$-module $\mathcal{M}$ is quasi-good if it is quasi-good as an $\mathcal{O}_X$-module.

Let E be an n-dimensional complex vector space. Let $j : E \to P$ denote the projective compactification of E. We regard E as complex algebraic variety. Let $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+,\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(E))$ be the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(E))$ consisting of objects whose cohomology groups are $\mathbb{R}^+$-conic and $\mathbb{R}$-constructible sheaves.

Let E^* be the dual space of E and let $\langle\,\cdot\,,\,\cdot\,\rangle : E \times E^* \to \mathbb{C}$ be the pairing map between E and E^*. We denote by φ the function $\varphi(z,w) := -\langle z,w\rangle$ on $E \times E^*$. We set

$$\begin{aligned} A &:= \{(z,w) \in E \times E^* : \Re\varphi(z,w) \geq 0\},\\ A' &:= \{(z,w) \in E \times E^* : \Re\varphi(z,w) \leq 0\}. \end{aligned}$$

Consider the diagram

$$E \xleftarrow{p_1} E \times E^* \xrightarrow{p_2} E^*.$$

Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+,\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(E))$ and let $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+,\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(E^*))$. Its Fourier-Sato transformation and inverse Fourier-Sato transformation are defined by

$$\begin{aligned} F^{\wedge} &:= Rp_{2!}(p_1^{-1}F)_{A'},\\ G^{\vee} &:= Rp_{1!}(p_2^{!}G)_{A}. \end{aligned}$$

Further information can be found in Kashiwara-Schapira [2].

We denote by $D(E)$ the Weyl algebra on E, and by $O(E)$ the polynomial ring on E.

We denote by $(\,\cdot\,)^{\wedge}$ the Fourier isomorphism :

$$(\,\cdot\,)^{\wedge} : D(E) \xrightarrow{\sim} D(E^*).$$

If $(z_1,\cdots,z_n)$ is a system of linear coordinates on E and $(w_1,\cdots,w_n)$ the dual coordinate system on E^*, then $(\,\cdot\,)^{\wedge}$ is given by :

$$(z_j)^{\wedge} = -\frac{\partial}{\partial w_j}, \quad \left(\frac{\partial}{\partial z_j}\right)^{\wedge} = w_j.$$

Let $(\,\cdot\,)^{\vee} : D(E^*) \xrightarrow{\sim} D(E)$ be the inverse of $(\,\cdot\,)^{\wedge}$. For a $D(E)$-module N, the ring isomorphism $(\,\cdot\,)^{\wedge} : D(E) \xrightarrow{\sim} D(E^*)$ makes N a $D(E^*)$-module, which we denote by $N^{\wedge}$. Thus it gives an equivalence of categories $(\,\cdot\,)^{\wedge} : \mathbf{D}^{\mathrm{b}}_{q-good}(D(E^*)) \to \mathbf{D}^{\mathrm{b}}_{q-good}(D(E))$, and similarly $(\,\cdot\,)^{\vee} : \mathbf{D}^{\mathrm{b}}_{q-good}(D(E)) \to \mathbf{D}^{\mathrm{b}}_{q-good}(D(E^*))$. For any $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+,\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(E))$, we put

$$\begin{aligned} \mathrm{THom}\,(F,\mathcal{O}_E) &:= \mathrm{R}\Gamma(P; T\mathcal{H}om\,(j_!F,\mathcal{O}_P)),\\ F \overset{\mathrm{W}}{\otimes} \mathcal{O}_E &:= \mathrm{R}\Gamma_c(P; j_!F \overset{\mathrm{w}}{\otimes} \mathcal{O}_P)). \end{aligned}$$

Let $N \in \mathbf{D}^{\mathrm{b}}_{q-good}(D(E^*))$. Then, Kashiwara-Schapira [5] defined the Laplace morphism

$$\mathrm{RHom}_{D(E)}(N^\wedge, F \overset{\mathrm{W}}{\otimes} \mathcal{O}_E) \xrightarrow{L} \mathrm{RHom}_{D(E^*)}(N, F^\wedge[n] \overset{\mathrm{W}}{\otimes} \mathcal{O}_{E^*}), \tag{6.1.1}$$

$$\mathrm{THom}\,(F^\wedge[n], \Omega_{E^*}) \otimes^L_{D(E^*)} N \xrightarrow{{}^tL} \mathrm{THom}\,(F, \Omega_E) \otimes^L_{D(E)} N^\wedge, \tag{6.1.2}$$

where Ω_E is the sheaf of n-forms. These Laplace morphisms are isomorphisms.

Finally, we recall the definition of $\mathcal{O}^t$. Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+,\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(E))$. Then, we put

$$\mathrm{THom}\,(F, \mathcal{D}b_E) := \mathrm{R}\Gamma(P; T\mathcal{H}om\,(j_!F, \mathcal{D}b_P)).$$

If U is a subanalytic cone in E, then the complex $\mathrm{THom}\,(\mathbb{C}_U, \mathcal{D}b_E)$ is concentrated in degree 0. Hence, by Proposition 5.1.1, the conic sheaf $\mathcal{D}b^t_E$ is defined as the associated sheaf to the conic presheaf

$$U \mapsto \mathrm{THom}\,(\mathbb{C}_U, \mathcal{D}b_E).$$

We put

$$\mathcal{O}^t_E := R\mathcal{H}om_{D(\overline{E})}(O(\overline{E}), \mathcal{D}b^t_E),$$

where $\overline{E}$ is the complex conjugate of E. This sheaf is called one of sheaves of tempered holomorphic functions, and (6.1.1) induces the following formula.

Theorem 6.1.1 (Kashiwara-Schapira [5]). *There is an isomorphism*

$$(\mathcal{O}^t_E)^\wedge[n] \simeq \mathcal{O}^t_{E^*}.$$

This isomorphism can also be found in Olivier Berni [1].

6.2 Main theorem

We define a cosheaf of the Whitney holomorphic functions, and we prove an analogy of Theorem 6.1.1 by using (6.1.2).

We first discuss the Fourier-Sato transformations on cosheaves.

Definition 6.2.1. We define the functors

$$\begin{array}{rcccl} (\,\cdot\,)^\wedge & : & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Csh}(E^*)) & \to & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Csh}(E)), \\ (\,\cdot\,)^\vee & : & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Csh}(E)) & \to & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Csh}(E^*)), \end{array}$$

as follows :

$$\begin{array}{rcl} F^\wedge & := & Rp_{1*}\mathrm{R}\Gamma_A p_2^{-1}F, \\ G^\vee & := & Rp_{2*}\mathrm{R}\Gamma_{A'} p_1^! G. \end{array}$$

Remark 6.2.2. The functors of Definition 6.2.1 induce the equivalences of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Csh}(E))$ and $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathrm{Csh}(E^*))$. The proof follows from Theorem 3.7.9 of [2] and Corollary 4.2.2.

Theorem 6.2.3. *Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}+}(\mathrm{Sh}(E))$ and $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}+}(\mathrm{Sh}(E^*))$. Then, we have*

$$\begin{aligned}\mathrm{Lc}(F^\wedge) &\simeq \mathrm{Lc}(F)^\vee,\\ \mathrm{Lc}(G^\vee) &\simeq \mathrm{Lc}(G)^\wedge.\end{aligned}$$

Proof. Applying Theorem 5.2.9, Remark 5.2.10 and Theorem 5.2.11, we have

$$\mathrm{Lc}(F^\wedge) \simeq \mathrm{Lc}(Rp_{2!}(p_1^{-1}F)_{A'}) \simeq Rp_{2*}\mathrm{R}\Gamma_{A'}p_1^!\mathrm{Lc}(F) \simeq \mathrm{Lc}(F)^\vee.$$

Hence, the first isomorphism holds. The second isomorphism follows from the first isomorphism. q.e.d.

Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}+,\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(E))$. Then, by putting

$$\mathrm{CWHom}\,(F, \mathcal{C}^\infty_E) := \mathrm{R}\Gamma(P; \mathrm{Lc}(j_!F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_P)),$$

we get the following functor

$$\mathrm{CWHom}\,(\,\cdot\,, \mathcal{C}^\infty_E) : \mathbf{D}^{\mathrm{b}}_{\mathbb{R}+,\mathbb{R}-\mathrm{c}}(\mathrm{Sh}(E)) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Csh}(D(E)^{\mathrm{op}})).$$

If U is a subanalytic cone in E, then the complex $\mathrm{CWHom}\,(\mathbb{C}_U, \mathcal{C}^\infty_E)$ is concentrated in degree 0. There is a natural isomorphism $\mathrm{CWHom}\,(\mathbb{C}_U, \mathcal{C}^\infty_E) \simeq \Gamma_c(P; j_!\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_E)^{\mathrm{op}}$.

Definition 6.2.4. We define the conic cosheaf $\mathcal{C}^{\infty\mathrm{cw}}_E$ as the associated cosheaf to the precosheaf

$$U \mapsto \mathrm{CWHom}\,(\mathbb{C}_U, \mathcal{C}^\infty_E).$$

Proposition 6.2.5. *Let U be a conic open cone on E and let U' range the collection of open subanalytic cones. Then, we have*

(i) $\Gamma(U; \mathcal{C}^{\infty\mathrm{cw}}_E) = \varinjlim\limits_{U' \subset\subset U} \mathrm{CWHom}\,(\mathbb{C}_{U'}, \mathcal{C}^\infty_E)$.

(ii) *For any $j \neq 0$,* $\mathrm{R}^j\Gamma(U; \mathcal{C}^{\infty\mathrm{cw}}_E) = 0$.

(iii) $\mathrm{R}\Gamma(E; \mathcal{C}^{\infty\mathrm{cw}}_E) \simeq \mathrm{CWHom}\,(\mathbb{C}_E, \mathcal{C}^\infty_E)$.

(iv) $\mathrm{R}\Gamma_{\{0\}}(E; \mathcal{C}^{\infty\mathrm{cw}}_E) \simeq \mathrm{CWHom}\,(\mathbb{C}_{\{0\}}, \mathcal{C}^\infty_E)$.

(v) $\mathcal{C}^{\infty\mathrm{cw}}_E$ *is a conically soft cosheaf.*

Proof. By the theorem of Lojaciewicz [7] (see Malgrange [8]), for any two open subanalytic cones U, V of E, there is an exact sequence

$$0 \to \mathrm{CWHom}\,(\mathbb{C}_{U\cup V}, \mathcal{C}^\infty_E) \to \mathrm{CWHom}\,(\mathbb{C}_U \oplus \mathbb{C}_V, \mathcal{C}^\infty_E) \to \mathrm{CWHom}\,(\mathbb{C}_{U\cap V}, \mathcal{C}^\infty_E) \to 0.$$

Hence, by Theorem 5.1.6, the proof follows. q.e.d.

Definition 6.2.6. We set

$$\begin{aligned}\mathrm{CWHom}\,(F, \mathcal{O}_E) &:= \mathrm{CWHom}\,(F, \mathcal{C}^\infty_E) \otimes^L_{D(\overline{E})} O(\overline{E}),\\ \mathcal{O}^{\mathrm{cw}}_E &:= \mathcal{C}^{\infty\mathrm{cw}}_E \otimes^L_{D(\overline{E})} O(\overline{E}).\end{aligned}$$

Our main theorem is as follows.

Theorem 6.2.7.

(i) *The Fourier-Sato transfromation* $(\mathcal{O}_E^{\mathrm{cw}})^\wedge$ *is concentrated in degree* 0.

(ii) *The Laplace transform induces an isomorphism* $(\mathcal{O}_E^{\mathrm{cw}})^\wedge[n] \simeq \mathcal{O}_{E^*}^{\mathrm{cw}}$ *of* $D(E^*)$*-linear.*

To prove the above theorem, we need some results.

Lemma 6.2.8. *We have*

$$\begin{aligned} \mathrm{R}\Gamma(E;\mathcal{O}_E^{\mathrm{cw}}) &\simeq \mathrm{CWHom}\,(\mathbb{C}_E,\mathcal{O}_E), \\ \mathrm{R}\Gamma_{\{0\}}(E;\mathcal{O}_E^{\mathrm{cw}}) &\simeq \mathrm{CWHom}\,(\mathbb{C}_{\{0\}},\mathcal{O}_E). \end{aligned}$$

Proof. Take a Spencer resolution $M^\bullet(\overline{E}) \xrightarrow{\mathrm{qis}} O(\overline{E})$, where $M^\bullet(\overline{E})$ is a complex of finite free $D(\overline{E})$-modules. Then, applying Proposition 6.2.5, we have

$$\begin{aligned} \mathrm{R}\Gamma(E;\mathcal{O}_E^{\mathrm{cw}}) &= \mathrm{R}\Gamma(E;\mathcal{C}_E^{\infty\mathrm{cw}} \otimes^L_{D(\overline{E})} O(\overline{E})) \\ &\simeq \Gamma(E;\mathcal{C}_E^{\infty\mathrm{cw}} \otimes_{D(\overline{E})} M^\bullet(\overline{E})) \\ &\simeq \Gamma(E;\mathcal{C}_E^{\infty\mathrm{cw}}) \otimes_{D(\overline{E})} M^\bullet(\overline{E}) \\ &\simeq \mathrm{CWHom}\,(\mathbb{C}_E,\mathcal{C}_E^{\infty}) \otimes_{D(\overline{E})} M^\bullet(\overline{E}) \\ &\simeq \mathrm{CWHom}\,(\mathbb{C}_E,\mathcal{C}_E^{\infty}) \otimes^L_{D(\overline{E})} O(\overline{E}) \\ &= \mathrm{CWHom}\,(\mathbb{C}_E,\mathcal{O}_E). \end{aligned}$$

The second isomorphism is similar. q.e.d.

Lemma 6.2.9. *Let* $U_1' \subset\subset U_1 \subset\subset U_2' \subset\subset U_2$ *be open cones with* U_1' *and* U_2' *subanalytic. Then, there is a canonical commutative diagram :*

$$\begin{array}{ccc} \mathrm{R}\Gamma(U_2;\mathcal{O}_E^{\mathrm{cw}}) & \longrightarrow & \mathrm{CWHom}\,(\mathbb{C}_{U_2'},\mathcal{O}_E) \\ \downarrow & \swarrow & \downarrow \\ \mathrm{R}\Gamma(U_1;\mathcal{O}_E^{\mathrm{cw}}) & \longrightarrow & \mathrm{CWHom}\,(\mathbb{C}_{U_1'},\mathcal{O}_E). \end{array}$$

Proof. This follows from the commutative diagram

$$\begin{array}{ccc} \Gamma(U_2;\mathcal{C}_E^{\infty\mathrm{cw}}) & \longrightarrow & \mathrm{CWHom}\,(\mathbb{C}_{U_2'},\mathcal{C}_E^{\infty}) \\ \downarrow & \swarrow & \downarrow \\ \Gamma(U_1;\mathcal{C}_E^{\infty\mathrm{cw}}) & \longrightarrow & \mathrm{CWHom}\,(\mathbb{C}_{U_1'},\mathcal{C}_E^{\infty}). \end{array}$$

q.e.d.

Remark 6.2.10. Let $Z_2 \subset Z_2' \subset Z_1 \subset Z_1'$ be closed cones in E with Z_1' and Z_2' subanalytic and $E\setminus Z_1' \subset\subset E\setminus Z_1 \subset\subset E\setminus Z_2' \subset\subset E\setminus Z_2$. Then there is a canonical commutative diagram:

$$\begin{array}{ccc} \mathrm{R}\Gamma_{Z_2}(E;\mathcal{O}_E^{\mathrm{cw}}) & \longrightarrow & \mathrm{CWHom}\,(\mathbb{C}_{Z_2'},\mathcal{O}_E) \\ \downarrow & \swarrow & \downarrow \\ \mathrm{R}\Gamma_{Z_1}(E;\mathcal{O}_E^{\mathrm{cw}}) & \longrightarrow & \mathrm{CWHom}\,(\mathbb{C}_{Z_1'},\mathcal{O}_E). \end{array}$$

The proof is similar to Lemma 6.2.9.

Lemma 6.2.11. *Let U be an open convex subanalytic cone in E and $Z := U^\circ$. Then, there is an isomorphism*

$$\mathrm{CWHom}\,(\mathbb{C}_U, \mathcal{O}_{E^*})[-n] \quad \xrightarrow{\sim} \quad \mathrm{CWHom}\,(\mathbb{C}_Z, \mathcal{O}_E),$$

and both sides are concentrated in degree 0.

Proof. In (6.1.1), by putting $F := \mathbb{C}_Z$, we get

$$L : \mathbb{C}_Z \overset{\mathrm{W}}{\otimes} \mathcal{O}_E \xrightarrow{\sim} \mathbb{C}_U \overset{\mathrm{W}}{\otimes} \mathcal{O}_{E^*}[n].$$

So, the desired results follows. (See also (6.1.5) of [5]). q.e.d.

Lemma 6.2.12. (i) *The conic object $\mathcal{O}_E^{\mathrm{cw}}$ is concentrated in degree $-n$.*

(ii) *The conic cosheaf* $\mathrm{H}^{-n}(\mathcal{O}_E^{\mathrm{cw}})$ *is the conic cosheaf associated to the precosheaf*

$$U \mapsto \mathrm{H}^{-n}\mathrm{CWHom}\,(\mathbb{C}_U, \mathcal{O}_E).$$

Proof. By Lemma 6.2.8 and Lemma 6.2.9, it is enough to prove this assertion at each point of $E \setminus \{0\}$. Applying Lemma 6.2.11, $\mathrm{CWHom}\,(\mathbb{C}_U, \mathcal{O}_E)$ is concentrated in degree $-n$ for any open subanalytic cone U, so the proof follows. q.e.d.

We shall begin to prove the Theorem 6.2.7.

Proof. Choose open convex cones $U_1 \subset\subset U_2 \subset\subset U_3$ with U_2 subalalytic, and put $Z_2 := U_2^\circ$. By Remark 6.2.10, we have

$$\mathrm{R}\Gamma(U_3; (\mathcal{O}_E^{\mathrm{cw}})^\wedge) \to \mathrm{CWHom}\,(\mathbb{C}_{Z_2}, \mathcal{O}_E) \to \mathrm{R}\Gamma(U_1; (\mathcal{O}_E^{\mathrm{cw}})^\wedge)$$

and

$$\mathrm{R}\Gamma(E; (\mathcal{O}_E^{\mathrm{cw}})^\wedge) \simeq \mathrm{R}\Gamma_{\{0\}}(E; \mathcal{O}_E^{\mathrm{cw}}) \simeq \mathrm{CWHom}\,(\mathbb{C}_{\{0\}}, \mathcal{O}_E).$$

Applying Lemma 6.2.11, $(\mathcal{O}_E^{\mathrm{cw}})^\wedge$ is isomorphic to the conic cosheaf associated with the precosheaf $U \mapsto \mathrm{H}^{-n}\mathrm{CWHom}\,(\mathbb{C}_U, \mathcal{O}_{E^*})$ for an open subanalytic convex cone U. It is obvious that $(\mathcal{O}_E^{\mathrm{cw}})^\wedge$ is concentrated in degree 0. q.e.d.

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