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Uniqueness in identification of the support of a source term in an elliptic equation

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UNIQUENESS IN IDENTIFICATION OF THE SUPPORT OF A SOURCE TERM IN AN ELLIPTIC EQUATION

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ABSTRACT. We consider an inverse problem of identifying the support D of a source term in an elliptic equation

$$-\Delta u(x) + q(x)\chi_D(x)u(x) = 0, \quad x \in \Omega \qquad \text{and} \qquad u(x) = f(x), \quad x \in \partial\Omega.$$

Here q is a given positive function and χ_D is the characteristic function of a subdomain D such that $\overline{D} \subset \Omega$. By using a Carleman estimate, we prove the global uniqueness in this inverse problem within convex hulls of polygons D's.

1. INTRODUCTION

We consider an inverse problem of recovering the shape and location of an unknown stationary heat source F. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and D a subdomain of Ω with Lipschitz boundary.

In this paper, we assume that the source F at $x = (x_1, x_2)$ is limited to D and propotional to the temperature u at x, that is, $F(x,t,u) = q(x)\chi_D(x)u(x,t)$. Here and henceforth, χ_D is the characteristic function of the subdomain $D \subset \Omega$, and $q \in C^2(\overline{\Omega})$, $q > 0$ on $\overline{\Omega}$.

If we apply a potential f on the boundary $\partial\Omega$ of Ω, then the resulting temperature u satisfies the Dirichlet problem

$$\begin{cases} -\Delta u + & q\chi_D u & = 0 & \text{in} \quad \Omega \\ & u & = f & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

It is well known that for a given domain D and $f \in H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ to (1.1). Thus we can define the Dirichlet-to-Neumann map $\Lambda_D : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ by

$$\Lambda_D(f) := \frac{\partial u}{\partial \nu}|_{\partial\Omega}, \tag{1.2}$$

where ν is the unit outward normal vector to $\partial\Omega$.

Restricting D to a polygon such that $\overline{D} \subset \Omega$, we discuss an inverse problem of determining D by a single boundary measurement $(f, \Lambda_D(f))$.

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There have been researches related to our inverse problem, which is motivated by determination of transistor contact resistivity and contact window location in the equation $-\Delta u + \chi_D u = 0$ in Ω. See [3], [5], [13]. In particular, a uniqueness result within a one-parameter monotone family from a one-point boundary measurement of the potential was obtained in [3]. Moreover [13] provides a global uniqueness result and a reconstruction scheme within the class of two- or three-dimensional balls from a single boundary measurement.

As for related inverse problems of determining piecewise continuous $\gamma = \gamma(x)$ in $\nabla \cdot (\gamma \nabla u) = 0$ in Ω, we can refer to [2], [7], [14] - [16]. Our inverse problem is concerned with the determination of shapes of domains and is of a character similar to the classical inverse source problem or the inverse gravimetry where we are required to determine a domain D in $-\Delta u = \chi_D$ by a single measurement of an exterior potential. As for the inverse source problem, we refer to the books [1], [8], [9] and the references therein. Our method is applicable also to the inverse source problem.

The main purpose of this paper is to prove global uniqueness results within polygons under extra conditions. We always assume that the boundary of a polygon under consideration is a simple closed curve, and by a polygon we mean its interior. Moreover, throughout this paper, we assume

$$f \geq 0, \quad \not\equiv 0 \quad \text{on } \partial\Omega, \quad q > 0 \quad \text{on } \overline{\Omega}. \tag{1.3}$$

We state our first main theorem. For $D \subset \mathbb{R}^2$, we denote the convex hull (i.e., the smallest convex set containing D) by $\mathrm{co}\,(D)$.

Theorem 1.1. *If D_1 and D_2 are polygons such that $\overline{D_1}, \overline{D_2} \subset \Omega$ and $\Lambda_{D_1}(f) = \Lambda_{D_2}(f)$, then $co\,(D_1) = co\,(D_2)$.*

From Theorem 1.1, we can readily derive

Corollary 1.2. *If D_1 and D_2 are convex polygons such that $\overline{D_1}, \overline{D_2} \subset \Omega$ and $\Lambda_{D_1}(f) = \Lambda_{D_2}(f)$, then $D_1 = D_2$.*

In Theorem 1.1, we cannot conclude that $D_1 = D_2$ without convexity. In the case of Figure 1, our argument does not work, and we do not know the uniqueness.

Next we show some uniqueness results for non-convex polygons, and we think that the uniqueness results for non-convex cases obtained so far, are not comprehensive and should be improved. Our results in non-convex cases are stated as follows. First we show the uniqueness in a case where D_1 and D_2 have a common contact edge. For any domains D, E compactly contained in Ω, we denote the outer most boundary of $D \cup E$ by $\partial_{\text{out}}(D \cup E)$, i.e.,

$$\begin{aligned}\partial_{\text{out}}(D \cup E) = \{&x \in \partial(D \cup E) |\, \text{there exists a continuous curve} \\ &\text{in } \Omega \setminus \overline{(D \cup E)} \text{ joining } x \text{ with some point of } \partial\Omega\}.\end{aligned}$$

Here and henceforth, by a curve, we exclude the end points.

Theorem 1.3. *Assume that D_1 and D_2 are polygons and that a line segment $\overline{A_0B_0} \subset \partial D_1 \cap \partial D_2$ lies on $\partial_{out}(D_1 \cup D_2)$. Then $\Lambda_{D_1}(f) = \Lambda_{D_2}(f)$ yields $D_1 = D_2$.*

Finally we show the uniqueness in a case where all edges of D_1 and D_2 are parallel to two independent vectors.

Theorem 1.4. *Assume that D_1 and D_2 are polygons such that there exist two independent vectors $\vec{a}$ and $\vec{b}$ such that all the edges of D_1 and D_2 are parallel to $\vec{a}$ or $\vec{b}$. Then $\Lambda_{D_1}(f) = \Lambda_{D_2}(f)$ yields $D_1 = D_2$.*

In particular, if polygons D_1 and D_2 are composed of rectangles in the forms of $\{(x_1, x_2) | \, a_1 < x_1 < b_1, a_2 < x_2 < b_2\}$, then Theorem 1.4 is applicable. Our argument does not work even if all the vertex angles are the right angle but if all the edges are not parallel to one of the fixed two direction. See Figure 2.

Let u_j, $j = 1, 2$, be the solution to (1.1) corresponding to the domain D_j. It is well known that for any subdomain Ω' compactly contained in Ω, the solutions u_j, $j = 1, 2$, satisfy

$$u_j \in H^2(\Omega') \cap C^{0,\kappa}(\overline{\Omega'}) \qquad \text{for some} \quad 0 < \kappa < 1. \tag{1.4}$$

See, e.g., [4], [12]. Moreover the maximum principle applied to u_j shows that

$$u_j > 0 \qquad \text{in} \quad \Omega, \quad j = 1, 2. \tag{1.5}$$

In the next section, we describe a Carleman estimate. We show one proposition by using that Carleman estimate, and our main theorem is derived from the proposition.

We can apply our argument to obtain similar uniqueness results in the case where $-\Delta u$ in (1.1) is replaced by a uniformly elliptic operator

$$-\sum_{i,j=1}^{2} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) + \sum_{i=1}^{2} b_i(x)\partial_{x_i} u$$

with smooth coefficients. For simplicity, however, we will consider only $-\Delta u$.

2. Non-existence of an H^2- solution to a Cauchy problem for the Laplace equation

We present a Carleman estimate for an elliptic operator. The proof of our Carleman estimate is based on [6] and the usual density argument. For convenience, we will give the proof in Appendix. As for Carleman estimates, we refer further to [10], [17].

For $\beta > 0$, we define the functions $\psi = \psi(x_1, x_2)$ and $\varphi = \varphi(x_1, x_2)$ by

$$\psi(x_1, x_2) = x_1 + \beta x_2^2 \qquad \text{and} \qquad \varphi(x_1, x_2) = e^{-\lambda\psi(x_1,x_2)} \tag{2.1}$$

with a parameter $\lambda > 0$. Moreover we introduce an elliptic operator in the following form

$$Pv = \Delta v + \alpha \partial_{x_1} \partial_{x_2} v, \tag{2.2}$$

where a constant α satisfies

$$|\alpha| < 2.$$

We set $\nabla = (\partial_{x_1}, \partial_{x_2})$.

Proposition 2.1. *Let $Q := (0,R) \times (-T,T)$ be an open rectangle in $\mathbb{R}^2$. Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there exist constants $s_0 = s_0(\lambda) > 0$ and $C = C(s_0, \lambda_0, R, T)$ such that*

$$\int_Q (s|\nabla y|^2 + s^3 y^2) e^{2s\varphi} dx \le C \int_Q |Py|^2 e^{2s\varphi} dx \tag{2.3}$$

for all $s > s_0(\lambda)$, provided that

$$\begin{cases} Py \in L^2(Q), \qquad y \in H^1(Q) \\ y(0,\cdot) = y(R,\cdot) = 0 \qquad in\ L^2(-T,T) \\ \partial_{x_1} y(0,\cdot) = \partial_{x_1} y(R,\cdot) = 0 \qquad in\ H^{-\frac{1}{2}}(-T,T) \\ y(\cdot,T) = y(\cdot,-T) = 0 \qquad in\ L^2(0,R) \\ \partial_{x_2} y(\cdot,T) = \partial_{x_2} y(\cdot,-T) = 0 \qquad in\ H^{-\frac{1}{2}}(0,R). \end{cases} \tag{2.4}$$

Applying the above Carleman estimate, we can show a proposition about some non-existence of an H^2- solution to a Cauchy problem of the Laplace equation. This proposition plays the essential role in proving our theorems.

Proposition 2.2. *By D, let us denote an interior of a triangle $\triangle AOB$ that has three vertices O (the origin), A, $B \in \mathbb{R}^2$, and by Γ, the union of the edges $\overline{OA}$ and $\overline{OB}$ of $\triangle AOB$.*

Let $G \in H^1(D)$ and G be strictly positive along the edges $\overline{OA}$ and $\overline{OB}$. Then there exists no solution $y \in H^2(D)$ to

$$\begin{cases} \Delta y = G \qquad in \quad D \\ y = |\nabla y| = 0 \qquad on \quad \Gamma. \end{cases} \tag{2.5}$$

Remark 2.1. Within $y \in C^2(\overline{D})$, the proof of the proposition is straightforward. That is, let $A = (a_1, a_2)$, $a_2 \neq 0$, and $B = (b_1, 0)$. Then $y(a_1 t, a_2 t) = 0$ and $y(b_1 t, 0) = \partial_{x_2} y(b_1 t, 0) = 0$ for $0 \le t \le t_0$: some constant. Therefore

$$(\partial_{x_1}\partial_{x_2} y)(b_1 t, 0) = (\partial_{x_1}^2 y)(b_1 t, 0) = 0$$

and

$$0 = \frac{d^2 y(a_1 t, a_2 t)}{dt^2} = a_1^2 (\partial_{x_1}^2 y)(a_1 t, a_2 t) + 2a_1 a_2 (\partial_{x_1}\partial_{x_2} y)(a_1 t, a_2 t) + a_2^2 (\partial_{x_2}^2 y)(a_1 t, a_2 t)$$

for $0 \le t \le t_0$. Hence, by $y \in C^2(\overline{D})$, we have

$$\partial_{x_1}^2 y(0,0) = \partial_{x_2}^2 y(0,0) = \partial_{x_1}\partial_{x_2} y(0,0) = 0,$$

so that $\Delta y(0,0) = G(0,0) = 0$, which contradicts that $G > 0$ on $\overline{OB}$.

However the non-existence within $C^2(\overline{D})$ is not helpful for the proofs of our theorems.

Remark 2.2. In Proposition 2.2, it is essential that $\overline{OA}$ and $\overline{OB}$ intersects at O transversally. In fact, in the case where a curve $\Gamma = OA \cup OB$ is smooth at O, there may exist a solution $y \in H^2(D)$ for some $G \in L^2(D)$ with $\partial_{x_2} G \in L^2(D)$. We note that the example with $G, \partial_{x_2} G \in L^2(D)$ is sufficient as a counterexample against the non-existence. In fact, as is seen from the proof below, in the case where $\overline{OB} \subset \{(x_1, x_2) | x_2 = 0\}$, we will use only the regularity G, $\partial_{x_2} G \in L^2(D)$ for the non-existence. In other words, $G \in H^1(D)$ is a superfluous assumption in the proposition.

Example for existence for a smooth curve Γ. Let

$$\begin{aligned} D = & \left\{(x_1, x_2) |\, 0 \le x_1 < \frac{1}{2},\, 0 < x_2 < -\frac{1}{4}\left(x_1 - \frac{1}{2}\right)\right\} \\ \cup & \left\{(x_1, x_2) |\, -\frac{1}{2} < x_1 < 0,\, x_1^2 < x_2 < -\frac{1}{4}\left(x_1 - \frac{1}{2}\right)\right\} \end{aligned}$$

and

$$y(x_1, x_2) = \begin{cases} x_2^2, & x_1 \ge 0, \\ (x_1^2 - x_2)^2, & x_1 < 0, \end{cases}$$

$$G(x_1, x_2) = \begin{cases} 2, & x_1 \ge 0, \\ 2 - 4x_2 + 12x_1^2, & x_1 < 0. \end{cases}$$

We regard $\left\{(x_1, 0) | 0 \le x_1 < \frac{1}{2}\right\} \cup \left\{(x_1, x_1^2) | -\frac{1}{2} < x_1 < 0\right\}$ as Γ. Then the two parts of Γ connect at O smoothly. Moreover we can directly verify that $y \in C^1(\overline{D}) \cap H^2(D)$, $G, \partial_{x_2} G \in L^2(D)$, $y = |\nabla y| = 0$ on Γ and $\Delta y = G > 0$ in D.

Proof of Proposition 2.2. Suppose that $y \in H^2(D)$ satisfies (2.5). Let $A := (a_1, a_2)$ and $B := (b_1, b_2)$. We can take a suitable rotation and a shorter edge as $\overline{OA}$, if necessary, so that we may assume that $a_2 > 0$, $a_1 < b_1$ and $b_2 = 0$. Let us denote the angle $\measuredangle AOB$ by θ. We consider two cases: $0 < \theta \le \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \pi$.

Case: $0 < \theta \le \frac{\pi}{2}$**.** Let $A_- = (a_1, -a_2)$,

$$D_- := \{(x_1, -x_2) \in \mathbb{R}^2 | (x_1, x_2) \in D\}$$

and

$$D_E = D \cup D_- \cup \{(x_1, 0) | 0 < x_1 < b_1\}.$$

We extend the function y in D by the formula $y(x_1, x_2) := y(x_1, -x_2)$ for all $(x_1, x_2) \in D_-$. The equality $y(x_1, 0) = \partial_{x_2} y(x_1, 0) = 0$ yields

$$y \in H^2(D_E). \tag{2.6}$$

Moreover we extend the function $\partial_{x_2}G$ in D to a function in D_E as the even function in x_2 and denote the extension by the same symbol $\partial_{x_2}G$, because there is no fear of confusion. Then

$$\partial_{x_2}G \in L^2(D_E). \tag{2.7}$$

Let $Q(\varepsilon) := \{(x_1,x_2) \in \mathbb{R}^2 | \varphi(x_1,x_2) > \varepsilon\}$ for $\varepsilon > 0$. We choose $\varepsilon_0 \in (0,1)$ and a sufficiently large $\lambda > 0$ such that $\frac{1}{\lambda}\log\frac{1}{\varepsilon_0} < b_1$ and $\frac{1}{\beta\lambda}\log\frac{1}{\varepsilon_0} < a_2^2$. Then the boundary $\partial Q(\varepsilon_0)$ passes through the edge $\overline{OA}$ but does not through the edge $\overline{AB}$. Fix ε_1, $\varepsilon_2 > 0$ with $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < 1$. By the definition, we can find $Q(\varepsilon_2) \subsetneqq Q(\varepsilon_1) \subsetneqq Q(\varepsilon_0)$.

In order to apply Proposition 2.1, we have to introduce a cut-off function χ satisfying $0 \le \chi \le 1$, $\chi \in C^\infty(\mathbb{R}^2)$, and

$$\chi(x) := \begin{cases} 1 & \text{if} \quad x \in Q(\varepsilon_1) \\ 0 & \text{if} \quad x \in \mathbb{R}^2 \setminus \overline{Q(\varepsilon_0)}. \end{cases} \tag{2.8}$$

We set

$$z := (\partial_{x_2}y)e^{s\varphi}\chi \in H^1(D_E). \tag{2.9}$$

By (2.5), the function z satisfies the equation

$$\begin{aligned} \Delta z = \ & (\partial_{x_2}G)e^{s\varphi}\chi + 2s\nabla\varphi\cdot\nabla z + sz(\Delta\varphi) - s^2 z|\nabla\varphi|^2 \\ & + 2e^{s\varphi}\nabla(\partial_{x_2}y)\cdot\nabla\chi + (\partial_{x_2}y)e^{s\varphi}(\Delta\chi) \end{aligned} \tag{2.10}$$

in D_E. In fact,

$$\partial_{x_1}z = (\partial_{x_1}\partial_{x_2}y)e^{s\varphi}\chi + s(\partial_{x_1}\varphi)z + (\partial_{x_2}y)e^{s\varphi}\partial_{x_1}\chi,$$

that is,

$$(\partial_{x_1}\partial_{x_2}y)e^{s\varphi}\chi = \partial_{x_1}z - s(\partial_{x_1}\varphi)z - (\partial_{x_2}y)e^{s\varphi}\partial_{x_1}\chi.$$

Therefore

$$\begin{aligned} & \partial_{x_1}^2 z = (\partial_{x_1}^2\partial_{x_2}y)e^{s\varphi}\chi + s(\partial_{x_1}\varphi)(\partial_{x_1}\partial_{x_2}y)e^{s\varphi}\chi + (\partial_{x_1}\partial_{x_2}y)e^{s\varphi}\partial_{x_1}\chi \\ + \quad & s(\partial_{x_1}^2\varphi)z + s(\partial_{x_1}\varphi)(\partial_{x_1}z) + (\partial_{x_1}\partial_{x_2}y)e^{s\varphi}\partial_{x_1}\chi + (\partial_{x_2}y)s(\partial_{x_1}\varphi)e^{s\varphi}\partial_{x_1}\chi \\ + \quad & (\partial_{x_2}y)e^{s\varphi}\partial_{x_1}^2\chi \\ = \quad & (\partial_{x_1}^2\partial_{x_2}y)e^{s\varphi}\chi + s(\partial_{x_1}\varphi)(\partial_{x_1}z - s(\partial_{x_1}\varphi)z - (\partial_{x_2}y)e^{s\varphi}\partial_{x_1}\chi) \\ + \quad & (\partial_{x_1}\partial_{x_2}y)e^{s\varphi}\partial_{x_1}\chi + s(\partial_{x_1}^2\varphi)z + s(\partial_{x_1}\varphi)(\partial_{x_1}z) \\ + \quad & (\partial_{x_1}\partial_{x_2}y)e^{s\varphi}\partial_{x_1}\chi + (\partial_{x_2}y)s(\partial_{x_1}\varphi)e^{s\varphi}\partial_{x_1}\chi \\ + \quad & (\partial_{x_2}y)e^{s\varphi}\partial_{x_1}^2\chi \\ = \quad & (\partial_{x_1}^2\partial_{x_2}y)e^{s\varphi}\chi + 2s(\partial_{x_1}\varphi)(\partial_{x_1}z) - s^2(\partial_{x_1}\varphi)^2 z \\ + \quad & s(\partial_{x_1}^2\varphi)z + 2(\partial_{x_1}\partial_{x_2}y)e^{s\varphi}\partial_{x_1}\chi + (\partial_{x_2}y)e^{s\varphi}\partial_{x_1}^2\chi. \end{aligned}$$

Similarly we have

$$\begin{aligned} & \partial_{x_2}^2 z = (\partial_{x_2}^3 y)e^{s\varphi}\chi + 2s(\partial_{x_2}\varphi)(\partial_{x_2} z) - s^2(\partial_{x_2}\varphi)^2 z \\ + \quad & s(\partial_{x_2}^2\varphi)z + 2(\partial_{x_2}^2 y)e^{s\varphi}\partial_{x_2}\chi + (\partial_{x_2} y)e^{s\varphi}\partial_{x_2}^2\chi. \end{aligned}$$

Therefore (2.10) is seen.

In particular, setting $w = \chi(\partial_{x_2} y)$ and $s = 0$ in (2.9) and (2.10), we have

$$\Delta w = \chi(\partial_{x_2} G) + 2\nabla(\partial_{x_2} y)\cdot\nabla\chi + (\partial_{x_2} y)(\Delta\chi) \tag{2.11}$$

in D_E. Now we will apply Proposition 2.1 to the equation (2.11). Let us take a rectangle $Q := (0, R) \times (-T, T)$ in $\mathbb{R}^2$ containing D_E and extend the functions w and $\partial_{x_2} G$ in Q by defining $w = \partial_{x_2} G = 0$ in $Q \setminus \overline{D_E}$. By (2.5), (2.6), (2.8), and (2.11), we see that the extension $w \in H^1(Q)$ satisfies all the conditions in Proposition 2.1. Hence by Proposition 2.1 and the definition of the extension w, we obtain

$$\begin{aligned} & \int_{D_E} (s|\nabla w|^2 + s^3 w^2)e^{2s\varphi}dx \\ \le \quad & C\int_{D_E} |\partial_{x_2} G|^2 e^{2s\varphi}\chi^2 dx + C\int_{D_E} \{2\nabla(\partial_{x_2} y)\cdot\nabla\chi + (\partial_{x_2} y)(\Delta\chi)\}^2 e^{2s\varphi}dx. \end{aligned} \tag{2.12}$$

By (2.8), we have

$$\begin{aligned} & \left|\int_{D_E} \{2\nabla(\partial_{x_2} y)\cdot\nabla\chi + (\partial_{x_2} y)(\Delta\chi)\}^2 e^{2s\varphi}dx\right| \\ = \quad & \left|\int_{Q(\varepsilon_0)\setminus Q(\varepsilon_1)} \{2\nabla(\partial_{x_2} y)\cdot\nabla\chi + (\partial_{x_2} y)(\Delta\chi)\}^2 e^{2s\varphi}dx\right| \\ \le \quad & Ce^{2s\varepsilon_1}\|y\|^2_{H^2(D_E)}. \end{aligned} \tag{2.13}$$

Noting that $z = we^{s\varphi}$, we have

$$s^3 z^2 = s^3 w^2 e^{2s\varphi}, \quad s|\nabla z|^2 = s|\nabla w + sw\nabla\varphi|^2 e^{2s\varphi} \le C(s|\nabla w|^2 + s^3 w^2)e^{2s\varphi}. \tag{2.14}$$

Therefore, by (2.12) - (2.14), we obtain

$$\begin{aligned} & \int_{D_E} (s|\nabla z|^2 + s^3 z^2)dx \\ \le \quad & C\int_{D_E} |\partial_{x_2} G|^2 e^{2s\varphi}\chi^2 dx + Ce^{2s\varepsilon_1}\|y\|_{H^2(D_E)}. \end{aligned} \tag{2.15}$$

Take a vector $\overrightarrow{d} = (a, 1)$ parallel to the vector $(a_1, -a_2)$. We multiply (2.10) by $\nabla z \cdot \overrightarrow{d}$ and integrate it over D_-

$$\begin{aligned} & \int_{D_-} (\Delta z)(\nabla z\cdot\overrightarrow{d})dx \\ = \quad & \int_{D_-} (\partial_{x_2} G)e^{s\varphi}\chi(\nabla z\cdot\overrightarrow{d})dx \\ & + \int_{D_-} (2s\nabla\varphi\cdot\nabla z - s^2 z|\nabla\varphi|^2 + sz(\Delta\varphi))(\nabla z\cdot\overrightarrow{d})dx \\ & + \int_{D_-} \{2(\nabla(\partial_{x_2} y)\cdot\nabla\chi)(\nabla z\cdot\overrightarrow{d}) + (\partial_{x_2} y)(\Delta\chi)(\nabla z\cdot\overrightarrow{d})\}e^{s\varphi}dx. \end{aligned} \tag{2.16}$$

Henceforth $\nu = (\nu_1, \nu_2)$ denotes the unit outward normal vector to ∂D_-.

We denote the left and the right hand sides of (2.16) respectively by I_1 and I_2. We integrate by parts and apply the boundary condition of z, so that we have

$$\begin{aligned}
I_1 &= \int_{D_-}(a(\partial_{x_1}^2 z)\partial_{x_1}z + a(\partial_{x_2}^2 z)\partial_{x_1}z + (\partial_{x_1}^2 z)\partial_{x_2}z + (\partial_{x_2}^2 z)\partial_{x_2}z)dx \\
&= \tfrac{1}{2}\int_{D_-}\nabla(|\nabla z|^2)\cdot \overrightarrow{d}\,dx \\
&+ \int_{D_-}\{a((\partial_{x_2}^2 z)\partial_{x_1}z - (\partial_{x_2}z)\partial_{x_1}\partial_{x_2}z) + (\partial_{x_1}^2 z)\partial_{x_2}z - (\partial_{x_1}z)\partial_{x_1}\partial_{x_2}z\}dx \\
&= \tfrac{1}{2}\int_{D_-}\nabla(|\nabla z|^2)\cdot \overrightarrow{d}\,dx - 2\int_{D_-}(a(\partial_{x_2}z)\partial_{x_1}\partial_{x_2}z + (\partial_{x_1}z)\partial_{x_1}\partial_{x_2}z)dx \\
&\quad + \int_{\partial D_-}(a\nu_2+\nu_1)(\partial_{x_1}z)\partial_{x_2}z d\sigma \\
&= \tfrac{1}{2}\int_{\partial D_-}|\nabla z|^2\overrightarrow{d}\cdot\nu d\sigma - \int_{D_-}\{a\partial_{x_1}(|\partial_{x_2}z|^2) + \partial_{x_2}(|\partial_{x_1}z|^2)\}dx \\
&\quad + \int_{\partial D_-}(a\nu_2+\nu_1)(\partial_{x_1}z)\partial_{x_2}z d\sigma \\
&= \tfrac{1}{2}\int_{\overline{OB}}|\partial_{x_2}z|^2 d\sigma - \int_{\partial D_-}(a\nu_1(\partial_{x_2}z)^2 + \nu_2(\partial_{x_1}z)^2)d\sigma \\
&\quad + \int_{\partial D_-}(a\nu_2+\nu_1)(\partial_{x_1}z)\partial_{x_2}z d\sigma \\
&= \tfrac{1}{2}\int_{\overline{OB}}|\partial_{x_2}z|^2 d\sigma + \int_{\overline{OB}\cup\overline{OA_-}}(\partial_{x_1}z - a\partial_{x_2}z)(\nabla z\cdot(-\nu_2,\nu_1))d\sigma.
\end{aligned} \tag{2.17}$$

Here $\int_{\overline{OB}\cup\overline{OA_-}}\cdots d\sigma$ is the line integral with the orientation $B\to O\to A_-$.

By the boundary condition of z, we have $\nabla z = \frac{\partial z}{\partial\nu}(\nu_1,\nu_2)$ on $\overline{OA_-}$ and so $\nabla z\cdot(-\nu_2,\nu_1) = 0$ on $\overline{OA_-}$. Consequently

$$\int_{\overline{OB}\cup\overline{OA_-}}(\partial_{x_1}z - a\partial_{x_2}z)(\nabla z\cdot(-\nu_2,\nu_1))d\sigma = 0. \tag{2.18}$$

Therefore, by (2.5), (2.9), (2.17) and (2.18), we have

$$I_1 = \frac{1}{2}\int_0^{b_1}|\partial_{x_2}^2 y(x_1,0)|^2 e^{2s\varphi}\chi^2 dx_1 = \frac{1}{2}\int_0^{b_1}|G(x_1,0)|^2 e^{2s\varphi}\chi^2 dx_1. \tag{2.19}$$

Furthermore, by the Cauchy-Bunyakovskii inequality, we have

$$\begin{aligned}
I_2 \le\ & C\int_{D_-}|\partial_{x_2}G|^2 e^{2s\varphi}\chi^2 dx + C\int_{D_-}(s|\nabla z|^2 + s^3 z^2)dx \\
&+C\int_{D_-}\{2\nabla(\partial_{x_2}y)\cdot\nabla\chi + (\partial_{x_2}y)(\Delta\chi)\}^2 e^{2s\varphi}dx.
\end{aligned} \tag{2.20}$$

Application of (2.8) and (2.15) yields

$$\begin{aligned}
I_2 &\le C\int_{D_E}|\partial_{x_2}G|^2 e^{2s\varphi}\chi^2 dx + Ce^{2s\varepsilon_1}\|y\|^2_{H^2(D_E)} \\
&+ C\int_{Q(\varepsilon_0)\setminus Q(\varepsilon_1)}\{|\nabla(\partial_{x_2}y)|^2 + |\partial_{x_2}y|^2\}e^{2s\varphi}dx \\
&\le C\int_{D_E}|\partial_{x_2}G|^2 e^{2s\varphi}\chi^2 dx + Ce^{2s\varepsilon_1}\|y\|^2_{H^2(D_E)}.
\end{aligned} \tag{2.21}$$

Consequently (2.19) and (2.21) imply

$$\begin{aligned}
&\frac{1}{2}\int_0^{b_1}|G(x_1,0)|^2 e^{2s\varphi(x_1,0)}\chi^2(x_1,0)dx_1 \\
\le\ & C\int_{D_E}|\partial_{x_2}G|^2 e^{2s\varphi}\chi^2 dx + Ce^{2s\varepsilon_1}\|y\|^2_{H^2(D_E)}.
\end{aligned}$$

Moreover, by (2.8), we have

$$\begin{aligned}
&\frac{1}{2}\int_0^{b_1}|G(x_1,0)|^2e^{2s\varphi(x_1,0)}\chi^2(x_1,0)dx_1\\
\ge\ &\frac{1}{2}\int_{(0,b_1)\cap Q(\varepsilon_1)}|G(x_1,0)|^2e^{2s\varphi(x_1,0)}\chi^2(x_1,0)dx_1\\
=\ &\frac{1}{2}\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}}|G(x_1,0)|^2e^{2s\varphi(x_1,0)}\chi^2(x_1,0)dx_1\\
=\ &\frac{1}{2}\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}}|G(x_1,0)|^2e^{2s\varphi(x_1,0)}dx_1
\end{aligned}$$

and

$$\begin{aligned}
&\int_{D_E}|\partial_{x_2}G|^2e^{2s\varphi}\chi^2dx=\left(\int_{D_E\cap Q(\varepsilon_1)}+\int_{D_E\setminus Q(\varepsilon_1)}\right)|\partial_{x_2}G|^2e^{2s\varphi}\chi^2dx\\
\le\ &\int_{-a_2}^{a_2}\left(\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}}|\partial_{x_2}G|^2e^{2s\varphi}dx_1\right)dx_2+Ce^{2s\varepsilon_1}\|\partial_{x_2}G\|^2_{L^2(D_E)}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\frac{1}{2}\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}}|G(x_1,0)|^2e^{2s\varphi(x_1,0)}dx_1\\
\le\ &C\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}}\left(\int_{-a_2}^{a_2}|\partial_{x_2}G|^2e^{2s\varphi}dx_2\right)dx_1+Ce^{2s\varepsilon_1}(\|\partial_{x_2}G\|^2_{L^2(D_E)}+\|y\|^2_{H^2(D_E)}).
\end{aligned}\tag{2.22}$$

Next we will estimate the first term of the right hand side of (2.22). Define the function g_0 by

$$g_0(x_2)|G(x_1,0)|=|\partial_{x_2}G(x_1,x_2)|,\qquad (x_1,x_2)\in(-a_2,a_2)\times(0,b_1).\tag{2.23}$$

Since G is strictly positive on $\overline{OB}$ and $\partial_{x_2}G\in L^2(D_E)$, we can find that g_0 is well-defined and belongs to $L^2(-a_2,a_2)$. By (2.23), we have

$$\begin{aligned}
&\textstyle\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}}\left(\int_{-a_2}^{a_2}|\partial_{x_2}G|^2\exp(2se^{-\lambda x_1-\lambda\beta x_2{}^2})dx_2\right)dx_1\\
\le\ &\textstyle\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}}\left\{\int_{-a_2}^{a_2}|g_0(x_2)|^2\exp(2se^{-\lambda x_1}(e^{-\lambda\beta x_2{}^2}-1))dx_2\right\}|G(x_1,0)|^2e^{2s\varphi(x_1,0)}dx_1\\
\le\ &\textstyle\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}}\left(\int_{-a_2}^{a_2}\eta_s(x_2)dx_2\right)|G(x_1,0)|^2e^{2s\varphi(x_1,0)}dx_1.
\end{aligned}\tag{2.24}$$

Here and henceforth, we define a function η_s in x_2 by

$$\eta_s(x_2):=(g_0(x_2))^2\exp(2se^{-\lambda b_1}(e^{-\lambda\beta x_2{}^2}-1)).$$

Then, by (2.23), we see that $\eta_s \in L^1(-a_2, a_2)$, and $\lim_{s\to\infty} \eta_s(x_2) = 0$ for $x_2 \neq 0$ and

$$|\eta_s(\cdot)| \le |g_0(\cdot)|^2 \in L^1(-a_2, a_2).$$

Hence the Lebesgue convergence theorem implies

$$\int_{-a_2}^{a_2} \eta_s(x_2)dx_2 = o(1) \qquad \text{as} \quad s \to \infty. \tag{2.25}$$

Hence, (2.22), (2.24) and (2.25) yield

$$\begin{aligned} &\tfrac{1}{2}\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}} |G(x_1,0)|^2 e^{2s\varphi(x_1,0)}dx_1 \\ \le\ & o(1)\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}} |G(x_1,0)|^2 e^{2s\varphi(x_1,0)}dx_1 + Ce^{2s\varepsilon_1}(\|y\|^2_{H^2(D_E)} + \|\partial_{x_2}G\|^2_{L^2(D_E)}) \end{aligned} \tag{2.26}$$

as $s \to \infty$. Hence, by $Q(\varepsilon_2) \subset Q(\varepsilon_1)$, we obtain

$$\begin{aligned} & (\tfrac{1}{2} - o(1))e^{2s\varepsilon_2}\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_2}} |G(x_1,0)|^2 dx_1 \\ \le\ & (\tfrac{1}{2} - o(1))\int_{Q(\varepsilon_2)\cap(0,b_1)} |G(x_1,0)|^2 e^{2s\varphi(x_1,0)}dx_1 \\ \le\ & (\tfrac{1}{2} - o(1))\int_{Q(\varepsilon_1)\cap(0,b_1)} |G(x_1,0)|^2 e^{2s\varphi(x_1,0)}dx_1 \\ =\ & (\tfrac{1}{2} - o(1))\int_0^{\frac{1}{\lambda}\log\frac{1}{\varepsilon_1}} |G(x_1,0)|^2 e^{2s\varphi(x_1,0)}dx_1 \\ \le\ & Ce^{2s\varepsilon_1}(\|y\|^2_{H^2(D_E)} + \|\partial_{x_2}G\|^2_{L^2(D_E)}). \end{aligned}$$

Since $\varepsilon_1 < \varepsilon_2$, as $s \to \infty$, we have $G(x_1, 0) = 0$, $0 < x_1 < \frac{1}{\lambda}\log\frac{1}{\varepsilon_2}$. This contradicts that $G(x_1, 0) \neq 0$ for $0 < x_1 < b_1$.

Case: $\frac{\pi}{2} < \theta < \pi$**.** To orthogonalize the triangle D, we introduce a transformation Ψ from the x_1x_2- plane into the $\eta_1\eta_2-$plane

$$\Psi(x_1, x_2) := (a_2x_1 - a_1x_2, \sqrt{a_1^2 + a_2^2}x_2). \tag{2.27}$$

Then the transformation Ψ maps our obtuse-angle triangle D onto a right-angle one $\Psi(D)$. Here we note that the vertices of $\Psi(D)$ are $A' = (0, \sqrt{a_1^2 + a_2^2}a_2)$, O, $B' = (a_2b_1, 0)$.

Defining $Y(\eta_1, \eta_2) := y \circ \Psi^{-1}(\eta_1, \eta_2)$ in $\Psi(D)$, we see that the function Y satisfies

$$\begin{cases} Y \in H^2(\Psi(D)) \\ \partial_{\eta_1}^2 Y + \partial_{\eta_2}^2 Y - \frac{2a_1}{\sqrt{a_1^2+a_2^2}}\partial_{\eta_1}\partial_{\eta_2}Y = \frac{1}{a_1^2+a_2^2}G \circ \Psi^{-1} \qquad \text{in} \quad \Psi(D) \\ Y = \nabla Y = 0 \qquad \text{on} \quad \Psi(\Gamma). \end{cases} \tag{2.28}$$

Here we note that $\alpha \equiv \frac{-2a_1}{\sqrt{a_1^2+a_2^2}}$ satisfies $|\alpha| < 2$ by $a_2 > 0$.

Repeating the previous calculations for the right-angle case, we are led to a contradiction, which implies that there is no solution in $H^2(\Psi(D))$ of (2.28). Thus the proof of Proposition 2.2 is complete.

Remark. The proof of the proposition is inspired by [11] which treats a different inverse problem by a Carleman estimate.

3. Proof of Theorem 1.1

Let us define $y := u_1 - u_2$ in Ω. Then by (1.1) and (1.5), the function y satisfies

$$\Delta y = 0 \quad \text{in} \quad \Omega \setminus (\overline{D_1 \cup D_2}), \tag{3.1}$$

$$\Delta y = qu_1 > 0 \quad \text{in} \quad D_1 \setminus \overline{D_2}, \tag{3.2}$$

$$\Delta y = -qu_2 < 0 \quad \text{in} \quad D_2 \setminus \overline{D_1}, \tag{3.3}$$

$$\Delta y = qy \quad \text{in} \quad D_1 \cap D_2, \tag{3.4}$$

$$y = |\nabla y| = 0 \quad \text{on} \quad \partial\Omega. \tag{3.5}$$

Henceforth F is the component of $\Omega \setminus (\overline{D_1 \cup D_2})$ which is connected with $\partial\Omega$. Since y is harmonic in $\Omega \setminus (\overline{D_1 \cup D_2})$ and $y = \frac{\partial y}{\partial \nu} = 0$ on $\partial\Omega$, the unique continuation (e.g., [6], [8]) implies that

$$y \equiv 0 \quad \text{on} \quad \overline{F}. \tag{3.6}$$

Then we note that

$$\begin{gathered} \text{If } \overline{D}, \overline{E} \subset \Omega \text{ are convex polygons and } D \neq E, \text{ then there exists} \\ \text{a vertex } O \text{ of } D \text{ such that } O \in \Omega \setminus \overline{E} \\ \text{or a vertex } O \text{ of } E \text{ such that } O \in \Omega \setminus \overline{D}. \end{gathered} \tag{3.7}$$

In fact, we contrarily suppose that the conclusion is not true. Then any vertex of D is in $\overline{E}$ and any vertex of E is in $\overline{D}$. By the convexity of D and E, this means that $\overline{D} \subset \overline{E}$ and $\overline{E} \subset \overline{D}$. Therefore $\overline{D} = \overline{E}$, which contradicts that $D \neq E$.

Moreover

$$\text{If } \overline{D}, \overline{E} \subset \Omega \text{ are convex polygons, then } \Omega \setminus \overline{(D \cup E)} \text{ is connected.} \tag{3.8}$$

If $\overline{D} \cap \overline{E} = \emptyset$, then (3.8) is clear. Suppose that $\overline{D} \cap \overline{E} \neq \emptyset$ and fix a point $P \in \overline{D} \cap \overline{E}$. For any two distinct points $A_1, A_2 \in \Omega \setminus \overline{(D \cup E)}$, since $\overline{D}, \overline{E} \subset \Omega$, there exists a small $\epsilon > 0$ such that

$$\overline{D}, \overline{E} \subset \Omega_\epsilon \quad \text{and} \quad A_1, A_2 \in \Omega_\epsilon,$$

where $\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \epsilon\}$. Let $\alpha_j : [0, \infty) \to \mathbb{R}^2, j = 1, 2$, be the straight line such that

$$\alpha_j(0) = P, \quad \alpha_j(1) = A_j, \quad \text{and} \quad \lim_{t \to \infty} |\alpha_j(t)| = \infty.$$

Since $P \in \overline{D} \cap \overline{E}$ and $A_1, A_2 \in \Omega \setminus \overline{(D \cup E)}$, the convexity of $\overline{D}$ and $\overline{E}$ implies

$$\overline{(D \cup E)} \cap \alpha_j(1, \infty) = \emptyset \qquad \text{for all} \quad j = 1, 2.$$

Here and henceforth, we set $\alpha_j(1, \infty) = \{\alpha_j(t) | t > 1\}$ and $\alpha_j[c, d] = \{\alpha_j(t) | t \in [c, d]\}$, etc.

Let $t_j = \inf\{t \in (1,\infty) \mid \alpha_j(t) \in \partial\Omega_\epsilon\}$, and let a continuous curve $\alpha : [0,1] \to \partial\Omega_\epsilon$ start at $\alpha_1(t_1)$ and end at $\alpha_2(t_2)$. Setting

$$\beta(t) = \begin{cases} \alpha_1((t_1-1)t+1), & t \in [0,1) \\ \alpha(t-1), & t \in [1,2] \\ \alpha_2(t(1-t_2)+3t_2-2), & t \in (2,3], \end{cases}$$

we see that β is a continuous curve connecting A_1 with A_2 and that

$$\beta[0,3] \subset \Omega \setminus \overline{(D \cup E)}.$$

Therefore $\Omega \setminus \overline{(D \cup E)}$ is connected.

Now we will complete the proof of Theorem 1.1. Assume contrarily that $\mathrm{co}\,(D_1) \neq \mathrm{co}\,(D_2)$. Then, by (3.7), there exists a vertex O of $\mathrm{co}\,(D_1)$ such that $O \in \Omega \setminus \overline{\mathrm{co}\,(D_2)}$ or a vertex O of $\mathrm{co}\,(D_2)$ such that $O \in \Omega \setminus \overline{\mathrm{co}\,(D_1)}$. Without loss of generality, we may assume the former case. Then, since $O \in \Omega \setminus \overline{\mathrm{co}\,(D_2)}$, we can take a sufficiently small triange $\triangle OAB$ such that

$$\overline{OA} \cup \overline{OB} \subset \partial(\mathrm{co}\,(D_1)) \qquad \text{and} \qquad \triangle OAB \subset \mathrm{co}\,(D_1) \setminus \overline{\mathrm{co}\,(D_2)}.$$

By (3.8), we have

$$\overline{OA} \cup \overline{OB} \subset \Omega \setminus (\mathrm{co}\,(D_1) \cup \mathrm{co}\,(D_2)) \subset \overline{F}. \tag{3.9}$$

Any vertex of $\mathrm{co}\,(D_1)$ is a convex vertex of D_1, that is, in a neighbourhood of it, D_1 is convex. Therefore O is a convex vertex of D_1. By $\mathrm{co}\,(D_1) \supset D_1$, we can take $\triangle OA'B'$ such that

$$\overline{OA'} \cup \overline{OB'} \subset \partial D_1 \qquad \text{and} \qquad \triangle OA'B' \subset \triangle OAB.$$

Hence it follows from $\triangle OAB \subset \mathrm{co}\,(D_1)\setminus\overline{\mathrm{co}\,(D_2)}$ that $\triangle OA'B' \subset \mathrm{co}\,(D_1)\setminus\overline{\mathrm{co}\,(D_2)}$. Moreover, by (3.9), we see that $\overline{OA'} \cup \overline{OB'}$ is included in $\overline{F}$. Therefore, by (3.2) and (3.6), we have $\Delta y = qu_1 > 0$ in $\triangle OA'B'$ and $y = |\nabla y| = 0$ on $\overline{OA'} \cup \overline{OB'}$. Again by (1.4), we see that $qu_1 \in H^1(\triangle OA'B')$, and so we apply Proposition 2.2, which yields a contradiction. Hence $\mathrm{co}\,(D_1) = \mathrm{co}\,(D_2)$ follows. Thus the proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.3

Let E be the connected component of $D_1 \cap D_2$ such that $\overline{A_0B_0} \subset \partial E$. Since $\overline{A_0B_0} \subset \partial_{\mathrm{out}}(D_1 \cup D_2)$ and $\Delta y - qy = 0$ in E, the unique continuation implies that

$$y = 0 \quad \text{in } E. \tag{4.1}$$

We represent the boundary ∂D_j, $j = 1,2$, by a continuous curve $\alpha_j : [0,1] \to \partial D_j$ such that α_j is injective in $[0,1)$, $\alpha_j(0) = A_0$, $\alpha_j(\frac{1}{2}) = B_0$, and $\alpha_j(1) = \alpha_j(0)$. Exchanging A_0 with B_0 if necessary, we may assume that the curves α_j are oriented in the positive direction, that is, the outward normal vector to ∂D_j and the oriented tangential vector of ∂D_j form a right-handed system at any point of ∂D_j.

Let

$$a = \inf\{t \in [0,1] |\, \alpha_1(t) \neq \alpha_2(t)\}.$$

Then we note that $\alpha_1(t) = \alpha_2(t)$ if $0 \le t \le a$.

We will prove the theorem by reduction to absurdity. That is, assume that $D_1 \neq D_2$. Then, by $\alpha_1(1/2) = \alpha_2(1/2)$ and $\alpha_1(1) = \alpha_2(1)$, we can take a number $\frac{1}{2} \le a < b \le 1$ such that $\alpha_1(t) \neq \alpha_2(t)$ for $t \in (a,b)$ and $\alpha_1(b) = \alpha_2(b)$.

Since $\alpha_1(t) = \alpha_2(t)$ for $0 \le t \le a$ and $\alpha_1(t) \neq \alpha_2(t)$ for $t \in (a,b)$, the point $\alpha_1(a)$ is a vertex of D_1 or a vertex of D_2. Therefore we see that $\alpha_1(a,b)$ is outside $\overline{D_2}$ or $\alpha_2(a,b)$ is outside $\overline{D_1}$. Therefore either $\alpha_1[a,b]$ or $\alpha_2[a,b]$ is on $\partial_{\rm out}(D_1 \cup D_2)$.

In fact, let $\alpha_1(a,b)$ be outside $\overline{D_2}$. For any $x \in \alpha_1[a,b]$, there exists a continuous curve γ_1 connecting x and some $y \in \alpha_1[0,\frac{1}{2}]$ such that $\gamma_1 \setminus \{x,y\} \subset \Omega \setminus \overline{(D_1 \cup D_2)}$. Since $\alpha_1[0,\frac{1}{2}] \subset \partial_{\rm out}(D_1 \cup D_2)$, we can take a continuous curve γ_2 connecting y and some $x_0 \in \partial\Omega$ such that $\gamma_2 \setminus \{y\} \subset \Omega \setminus \overline{(D_1 \cup D_2)}$. Hence we can choose a continuous curve γ such that γ is sufficiently close to $\gamma_1 \cup \gamma_2$, $\gamma \subset \Omega \setminus \overline{(D_1 \cup D_2)}$ and γ connects x and x_0. Thus $\alpha_1[a,b] \subset \partial_{\rm out}(D_1 \cup D_2)$.

Without loss of generality, we may assume that

$$\alpha_1[a,b] \text{ is contained in } \partial_{\rm out}(D_1 \cup D_2). \tag{4.2}$$

Let $a < t_1^j < \cdots < t_{k_j}^j < b$, $j = 1,2$, be a partition of $[a,b]$ such that $\alpha_j(t_1^j), \cdots, \alpha_j(t_{k_j}^j)$ are all the vertices of D_j on $\alpha_j(a,b)$.

We will claim that

$$\alpha_1(a) \text{ is a vertex of both } D_1 \text{ and } D_2. \tag{4.3}$$

In fact, since $\alpha_1(t) = \alpha_2(t)$ for $t \in [0,a]$ and $\alpha_1(t) \neq \alpha_2(t)$ for $t \in (a,b)$, the point $\alpha_1(a)$ can not be simultaneously on an edge of D_1 and on an edge of D_2. Here and henceforth, by an edge, we mean that it does not contain any vertices.

Moreover, if $\alpha_1(a)$ is on an edge of one domain and is a vertex of the other, then, in terms of (4.2), we can take a triangle $\triangle\alpha_2(t_1^2)\alpha_1(a)\alpha_1(t_1^1)$, so that

$$\begin{cases} \overline{\alpha_1(a)\alpha_2(t_1^2)} \subset \partial E, \quad \overline{\alpha_1(a)\alpha_1(t_1^1)} \subset \partial_{\rm out}(D_1 \cup D_2), \\ \text{and the interior of this triangle is contained in} \quad D_1 \setminus \overline{D_2}. \end{cases} \tag{4.4}$$

By (4.1) and (3.6), we apply Proposition 2.2 to be led to a contradiction. Thus we have proved (4.3).

We choose small $\varepsilon > 0$, so that $\alpha_1(t) = \alpha_2(t)$ is on an edge of D_j, $j = 1,2$, for $t \in [a-\varepsilon, a]$. Furthermore we can take a suitable rotation, if necessary, so that $\alpha_1(t)$ is on the x_1- axis for $t \in [a-\varepsilon, a]$ and the x_1- component of $\alpha_1(a-\varepsilon)$ is smaller than the one of $\alpha_1(a)$. Then, by the orientation of α_1 and α_2, the domains D_1 and D_2 are located in the upper half plane $\mathbb{R}^2_+ := \{(x_1,x_2)|x_2 > 0\}$ locally near the edge $\overline{\alpha_1(a-\varepsilon)\alpha_1(a)}$.

Furthermore

$$\text{the edge } \overline{\alpha_1(a)\alpha_1(t_1^1)} \text{ lies in the lower half plane } \mathbb{R}^2_- \\ := \{(x_1, x_2)| x_2 < 0\} \text{ and the edge } \overline{\alpha_1(a)\alpha_2(t_1^2)} \text{ in } \mathbb{R}^2_+. \tag{4.5}$$

In fact, assume contrarily. Then, by (4.3), we alternatively have two cases:

$$\text{(i)} \quad \overline{\alpha_1(a)\alpha_1(t_1^1)} \subset \mathbb{R}^2_+, \quad \overline{\alpha_1(a)\alpha_2(t_1^2)} \subset \mathbb{R}^2_-.$$

$$\text{(ii)} \quad \overline{\alpha_1(a)\alpha_1(t_1^1)} \cup \overline{\alpha_1(a)\alpha_2(t_1^2)} \subset \mathbb{R}^2_+ \quad \text{or} \quad \mathbb{R}^2_-.$$

The case (i) is impossible. Because the domains D_1 and D_2 are located in $\mathbb{R}^2_+$ locally near $\overline{\alpha_1(a-\varepsilon)\alpha_1(a)}$, and so, if (i) occurs, then $\overline{\alpha_1(a)\alpha_1(t_1^1)} \subset \partial_{\rm out}(D_1 \cup D_2)$ does not hold. This contradicts (4.2). The case (ii) is impossible also. In fact, assume that the case (ii) occurs. Then, by (4.2), we can take a triangle $\triangle\alpha_2(t_1^2)\alpha_1(a)\alpha_1(t_1^1)$ satisfying (4.4). This is again a contradiction to Proposition 2.2. Hence we have proved (4.5).

By (4.2), we have $\alpha_2[a, b] \subset \partial E$, so that Proposition 2.2 implies that

$$\alpha_1(t_i^1) \notin \mathrm{CV}(D_1) \qquad , \quad i = 1, \cdots, k_1 \tag{4.6}$$

and

$$\alpha_2(t_i^2) \in \mathrm{CV}(E) \qquad , \quad i = 1, \cdots, k_2. \tag{4.7}$$

Here $\mathrm{CV}(D)$ denotes the set of all convex vertices of a polygon D.

We will prove (4.6) and (4.7). In fact, otherwise, there is a vertex $\alpha_1(t_{i_0}^1) \in CV(D_1)$ or $\alpha_2(t_{j_0}^2) \notin CV(E)$. Firstly let $\alpha_1(t_{i_0}^1) \in CV(D_1)$ for some i_0. Then, by (4.2), we can take a triangle $\triangle P_1\alpha_1(t_{i_0}^1)Q_1 \subset D_1 \setminus \overline{D_2}$ such that $y = |\nabla y| = 0$ on the parts $\overline{P_1\alpha_1(t_{i_0}^1)}$ and $\overline{\alpha_1(t_{i_0}^1)Q_1}$ of the edges of D_1. This is a contradiction by Proposition 2.2. Therefore (4.6) has to hold. Secondly let $\alpha_2(t_{j_0}^2) \notin CV(E)$ for some j_0. By (4.1), $y = |\nabla y| = 0$ on the parts $\overline{P_2\alpha_2(t_{j_0}^2)}$ and $\overline{\alpha_2(t_{j_0}^2)Q_2}$ of the edges of D_2. Moreover, by (4.2), we see that $\triangle P_2\alpha_2(t_{j_0}^2)Q_2 \subset D_1 \setminus \overline{D_2}$. This is a contradiction again by Proposition 2.2. Thus the proof of (4.7) is complete.

Let us trace the curves $\Gamma_1 := \alpha_1[a-\varepsilon, b]$ and $\Gamma_2 := \alpha_2[a-\varepsilon, b]$. The both curves coincide from $t = a-\varepsilon$ to $t = a$. By (4.6) and (4.7), the former is oriented clockwise, while the latter counterclockwise. By $\alpha_1(b) = \alpha_2(b)$, the curves $(-\Gamma_1) \cup \Gamma_2 \setminus \alpha_1[a-\varepsilon, a]$ is a closed curve and surrounds a polygon $\widetilde{D}$. Here we regard $-\Gamma_1$ as a curve oriented from $\alpha_1(b)$ to $\alpha_1(a-\varepsilon)$. Moreover the intersection of $\widetilde{D}$ and some neighbourhood of Γ_2 is in D_2, while the intersection of $\Omega \setminus \overline{\widetilde{D}}$ and some neighbourhood of $-\Gamma_1$ is in D_1 (Figure 3). Therefore Γ_1 cannot be connected to $\partial\Omega$ by any continuous curve in $\partial_{\rm out}(D_1 \cup D_2)$. In fact, for any $x \in \partial\Omega$ and $\widetilde{x} \in \Gamma_1$, let γ be an arbitrary continuous curve connecting x and $\widetilde{x}$. Then γ must intersect Γ_1 or Γ_2 transversally. If γ intersects Γ_1 transversally, then γ must pass in D_1. If γ intersects Γ_2 transversally, then γ must pass in D_2. Therefore $\gamma \not\subset \Omega \setminus \overline{(D_1 \cup D_2)}$. This contradicts (4.2). Thus, by reduction to absurdity, the proof of Theorem 1.3 is complete.

5. Proof of Theorem 1.4

Without loss of generality, we may assume that $\vec{b} = (1, 0)$. Since $a_2 \neq 0$ by the linear independency of $\vec{a}$ and $\vec{b}$, we can choose $a_2 = 1$. Let us set $\vec{\mu} = (1, -a_2)$. We set $x = (x_1, x_2) \in \mathbb{R}^2$, and

$$t_0 = \min\{x \cdot \vec{\mu} | x \in \overline{D_1}\} \quad \text{and} \quad s_0 = \min\{x \cdot \vec{\mu} | x \in \overline{D_2}\}. \tag{5.1}$$

Here and henceforth, $x \cdot \vec{\mu}$ denotes the scalar product of $x, \vec{\mu} \in \mathbb{R}^2$. Then

$$t_0 = s_0. \tag{5.2}$$

In fact, otherwise, we may assume that $t_0 < s_0$. Then

$$\{x \in D_1 | t_0 < x \cdot \vec{\mu} < s_0\} \subset \Omega \setminus \overline{D_2},$$

and there exists a vertex O of D_1 with $O \cdot \vec{\mu} = t_0$. Therefore we can take a small triangle such that

$$\overline{OA} \cup \overline{OB} \subset \partial D_1 \quad \text{and} \quad \triangle OAB \subset \{x \in D_1 | t_0 < x \cdot \vec{\mu} < s_0\}.$$

We recall that F is the connected component of $\Omega \setminus \overline{(D_1 \cup D_2)}$ with $\partial\Omega$. Then we see that $\overline{OA} \cup \overline{OB} \subset \overline{F}$. Hence, by (3.6), we have $y = |\nabla y| = 0$ on $\overline{OA} \cup \overline{OB}$. In term of (3.2), we apply Proposition 2.2, so that non-existence of y is shown, which is a contradiction. Thus (5.2) has been proved.

Next for $j = 1$, 2, let $q_j = \sup\{x_2 | x = (x_1, x_2) \in \partial D_j \text{ and } x \cdot \vec{\mu} = t_0\}$ and let $P_j = (p_j, q_j)$ be the intersection point of $x_2 = q_j$ and $x \cdot \vec{\mu} = t_0$. If $q_1 \neq q_2$, then we may assume that $q_1 > q_2$. Then we can take a small triangle $\triangle P_1QR$ such that

$$\overline{P_1Q} \cup \overline{P_1R} \subset \partial D_1 \quad \text{and} \quad \triangle P_1QR \subset D_1 \setminus \overline{D_2}.$$

Then $\overline{P_1Q} \cup \overline{P_1R} \subset \overline{F}$, by (3.2) and (3.6), we apply Proposition 2.2, so that non-existence of y is shown, which is a contradiction. Therefore

$$q_1 = q_2. \tag{5.3}$$

The relations (5.2) and (5.3) impliy that $\partial D_1 \cap \partial D_2 \cap \{x \mid x \cdot \vec{\mu} = t_0\}$ must contain a common line segment in $\partial_{\text{out}}(D_1 \cup D_2)$. By Theorem 1.3, we can conclude that $D_1 = D_2$.

Appendix. Proof of Proposition 2.1

We can prove Proposition 2.1 by Theorem 8.3.1 in [6] for example. For this, we set

$$P_m(x, \zeta) = -\zeta_1^2 - \zeta_2^2 - \alpha\zeta_1\zeta_2, \quad \zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2.$$

It is sufficient to verify that if

$$\zeta = \xi + i\tau\nabla\varphi(x) = (\xi_1 - \lambda\tau\varphi i, \xi_2 - 2\beta x_2\lambda\tau\varphi i), \neq 0, \quad x \in \overline{Q},\ \xi \in \mathbb{R}^2,\ \tau \in \mathbb{R} \tag{1}$$

satisfies

$$P_m(x, \zeta) = 0, \tag{2}$$

then

$$\sum_{j,k=1}^{2} \partial_{x_j}\partial_{x_k}\varphi \frac{\partial P_m}{\partial \zeta_j}\overline{\frac{\partial P_m}{\partial \zeta_k}} > 0. \tag{3}$$

By direct calculations, from (2) we obtain

$$\xi_1^2 + \xi_2^2 + \alpha\xi_1\xi_2 = \tau^2\lambda^2\varphi^2(1 + 4\beta^2x_2^2 + 2\alpha\beta x_2) \tag{4}$$

and

$$(2 + 2\alpha\beta x_2)\xi_1 + (4\beta x_2 + \alpha)\xi_2 = 0. \tag{5}$$

Moreover we can directly see

$$\begin{aligned}
&\sum_{j,k=1}^{2} \partial_{x_j}\partial_{x_k}\varphi \frac{\partial P_m}{\partial \zeta_j}\overline{\frac{\partial P_m}{\partial \zeta_k}} \\
= \quad & (\partial_{x_1}^2\varphi)|2\zeta_1 + \alpha\zeta_2|^2 + (\partial_{x_2}^2\varphi)|2\zeta_2 + \alpha\zeta_1|^2 \\
+ \quad & (\partial_{x_1}\partial_{x_2}\varphi)\{(2\zeta_1 + \alpha\zeta_2)\overline{(2\zeta_2 + \alpha\zeta_1)} + (2\zeta_2 + \alpha\zeta_1)\overline{(2\zeta_1 + \alpha\zeta_2)}\} \\
= \quad & \{\lambda^2|2\zeta_1 + \alpha\zeta_2|^2 + 4\lambda^2\beta^2x_2^2|2\zeta_2 + \alpha\zeta_1|^2\}\varphi \\
+ \quad & 2\beta\lambda^2x_2\varphi\{(2\zeta_1 + \alpha\zeta_2)\overline{(2\zeta_2 + \alpha\zeta_1)} + (2\zeta_2 + \alpha\zeta_1)\overline{(2\zeta_1 + \alpha\zeta_2)}\} \\
- \quad & 2\lambda\beta\varphi|2\zeta_2 + \alpha\zeta_1|^2 \\
= \quad & \lambda^2\varphi|2\zeta_1 + \alpha\zeta_2 + 2\beta x_2(2\zeta_2 + \alpha\zeta_1)|^2 \\
- \quad & 2\beta\lambda\varphi|2\zeta_2 + \alpha\zeta_1|^2. \\
= \quad & 4\lambda^4\tau^2\varphi^3(4\beta^2x_2^2 + 2\alpha\beta x_2 + 1)^2 - 2\beta\lambda^3\tau^2\varphi^3(\alpha + 4\beta x_2)^2 \\
- \quad & 2\beta\lambda\varphi(2\xi_2 + \alpha\xi_1)^2.
\end{aligned}$$

Here, for the calculation of $|2\zeta_1 + \alpha\zeta_2 + 2\beta x_2(2\zeta_2 + \alpha\zeta_1)|^2$ at the last equality, we have used (5).

By $|\alpha| < 2$, we have

$$4\beta^2x_2^2 + 2\alpha\beta x_2 + 1 \ge 1 - \frac{\alpha^2}{4} > 0.$$

Therefore

$$\sum_{j,k=1}^{2} \partial_{x_j}\partial_{x_k}\varphi \frac{\partial P_m}{\partial \zeta_j}\overline{\frac{\partial P_m}{\partial \zeta_k}} \ge (4 - \alpha^2)\lambda^4\tau^2\varphi^3 - C\lambda^3\tau^2\varphi^3 - C\lambda\varphi(2\xi_2 + \alpha\xi_1)^2. \tag{6}$$

Here and henceforth $C > 0$, $C_k > 0$, $1 \le k \le 4$, denote constants which are dependent on T, R, α, β, but independent of λ, τ and φ.

Moreover, by (4) and $|\alpha| < 2$, we have

$$\tau^2\lambda^2\varphi^2 = \frac{\xi_1^2 + \xi_2^2 + \alpha\xi_1\xi_2}{4\beta^2x_2^2 + 2\alpha\beta x_2 + 1},$$

and so

$$\frac{\xi_1^2+\xi_2^2+\alpha\xi_1\xi_2}{C_0} \le \tau^2\lambda^2\varphi^2 \le \frac{4(\xi_1^2+\xi_2^2+\alpha\xi_1\xi_2)}{4-\alpha^2}, \qquad x=(x_1,x_2)\in\overline{Q},$$

where we set $C_0 = 4\beta^2T^2 + 2|\alpha|\beta T + 1$. Hence (6) yields

$$\sum_{j,k=1}^{2} \partial_{x_j}\partial_{x_k}\varphi \frac{\partial P_m}{\partial \zeta_j}\overline{\frac{\partial P_m}{\partial \zeta_k}} \ge (C_1\lambda^2\varphi - C_2\lambda\varphi)(\xi_1^2+\xi_2^2+\alpha\xi_1\xi_2) - C\lambda\varphi(2\xi_2+\alpha\xi_1)^2. \quad (7)$$

By the homogeneity, we may assume that

$$\xi_1^1 + \xi_2^2 = 1 \quad (8)$$

or

$$\xi_1 = \xi_2 = 0, \quad \tau \ne 0. \quad (9)$$

Case (8). By (7), we have

$$\sum_{j,k=1}^{2} \partial_{x_j}\partial_{x_k}\varphi \frac{\partial P_m}{\partial \zeta_j}\overline{\frac{\partial P_m}{\partial \zeta_k}}$$

$$\ge C_1\lambda^2\varphi\left(1 - \frac{C_2}{C_1}\frac{1}{\lambda}\right)\min_{\xi_1^2+\xi_2^2=1}|\xi_1^2+\xi_2^2+\alpha\xi_1\xi_2| - C\lambda\varphi(2+|\alpha|)^2. \quad (10)$$

The minimum is not zero, because of $|\alpha| < 2$. Consequently

$$\sum_{j,k=1}^{2} \partial_{x_j}\partial_{x_k}\varphi \frac{\partial P_m}{\partial \zeta_j}\overline{\frac{\partial P_m}{\partial \zeta_k}} \ge C_3\lambda^2\varphi\left(1 - \frac{C_4}{\lambda}\right).$$

Hence, for sufficiently large $\lambda > 0$, we have (3) in the case (8).

Case (9). By (4), we have

$$1 + 4(\beta x_2)^2 + 2\alpha(\beta x_2) = 0.$$

This is impossible for $\beta x_2 \in \mathbb{R}$ because $|\alpha| < 2$. Thus the verification of the conditions of Theorem 8.3.1 in [6] is complete.

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