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Log smooth extension of family of curves and semi-stable reduction

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Summery: We show that a family of smooth stable curves defined on the interior of a log regular scheme is extended to a log smooth scheme over the whole log regular scheme, if it is so at each generic point of the boundary, under a very mild assumption. We also include a proof of the fact that a log smooth scheme over a discrete valuation ring has potentially a semi-stable model. As a consequence, we show that a hyperbolic polycurve in the sense of [9] over a discrete valuation field has potentially a proper semi-stable model if the characteristic of the residue field is sufficiently large.

A. de Jong and F. Oort proved in [4] that a family of smooth stable curves defined on the complement of a divisor with simple normal crossing of a regular scheme is extended uniquely to a family of stable curves over the whole regular scheme, if it is so at each generic point of the boundary. S. Mochizuki generalized their result to the case where the base scheme is assumed a log regular scheme in [9]. In this paper, we study a log smooth extension instead of a stable extension. The main result is the following.

Theorem 1 (See **Corollary 1.4**) *Let Y be a regular noetherian scheme and D_Y be a divisor with normal crossings. Let $f_{U_Y} : X_{U_Y} \to U_Y = Y - D_Y$ be a proper smooth and geometrically connected curve and D_{U_Y} be a divisor of X_{U_Y} finite and etale over U_Y. Let g be the genus of $X_{U_Y} \to U_Y$ and r be the degree of D_{U_Y} over U_Y and assume $2g - 2 + r > 0$. We put $U_X = X_{U_Y} - D_{U_Y}$. We consider the following conditions.*

(1) There exists a projective and regular scheme $f : X \to Y$ extending $X_{U_Y} \to U_Y$ such that $U_X (\subset X_{U_Y}) \subset X$ is the complement of a divisor of X with normal crossings and that the pair (X, U_X) is log smooth over (Y, U_Y).

(2) For a generic point η_i of D_Y, let K_i be the completion of the function field of Y at η_i. Then, for each η_i, there exists a projective and regular scheme $X_{O_{K_i}}$ over O_{K_i} extending the base change $X_{K_i} = X_{U_Y} \times_{U_Y} K_i$ such that $U_{K_i} = U_X \times_{U_Y} K_i (\subset X_{K_i}) \subset X_{O_{K_i}}$ is the complement of a divisor of $X_{O_{K_i}}$ with normal crossings and that the pair $(X_{O_{K_i}}, U_{K_i})$ is log smooth over O_{K_i}.

(3) Let $N \geq 1$ be an integer invertible on Y. The finite etale covering D_{U_Y} of U_Y and the finite group scheme J_{N,U_Y} over U_Y of N-torsion points of the Jacobian J_{U_Y} of X_{U_Y} are tamely ramified at each generic point η_i of D_Y.

Then we have (1)⇒(2) ⇒(3). We have (3)⇒(2) if $N \geq 3$. We have (2)⇒(1) if one of the following conditions (a), (b_1) and (b_2) is satisfied.

(a) 2 is invertible on Y.

(b_1) $2 = 0$ on Y.

(b_2) Y is proper over a discrete valuation ring of residue characteristic 2.

A characterization of log smooth morphisms is recalled in Proposition 1.2.

Applying Theorem 1 to a polycurve over a discrete valuation field, we obtain the following consequence.

Corollary 2 (See **Corollary 1.9**) *Let U be a smooth scheme of dimension n of finite type over a discrete valuation field K of residue characteristic $p > 0$. Assume that there exists a sequence of morphisms of smooth schemes $U = U_n \to U_{n-1} \to \cdots \to U_0 = \mathrm{Spec}\ K$ satisfying the following condition:*

For $0 \leq i < n$, there exist a proper smooth and geometricall connected curve X_{i+1} over U_i and a divisor D_{i+1} of X_{i+1} finite and etale over U_i such that U_{i+1} is the complement $X_{i+1} - D_{i+1}$.

Let g_i be the genus of the curve X_{i+1} and r_i be the degree of D_{i+1} over U_i. Assume further that $2g_i - 2 + r_i > 0, p \geq 2g_i + 2$ and $p > r_i$ for $1 \leq i < n$. Then there exist a finite separable extension L of K, a projective and semi-stable scheme X_{O_L} over O_L and an open immersion $U_L = U \otimes_K L \to X_{O_L}$ such that the pair (X_{O_L}, U_L) is semi-stable over O_L.

In a paper in preparation, we plan to discuss its application to semi-stable reduction of surfaces of various types.

To deduce Corollary 2 from Theorem 1, we show that a log smooth scheme over a discrete valuation ring has potentially a semi-stable reduction.

Theorem 3 (See **Theorem 1.8**) *Let X be a log scheme log smooth and of finite type over a discrete valuation ring O_K and U_X be the interior of X. Then there exist an integer $e \geq 1$ such that, if L is a finite separable extension and the ramification index $e_{L/K}$ is divisible by e, there exists a projective and log etale morphism $f : W \to X \otimes^{\log}_{O_K} O_L$ such that $U_W = f^{-1}(U_{X,L}) \to U_{X,L} = U_X \otimes_K L$ is an isomorphism and (W, U_W) is semi-stable over O_L.*

The terminologies on log schemes in the statement will be recalled in Section 1.1.

The case where $U_X = X_K$ is proved by H.Yoshioka [14]. We give a proof of Theorem 3 which is very close to the original proof of him because it is basic to the application above and his proof is not easily accessible in a written form.

The ideas of the proof of the main results are the following. In the proof of Theorem 1, the essential part is the implication (3)⇒(1). By the extension theorem and the log purity theorem in [9], the assumption (3) implies that there is a finite and log etale Galois covering Y' of Y where the pull-back of X_{U_Y} is extended uniquely to a stable curve $X' \to Y'$. If the quotient X'/G by the Galois group G was log smooth over $Y = Y'/G$, it would be done. However, the action of G on X' is not toroidal in general in the sense defined in Definition 3.4 and the quotient may not have the required property. We modify X' so that the action of G on the modification $X_{Y'}$ is toroidal. The notion of toroidal action and the modification process played essential roles in [1] in the language of toroidal geometry. Taking the quotient $X = X_{Y'}/G$, we obtain the required extension. In practice, we need to make this construction etale locally on Y' and need some argument to descend it.

The idea of the proof of Theorem 3 is to follow the proof in the case of characteristic 0 using toroidal embeddings in [8]. In this paper, we stick to the language of log schemes and fans in the sense of Kato [7].

The contents of the paper are as follows. In Section 1, we recall basic definitions on log schemes and semi-stable schemes and state the main results. At the end of the section, we recall basic definitions on families of curves. In Section 2, we recall basic properties of log blow-ups and fans. We give a dictionary between polyhedral complexes with integral structures in [8] and fans in [7] and give a proof of Theorem 3. After establishing some basic facts on actions of finite groups on schemes, we define the notion of toroidal action in Section 3. It is an analogue of what defined in [1]. In the second half of the section, we study the locus where a tame group action on a nodal curve is not toroidal and how one can modify it to make the action toroidal. In the last Section 4, applying the results in Section 3, we complete the proof of Theorem 1. At the end of proof, we give equivalent conditions in Theorem 4.2 for a curve over a discrete valuation field to have a log smooth model over the integer ring. It is a generalization of a result in [12].

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1. Main results.

1.1 Log smooth extension of family of curves.

We briefly recall some generalities on log structures. For the details, we refer to [6], [7], [11] and [13]. In this paper, a monoid means a commutative monoid. A finitely generated monoid P is called an fs-monoid if the canonical map to the associated group $P \to P^{\mathrm{gp}} = \{ab^{-1} | a, b \in P\}$ is injective and its image is equal to the saturation $P^{\mathrm{sat}} = \{a \in P^{\mathrm{gp}} | a^n \in \mathrm{Im}\ P \text{ for some } n \geq 1\}$. We identify an fs-monoid P with its image in P^{gp}.

In this paper, a log structure $\alpha : M_X \to O_X$ on a scheme X means an fs-log structure defined on the etale site of X. Namely, it is defined etale locally on X by charts by fs-monoids. A scheme X equipped with a log structure is called a log scheme. If a log structure M_X is defined Zariski locally, we say M_X is a Zariski log structure and X is a Zariski log scheme (cf. [11] Proposition 2.1). We put $\bar{M}_X = M_X/O_X^\times$.

A locally noetherian log scheme X is log regular if, for $x \in X$, the quotient $O_{X,\bar{x}}/I_{\bar{x}}$ by the ideal $I_{\bar{x}}$ generated by the complement $M_{X,\bar{x}} - O_{X,\bar{x}}$ is regular and $\dim O_{X,\bar{x}} = \dim(O_{X,\bar{x}}/I_{\bar{x}}) + \operatorname{rank} \bar{M}^{\mathrm{gp}}_{X,\bar{x}}$ [13] 4.4.5, [11] Definition 2.2. Let X be a locally noetherian log regular log scheme. The scheme X is normal and the maximum open subset $U \subset X$ where $M_X|_U = O_U^\times$ is dense. We have $M_X = O_X \cap j_* O_U^\times$ where $j : U \to X$ is the open immersion [11] Proposition 2.2. We call U the interior of X and M_X the log structure defined by U. If X is a locally noetherian log regular log scheme and U is the interior of X, we say the pair (X, U) is a toric pair as in [9]. When (X, U) is a toric pair, by abuse of notation, we write (X, U) to denote the log scheme (X, M_X) where M_X is the log structure defined by U. If X is regular and $U = X - D$ is the complement of a divisor $D \subset X$ with normal crossings, the pair (X, U) is a toric pair. A toric pair has a resolution in the following sense.

Proposition 1.1 ([11] Theorems 5.2 and 5.3, cf. Lemma 2.1.2) *Let (X, U) be a toric pair. Then, there exist a regular noetherian scheme X', a divisor $D' \subset X'$ with simple normal crossings and a projective surjective morphism $f : X' \to X$ such that $U' = X' - D'$ is equal to $f^{-1}(U)$, that the induced map $U' \to U$ is an isomorphism and that the map $(X', U') \to (X, U)$ is log etale.*

Remark. As a converse of Proposition 1.1, we expect to have the following statement: On a noetherian log scheme X, if there exists a coherent ideal $\mathcal{I}$ such that the log blow-up (see §2.1) is log regular, then the log scheme X is log regular.

We recall a characterization of log smooth morphisms.

Proposition 1.2 ([6] Theorem (3.5)) *Let S be a log scheme and $f : X \to S$ be a log scheme over S. Let x be a point of X, $V \to S$ be an etale neighborhood of $s = f(x)$ and $Q \to \Gamma(V, O_S)$ be a chart. Then, the following conditions (1) and (2) are equivalent.*

(1) f is log smooth (resp. log etale) over S at x.

(2) There exist an affine etale neighborhood $\varphi : U = \operatorname{Spec} A \to X$ of x, a map $U \to V$ over $f : X \to S$, a chart $P \to \Gamma(U, O_X)$ and a morphism $Q \to P$ of charts satisfying the following conditions (a) and (b).

(a) The induced map $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective and the order of the torsion part of its cokernel (resp. the cokernel is finite and its order) is invertible on U.

(b) The map $U \to V \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P]$ is (classically) smooth (resp. etale).

A log scheme log smooth over a log regular scheme is log regular.

Now we state the main result. In the following, if it is not stated otherwise explicitly, we regard $\operatorname{Spec} O_K$ for a discrete valuation ring O_K as a log scheme with the interior $\operatorname{Spec} K$.

Theorem 1.3 *Let (Y, U_Y) be a toric pair. Let $f_{U_Y} : X_{U_Y} \to U_Y$ be a proper smooth and geometrically connected curve and D_{U_Y} be a divisor of X_{U_Y} finite and etale over U_Y. Let g be the genus of $X_{U_Y} \to U_Y$ and r be the degree of $D_{U_Y} \to U_Y$ and assume*

$2g-2+r>0$. Let U_X be the complement $X_{U_Y}-D_{U_Y}$. We consider the following conditions.

(1) There exists a projective and log smooth scheme $f: X \to Y$ extending the toric pair (X_{U_Y}, U_X) over U_Y.

(2) For a generic point η_i of D_Y, let K_i be the completion of the function field of Y at η_i. Then, for each η_i, there exists a projective and log smooth scheme $X_{O_{K_i}}$ over the integer ring O_{K_i} extending the toric pair $(X_{K_i}, U_{X_{K_i}}) = (X_{U_Y} \times_{U_Y} K_i, U_X \times_{U_Y} K_i)$.

(3) Let $N \geq 1$ be an integer invertible on Y. The finite etale covering D_{U_Y} of U_Y and the finite group scheme J_{N,U_Y} over U_Y of N-torsion points of the Jacobian J_{U_Y} of X_{U_Y} are tamely ramified at each generic point η_i of D_Y.

Then we have (1)⇒(2) ⇒(3). We have (3)⇒(2) if $N \geq 3$. We have (2)⇒(1) if either of the following conditions (a) and (b) is satisfied.

(a) 2 is invertible on Y and Y is quasi-compact.

(b) There exists an integer $N \geq 3$ invertible on Y and there is no closed subset of $D_Y = Y - U$ which is a subset of $D_Y[\frac{1}{2}]$.

Proof will be given in Section 4.

Remark. 1. The condition (b) is satisfied if either of the following conditions are satisfied.

(b_1) D_Y is a scheme over a field of characteristic 2.

(b_2) D_Y is proper over a discrete valuation ring of residue characteristic 2.

2. In [9] Theorems A and B, analogous statements for stable extension of a family of curves and log etale extension of a finite etale covering are proved. Contrary to them, we do not have the uniqueness of the extension in Theorem 1.3 because a log blow-up (see §2.1) will give another extension.

By Proposition 1.1, we have the following Corollary.

Corollary 1.4 *Let Y be a regular noetherian scheme and D_Y be a divisor with normal crossings. As in Theorem 1.3, let $f_{U_Y}: X_{U_Y} \to U_Y$ be a proper and smooth curve and D_{U_Y} be a finite and etale divisor satisfying $2g-2+r>0$. We put $U_X = X_{U_Y} - D_{U_Y}$. We consider the following conditions.*

(1) There exists a projective and regular scheme $f: X \to Y$ extending $X_{U_Y} \to U_Y$ such that U_X is the complement a divisor with normal crossing and that the toric pair (X, U_X) is log smooth over (Y, U_Y).

(2) For a generic point η_i of D_Y, let K_i be the completion of the function field of Y at η_i. Then, for each η_i, there exists a projective and regular scheme $X_{O_{K_i}}$ over O_{K_i} extending $X_{K_i} \to K_i$ such that $U_{K_i} = U_X \times_{U_Y} K_i$ is the complement of a divisor of $X_{O_{K_i}}$ with normal crossing and that the toric pair $(X_{O_{K_i}}, U_{K_i})$ is log smooth over O_{K_i}.

(3) Let $N \geq 1$ be an integer invertible on Y. The finite etale covering D_{U_Y} and the finite group scheme J_{N,U_Y} of N-torsion points of the Jacobian J_{U_Y} of X_{U_Y} are tamely ramified at each generic point η_i of D_Y.

Then we have (1)⇒(2) ⇒(3). We have (3)⇒(2) if $N \geq 3$. We have (2)⇒(1) if either of the (a) and (b) in Theorem 1.3 is satisfied.

As a consequence of Proposition 1.1 and Theorem 1.3, a hyperbolic polycurve has a smooth compactification if the characteristic is sufficiently large.

Corollary 1.5 *Let U be a smooth scheme of dimension n of finite type over a perfect field F of characteritic $p \geq 0$. Assume that there exists a sequence of morphisms of smooth schemes $U = U_n \to U_{n-1} \to \cdots \to U_2$ over F satisfying the following condition:*

The scheme U_2 is a smooth surface of finite type over F. For $2 \leq i < n$, there exist a proper smooth and geometricall connected curve X_{i+1} over U_i and a divisor D_{i+1} of X_{i+1} finite and etale over U_i such that U_{i+1} is the complement $X_{i+1} - D_{i+1}$.

Let g_i be the genus of the curve X_{i+1} and r_i be the degree of D_{i+1} over U_i. Assume further that $2g_i - 2 + r_i > 0, p \geq 2g_i + 2$ and $p > r_i$ for $2 \leq i < n$. Then there exist a projective smooth scheme X and an open immersion $U \to X$ such that the complement $X - U$ is a divisor with simple normal crossings.

Proof of Corollary 1.5. By Proposition 1.1, it is sufficient to show that there exists a projective and log smooth scheme X' over F whose interior is U. If $n = 2$, it is well-known. We show the general case by induction applying Theorem 1.3 (3)⇒(1). Let X'_i be a projective and log smooth scheme over F whose interior is U_i. By the assumption that $p \geq 2g_i + 2$, the action of the inertia group I_{η_j} at each generic point η_j of $X'_i - U_i$ on the N-torsion $J_{i+1,N}$ of the Jacobian of $X_{i+1,L}$ over $U_{i,L}$ is tamely ramified for an integer N invertible in F. By the assumption that $p > d_i$, the finite covering $D_{i+1,L} \to U_{i,L}$ is tamely ramified at each generic point η_j. Hence the curve X_{i+1} and the divisor D_{i+1} satisfy the condition (3). Applying (3)⇒(1) of Theorem 1.3, we obtain a projective and log regular scheme X'_{i+1} log smooth over X'_i such that the interior is U_{i+1}. Hence the assertion follows by induction.

1.2 Log smoothness and semi-stable reduction.

Let K be a discrete valuation field.

Definition 1.6 *Let X be a scheme locally of finite presentation over O_K and U be an open subscheme of X. We say a pair (X, U) of X is semi-stable over O_K of relative dimension n if the following condition is satisfied.*

Etale locally on X, the scheme X is etale over Spec $O_K[T_0, \ldots, T_n]/(T_0 \cdots T_r - \pi)$ *and U is the inverse image of* Spec $O_K[T_0, \ldots, T_n, T_0^{-1}, \ldots T_m^{-1}]/(T_0 \cdots T_r - \pi)$ *for some $0 \leq r \leq m \leq n$ and a prime element π of K.*

If the pair (X, X_K) is semi-stable, we say X is semi-stable.

If "Etale locally" in the condition is replaced by "Zariski locally", we say a pair (X, U) is strictly semi-stable. If (X, U) is semi-stable, U is a subscheme of the generic fiber X_K and is the complement of a divisor of a regular scheme X with normal crossings. If the residue field is perfect, a scheme X locally of finite presentation over O_K is semi-stable if and only if the following conditions (1) and (2) are satisfied.

(1) X is regular and flat over O_K and U is the complement of a divisor D with normal crossings.

(2) The generic fiber X_K is smooth and D_K is a divisor of X_K with relative normal crossings.

We regard Spec O_K as a regular log scheme defined by the interior Spec K.

Lemma 1.7 *If (X, U) is semi-stable over O_K, it is log smooth over O_K.*

Proof. The question is etale local. We take a chart $\mathbf{N} \to O_K$ sending 1 to a prime element π. Then, it suffices to apply Proposition 1.2 (2)⇒(1) to the map $\mathbf{N} \to \mathbf{N}^m$ sending 1 to $(1, \dots, 1, 0, \dots, 0)$ with 1 in the first r-components.

We recall a local description of the fiber product in the category of log schemes. Let X and Y be log schemes over a log scheme S. Let $P \to \Gamma(X, O_X), Q \to \Gamma(Y, O_Y)$ and $R \to \Gamma(S, O_S)$ be charts and $\varphi : R \to P$ and $\psi : R \to Q$ be morphisms compatible with $X \to S$ and $Y \to S$. Let $P +_R^{\mathrm{sat}} Q$ be the saturation of the image of $P + Q$ in the amalgamete sum $P^{\mathrm{gp}} +_{R^{\mathrm{gp}}} Q^{\mathrm{gp}} = \mathrm{Coker}(\varphi - \psi : R^{\mathrm{gp}} \to P^{\mathrm{gp}} \oplus Q^{\mathrm{gp}})$. Then the fiber product $X \times_S^{\log} Y$ as a log scheme is the scheme $(X \times_S Y) \otimes_{\mathbf{Z}[P+Q]} \mathbf{Z}[P +_R^{\mathrm{sat}} Q]$ with the log structure defined by the chart $P +_R^{\mathrm{sat}} Q$.

Conversely to Lemma 1.7, a log smooth scheme over a discrete valuation ring has potentially a semi-stable model.

Theorem 1.8 *Let X be a log scheme log smooth and of finite type over O_K and U_K be the interior of X. Then there exist an integer $e \geq 1$ such that, if L is a finite separable extension and the ramification index $e_{L/K}$ is divisible by e, there exists a projective and log etale morphism $f : W \to X \otimes_{O_K}^{\log} O_L$ such that $U_W = f^{-1}(U_{X,L}) \to U_{X,L} = U_X \otimes_K L$ is an isomorphism and (W, U_W) is semi-stable over O_L.*

Proof of Theorem 1.8 will be given at the end of Section 2. The case where $U_X = X_K$ is proved by H.Yoshioka [14].

Remark. As a converse of Theorem 1.8, K.Kato informed me of the author that he has a proof of the following statement: Let X be a log scheme of finite type over O_K. If there exist a finite separable extension L of K and a coherent ideal $\mathcal{I}$ on the base change $X \otimes_{O_K}^{\log} O_L$ such that the log blow-up (see §2.1) is log smooth over O_L, then X is log smooth over O_K.

Corollary 1.9 *Let U be a smooth scheme of dimension n of finite type over a discrete valuation field K of residue characteristic $p > 0$. Assume that there exists a sequence of morphisms of smooth schemes $U = U_n \to U_{n-1} \to \cdots \to U_0 =$ Spec K satisfying the following condition:*

For $0 \leq i < n$, there exist a proper smooth and geometricall connected curve X_{i+1} over U_i and a divisor D_{i+1} of X_{i+1} finite and etale over U_i such that U_{i+1} is the complement $X_{i+1} - D_{i+1}$.

Let g_i be the genus of the curve X_{i+1} and r_i be the degree of D_{i+1} over U_i. Assume further that $2g_i - 2 + r_i > 0, p \geq 2g_i + 2$ and $p > r_i$ for $1 \leq i < n$. Then there exist a finite separable extension L of K, a projective and semi-stable scheme X_{O_L} over O_L and an open immersion $U_L = U \otimes_K L \to X_{O_L}$ such that the pair (X_{O_L}, U_L) is semi-stable over O_L.

Proof of Corollary 1.9. We deduce Corollary 1.9 from Theorems 1.3 and 1.8. By Theorem 1.8, it is sufficient to show that there exists a projective and log smooth scheme X'_{O_L} over O_L whose interior is the base change U_L. If $n = 1$, it follows from Proposition 1.14 (3)⇒(1) in the following subsection. The general case is proved similarly as Corollary 1.5 by induction applying Theorem 1.3 (3)⇒(1).

1.3 Nodal curves and stable curves.

In this subsection, we recall some basic definitions on families of curves.

Definition 1.10 *We say a scheme $f : X \to Y$ flat and locally of finite presentation over a scheme Y is a nodal curve over Y, if the fiber $f^{-1}(y)$ is a reduced curve with at most ordinary double points for each $y \in Y$. We say a pair $(f : X \to Y, D)$ of a nodal curve $f : X \to Y$ and a closed subscheme D of X is a pointed nodal curve if D is etale over Y and X is smooth over Y on a neighborhood of D.*

A nodal curve is called a locally stable curve in [9] when it has geometrically connected fibers. For a nodal curve $f : X \to Y$ over Y, let $\Sigma = \Sigma(X)$ denote the set of non-smooth points. It is a closed subset of X and the intersection $D \cap \Sigma$ is empty. For a geometric point $\bar{x}$ above $x \in \Sigma$, let $Br(\bar{x}) = \text{Spec } O^{sh}_{X_{f(\bar{x})},\bar{x}} - \{\bar{x}\}$ denote the set of branches of the geometric fiber at $\bar{x}$. The set $Br(\bar{x}) = \text{Spec } O^{sh}_{X_{f(\bar{x})},\bar{x}} - \{\bar{x}\}$ consists of two elements.

Lemma 1.11 ([2] Corollaire 1.3.2 (i)) *Let $f : X \to Y$ be a nodal curve over Y and $x \in X$. Then there exist etale neighborhoods U of x and V of $y = f(x)$, a section $w \in \Gamma(V, O_Y)$ and an etale morphism $U \to V[S,T]/(ST - w)$ over Y.*

By Lemma 1.11, a nodal curve X over Y is locally of relative complete intersection of relative dimension 1. The relative dualizing sheaf $\omega_{X/Y}$ is an invertible O_X-module. If $(X \to Y, D)$ is a pointed nodal curve, $\omega_{X/Y}(\log D) = \omega_{X/Y} \otimes O(D)$ is an invertible O_X-module.

Lemma 1.12 (cf. [9] Lemma 4.2) *Let (Y, U_Y) be a toric pair and $(f : X \to Y, D)$ be a pointed nodal curve such that f is smooth over U_Y. We put $U_X = f^{-1}(U_Y) \cap (X - D)$. Then (X, U_X) is a toric pair and the map $f : (X, U_X) \to (Y, U_Y)$ is log smooth.*

We call the log structure on X defined by U_X the standard log structure.

Proof. The assertion is etale local on X. By Lemma 1.11, it is sufficient to consider the following two cases. In one case, $Y = \text{Spec } R$, $X = \text{Spec } R[S,T]/(ST - w)$ where $w \in R$ is invertible on U_Y and $D = \emptyset$. In the other case, X is classically smooth over Y and D is a section of f. The assertion is clear in the second case and we show the first case. We consider the universal case where $R_0 = \mathbf{Z}[w_0]$ is the polynomial ring, the log structure on $Y_0 = \text{Spec } R_0$ is defined by the chart $\mathbf{N} \to R_0$ sending 1 to w_0 and $X_0 = \text{Spec } R[S,T]/(ST - w_0)$. Then, if we define a map $\mathbf{N} \to \mathbf{N}^2$ by sending 1 to $(1,1)$, we have $X_0 = \text{Spec } R_0 \otimes_{\mathbf{Z}[\mathbf{N}]} \mathbf{Z}[\mathbf{N}^2]$. With the log structure M_{X_0} on X_0 defined by the canonical map $\mathbf{N}^2 \to R_0 \otimes_{\mathbf{Z}[\mathbf{N}]} \mathbf{Z}[\mathbf{N}^2]$, the log scheme X_0 is log smooth over Y_0 by Lemma 1.2 and the interior of X_0 is the inverse image U_{X_0} of U_{Y_0}. Hence the assertion follows in this case. We consider the general case. Since $w \in R$ is invertible on U_Y, it is in the image of $\Gamma(Y, M_Y)$. Hence, by localizing if necessary, we may take a chart $Q \to R$ and an element $W \in Q$ whose image is w. We define a map of log schemes $Y \to Y_0$ by sending w_0 to w and 1 to W. Since the map $\mathbf{N} \to \mathbf{N}^2$ of monoids is saturated, the amalgamate sum $P = Q +_{\mathbf{N}} \mathbf{N}^2$ is equal to its saturation and the map $X \to Y \times^{\log}_{Y_0} X_0$ of the underlying schemes is an isomorphism. Hence the assertion follows.

For a morphism of log schemes $f : X \to Y$, we define a quasi-coherent O_X-module $\Omega^1_{X/Y}(\log / \log)$ by

$$\Omega^1_{X/Y}(\log / \log) = \left(\Omega^1_{X/Y} \oplus O_X \otimes_{\mathbf{Z}} M_X^{\text{gp}} / f^* M_Y^{\text{gp}}\right) / (d\alpha(a) - \alpha(a) \otimes a; a \in M_X).$$

For a section $a \in M_X$, the image of $1 \otimes a$ is denoted by $d \log a$. For a pointed nodal curve $(X \to Y, D)$ over a log regular scheme Y with the standard log structure, we have a canonical isomorphism $\omega_{X/Y}(\log D) = \Omega^1_{X/Y}(\log / \log)$.

We recall the definition of stable curves.

Definition 1.13 *We say a pointed nodal curve $(f : X \to Y, D)$ is stable if f is proper, $f_* O_X = O_Y$ and if $\omega_{X/Y}(\log D)$ is f-ample.*

Under the other conditions, the last condition that $\omega_{X/Y}(\log D)$ is f-ample is equivalent to the following condition.

Let $\bar{y} \to Y$ be a geometric point and C be an irreducible component of a geometric fiber $X_{\bar{y}}$. Let g be the geometric genus of C and r be the number of points of $C \cap (\overline{X_{\bar{y}} - C} \cup D_{\bar{y}})$. Then we have $2g - 2 + r > 0$.

For the extension of a smooth stable curve over a discrete valuation field, the following result is well-known.

Proposition 1.14 ([3] Theorem (2,4) and Proof of Lemma (1.12)) *Let K be a discrete valuation field, $f_K : X_K \to K$ be a proper smooth and geometrically connected curve and D_K be a divisor of X_K such that the pair $(f_K : X_K \to K, D_K)$ is a stable curve. Let $N \geq 1$ be an integer invertible in O_K. We consider the following conditions.*

(1) There exists a semi-stable scheme (X_{O_K}, U_K) extending $(X_K, U_K = X_K - D_K)$.

(2) There exists a stable curve $(f_{O_K} : X_{O_K} \to O_K, D_{O_K})$ extending $(f_K : X_K \to K, D_K)$.

(3) The finite covering D_K over K is unramified and the action of the inertia I_K on the finite group scheme $J_{N,K}$ of N-torsion of the Jacobian J_K is unipotent.

Then we have (1)⇔(2)⇒(3). We have (3)⇒(1) if $N \geq 3$.

2. Log blow-up and Fan.

2.1 Log blow-up.

We briefly recall the definition and basic properties of log blow-ups [13] 1.6. Let P be a monoid. An ideal of P is a subset I of P satisfying $PI \subset I$. For an fs-monoid P, we call a subset I of $P^{\rm gp}$ a fractional ideal of P if there exist a finite number of elements $a_1, \ldots, a_r \subset I$ such that $I = \bigcup_{i=1}^r a_i P$ and I is not empty.

Let X be a log scheme. We say a sheaf $\mathcal{I}$ of ideals of M_X is coherent if, for $x \in X$, there exist a chart $P \to \Gamma(U, M_X)$ on an etale neighborhood U of x and an ideal I such that $\mathcal{I}|_U = IM_X|_U$. We say a subsheaf $\mathcal{I}$ of $M_X^{\rm gp}$ is a fractional ideal if $M_X \cdot \mathcal{I} \subset \mathcal{I}$ and, for $x \in X$, there exist a chart $P \to \Gamma(U, M_X)$ on an etale neighborhood U of x and a fractional ideal $I \subset P^{\rm gp}$ such that $\mathcal{I}|_U = IM_X|_U$. A coherent ideal $\mathcal{I}$ is a fractional ideal if and only if, for each $x \in X$, the stalk $\mathcal{I}_{\bar{x}}$ is not empty. A fractional ideal $\mathcal{I}$ is a coherent ideal if and only if $\mathcal{I} \subset M_X$.

Let X be a log scheme and $\mathcal{I}$ be a fractional ideal of M_X. Then the log blow-up is locally described as follows. Let $P \to \Gamma(U, M_X)$ be a chart on an etale neighborhood and I be a fractional ideal of P such that $\mathcal{I}|_U = IM_X|_U$. Let I^c be the saturation $\{a \in P^{\rm gp} | a^n \in I^n \text{ for some } n \geq 1\}$. Then, the log blow-up $X_{\mathcal{I}} \to X$ is defined by patching $U \times_{\mathrm{Spec}\ \mathbf{Z}[P]} \mathrm{Proj}\ \mathbf{Z}[\bigoplus_{n=0}^\infty (I^c)^n]$. The canonical map $X_{\mathcal{I}} \to X$ is projective. We show that it is log etale. For $a \in I$, let P_a be the submonoid $\bigcup_{n=0}^\infty a^{-n} I^n$ of $P^{\rm gp}$ and $P_a^{\rm sat}$ be the saturation. Then $\mathrm{Proj}\ \mathbf{Z}[\bigoplus_{n=0}^\infty (I^c)^n]$ has an open covering by $\mathrm{Spec}\ \mathbf{Z}[P_a^{\rm sat}]$ for $a \in I$. Hence $X_{\mathcal{I}} \to X$ is log etale.

Lemma 2.1 *Let X be a noetherian log regular scheme.*

1. ([11] Proposition 4.2) *Let $\mathcal{I}$ be a fractional ideal of M_X. Then the log blow-up $X_{\mathcal{I}}$ of X by $\mathcal{I}$ is canonically identified with the normalization of the blowing-up of X by the ideal $\mathcal{I}_X = \alpha(\mathcal{I})O_X$ generated by the image of $\mathcal{I}$.*

2. ([11] Theorems 5.2 and 5.3, cf. Lemma 2.3.1) *There exists a coherent ideal $\mathcal{I}$ of M_X such that the log blow-up $X_{\mathcal{I}}$ is a regular scheme and that the interior $U_{X_{\mathcal{I}}}$ is the complement of a divisor with simple normal crossings.*

2.2 Fan.

We recall the definition of fans, [7] Sections 5 and 9. Let P be a monoid. A prime ideal of P is an ideal p of P such that the complement $P - p$ is a submonoid of P. The set of prime ideals of P is denoted by Spec P. For example, for the additive monoid $\mathbf{N}$ of non-negative integers, we have Spec $\mathbf{N} = \{\emptyset, \{n \in \mathbf{N} | n \geq 1\}\}$. The topology of the set $F = \mathrm{Spec}\ P$ is defined by the open basis consisting of subsets $U_a = \{p \in F | a \notin p\}$ for $a \in P$. The sheaf M_F of monoids on F is defined by requiring $M_F(U_a) = P[a^{-1}]/P[a^{-1}]^\times$.

We say a pair (F, M_F) of a topological space F with a sheaf M_F of monoids is a fan if it has an open covering by affine fans. A map $f : F \to F'$ of fans (F, M_F) and $(F', M_{F'})$ consists of a continuous map $f : F \to F'$ and a map of sheaves $\varphi : f^*M_{F'} \to M_F$ of monoids satisfying $\varphi_t^{-1}(1) = 1$ for $t \in F$. We say a fan F is an fs-fan if it has a finite open covering by affine fans that are isomorphic to the spectrums of fs-monoids. If F is an fs-fan and $x \in F$, the stalk $M_{F,x}$ is an fs-monoid and the set $U_x = \{y \in F | x \in \overline{\{y\}}\}$ is an affine open subset canonically isomorphic to Spec $M_{F,x}$. In the rest of the paper, we consider only fs-fans and the word fan will mean fs-fan.

Let $f : F' \to F$ and $g : F'' \to F$ be morphisms of fans. Then the fiber product $F' \times_F F''$ in the category of fs-fans is locally described as follows. Let $U \subset F, V \subset F'$ and $W \subset F''$ be affine open subfans satisfying $f(V) \subset U$ and $g(W) \subset U$. We put $P = \Gamma(U, M_F), Q = \Gamma(V, M_{F'}), R = \Gamma(W, M_{F''})$ and let $\varphi : P \to Q$ and $\psi : P \to R$ be the induced map. Let $Q +_P^{\mathrm{sat}} R$ be the saturation of the image of $Q + R$ in the quotient of $Q^{\mathrm{gp}} \oplus R^{\mathrm{gp}}/((\varphi(a), -\psi(a)), a \in P^{\mathrm{gp}})$. Then the fiber product $F' \times_F F''$ is obtained by patching Spec $Q +_P^{\mathrm{sat}} R$.

Let F be an fan. We say a sheaf $\mathcal{I}$ of ideals of M_F is coherent if, for an affine open subfan $U \subset F$, the ideal $\mathcal{I}|_U$ is generated by the ideal $\Gamma(U, \mathcal{I})$ of $\Gamma(U, M_F)$. We say a subsheaf $\mathcal{I}$ of M_F^{gp} is a fractional ideal if $M_F \cdot \mathcal{I} \subset \mathcal{I}$ and, for an affine open subfan $U \subset F$ the subset $\Gamma(U, \mathcal{I}) \subset \Gamma(U, M_F^{\mathrm{gp}})$ is a fractional ideal of $\Gamma(U, M_F)$ and we have $\mathcal{I}|_U = \Gamma(U, \mathcal{I}) M_F|_U$. A coherent ideal $\mathcal{I}$ is a fractional ideal if and only if the stalk $\mathcal{I}_x$ is not empty for each $x \in F$.

Let F be a fan and $\mathcal{I}$ be a fractional ideal of M_F. The blow-up $F_{\mathcal{I}}$ of F by $\mathcal{I}$ is defined as follows. First we consider the case where $F = \mathrm{Spec}\ P$ for an fs-monoid P. Let $I = \Gamma(F, \mathcal{I})$ be the fractional ideal of P satisfying $\mathcal{I} = \Gamma(F, \mathcal{I}) M_F$. For $a \in I$, we define $P_a = \bigcup_{n=0}^\infty a^{-n} I^n$ as a submonoid of P^{gp}. For $a, b \in I$, we have $P_a[(b/a)^{-1}] = P_b[(a/b)^{-1}]$. By patching the saturation Spec P_a^{sat} for $a \in I$ by the identity Spec $P_a^{\mathrm{sat}}[(b/a)^{-1}] = \mathrm{Spec}\ P_b^{\mathrm{sat}}[(a/b)^{-1}]$, we define $F_{\mathcal{I}} = \mathrm{Proj}(\coprod_{n \in \mathbf{N}} I^n)$. In the general case, the blow-up $F_{\mathcal{I}}$ is defined by patching.

Definition 2.2 *1. Let F be an fan. We say a fan F is regular if, for $x \in F$, there is an isomorphism $M_{F,x} \to \mathbf{N}^{r(x)}$ where $r(x) = \mathrm{rank}\ M_{F,x}^{\mathrm{gp}}$.*

2. Let F be a fan over $\mathbf{N}$. We say F is saturated over $\mathbf{N}$ if, for $x \in F$ such that $r(x) = 1$, the map $\mathbf{N} \to \Gamma(F, M_F) \to M_{F,x}$ is either an isomorphism or the 0-map.

We say F is semi-stable if it is regular and is saturated over $\mathbf{N}$.

The terminology will be justified by Lemma 2.7. Let F be a fan over $\mathbf{N}$. For an integer $e \geq 1$, let N_e be monoid $\mathbf{N}$ regarded as a monoid over $\mathbf{N}$ by the map $e\times : \mathbf{N} \to \mathbf{N} = N_e$ and F_e be the base change $F_e = F \times_{\mathrm{Spec}\ \mathbf{N}} \mathrm{Spec}\ N_e$ defined as an fs-fan.

The following Lemma is crucial in the proof of Theorem 1.8.

Lemma 2.3 *1. Let F be a fan. Then there exists a coherent ideal $\mathcal{I}$ on F such that the blow-up $F_{\mathcal{I}}$ is regular.*

2. Let F be a fan over $\mathbf{N}$. Then there exist an integer $e \geq 1$ and a fractional ideal $\mathcal{I}$ on the base change F_e such that the blow-up $F_{e,\mathcal{I}}$ is semi-stable over N_e.

3. Let F be a fan semi-stable over $\mathbf{N}$ and $e \geq 1$ be an integer. Then there exists a coherent ideal $\mathcal{I}$ on F_e such that the blow-up $F_{e,\mathcal{I}}$ is semi-stable over N_e.

The rest of this subsection is devoted to a proof of Lemma 2.3. Before starting the proof, we recall the dictionary between fractional ideals on fans and good functions on polyhedral complexes. We call the point $\eta = \emptyset \in \operatorname{Spec} \mathbf{N}$ the generic point and $s = \{n \geq 1 | n \in \mathbf{N}\} \in \operatorname{Spec} \mathbf{N}$ the closed point. Let $f : F \to \operatorname{Spec} \mathbf{N}$ be a fan over $\mathbf{N}$. We say F is generically trivial, if, for $t \in f^{-1}(\eta)$, the stalk $M_{F,t}$ is the trivial monoid $\{1\}$.

In the following, we use the terminology in [8] Chapter II Definitions 5 and 6 p.69-70, with slight modifications. We begin with attaching a conical polyhedral complex with an integral structure to a fan and a compact polyhedral complex with an integral structure to a fan generically trivial and saturated over $\mathbf{N}$. Let $\mathbf{R}^+$ be the monoid of non-negative real numbers.

Let F be a fan. For an affine open subfan $U_\alpha \subset F$, we put $P_\alpha = \Gamma(U_\alpha, M_X)$ and $\sigma_\alpha = Hom_{\text{monoid}}(P_\alpha, \mathbf{R}^+)$. The set σ_α is a conical convex polyhedron in the dual space V_α^* of the $\mathbf{R}$-vector space $V_\alpha = P_\alpha^{\text{gp}} \otimes_{\mathbf{Z}} \mathbf{R}$. The polyhedron σ_α is not contained in a hyperplane. Let $L_\alpha \subset V_\alpha$ be the finitely generated free abelian group P_α^{gp}. By patching σ_α's, we obtain a conical polyhedral complex $\Delta_F = \bigcup_\alpha \sigma_\alpha$. Modifying the terminology loc.cit., we call the family $(N_\alpha)_\alpha$ of the finitely generated abelian groups $N_\alpha = Hom(L_\alpha, \mathbf{Z}) \subset V_\alpha^*$ the integral structure of Δ_F. For $a \in V_\alpha$ and $x \in V_\alpha^*$, let $\langle a, x\rangle \in \mathbf{R}$ denote the canonical pairing.

Let $F = \bigcup_\alpha U_\alpha$ be a fan and $\Delta_F = \bigcup_\alpha \sigma_\alpha$ be the corresponding conical polyhedral complex constructed above. If a function $f : \Delta_F \to \mathbf{R}$ satisfies the following properties (i)-(iv), we say f satisfies the condition $(*)$.

(i) $f(\lambda x) = \lambda f(x)$ for $\lambda \in \mathbf{R}^+$ and $x \in \Delta_F$,

(ii) f is continuous and piecewise-linear,

(iii) $f(\sigma_\alpha \cap N_\alpha) \subset \mathbf{Z}$ for all α,

(iv) f is convex on each σ_α; $f(\lambda x + \mu y) \geq \lambda f(x) + \mu f(y)$ for $x, y \in \sigma_\alpha$ and $\lambda, \mu \in \mathbf{R}^+$.

For a function $f : \Delta_F \to \mathbf{R}$ satisfying the condition $(*)$, we define a fractional ideal $\mathcal{I}_f$ of M_F as follows. Let $U_\alpha \subset F$ and P_α be as above. We define a fractional ideal $I_{\alpha,f}$ of P_α by $I_{\alpha,f} = \{a \in P_\alpha^{\text{gp}} | \langle a, x\rangle \geq f(x) \text{ for all } x \in \sigma_\alpha\}$. Then it is easily verified that there exists a unique fractional ideal $\mathcal{I}_f$ characterized by the condition $\Gamma(U_\alpha, \mathcal{I}_f) = I_{\alpha,f}$ for all α. If f has values in $\mathbf{R}^+$, the corresponding fractional ideal $\mathcal{I}_f$ is integral.

Let $f : \Delta_F \to \mathbf{R}$ be a function satisfying the condition $(*)$. We say a subpolyhedron σ of some σ_α is associated to f if it is a maximal subpolyhedron on which f is linear. We say a polyhedron $\sigma \subset \sigma_\alpha$ associated to f is of multiplicity 1, if there exists a basis $(x_1, \ldots, x_r)$ of N_α such that σ is spanned by $x_1, \ldots, x_r$.

Let $F \to \mathbf{N}$ be a fan over $\mathbf{N}$. We assume F is generically trivial and saturated over $\mathbf{N}$. Let $\Delta_F = \bigcup_\alpha \sigma_\alpha$ be the conical polyhedral complex with the integral structure $(N_\alpha)_\alpha$ defined above. Let $U_\alpha \subset F$ and P_α be as above and let π denote the image of 1 by the map $\mathbf{N} \to P_\alpha$. If $P_\alpha \neq \{1\}$, we define a subspace $V_\alpha^{*0} \subset V_\alpha^*$ of codimension 1 to be $\{x \in V_\alpha^* | \langle \pi, x\rangle = 0\}$ and a V_α^{*0}-torsor V_α^{*1} to be $\{x \in V_\alpha^* | \langle \pi, x\rangle = 1\}$. We define a compact convex polyhedron $\sigma_\alpha^1 \subset \sigma_\alpha$ to be $\sigma_\alpha^1 = \sigma_\alpha \cap V_\alpha^{*1}$. By patching σ_α^1's for $P_\alpha \neq \{1\}$, we obtain a compact polyhedral complex $\Delta_F^1 = \bigcup_\alpha \sigma_\alpha^1 \subset \Delta_F$. Modifying the terminology loc.cit., we call the family $(N_\alpha^1)_\alpha$ of the $N_\alpha^0 = \{x \in N_\alpha | \langle \pi, x\rangle = 0\}$-torsors $N_\alpha^1 = N_\alpha \cap V_\alpha^{*1}$ the integral structure of Δ_F^1. Note that by the assumption that F is saturated over $\mathbf{N}$, we have $N_\alpha = N_\alpha^0 \oplus \mathbf{Z}x$ for $x \in N_\alpha^1$.

Let $F \to \mathbf{N}$ be a fan generically trivial and saturated over $\mathbf{N}$ and $\Delta_F^1 = \bigcup_\alpha \sigma_\alpha^1$ be the corresponding compact polyhedral complex constructed above. If a function $f : \Delta_F^1 \to \mathbf{R}$ satisfies the following properties (ii^1)-(iv^1), we say f satisfies the condition $(*^1)$.

(ii^1) f is continuous and piecewise-affine; there exists a subdivision of each σ_α by subpolyhedrons and on each polyhedron we have $f(\lambda x+(1-\lambda)y) = \lambda f(x)+(1-\lambda)f(y)$ for $0 \leq \lambda \leq 1$,

(iii^1) $f(\sigma_\alpha^1 \cap N_\alpha^1) \subset \mathbf{Z}$ for all α,

(iv^1) f is convex on each σ_α.

If a function $f : \Delta_F^1 \to \mathbf{R}$ satisfies the condition $(*^1)$, its linear extension $\tilde{f} : \Delta_F \to \mathbf{R}$ defined by $\tilde{f}(\lambda x) = \lambda f(x)$ for $x \in \Delta_F^1$ and $\lambda \in \mathbf{R}^+$ satisfies the condition $(*)$. We define a fractional ideal $\mathcal{I}_f$ of M_F to be $\mathcal{I}_{\tilde{f}}$ defined above.

Let $f : \Delta_F^1 \to \mathbf{R}$ be a function satisfies the condition $(*^1)$. We say a subpolyhedron σ^1 of some σ_α^1 is associated to f if it is a maximal subpolyhedron on which f is affine. We say a polyhedron $\sigma \subset \sigma_\alpha^1$ associated to f is of multiplicity 1, if there exist elements $x_1, \ldots, x_r$ of N_α^1 such that $(x_2 - x_1, \ldots, x_r - x_1)$ is a basis of N_α^0 and σ is spanned by $x_1, \ldots, x_r$.

The following Lemma is clear from the definition.

Lemma 2.4 *1. Let F be a fan and f be a $\mathbf{R}$-valued function on the conical polyhedral complex Δ_F satisfying the condition $(*)$. Then the following conditions are equivalent.*

(1) The blow-up $F_{\mathcal{I}_f}$ of F by the fractional ideal $\mathcal{I}_f$ is regular.

(2) The polyhedra σ associated to f are of multiplicity 1.

2. Let F be a fan over $\mathbf{N}$. Assume F is generically trivial and saturated over $\mathbf{N}$. Let f be a $\mathbf{R}$-valued function on the compact polyhedral complex Δ_F^1 satisfying the condition $(*^1)$. Then the following conditions are equivalent.

(1) The blow-up $F_{\mathcal{I}_f}$ of F by the fractional ideal $\mathcal{I}_f$ is semi-stable over $\mathbf{N}$.

(2) The polyhedra σ associated to f are of multiplicity 1.

Let F be a fan generically trivial and saturated over $\mathbf{N}$ and an integer $e \geq 1$. We define a canonical identification of the compact polyhedral complex $\Delta_{F_e}^1$ with Δ_F^1. For an affine open subfan $U_\alpha \subset F$, we identify the amalgamete sum $P_{\alpha,e} = P_\alpha +_{\mathbf{N}} N_e$ with the submonoid $P_\alpha + \langle \pi/e\rangle \subset V_\alpha$. We define an isomorphism $\sigma_\alpha = Hom(P_\alpha, \mathbf{R}^+) \to$

$\sigma_{\alpha,e} = Hom(P_{\alpha,e}, \mathbf{R}^+)$ by sending $x \in \sigma_\alpha$ to the map characterized by $a \mapsto e\langle a, x\rangle$ for $a \in P_\alpha$. By the isomorphism, the polyhedron $\sigma^1_{\alpha,e}$ is identified with σ^1_α. It induces an isomorphism $\Delta^1_F \to \Delta^1_{F_e}$. By the isomorphism $\sigma^1_\alpha \to \sigma^1_{\alpha,e}$, the integral structure $N^1_{\alpha,e}$ is identified with the $N^0_{\alpha,e} = \frac{1}{e}N^0_\alpha$-torsor induced by the N^0_α-torsor torsor N^1_α.

The proof of [8] Chapter I Theorem 11 shows Lemma 2.5.1 below. Similarly, [8] Chapter III Theorem 4.1 means Lemma 2.5.2 below.

Lemma 2.5 *1. Let F be a fan and Δ_F be the corresponding conical polyhedral complex with integral structure. Then there exists a function $f : \Delta_F \to \mathbf{R}^+$ satisfying the condition $(*)$ such that the polyhedra σ associated to f are of multiplicity 1.*

2. Let F be a fan generically trivial and saturated over $\mathbf{N}$ and Δ^1_F be the corresponding compact polyhedral complex. Then there exist an integer $e \geq 1$ and a function $f : \Delta^1_F \to \mathbf{R}$ satisfying the conditions $(*^1)$ with respect to the integral structure defined by F_e such that the polyhedra σ associated to f are of multiplicity 1 with respect to the integral structure defined by F_e.

Proof of Lemma 2.3. 1. It follows from Lemmas 2.5.1 and 2.4.1.

2. If F is generically trivial and saturated over $\mathbf{N}$, it also follows from Lemmas 2.5.2 and 2.4.2. We reduce the general case to this case.

First we reduce it to the case where F is generically trivial. Let $f : F \to \operatorname{Spec} \mathbf{N}$ be a fan over $\operatorname{Spec} \mathbf{N}$. By 1, replacing F by some blow-up, we may assume F is regular. Let F_0 be the maximum open subfan containing the closed fiber $f^{-1}(s)$ and $f_0 : F_0 \to \operatorname{Spec} \mathbf{N}$ be the restriction. We show that F_0 is generically trivial and we define a map $g : F \to F_0$ satisfying $f = f_0 \circ g$. Let $U \subset F$ be an affine open subfan. Then U is isomorphic to $\mathbf{N}^m$ for some integer $m \geq 0$ and the map $U \to \operatorname{Spec} \mathbf{N}$ is defined by the map $\mathbf{N} \to \mathbf{N}^m$ sending 1 to $(1, \ldots, 1, 0, \ldots, 0)$ where the first r components are 1 upto numbering. The open immersion $U \cap F_0 \to U$ is corresponding to the projection $\mathbf{N}^m \to \mathbf{N}^r$ to the first r-components. Hence F_0 is generically trivial. We define a map $U \to U \cap F_0$ to be that corresponding to the inclusion $\mathbf{N}^r \to \mathbf{N}^m$ of the first r-components. Then it is easy to check that they are glued to define a map $g : F \to F_0$ and we have $f = f_0 \circ g$. Let $e \geq 1$ be an integer and assume that there exists a fractional ideal $\mathcal{I}_0$ on $F_{0,e}$ such that the blow-up $F_{0,e,\mathcal{I}_0}$ is semi-stable over N_e. We define a fractional ideal $\mathcal{I}$ on F_e to be that generated by the pull-back of $\mathcal{I}_0$. Then the blow-up $F_{e,\mathcal{I}}$ is isomorphic to the fiber product $F_{0,e,\mathcal{I}_0} \times_{F_0} F$ and is semi-stable over N_e. Thus it is reduced to the generically trivial case.

Next, we reduce it to the case where F is saturated over $\mathbf{N}$. Let $t_1, \ldots, t_r$ be the points in the closed fiber $f^{-1}(s)$ and m be a common multiple of the images l_i of 1 by the maps $\mathbf{N} \to M_{F,t_i} \to \mathbf{N}$ for $i = 1, \ldots, r$. It is sufficient to show that the base change $F_m = F \times_{\mathbf{N}} N_m$ is saturated over N_m. By localizing, we may assume $F = \operatorname{Spec} \mathbf{N}$ and the map $F \to \mathbf{N}$ is defined by the map $\times l : \mathbf{N} \to \mathbf{N}$. Then, since m is a multiple of l, the saturation $N_l +^{\mathrm{sat}}_{\mathbf{N}} N_m$ is isomorphic to $N_m \times \mathbf{Z}/l\mathbf{Z}$ and the assertion follows.

3. We reduced it to the case where $F = F_0 = \operatorname{Spec} \mathbf{N}^m$ and the map $f_0 : F_0 \to \operatorname{Spec} \mathbf{N}$ is corresponding to the map $\mathbf{N} \to \mathbf{N}^m$ sending 1 to $(1, \ldots, 1)$. By the same

argument as above, we may assume F is generically trivial. Let $t_1, \dots, t_m$ be the points of F such that M_{F,t_i} is isomorphic to $\mathbf{N}$. We define a local isomorphism $g : F \to F_0 = \mathrm{Spec}\ \mathbf{N}^m$ such that $f = f_0 \circ g$. Let U be an affine open subfan and put $I = \{i | t_i \in U\}$. Then we have a canonical isomorphism $U \to \mathrm{Spec}\ \mathbf{N}^I$. Patching the compositions $U \to \mathrm{Spec}\ \mathbf{N}^I \to \mathrm{Spec}\ \mathbf{N}^m = F_0$ with the map defined by the projection $\mathbf{N}^m \to \mathbf{N}^I$, we define a map $g : F \to F_0$. Then we have $f = f_0 \circ g$. Hence it is reduced to the case where $F = F_0$ and $f = f_0$. In this case, it is proved in [8] Chapter III Example 2.3.

2.3 Fans associated to log regular schemes. Proof of Theorem 1.8.

We recall the definition of the fan associated to a log regular scheme, [7] Section 10. Let X be a noetherian log regular Zariski log scheme. Let $F(X) \subset X$ be the subspace consisting of the points x such that the maximal ideal m_x is generated by the complement $M_{X,x} - O_{X,x}^\times$. We define a sheaf $M_{F(X)}$ of monoids on $F(X)$ as the restriction of $\bar{M}_X$. If $P \to \Gamma(U, O_X)$ is a chart on an open subscheme U, there is a canonical map $F(U) = U \cap F(X) \to \mathrm{Spec}\ P$ and it is a local isomorphism. For $x \in X$ and $P = \bar{M}_{X,x}$, there exists a chart $P \to \Gamma(U, O_X)$ on an open neighborhood U such that the canonical map $F(U) \to \mathrm{Spec}\ P$ is an isomorphism [7] Proposition (10.1). Hence the topological space $F(X)$ together with the sheaf $M_{F(X)}$ of monoids is a fan. We call the fan $F(X)$ the fan associated to X.

Lemma 2.6 *Let X be a noetherian log regular Zariski log scheme and F be its associated fan. For a fractional ideal $\mathcal{I}$ of M_X, let $\mathcal{I}_F$ dente the restriction of the image of $\mathcal{I}$ in $\bar{M}_X^{\mathrm{gp}}$ to F. Then the map $\{$fractional ideals of $M_X\} \to \{$fractional ideals of $M_F\}$ sending $\mathcal{I}$ to $\mathcal{I}_F$ is a bijection.*

Proof. Since the question is local, we may assume $F(X) = \mathrm{Spec}\ P$ where $P = \bar{M}_{X,x}$ for some $x \in X$. Then both the coherent ideals of M_X and the coherent ideals of M_F are in one-to-one correspondences with the ideals of P and the assertion follows.

The fan $F(\mathrm{Spec}\ O_K)$ for a discrete valuation ring O_K is canonically identified with $\mathrm{Spec}\ \mathbf{N}$. The semi-stability of a log smooth scheme over O_K is described in terms of a fan as follows.

Lemma 2.7 *Let X be a noetherian log regular Zariski log scheme.*
1. The following conditions are equivalent.

(1) The scheme X is regular and the interior U is the complement of a divisor with normal crossings.

(2) The fan $F(X)$ is regular.

2. If X is log smooth over a discrete valuation ring O_K, the following conditions are equivalent.

(1) The scheme X is semi-stable over O_K.

(2) The fan $F(X)$ is semi-stable over $F(\mathrm{Spec}\ O_K)$.

Proof. 1. Follows immediately from the definitions.

2. (1)⇒(2) is clear from the proof of Lemma 1.7. We show (2)⇒(1). The question is etale local on X. Let $Q = \mathbf{N} \to O_K$ be the chart sending 1 to a prime element π. Shrinking X, we take a chart $P \to \Gamma(X, O_X)$ and a map $Q \to P$ as in Proposition 1.2 (2). The map $X \to \mathrm{Spec}\ O_K \otimes_{\mathbf{Z}[\mathbf{N}]} \mathbf{Z}[P]$ is smooth and U is the inverse image of $\mathrm{Spec}\ O_K \otimes_{\mathbf{Z}[\mathbf{N}]} \mathbf{Z}[P^{\mathrm{gp}}]$. If $a \in P$ denote the image of 1, we have $O_K \otimes_{\mathbf{Z}[\mathbf{N}]} \mathbf{Z}[P] = O_K[P]/(a-\pi)$. Shrinking X and localizing P, we may assume that $F(X)$ is isomorphic to Spec P. By 1, the quotient $P/P^\times$ is isomorphic to $\mathbf{N}^m$ for some integer $m \geq 0$. By the assumption that F is semi-stable, we may assume that the image of 1 by the composition $\mathbf{N} \to P/P^\times \to \mathbf{N}^m$ is $(1, \ldots, 1, 0, \ldots, 0)$ where the first r coordinates are 1 upto numbering. By taking a splitting and an isomorphism $P^\times \to \mathbf{Z}^{n-m}$, we have an isomorphism $P \to \mathbf{N}^m \times \mathbf{Z}^{n-m}$. If $r \neq 0$, by modifying the splitting, we may assume that the image of 1 by the composition $\mathbf{N} \to P \to \mathbf{N}^m \times \mathbf{Z}^{n-m}$ is $(1, \ldots, 1, 0, \ldots, 0)$ where the first r coordinates are 1. Thus in this case, the assertion is proved. Assume $r = 0$. Then we have $X = X_K$. By the same argument as above, localizing X if necessary, there is a smooth map $X \to \mathrm{Spec}\ K[P]$ such that U is the inverse image of $\mathrm{Spec}\ K[P^{\mathrm{gp}}]$ and $P/P^\times$ is isomorphic to $\mathbf{N}^r$ for some integer r. Thus also in this case, the assertion is proved.

Lemma 2.8 *1. Let S be a noetherian log regular Zariski log scheme, X be a noetherian Zariski log scheme log smoooth over S and T be a noetheran log regular Zariski log scheme over S. Let $X \times_S^{\log} T$ be the fiber product as an fs-log scheme and let $F(S), F(X), F(T)$ and $F(X \times_S^{\log} T)$ be the associated fans. Then there is a canonical map $F(X \times_S^{\log} T) \to F(X) \times_{F(S)} F(T)$ and it is locally an isomorphism.*

2. Let X be a notherian log regular Zariski log scheme and $F = F(X)$ be its fan. Let $\mathcal{I}$ be a coherent ideal of M_X and $\mathcal{I}_F$ be the corresponding coherent ideal of M_F. Then the log blow-up $X_{\mathcal{I}}$ is Zariski log regular and there is a canonical map $F(X_{\mathcal{I}}) \to F(X)_{\mathcal{I}_F}$. Further the canonical map $F(X_{\mathcal{I}}) \to F(X)_{\mathcal{I}_F}$ is locally an isomorphism.

Proof. 1. Since the question is local on X, we may assume that there exist charts $P \to \Gamma(X, M_X), Q \to \Gamma(S, M_S)$ and $R \to \Gamma(T, M_T)$ such that $F(X) = \mathrm{Spec}\ P, F(S) = \mathrm{Spec}\ Q$ and $F(T) = \mathrm{Spec}\ R$ and morphisms of charts $Q \to P$ and $Q \to R$. Then since $P +_Q^{\mathrm{sat}} R \to \Gamma(X \times_S^{\log} T, M_{X \times_S^{\log} T})$ is a chart, there is a canonical map $F(X \times_S^{\log} T) \to \mathrm{Spec}\ P +_Q^{\mathrm{sat}} R = F(X) \times_{F(S)} F(T)$ and it is locally an isomorphism. Thus the assertion follows.

2. Since the question is local on X, we may assume that there exist a chart $P \to \Gamma(X, M_X)$ such that $F(X) = \mathrm{Spec}\ P$ and an ideal $I \subset P$ defining $\mathcal{I}$. For $a \in I$, let U_a be the open subscheme $X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P_a^{\mathrm{sat}}] \subset X_{\mathcal{I}}$. Since the map $P_a \to \Gamma(U_a, M_{X_{\mathcal{I}}})$ is a chart on U_a, we have a canonical map $F(U_a) \to \mathrm{Spec}\ P_a^{\mathrm{sat}}$ and it is locally an isomorphism. By patching, we obtain the required map.

We prove Theorem 1.8. More precisely, we show the following.

Theorem 2.9 *1. Let X be a log scheme log smooth and of finite type over O_K. Then there exists an integer $e \geq 1$ such that, if L is a finite separable extension of ramification* *index e, there exists a fractional ideal $\mathcal{I}$ on the base change $X_{O_L} = X \otimes^{\log}_{O_K} O_L$ such that the pair (W, U_W) of log blow-up $W = X_{O_L,\mathcal{I}}$ and its interior U_W is strictly semi-stable over O_L.*

2. Let (X, U_X) be a strictly semi-stable pair over O_K and L be a finite separable extension. Then there exists a fractional ideal $\mathcal{I}$ on the base change $X_{O_L} = X \otimes^{\log}_{O_K} O_L$ such that the pair (W, U_W) of log blow-up $W = X_{O_L,\mathcal{I}}$ and its interior U_W is strictly semi-stable over O_L.

Proof. 1. By Lemma 2.1.2, we may assume X is a Zariski log scheme. Then by Lemmas 2.6, 2.7.2 and 2.8, it follows from Lemma 2.3.2.

2. Similarly as above, by Lemmas 2.6, 2.7.2 and 2.8, it follows from Lemma 2.3.3.

3. Toroidal action on log schemes.

3.1 Complement to SGA 1 Exposé V.

Let X be a scheme over a scheme S and G be a finite group acting on X over S. For a point $x \in X$, the inertia group I_x is the subgroup $\{g \in G | g(x) = x \text{ and } g|_{\kappa(x)} = \mathrm{id}_{\kappa(x)}\}$. For $\sigma \in G$, the fixed locus X^σ is the intersection of the diagonal X with the graph Γ_σ of σ in $X \times_S X$. With this notation, we have $I_x = \{\sigma \in G | x \in X^\sigma\}$. If X is separated over S and $x \in X$, there exists an affine open neighborhood U of x stable under I_x such that the inertia $I_{x'}$ for $x' \in U$ is a subgroup of I_x.

Let $f : X \to Y$ be a morphism of schemes over a scheme S and G be a finite group acting on X and on Y over S. Assume that the actions of G are compatible with f. Then for $x \in X$ and $y = f(x)$, we have $I_x \subset I_y$. If we have an equality $I_x = I_{f(x)}$, we say f preserves the inertia group at x. If f preserves the inertia group at all the points of X, we say f preserves the inertia groups.

Let X be a scheme over a scheme S and G be a finite group acting on X over S. We say the action of G on X is admissible if the ringed space (Y, O_Y), where $\pi : X \to Y = X/G$ is the quotient space and O_Y is the G-fixed part $\pi_* O_X^G$, is a scheme. We call $Y = X/G$ the quotient of X by G. If X is locally of finite presentation over S, the canonical map $\pi : X \to Y$ is finite and Y is locally of finite presentation over S. An action of G is admissible if and only if there exists an open covering $X = \bigcup_{i \in I} U_i$ by affine open subschemes $U_i \subset X$ stable by G.

Lemma 3.1 *Let X be a scheme over a scheme S and G be a finite group acting on X over S.*

1. Let $\bar{x}$ be a geometric point of X and $\bar{x} \to U \to X$ be an etale neighborhood of $\bar{x}$. Then there exists an etale neighborhood $\bar{x} \to U' \to U$ and an action of I_x on U' such that the composite map $U' \to X$ is compatible with the actions of I_x and, at the image x' of $\bar{x}$ in U', we have $I_{x'} = I_x$.

2. Let X' be another scheme over S with an action of G over S and $f : X \to X'$ be a morphism of schemes over S compatible with the actions of G. We assume that X and X' are locally of finite presentations over S. Let x be a point of X.

Then the following conditions (1) and (2) are equivalent.

(1) The morphism f is etale and preserves the inertia group at x.

(2) There exists affine open neighborhoods U of x and U' of $x' = f(x)$ stable under the action of I_x such that $f(U) \subset U'$ and that the restriction $U \to U'$ is etale and preserves the inertia groups with respect to the actions of I_x on U and on U'.

3. With the assumptions in 2, we assume further that the actions of G on X and on X' are admissible. Let $\pi : X \to Y = X/G$ and $\pi' : X' \to Y' = X'/G$ be the quotients. Then, the conditions (1) and (2) in 2 are equivalent to the following condition (3).

(3) The morphism $Y \to Y'$ is etale at $y = \pi(x)$ and the map $X \to X' \times_{Y'} Y$ is an isomorphism on a neighborhood of x.

Proof. 1. Replacing G by I_x, we may assume $G = I_x$. Shirinking X, we may assume $X = \text{Spec } B$ is affine. Let $\pi : X \to Y = X/G = \text{Spec } A$ be the quotient where $A = B^G$ and put $y = \pi(x)$. Let A' and B' be the strict henselizations of A and B respectively at y and x. Since $\pi^{-1}(y) = \{x\}$ and B is integral over A, we have $B' = B \otimes_A A'$. Since $A' = \varinjlim_{\bar{y} \to V \to Y} \Gamma(V, O_Y)$, there exist an etale neighborhood $V \to Y$ of $\bar{y}$ and a morphism $V \times_Y X \to U$ of etale neighborhoods. The group G acts on $U' = V \times_Y X$ via the second factor and we have $I_{x'} = I_x$.

2 and 3. The implications (2)⇒(1) in 2 and (3)⇒(2) in 3 are clear. We show (1)⇒(3) in 3. First we show $Y \to Y'$ is etale at $y = \pi(x)$. Since $X/I_x \to Y$ and $X'/I_x \to Y'$ are etale at the images of x and of x', we may assume $G = I_x$ by replacing G by I_x. By shrinking X and X', we may assume X and X' hence Y and Y' are affine. The induced map $Y \to Y'$ is of finite presentation. Since the question is etale local at y and at $y' = \pi(x')$, we may replace Y and Y' by their strict henselizations at y and at y' respectively. We put $Y = \text{Spec } A$ and $Y' = \text{Spec } A'$ where A and A' are strictly henselian and put $X = \text{Spec } B$ and $X' = \text{Spec } B'$. Since $\pi^{-1}(y) = \{x\}, \pi'^{-1}(y') = \{x'\}$ and B and B' are finite over A and A' respectively, the rings B and B' are strictly henselian local rings. By the assumption that the map $B \to B'$ is etale, it is an isomorphism. Hence the map $A \to A'$ is an isomorphism and the assertion is proved.

We show that the map $X \to Y \times_{Y'} X'$ is an isomorphism on a neighborhood of x. We may assume $Y \to Y'$ is etale. Then the map $X \to Y \times_{Y'} X'$ is finite and etale. Since the map $\pi^{-1}(\bar{y}) = G/I_x \to \pi^{-1}(\bar{y}') = G/I_{x'}$ induced on the inverse images is a bijection, it is an isomorphism on an neighborhood of x.

We show (1)⇒(2) in 2. Replacing G by I_x, we may assume $G = I_x$. Since the question is local, we may assume X and X' are affine and the map $f : X \to X'$ is etale. Then by (1)⇒(3) in 3, the map induced on the quotients $Y = X/G \to Y' = X'/G$ is etale and the map $X \to Y \times_{Y'} X'$ is an isomorphism. Thus the map $X \to X'$ preserves the inertia groups.

Lemma 3.2 *Let $f : X \to S$ be a quasi-projective scheme over a noetherian scheme S and G be a finite group acting on X over S.*

1. The action of G on X is admissible. The quotient $Y = X/G$ is quasi-projective over S. If X is projective over S, the quotient Y is also projective over S.

2. Let $\mathcal{L}$ be an f-ample invertible sheaf on X with an equivariant action of G. Then there exist an integer $N \geq 1$ and an f-ample invertible sheaf $\mathcal{M}$ on Y with a canonical isomorphism $\pi^*\mathcal{M} \to \mathcal{L}^{\otimes N}$ where $\pi : X \to Y$ is the canonical map.

Proof. 1. If $\mathcal{L}$ is an f-ample invertible sheaf, the sheaf $\bigotimes_{g\in G} g^*\mathcal{L}$ is also f-ample. Hence we assume $\mathcal{L}$ is an f-ample invertible sheaf on X with an equivariant action of G and show 1 and 2.

We put $\mathcal{R} = \bigoplus_{n\geq 0} f_*\mathcal{L}^{\otimes n}$ and let $\mathcal{R}^G = \bigoplus_{n\geq 0}(f_*\mathcal{L}^{\otimes n})^G$ be the G-fixed part. We show that the quotient $Y = X/G$ is constructed as an open subscheme of Proj $\mathcal{R}^G$. The question is local on S and we assume S is affine. We put $R = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ and $R^G = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})^G$. For a point $x \in X$, by [5] Corollary (4.5.4), there is an integer $n \geq 1$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that the open subscheme X_s where s is a basis of $\mathcal{L}^{\otimes n}$ is an affine neighborhood of the orbit $Z = Gx$ of x. By replacing s by the norm $\otimes_{g\in G} g^*s$, we may assume s is invariant by G. Hence X has an open covering by G-stable affine open subschemes X_s and the action of G on X is admissible. Since X is quasi-compact, there exist an integer $N \geq 1$ and G-invariant sections $s_1, \ldots, s_r \in \Gamma(X, \mathcal{L}^{\otimes N})^G$ such that $X_{s_1}, \ldots, X_{s_r}$ is an affine open covering of X. Since $X_{s_i} =$ Spec $R[1/s_i]_{\deg 0}$, the quotient $Y = X/G$ is obtained by patching $Y_{s_i} =$ Spec $R^G[1/s_i]_{\deg 0}$ and is an open subscheme of Proj R^G. Since Y has an affine open covering $Y_{s_1}, \ldots, Y_{s_r}$, the sheaf $\mathcal{M} = O(N)$ is an ample invertible O_Y-module and there is a canonical isomorphism $\pi^*\mathcal{M} \to \mathcal{L}^{\otimes N}$. Hence Y is quasi-projective over S. If X is projective, Y is also projective.

Lemma 3.3 *Let $f : X \to S$ be a smooth scheme over a scheme S and let G be a finite group with actions on X and S compatible with f. Let $x \in X$. Assume that the order N of the inertia group I_x is invertible at x and that the action of I_x on the fiber $\Omega^1_{X/S,x} \otimes \kappa(x)$ is the sum of characters $\chi_1, \ldots, \chi_n : I_x \to \mu_d(x)$ for an integer $d \geq 1$ invertible at x. Assume further that the local ring $O_{S,f(x)}$ contains a primitive d-th root of 1 and regard χ_i as characters $I_x \to \mu_d(O_{S,f(x)})$. Then there exist an open neighborhood U of x stable by I_x and functions $t_1, \ldots, t_n \in \Gamma(U, O)$ such that $g(t_i) = \chi_i(g)t_i$ for $g \in I_x$ and $i = 1, \ldots, n$ and that the map $U \to \mathbf{A}^n_S$ defined by $t_1, \ldots, t_n$ is etale.*

Proof. For a character $\chi : I_x \to \mu_d(O_{X,x})$, let e_χ denote the projector $\frac{1}{N}\sum_{g\in I_x} \chi^{-1}(g)g$ and let $e_\chi \cdot O_{X,x}$ and $e_\chi \cdot (\Omega^1_{X/S,x} \otimes \kappa(x))$ be the χ-parts. Then, the $\kappa(x)$-vector space $e_\chi \cdot (\Omega^1_{X/S,x} \otimes \kappa(x))$ is generated by the image of $e_\chi \cdot O_{X,x}$ by the derivation $d : O_{X,x} \to \Omega^1_{X/S,x} \otimes \kappa(x)$. Hence there exist $t_1, \ldots, t_n \in O_{X,x}$ such that $g(t_i) = \chi_i(g)t_i$ and $(dt_1, \ldots, dt_n)$ is a basis of $\Omega^1_{X/S,x} \otimes \kappa(x)$. The assertion follows from this immediately.

3.2 Toroidal action and quotient.

Definition 3.4 *1. Let X be a log schemes and G be a finite group acting on X by automorphisms of a log scheme. We say the action of G is tame at $x \in X$ if the following condition (i) is satisfied.*

(i) The order of the image $\mathrm{Im}(I_x \to \mathrm{Aut}(O_{X,\bar{x}}) \times \mathrm{Aut}(M_{X,\bar{x}}))$ *of the inertia group* I_x *is invertible at* x.

If the action of G *on* X *is tame at every point* $x \in X$, *we say the action of* G *on* X *is tame.*

2. Let $f : X \to Y$ *be a log smooth morphism of log schemes and* G *be a finite group acting on* X *and* Y *as automorphisms of log schemes compatible with* f. *We say the action of* G *is toroidal relative to* f *at* $x \in X$ *if the action is tame at* x *and if the following condition (ii) is satisfied.*

(ii) The action of I_x *on* $\Omega^1_{X/Y}(\log/\log)_x \otimes \kappa(\bar{x})$ *is trivial.*

We say an action of G *on* X *is toroidal relative to* f, *if it is toroidal relative to* f *at each point* $x \in X$.

If X is log regular, the condition (i) is equivalent to the following apparently weaker condition (i')

(i') The order of the image $\mathrm{Im}(I_x \to \mathrm{Aut}(O_{X,\bar{x}}))$ is invertible at x.

Lemma 3.5 *Let* $f : X \to Y$ *be a log smooth morphism of log schemes over a scheme* S *and* G *be a finite group acting on* X *and* Y *as automorphisms of log schemes over* S *compatible with* f.

1. Assume X *is separated over* S. *Then if the action of* G *is toroidal relative to* f *at* $x \in X$, *there is an open neighborhood* U *of* x *stable under* G *such that the action of* G *is toroidal relative to* f *on* U.

2. Let Z *be a log scheme over* S *and* $g : Y \to Z$ *be a log smooth map over* S. *Assume* G *acts on* Z *as automorphisms of a log scheme over* S *and the action is compatible with* g. *For* $x \in X$, *if the action of* G *is toroidal relative to* f *at* $x \in X$ *and toroidal relative to* g *at* $y = f(x)$, *the action of* G *is toroidal relative to* $g \circ f$ *at* x.

Proof. 1. By the assumption that X is separated over S, on a neighborhood U of x, we have $I_{x'} \subset I_x$ for $x' \in U$. By considering $\bigcup_{g \in G} gU$, we may assume $G = I_x$. By replacing G by the image $\mathrm{Im}(I_x \to \mathrm{Aut}(O_{X,\bar{x}}) \times \mathrm{Aut}(M_{X,\bar{x}}))$ and shrinking X, we may assume the order N of G is invertible on X. Then the action of G is tame. Let $(\omega_1, \dots, \omega_n)$ be a basis of $\Omega^1_{X/Y}(\log/\log)$ at x. We put $\omega'_i = \frac{1}{N}\sum_{g\in G} g^*\omega_i$. Since the action of G on $\Omega^1_{X/Y}(\log/\log)_x \otimes \kappa(x)$ is trivial, we have $\omega_i \equiv \omega'_i$ and $(\omega'_1, \dots, \omega'_n)$ is a G-invariant basis of $\Omega^1_{X/Y}(\log/\log)$ at x. Thus the action of G is toroidal on an open neighborhood of x.

2. Assume that the action of G is toroidal relative to f at $x \in X$ and toroidal relative to g at $y = f(x)$. Then the action is tame at x. By the exact sequence $0 \to \Omega^1_{Y/Z}(\log/\log)_y \otimes \kappa(x) \to \Omega^1_{X/Z}(\log/\log)_x \otimes \kappa(x) \to \Omega^1_{X/Y}(\log/\log)_x \otimes \kappa(x) \to 0$, the action of I_x on $\Omega^1_{X/Z}(\log/\log)_x \otimes \kappa(x)$ is trivial since the action is tame at x.

We give an characterization of toroidal action, similar to Proposition 1.2.

Proposition 3.6 *Let* S *be a log scheme and* $f : X \to S$ *be a log scheme log smooth over* S. *Let* G *be a finite group acting on* X *as automorphisms of a log scheme* X *over*

S. Let x be a point of X, $V = \mathrm{Spec}\ R \to S$ be an affine etale neighborhood of $s = f(x)$ and $Q \to \Gamma(V, O_S)$ be a chart on V. Then the following conditions (1) and (2) are equivalent.

(1) The action of G on X is toroidal relative to f at x.

(2) There exist an affine etale neighborhood $\varphi : U = \mathrm{Spec}\ A \to X$ of x, a map $U \to V$ compatible with f and an action of I_x on U as automorphisms over V such that the map $U \to X$ is compatible with the actions of I_x and that the inertia group $I_{x'}$ at a point $x' \in \varphi^{-1}(x)$ is I_x. Further, there exist a chart $P \to \Gamma(U, O_X)$, a map of monoids $Q \to P$ compatible with the map $U \to V$ and a bimultiplicative map $(\ ,\ ) : I_x \times P \to \mu_d(R)$ where d is an integer invertible in R, such that the pairing $(\ ,\ )$ is trivial on the image of Q and that the following conditions (a) and (b) satisfied.

(a) $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective and the order of the torsion part of its cokernel is invertible on U.

(b) We define an action of I_x on $R \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P]$ by $g(a \otimes p) = (g, p)a \otimes p$. Then the map $U \to V \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P]$ is (classically) etale and compatible with the actions of I_x.

Proof of Proposition 3.6. (2)$\Rightarrow$(1). Since the question is etale local at x, we may replace G by I_x, X by U and S by V. We may further replace U by $V \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P]$. Then the image of $I_x \to \mathrm{Aut}(O_{X,\bar{x}}) \times \mathrm{Aut}(M_{X,\bar{x}})$ is a quotient of the image of $I_x \to Hom(P, \mu_d(R))$ and its order is invertible at x. The canonical map $(P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \otimes \kappa(\bar{x}) \to \Omega^1_{X/Y}(\log/\log)_x \otimes \kappa(\bar{x})$ is an isomorphism and hence the action of I_x is trivial.

(1)$\Rightarrow$(2). We follow the proof (3.13) of Theorem (3.5) [6]. By Theorem (3.5) and Remark (3.6) loc.cit., there exist an affine etale neighborhood $U = \mathrm{Spec}\ A \to X$ of x, a map $U \to V$ compatible with f, a chart P on U and a map of monoids $Q \to P$ compatible with the map $U \to V$ satisfying the conditions (a) above and

(b') The map $U \to V \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P]$ is (classically) etale.

By Lemma 3.1, we may assume that U has an action of I_x as automorphisms over V and that the map $U \to X$ is compatible and preserves the inertia groups with respect to the actions of I_x.

To modify the chart P so that the condition (b) is satisfied, we define a bimultiplicative map $(\ ,\ )_x : I_x \times \bar{M}_{\bar{x}} \to R^\times$. First, we show that the action of I_x on $\bar{M}_{\bar{x}}$ is trivial. Since there is a canonical surjection $\Omega^1_{X/S}(\log/\log)_x \otimes \kappa(\bar{x}) \to (\bar{M}^{\mathrm{gp}}_{\bar{x}}/\bar{M}^{\mathrm{gp}}_{f(\bar{x})}) \otimes \kappa(\bar{x})$, the action of I_x on $(\bar{M}^{\mathrm{gp}}_{\bar{x}}/\bar{M}^{\mathrm{gp}}_{f(\bar{x})}) \otimes \kappa(\bar{x})$ is trivial. Since the order of the inertia is invertible at x, the action of I_x on $\bar{M}^{\mathrm{gp}}_{\bar{x}}/\bar{M}^{\mathrm{gp}}_{f(\bar{x})}$ is trivial. Since the action of I_x on $\bar{M}^{\mathrm{gp}}_{f(\bar{x})}$ is trivial, the action of I_x on $\bar{M}^{\mathrm{gp}}_{\bar{x}}$ is also trivial. Since I_x acts trivially on $\bar{M}_{\bar{x}}$, the map $I_x \times M_{\bar{x}} \to O^\times_{X,\bar{x}} : (g, m) \mapsto g(\alpha(m))/\alpha(m)$ is defined. It induces a bimultiplicative map $(\ ,\ )_x : I_x \times \bar{M}^{\mathrm{gp}}_{\bar{x}} \to \kappa(\bar{x})^\times$. By the assumption that the order of the image $I_x \to \mathrm{Aut}(M_{X,\bar{x}})$ is invertible at x, the image of the pairing is $\mu_d(\bar{x})$ for an integer $d \geq 1$ invertible at x. We identify $\mu_d(\bar{x})$ with $\mu_d(R)$ by shrinking V. Thus we obtain a pairing $(\ ,\ ) : I_x \times M_{\bar{x}} \to \mu_d(R) \subset R^\times$.

We take $t_1, \ldots, t_r \in M_{\bar{x}}$ such that $(d\log t_1, \ldots, d\log t_r)$ is a basis of $\Omega^1_{X/S}(\log/\log)_{\bar{x}}$ as in the proof (3.13) of Theorem (3.5) [6]. By the argument there, it is sufficient to show that, by modifying $t_1, \dots, t_r$, we may assume $g(t_i) = (g, t_i)t_i$ for $i = 1, \dots, r$. We put $u_{g,i} = g(t_i)/t_i$. Since I_x acts on $\bar{M}_{\bar{x}}$ trivially, it is a unit and congruent to (g, t_i). Hence $u_i = \frac{1}{|I_x|} \sum_{g \in I_x} (g, t_i)^{-1} u_{g,i}$ is a unit congruent to 1. We put $t'_i = u_i t_i$. Then we have $g(t'_i) = (g, t'_i)t'_i$. Hence it is sufficient to show that $d \log t'_i \equiv d \log t_i$ in $\Omega^1_{X/S}(\log / \log)_{\bar{x}} \otimes \kappa(\bar{x})$. By the assumption that the action of I_x on $\Omega^1_{X/S}(\log / \log)_{\bar{x}} \otimes \kappa(\bar{x})$ is trivial, we have $d \log u_{g,i} \equiv 0$ in $\Omega^1_{X/S}(\log / \log)_{\bar{x}} \otimes \kappa(\bar{x})$. Hence we have $d \log t'_i - d \log t_i = d \log u_i \equiv du_i = \frac{1}{|I_x|} \sum_{g \in I_x} (g, t_i)^{-1} du_{g,i} \equiv 0$ and the assertion follows.

Proposition 3.7 *Let S and X be locally noetherian log regular schemes and $f : X \to S$ be a log smooth morphism. Let G be a finite group acting on X as automorphisms of a log scheme X over S. Assume that the action of G is admissible and let $\pi : X \to Y = X/G$ be the quotient. Let U_X be the interior of X and put $U_Y = \pi(U_X)$. Then the following conditions are equivalent.*

(1) The action of G on X is toroidal relative to f.

(2) The pair (Y, U_Y) is toric, the map $Y \to S$ is log smooth and the map $X \to Y$ is log etale.

We call the log structure on Y defined by U_Y the standard log structure.

Proof. (2)⇒(1). To show this we only need to assume that (Y, U_Y) is toric and that $X \to Y$ is log etale. The question is etale local. Let $\bar{x}$ be a geometric point of X. Replacing Y by the spectrum of the strict henselization at $\bar{y} = \pi(\bar{x})$, we may assume Y is strictly local. Replacing G by I_x, we may assume $G = I_x$. Then X is the spectrum of the strict henselization at $\bar{x}$. Let $P = \overline{M}_{X,\bar{x}}$ and $Q = \overline{M}_{Y,\bar{y}}$ and take a chart $Q \to O_{Y,\bar{y}}$. Then, by the proof of Theorem 3.3 [9] p.57, there is an isomorphism $O_{Y,\bar{y}} \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P] \to O_{X,\bar{x}}$ where the map $Q \to P$ is injective and the index $[P^{\rm gp} : Q^{\rm gp}]$ is finite and invertible at x. Since the order of $\mathrm{Im}(I_x \to \mathrm{Aut}(O_{X,x}))$ is equal to $[X : Y] = [P^{\rm gp} : Q^{\rm gp}]$, the action is tame. Since $X \to Y$ is log etale, the map $\pi^*\Omega^1_{Y/S}(\log / \log) \to \Omega^1_{X/S}(\log / \log)$ is an isomorphism. Hence the action is toroidal.

(1)⇒(2). The question is etale local on Y. Let $x \in X$. We take etale neighborhood $U = \mathrm{Spec}\ A \to X$ of x and $V = \mathrm{Spec}\ R \to S$ of $f(x)$ and charts P on U and Q on V as in Proposition 3.6. Since $U/I_x \to Y = X/G$ is etale, we may assume $X = U, S = V$ and $G = I_x$ by replacing X, S and G by U, V and I_x. By Lemma 3.1.1, we may assume $A = R \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P]$. Replacing G further by its image in $Hom(P, \mu_d(R))$, we may assume that the order of G is invertible in R. We define an fs-monoid $P_0 \subset P$ by $P_0 = \{p \in P | (g, p) = 1 \text{ for all } g \in G\}$. The natural map $G \to Hom(P^{\rm gp}/P_0^{\rm gp}, \mu_d(R))$ is an isomorphism.

We put $Y' = \mathrm{Spec}\ R \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P_0]$ and consider Y' as a log scheme with the log structure defined by the chart $P_0 \to R \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P_0]$. The log scheme Y' is log smooth over S and hence is log regular. By the natural map $X \to Y'$, the log scheme X is a finite and log etale Galois covering of Y' of Galois group $G \simeq Hom(P^{\rm gp}/P_0^{\rm gp}, \mu_d(R))$. Hence we have an isomorphism $Y \to Y'$ of schemes. Since the interior of Y' is the image of U_X, the assertion follows.

3.3 Nodal curves and toroidal action.

Let G be a finite group with a tame action on a nodal curve $f : X \to Y$ over a log regular scheme Y. Then under a certain mild hypothesis, the points of X where the action are not toroidal is classified as follows.

Proposition 3.8 *Let S be a scheme, (Y, U_Y) be a toric pair over S and $(f : X \to Y, D)$ be a separated pointed nodal curve over Y such that f is smooth over U_Y. We regard X as a log scheme with the standard log structure. Let G be a finite group with tame actions on the log schemes X and on Y over S compatible with f. Let $\Phi \subset X$ be the subset consisting of the points where the action of G is not toroidal relative to f. Then,*

1. The subset $\Phi \subset X$ is closed. The intersection $\Phi \cap D$ is empty. If $x \in \Phi \cap (X - \Sigma)$, the action of I_x on $\Omega^1_{X/Y,x} \otimes \kappa(x)$ is non trivial. If $x \in \Phi \cap \Sigma$, the action of I_x on $Br(\bar{x})$ is non trivial. The intersection $\Phi \cap \Sigma$ is a subset of $X[\frac{1}{2}]$.

2. Assume S is a locally noetherian log regular scheme, Y is log smooth over S, the action of G on Y is toroidal over S and $\Phi \cap \Sigma = \emptyset$. Then, there exists a coherent ideal $\mathcal{I}_\Phi$ of O_X characterized by the following conditions.

The support of $O_X/\mathcal{I}_\Phi$ is Φ. For $x \in \Phi$, let $\chi_x : I_x \to \kappa(x)^\times$ be the character defined by the action on $\Omega^1_{X/Y,x} \otimes \kappa(x)$. Then $\mathcal{I}_{\Phi,\bar{x}}$ is generated by $\{f \in O_{X,\bar{x}} | g(f) = \chi_x(g) f$ for all $g \in I_x\}$.
We regard Φ as a closed subscheme of X defined by the ideal $\mathcal{I}_\Phi$. Then Φ is neat over Y; $\Omega^1_{\Phi/Y} = 0$. The subsets $\Phi_1 = \{x \in \Phi | \Phi \to Y$ is etale at $x\}$ and the complements $\Phi_2 = \Phi - \Phi_1$ are open (and closed) subsets of Φ.

Proof. 1. By the assumption that X is separated over S, the complement $X - \Phi$ is open by Lemma 3.5.1.

On D, we have a canonical isomorphism $\Omega^1_{X/Y}(\log / \log)|_D \to O_D$. Hence $\Phi \cap D = \emptyset$.

On $X - \Sigma$, we have $\Omega^1_{X/Y}(\log / \log)|_{X-\Sigma} = \Omega^1_{X/Y}|_{X-\Sigma}$. Hence the assertion for $x \in \Phi \cap (X - \Sigma)$ follows.

On Σ, we have a canonical isomorphism $\Omega^1_{X/Y}(\log / \log)|_\Sigma \to (\bar{M}^{\rm gp}_X / f^* \bar{M}^{\rm gp}_Y)|_\Sigma \otimes_{\mathbf{Z}} O_\Sigma$. At a geometric point $\bar{x}$ of Σ, the stalk $\bar{M}^{\rm gp}_{X,\bar{x}} / \bar{M}^{\rm gp}_{X,f(\bar{x})}$ is identified with the quotient of $\mathbf{Z}^{Br(\bar{x})}$ divided by the diagonal. Thus the description of the points in $\Phi \cap \Sigma$ follows. We show $\Phi \cap \Sigma \subset \Sigma[\frac{1}{2}]$. If $x \in \Phi \cap \Sigma$, the map $I_x \to \mathrm{Aut}(\bar{M}^{\rm gp}_{X,\bar{x}} / \bar{M}^{\rm gp}_{X,f(\bar{x})}) = \{\pm 1\}$ is surjective and hence 2 is invertible at x by the assumption that the action is tame.

2. The question is etale local on X. First, we give a local description of Y. Since $\Phi \cap \Sigma = \emptyset$ and Φ is closed, by replacing X by $X - \Sigma$, we may assume X is smooth over Y. Let $x \in \Phi$. Shrinking X, we may assume that $I_{x'} \subset I_x$ for $x' \in X$. Replacing G by I_x, we may assume $G = I_x$. By Proposition 3.6 (1)$\Rightarrow$(2), we may assume the following: The scheme $S = \mathrm{Spec}\ R$ is affine and the log structure on S is defined by a chart $Q \to R$. There exist a map of fs-monoids $Q \to P$, an integer $d \geq 1$ invertible in R and a bimultiplicative map $(\ ,\ ) : G \times P \to \mu_d(R)$ which is trivial on the image of Q and satisfies the condition (a) in Proposition 3.6. The scheme Y is $\mathrm{Spec}\ R \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P]$ and the action of G on Y is defined by by $g(a \otimes p) = (g, p) a \otimes p$.

We give a local description of X. We put $A = R \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P]$ so that $Y = \mathrm{Spec}\ A$. Replacing d by a multiple if necessary, we may assume that the character $\chi_x : G \to \kappa(x)^\times$ has values in $\mu_d(x)$. Shrinking S if necessary, we identify $\mu_d(x) = \mu_d(R)$ and regard the character χ_x has values in $\mu_d(R)$. We put $B = A[t]$ and define an action of G on B extending the action on A by $g(t) = \chi(g)t$ for $g \in G$. By Lemma 3.3, shrinking X further, we may take $t_1 \in \Gamma(X, O)$ such that the map $\psi : X \to \mathbf{A}^1_Y = \mathrm{Spec}\ B$ defined by $t \mapsto t_1$ is etale and compatible with the action of G. Since $I_{\psi(x)} = I_x$, shrinking X if necessary, we may assume that the map ψ preserves the inertia groups by Lemma 3.1.2. By replacing X by $\mathbf{A}^1_Y$, we may assume $X = \mathrm{Spec}\ B$.

We put $\tilde{P} = P \times \mathbf{N}$ and identify $B = R \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[\tilde{P}]$. We will define an ideal I of $\tilde{P}$ and show $\mathcal{I}_\Phi = IO_X$. We define a bimultiplicative map $(\ ,\ ) : G \times \tilde{P} \to \mu_d(R)$ extending $G \times P \to \mu_d(R)$ by putting $(\ , t) = \chi_x$. Replacing G by its image in $Hom(\tilde{P}, \mu_d(R))$, we regard G as a subgroup of $Hom(\tilde{P}, \mu_d(R))$. For $b \in \tilde{P}$, let $\chi_b : G \to \mu_d(R)$ denote the character defined by $\chi_b(g) = (g, b)$. We define an ideal I of $\tilde{P}$ to be that generated by $\tilde{P}_\chi = \{b \in \tilde{P} | \chi_b = \chi\}$. The ideal I is generated by $P_\chi = \{b \in P | \chi_b = \chi\}$ and by t.

We show $\mathcal{I}_\Phi = IO_X$. Changing the notation, let x denote an arbitrary point of X and $y = f(x)$. First we consider the case $t(x) \neq 0$. In this case, we have $IO_{X,\bar{x}} = O_{X,\bar{x}}$. On the other hand, we have $I_x \subset \mathrm{Ker}\ \chi_t$ and $x \notin \Phi$. Hence we have also $\mathcal{I}_{\Phi,\bar{x}} = O_{X,\bar{x}}$. We assume $t(x) = 0$. We define a prime ideal p of $\tilde{P}$ by $p = \{b \in \tilde{P} | \alpha(b)(x) = 0\}$. Then the inertia group I_x is equal to $H = \{g \in G | (g, b) = 1 \text{ for all } b \in \tilde{P} - p\}$. We put $\chi = \chi_t$ and let χ' be the restriction of χ to H. Then the ideal $\mathcal{I}_{\Phi,\bar{x}}$ is generated by the image of $\tilde{P}_{p,\chi'} = \{b \in \tilde{P}_p | \chi_b|_H = \chi'\}$ where $\tilde{P}_p$ denotes the localization at p. Hence it suffices to show the equality $\tilde{P}_{p,\chi'} = \tilde{P}_\chi \cdot \tilde{P}_p^\times$. Since H is the kernel of the map $G \to Hom(\tilde{P} - p, \mu_d)$, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{P}_p^\times & \longrightarrow & \tilde{P}^{\mathrm{gp}} & \xrightarrow{\pi} & \tilde{P}^{\mathrm{gp}}/\tilde{P}_p^\times & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
 & & (\tilde{P} - p)^{\mathrm{gp}} & \longrightarrow & Hom(G, \mu_d) & \longrightarrow & Hom(H, \mu_d) & \longrightarrow & 0.
\end{array}$$

Since $\tilde{P}_p = \pi^{-1}(\pi(\tilde{P}))$, we have $\tilde{P}_{p,\chi'} = \tilde{P}_\chi \cdot \tilde{P}_p^\times$ by diagram chasing. Thus the assertion is proved.

Since Φ is a subscheme of a section of X over Y, it is neat over Y. It is clear that Φ_1 is open in Φ. The subset Φ_1 is the closure of the inverse images of generic points of Y and is also closed.

Let G be a finite group with a tame action on a nodal curve $f : X \to Y$ over a log regular scheme Y. Then under a certain mild hypothesis, we can modify X so that the action of G is toroidal relative to f.

Proposition 3.9 *Let the assumption be the same as in Proposition 3.8. Then,*

1. Assume that the following condition is satisfied for each $x \in \Sigma$.

(s) There exist an etale neighborhood U of x, an open neighborhood V of $y = f(x)$ containing the image of U, a section $w \in \Gamma(V, O_Y)$ and an etale morphism $U \to V[S,T]/(ST - w^2)$ over Y.

Then there exists a coherent ideal $\mathcal{I}_\Sigma$ of M_X characterized by the following condition at each $x \in X$.

The quotient $M_X/\mathcal{I}_\Sigma$ is supported on Σ. If $x \in \Sigma$, the stalk $\mathcal{I}_{\Sigma,\bar{x}}$ is generated by S, T, w in the notation above.

The log blow-up $X_\Sigma \to X$ by the ideal $\mathcal{I}_\Sigma$ is a nodal curve over Y and the action of G is canonically extended on X_Σ. We regard X_Σ as a log scheme over Y with the standard log structure. Then the set $\Phi(X_\Sigma) \cap \Sigma(X_\Sigma)$ is empty.

2. Assume that S is a locally noetherian log regular scheme, Y is log smooth over S, the action of G on Y is toroidal over S and that $\Phi(X) \cap \Sigma(X) = \emptyset$. Let $\mathcal{I}_\Phi$ be the ideal defined in Proposition 3.8.2 and $\varphi : X_\Phi \to X$ be the normalization of the blow-up by $\mathcal{I}_\Phi$. We put $U_{X_\Phi} = \varphi^{-1}(f^{-1}(U_Y) \cap (X - (D \cup \Phi_1)) \subset X_\Phi$. Then (X_Φ, U_{X_Φ}) is a toric pair. We regard X_Φ as a log regular scheme with the log structure M_{X_Φ} defined by U_{X_Φ}. Then the map $X_\Phi \to Y$ is log smooth, the action of G is canonically extended on X_Φ and the action of G on X_Φ is toroidal relatively to the composition $X_\Phi \to X \to Y$.

We call the log structure on X_Φ defined by U_{X_Φ} the standard log structure.

Proof. 1. We verify that the coherent ideal $\mathcal{I}_\Sigma$ is well-defined. Since the question is etale local, we may assume X is etale over $Y[S,T]/(ST - w^2)$. It is sufficient to show that S, T and w are uniquely determined modulo units upto ordering. Since X and Y are normal, functions on X and on Y are determined modulo units by the discrete valuations at the points of codimension 1. Shrinking Y, we may assume Y is the spectrum of a discrete valuation ring R. Then upto ordering, the divisors of S and T are the irreducible components of the closed fibers. The ideal $(w^2) \subset R$ is the annihilator of $\Omega^2_{X/Y}$. Hence the ideal $\mathcal{I}_\Sigma$ is well-defined and is coherent.

We show the assertions on X_Σ. It is clear that X_Σ is a nodal curve over Y and that the action of G is canonically extended on X_Σ. We show $\Phi(X_\Sigma) \cap \Sigma(X_\Sigma)$ is empty. Assume $x' \in \Phi(X_\Sigma) \cap \Sigma(X_\Sigma)$. Then the image x of x' is in $\Sigma(X)$. One of the branches at $\bar{x}'$ is exceptional and the other is the proper transform of a branch at x. Hence the action of $I_{x'}$ on $Br(\bar{x}')$ is trivial and the assertion follows.

2. The question is etale local on X. Since $\Phi \cap \Sigma = \emptyset$ and Φ is closed, by replacing X by $X - \Sigma$, we may assume X is smooth. Similarly, we may assume $D = \emptyset$. It is sufficient to show the claim for $X - \Phi_2$ and $X - \Phi_1$. In other words, it is sufficient to consider the cases $\Phi = \Phi_1$ and $\Phi = \Phi_2$ respectively. Assume $\Phi = \Phi_1$. Then Φ_1 is a divisor of X etale over Y. Hence the underlying scheme of X_Φ is the same as that of X and the log scheme X_Φ with the log structure M_X defined by $U_{X_\Phi} = f^{-1}(U_Y) \cap (X - \Phi_1)$ is log smooth over Y. Since the restriction $\Omega^1_{X_\Phi/Y}(\log/\log) \otimes_{O_X} O_\Phi$ is canonically isomorphic to O_Φ by the residue map, the action of G on X is toroidal relatively to Y.

We assume $\Phi = \Phi_2$. As in the proof of Proposition 3.8.2, we may assume $Y =$ Spec $A, X =$ Spec B where $A = R \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P], B = R \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[\tilde{P}]$ in the notation loc.cit. Let $I \subset \tilde{P}$ denote the ideal generated by P_χ and t as loc.cit. We define an ad hoc log structure M'_X on X by the chart $\tilde{P} = P \times \mathbf{N} \to B$. It is log smooth over Y. We define a coherent ideal $\mathcal{I}'$ of M'_X to be that generated by the ideal $I \subset \tilde{P}$. Since $\mathcal{I}_\Phi = \mathcal{I}' O_X$, the normalization X_Φ is identified with the underlying scheme of the log blow-up by the coherent ideal $\mathcal{I}'$ by Lemma 2.1.1. Hence the scheme X_Φ has an open covering by $U_t = X \otimes_{\mathbf{Z}[\tilde{P}]} \mathbf{Z}[\tilde{P}_t^{\rm sat}]$ and $U_a = X \otimes_{\mathbf{Z}[\tilde{P}]} \mathbf{Z}[\tilde{P}_a^{\rm sat}]$ for $a \in P_\chi$ where $\tilde{P}_t = \bigcup t^{-n}I^n = P\langle P_\chi/t\rangle$ and $\tilde{P}_a = \bigcup a^{-n}I^n = P\langle P_\chi/a\rangle \times \langle t/a\rangle$ are submonoids of $\tilde{P}^{\rm gp}$.

We show that the pair (X_Φ, U_{X_Φ}) is toric and is log smooth over Y. Since we are assuming $D = \Phi_1 = \emptyset$, we have U_{X_Φ} is the inverse image $(f\circ\varphi)^{-1}(U_Y)$ of U_Y. First, we show that the pair $(U_t, U_t \cap (f\circ\varphi)^{-1}(U_Y))$ is toric and is log smooth over Y. Let M'_{X_Φ} be the log structure on X_Φ defined as the log blow-up of X. Since X_Φ with M'_{X_Φ} is log smooth over X, it is sufficient to show that the interior of U_t with respect to the log structure M'_{X_Φ} is the inverse image of U_Y. By assumption $\Phi = \Phi_2$, the set P_χ is not empty and we have $\tilde{P}_t^{\rm gp} = \tilde{P}_t \cdot P^{\rm gp}$. Hence the interior of U_t with respect to the log structure M'_{X_Φ} is the same as the inverse image of U_Y.

We show that the pair $(U_a, U_a \cap (f\circ\varphi)^{-1}(U_Y))$ is toric and is log smooth over Y for $a \in P_\chi$. Let $I_0 = \langle P_\chi\rangle \subset P$ be the ideal $I \cap P$ and put $P_a = \bigcup a^{-n}I_0^n = P\langle P_\chi/a\rangle \subset P^{\rm gp}$. Then we have $\tilde{P}_a = P_a \times \langle t/a\rangle$. Hence U_a is isomorphic to $Y \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P_a^{\rm sat}][t/a]$. The scheme $Y_a = Y \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P_a^{\rm sat}]$ with the log structure defined by the chart $P_a^{\rm sat}$ is log etale over Y and the scheme U_a with the log structure M_{U_a} defined by the chart $P_a^{\rm sat}$ is log smooth over Y. The interior of U_a with respect to the log structure M_{U_a} is the same as the inverse image of U_Y.

We show that the action of G on X_Φ is relatively toroidal over Y. It is sufficient to show this on U_t and on U_a for $a \in P_\chi$. First, we consider U_t. The action of G on (X, M'_X) is relatively toroidal since the basis $d\log t$ of $\Omega^1_{(X,M'_X)/Y}(\log/\log)$ is invariant under G. Since the map $U_t \to (X, M'_X)$ is log etale, the pull-back of $d\log t$ is a basis of $\Omega^1_{U_t/Y}(\log/\log)$. Hence the action of G on U_t is relatively toroidal. We consider U_a for $a \in P_\chi$. Since Y_a is log etale over Y, the log differential $d\log(t/a)$ is a basis of of $\Omega^1_{U_a/Y}(\log/\log) = \Omega^1_{U_a/Y}$. Since t/a is invariant under G, the action of G on U_a is also relatively toroidal.

4 Proof of Theorem 1.3.

4.1 Proof of Theorem 1.3 (3)⇒(1).

In this subsection, we show the following.

Lemma 4.1 *In Theorem 1.3, we have (3)⇒(1) if $N \geq 3$ and if either of the following conditions is satisfied.*

(a) 2 is invertible and Y is quasi-compact.

(b') There is no closed subset of $D_Y = Y - U$ which is a subset of $D_Y[\frac{1}{2}]$.

Proof. Let $N \geq 3$ be an integer invertible on Y and assume the condition (3). First we show that there is a finite etale Galois covering $U_{Y'} \to U_Y$ such that the normalization $Y' \to Y$ in $U_{Y'}$ satisfies the condition: $(Y', U_{Y'})$ is a toric pair log etale over (Y, U_Y) and the pull-back $(f : X_U \to U, D_U) \times_U U_{Y'}$ is extended to a stable curve over Y'. By (3), there is a finite etale Galois covering $U_{Y'_1} \to U_Y$ that trivializes the finite etale covering $D_U \to U$ and is tamely ramified at each generic point of D_Y. Let $U_{Y'_2}$ be the finite etale Galois covering $\mathrm{Isom}_{U_Y}((\mathbf{Z}/N\mathbf{Z})^{2g}, J_{N,U_Y})$ of U_Y. Then by (3), the covering $U_{Y'_2}$ is tamely ramified at each generic point of D_Y. Let Y' be the normalization of Y in the finite etale Galois covering $U_{Y'_1} \times_{U_Y} U_{Y'_2}$ tamely ramified at each generic point of D_Y. Then by the log purity theorem, Theorem B [9], the pair $(Y', U_{Y'})$ is a toric pair and log etale over (Y, U_Y). Further, by the extension theorem, Theorem A loc.cit., the pull-back $(f : X_U \to U, D_U) \times_U U_{Y'}$ is uniquely extended to a stable curve $(f' : X' \to Y', D_{Y'})$ over Y'.

Let G be the Galois group of the covering $Y' \to Y$. By the uniqueness of the extension, the curve $f' : X' \to Y'$ has a natural action of G. We regard X' as a log scheme with the standard log structure. The map $f' : X' \to Y'$ is log smooth. We will construct X by taking the quotient by G after modifying X' so that the action of G is toroidal relative to the map to Y'. By Proposition 3.7 (2)⇒(1), the action of G on Y' is toroidal relative to $Y' \to Y$. Since $I_x \subset I_y$ for $x \in X'$ and $y = f'(x)$, the action of G on X' is tame. Hence the assumption of Proposition 3.8 is satisfied.

We construct $X \to Y$ assuming that the stable curve $f' : X' \to Y'$ satisfies the condition (s) in Proposition 3.9.1. Let $\Sigma \subset X'$ be the non-smooth locus of the nodal curve $X' \to Y'$ and $\sigma : X'_\Sigma \to X'$ be the log blow-up constructed in Proposition 3.9.1. The map $\sigma : X'_\Sigma \to X'$ is an isomorphism on the inverse image of $U_{Y'}$. We show that the etale part $\Phi_1 = \Phi_1(X'_\Sigma)$ in Proposition 3.8.2 is empty. Since the action of G on the pull-back $X_{U_{Y'}} = X_U \times_{U_Y} U_{Y'}$ is free, we have $\Phi \cap (X' \times_{Y'} U_{Y'}) = \emptyset$ and $\Phi_1 = \emptyset$. Let $\varphi : (X'_\Sigma)_\Phi \to X'_\Sigma$ be the blow-up at $\Phi = \Phi_2(X'_\Sigma)$ constructed in Proposition 3.9.2. The map $\varphi : (X'_\Sigma)_\Phi \to X'_\Sigma$ is also an isomorphism on the inverse image of $U_{Y'}$. We put $X_{Y'} = (X'_\Sigma)_\Phi$. Since $\Phi_1 = \emptyset$ as shown above, the standard log structure of $X_{Y'}$ is defined by $(\varphi \circ \sigma)^{-1}(X_{U_{Y'}} - D_{U_{Y'}}) \subset X_{Y'}$. The action of G on X' is canonically extended to an action of G on $X_{Y'}$. By Proposition 3.9, the action of G on $X_{Y'}$ is toroidal relatively to the composition map $f_{Y'} : X_{Y'} \to Y'$.

We show that the action of G on $X_{Y'}$ is admissible and the quotient $X = X_{Y'}/G$ with the induced map $f : X \to Y = Y'/G$ satisfies the required properties. The scheme X' is projective over Y' and hence over Y. By Lemma 3.2.1, the action of G on $X_{Y'}$ is admissible and the quotient $X = X_{Y'}/G$ is projective over Y. Applying Proposition 3.7(1)⇒(2) to the composition $X' \to Y' \to Y$, the map $f : X \to Y$ is log smooth with respect to the standard log structure on X. Since the log structure of $X_{Y'}$ is defined by $(\varphi \circ \sigma)^{-1}(X_{U_{Y'}} - D_{U_{Y'}})$, the standard log structure of X is defined by $X_{U_Y} - D_{U_Y}$. Thus $f : X \to Y$ satisfies the required properties.

For later use, we construct an f-ample invertible sheaf on X. Since $(f' : X' \to Y', D')$ is a stable curve, the invertible sheaf $\mathcal{L}_{X'} = \Omega^1_{X'/Y'}(\log / \log)$ is f'-ample. The sheaf $\mathcal{L}_{X'}$ has a natural equivariant action of G. Let $O_\Phi(1)$ and $O_\Sigma(1)$ be the relatively very ample sheaf on $X_{Y'} = (X'_\Sigma)_\Phi$ over X'_Σ and on X'_Σ over X' respectively and $\pi : X_{Y'} \to X$ be the canonical morphism. The invertible $O_{X_{Y'}}$-module $\mathcal{L}^{(\mu,\nu)}_{X_{Y'}} = O_\Phi(1) \otimes \varphi^*(O_\Sigma(1) \otimes \sigma^* \mathcal{L}^{\otimes \nu}_{X'})^{\otimes \mu}$ is relatively ample over Y' and hence over Y for sufficiently large $\mu, \nu \geq 1$ and has a natural equivariant action of G. By Lemma 3.2.2, there exist an integer $N \geq 1$ and an invertible O_X-module $\mathcal{M}^{(\mu,\nu,N)}$ with a canonical isomorphism $\pi^* \mathcal{M}^{(\mu,\nu,N)} \to \mathcal{L}^{(\mu,\nu)\otimes N}_{X_{Y'}}$.

We complete the proof of the case (b) by showing that the condition (b) for D_Y implies the condition (s) in Proposition 3.9.1 for $f' : X' \to Y'$. Since $\Sigma \cap \Phi$ is proper over D_Y and is a subset of $X'[\frac{1}{2}]$, its image is a closed subset of D and is a subset of $D[\frac{1}{2}]$. Hence, if the condition (b) is satisfied, the set $\Sigma \cap \Phi$ is empty and the condition (s) is satisfied.

To prove the case (a), we consider the condition (s') below. We introduce a notation. Let V be a locally noetherian log regular scheme, $P \to \Gamma(V, O_V)$ be a chart on V and n be an integer invertible on V. Then, let V_n denote the finite log etale covering $V_n = V \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P]$ of V defined by the map of monoids $\times n : P \to P$.

(s') There exist an integer $n \geq 1$ invertible on Y, an etale covering $(V_i \to Y)_{i=1,\dots,m}$ and charts $P_i \to \Gamma(V_i, O_Y)$ satisfying the following condition:

A primitive n-th root 1 is defined on V_i. For the covering $V_i' = V_{i,n} = V_i \otimes_{\mathbf{Z}[P_i]} \mathbf{Z}[P_i] \to V_i$ defined above, the pull-back of the stable curve $(f' : X' \to Y', D_{Y'})$ to $Y' \times_Y V_i'$ satisfies the condition (s) in Proposition 3.9.1.

Let n be 2-times the 2-part of the order of G. We show that the condition (a) implies (s') for this n. The condition (a) implies that 2 and hence n is invertible on Y. Let $Y' \to Y$ be the finite log etale Galois covering above. It is sufficient to find, for each $\bar{y}$, an etale neighborhood $V \to Y$ and a chart $P \to \Gamma(V, O_Y)$ such that $pr_1^* \bar{M}_{Y'} \subset \bar{M}_{Y' \times_Y V_n}$ is a subsheaf of $\bar{M}^2_{Y' \times_Y V_n}$. Let $\bar{y} \to Y$ be a geometric point and $\bar{y}' \to Y'$ be a geometric point above $\bar{y}$. By the proof of Theorem 3.3 [9] p.57, there are etale neighborhood $V \to Y$ of $\bar{y}$ and $V' \to Y'$ of $\bar{y}'$, charts $P \to \Gamma(V, M_Y)$ and $P' \to \Gamma(V', M_{Y'})$ and a morphism of charts $P \to P'$ satisfying the following property. The map $P \to P'$ is injective and $\frac{1}{\mathrm{Card} G} P' \subset P$. Here and in the rest of paragraph, we use the additive notation. It is sufficient to show $P' \subset 2(\frac{1}{n}P + P')^{\mathrm{sat}}$. Let $a \in P'$ and put $\mathrm{Card} G = \frac{2}{n}(2m+1)$. Then $b = \frac{n}{2}(2m+1)a$ is an element of P. Since $a = 2(\frac{1}{n}b - ma)$, we have $\frac{1}{n}b - ma \in (\frac{1}{n}P + P')^{\mathrm{sat}}$ and the assertion follows.

We construct $f : X \to Y$ assuming the condition (s'). Let $n \geq 1$ be an integer, $(V_i \to Y)_{i=1,\dots,m}$ be an etale covering and $P_i \to \Gamma(V_i, O_Y)$ for $i = 1, \dots, m$ be charts as in the condition (s'). First, we construct it etale locally on Y. The finite log etale covering $V_i' \to V_i$ is a Galois covering of Galois group $G_i = Hom(P_i, \mu_n)$. Let $(f_i' : X_i' \to Y' \times_Y V_i', D_i)$ be the stable curve over $Y' \times_Y V_i'$ extending the pull-back. It satisfies the condition (s) in Proposition 3.9.2. We apply the above construction to the Galois covering $Y' \times_Y V_i' \to V_i$ of Galois group $G \times G_i$ to obtain a log smooth extension $f_i : X_i \to V_i$ of the pull-back $X_{U_Y} \to U_Y$. We put $V_{ij} = V_i \times_Y V_j$. To descend the family $f_i : X_i \to V_i$ to get $f : X \to Y$, we define isomorphisms $\varphi_{ij} : X_i \times_{V_i} V_{ij} \to X_j \times_{V_j} V_{ij}$ satisfying the cocycle conditions.

Let V_{ij}' be the normalization of $V_i' \times_Y V_j'$. It is a finite and log etale Galois covering of V_{ij} of Galois group $G_{ij} = G_i \times G_j$. On the finite and log etale Galois covering $Y' \times_Y V_{ij}'$ of V_{ij} of Galois group $G \times G_{ij}$, the pull-back is extended to a stable curve $(f_{ij}' : X_{ij}' \to Y' \times_Y V_{ij}', D_{ij})$ and the extension satisfies the condition (s). Applying the construction above, we obtain a log smooth extension $f_{ij} : X_{ij} \to V_{ij}$ of the pull-back. By the uniqueness of the extension of stable curves, we have canonical isomorphisms $\psi_{ij}' : X_{ij}' \to X_i' \times_{V_i'} V_{ij}'$. The map ψ_{ij}' is compatible with the natural actions of $G \times G_{ij}$.

Let $X_{V'_i} = (X'_{i,\Sigma})_\Phi \to X'_i$ and $X_{V'_{ij}} = (X'_{ij,\Sigma})_\Phi \to X'_{ij}$ denote the blow-ups defined in Proposition 3.9. The scheme X_i is the quotient of $X_{V'_i}$ by $G \times G_i$ and X_{ij} is the quotient of $X_{V'_{ij}}$ by $G \times G_{ij}$. We show that the isomorphisms $\psi'_{ij} : X'_{ij} \to X'_i \times_{V'_i} V'_{ij}$ induces isomorphisms $\tilde{\psi}_{ij} : X_{V'_{ij}} \to X_{V'_i} \times_{V'_i} V'_{ij}$ and $\psi_{ij} : X_{ij} \to X_i \times_{V_i} V_{ij}$ to define $\varphi_{ij} = \psi_{ij} \circ \psi_{ji}^{-1} : X_j \times_{V_j} V_{ij} \to X_i \times_{V_i} V_{ij}$.

For this, we show that the map $V'_{ij} \to V'_i$ is etale. It is sufficient to show that the finite and log etale Galois covering $V'_{ij} \to V'_i \times_{V_i} V_{ij}$ of Galois group G_j is classically etale. By the definition, V'_{ij} is the normalization of $(V'_i \times_{V_i} V_{ij}) \otimes_{\mathbf{Z}[P_j]} \mathbf{Z}[P_j]$ induced by $\times n : P_j \to P_j$. By the definition of V'_i, the image of the map $P_j \to \bar{M}_{V'_i \times_{V_i} V_{ij}}$ is contained in the subsheaf of $\bar{M}^n_{V'_i \times_{V_i} V_{ij}}$. Hence, for each point of $V'_i \times_{V_i} V_{ij}$, there is an etale neighborhood $W \to V'_i \times_{V_i} V_{ij}$ and a map $P_j \to \Gamma(W, O_W^\times)$ such that the pull-back $V'_{ij} \times_{V'_i \times_{V_i} V_{ij}} W$ is isomorphic to $W \otimes_{\mathbf{Z}[P_j]} \mathbf{Z}[P_j] = W \otimes_{\mathbf{Z}[\frac{1}{n}][P_j^{\mathrm{gp}}]} \mathbf{Z}[\frac{1}{n}][P_j^{\mathrm{gp}}]$ and is etale over W. Thus the map $V'_{ij} \to V'_i$ is etale as required.

We prove that the isomorphisms $\psi'_{ij} : X'_{ij} \to X'_i \times_{V'_i} V'_{ij}$ are extended to isomorphisms $\tilde{\psi}_{ij} : X_{V'_{ij}} \to X_{V'_i} \times_{V'_i} V'_{ij}$. Since $V'_{ij} \to V'_i \times_{V_i} V_{ij}$ is a finite and etale Galois covering of Galois group G_j, the map $X'_{ij} \to X'_i \times_{V_i} V_{ij}$ is also a finite and etale Galois covering of Galois group G_j. Since G_j acts freely on X'_{ij}, if $x \in X'_i$ is the image of $x' \in X'_{ij}$ and if $I_x \subset G \times G_i$ is the inertia group, the inertia group $I_{x'} \subset G \times G_{ij} = G \times G_i \times G_j$ is $I_x \times \{1\}$. Hence the closed subscheme $\Sigma(X'_{ij})$ of X'_{ij} is the inverse image of $\Sigma(X'_i)$ and the closed subscheme $\Phi(X'_{ij,\Sigma})$ of the blow-up $X'_{ij,\Sigma}$ is the inverse image of $\Phi(X'_{i,\Sigma})$. Thus, by definition of $X_{V'_i}$ and $X_{V'_{ij}}$, the isomorphisms $\psi'_{ij} : X'_{ij} \to X'_i \times_{V'_i} V'_{ij}$ are extended to isomorphisms $\tilde{\psi}_{ij} : X_{V'_{ij}} \to X_{V'_i} \times_{V'_i} V'_{ij}$ as claimed. It is compatible with the induced actions of $G \times G_{ij}$.

By taking the quotients of the isomorphisms $\tilde{\psi}_{ij} : X_{V'_{ij}} \to X_{V'_i} \times_{V'_i} V'_{ij}$ by $G \times G_{ij}$, we obtain isomorphisms $\psi_{ij} : X_{ij} \to X_i \times_{V_i} V_{ij}$. In fact, by the definition, the quotient $X_{V'_{ij}}/(G \times G_{ij})$ is X_{ij}. On the other hand, since the map $X_{V'_i} \times_{V'_i} V'_{ij} \to X_{V'_i} \times_{V_i} V_{ij}$ is a finite and etale Galois covering of Galois group G_j, the quotient $(X_{V'_i} \times_{V'_i} V'_{ij})/(G \times G_{ij})$ is identified with $(X'_i \times_{V_i} V_{ij})/(G \times G_i) = X_i \times_{V_i} V_{ij}$. We define an isomorphism φ_{ij} to be $\psi_{ij} \circ \psi_{ji}^{-1}$. It is easy to see that the isomorphisms φ_{ij} satisfy the cocycle condition.

To descend the descent datum $(f_i : X_i \to V_i, \varphi_{ij} : X_i \times_{V_i} V_{ij} \to X_j \times_{V_j} V_{ij})$, we define a descent datum on it of a relatively ample invetible sheaf. For sufficiently large integers $\mu_i, \nu_i \geq 1$ and sufficiently divisible integers $N_i \geq 1$, let $\mathcal{M}_i^{(\mu_i,\nu_i,N_i)}$ be the f_i-ample invertible sheaves on X_i with a canonical isomorphism $\pi_i^* \mathcal{M}_i^{(\mu_i,\nu_i,N_i)} \to \mathcal{L}_{X_{V'_i}}^{(\mu_i,\nu_i)\otimes N_i}$ where $\pi_i : X_{V'_i} \to X_i$ are the canonical map. By replacing μ_i and ν_i by $\mu = \max_i \mu_i$ and $\nu = \max_i \nu_i$, we may assume $\mu_i = \mu$ and $\nu_i = \nu$ for all i. Further by replacing N_i by $N = \mathrm{lcm}_i N_i$, we may assume $N_i = N$ for all i. We write $\mathcal{M}_i^{(\mu,\nu,N)} = \mathcal{M}_i$ for short.

We define isomorphisms $\varphi_{ij}^* \mathcal{M}_j|_{X_j \times_{V_j} V_{ij}} \to \mathcal{M}_i|_{X_i \times_{V_i} V_{ij}}$. The isomorphisms $\tilde{\psi}_{ij} : X_{V'_{ij}} \to X_{V'_i} \times_{V'_i} V'_{ij}$ induce isomorphisms $\tilde{\psi}_{ij}^* \mathcal{L}'^{(\mu,\nu)}_{X_{V'_i}}|_{X_{V'_i} \times_{V'_i} V'_{ij}} \to \mathcal{L}'^{(\mu,\nu)}_{X_{V'_{ij}}}$. Similarly as above, we define an f_{ij}-ample invertible sheaf $\mathcal{M}_{ij}$ on X_{ij}. Then, they induce an isomorphism $\psi_{ij}^* \mathcal{M}_i|_{X_i \times_{V_i} V_{ij}} \to \mathcal{M}_{ij}$ and isomorphisms $\varphi_{ij}^* \mathcal{M}_j|_{X_j \times_{V_j} V_{ij}} \to \mathcal{M}_i|_{X_i \times_{V_i} V_{ij}}$.

It is easy to see that the isomorphisms satisfy the cocycle condition. Thus the proof of Lemma 4.1 is completed.

4.2 End of Proof of Theorem 1.3.

We complete the proof of Theorem 1.3 by showing the following.

Theorem 4.2 *Let K be a discrete valuation field, $f_K : X_K \to K$ be a proper smooth and geometrically connected curve and D_K be a divisor of X_K such that the pair $(f_K : X_K \to K, D_K)$ is a stable curve. Let $N \geq 1$ be an integer invertible in O_K. We consider the conditions.*

(1) There exists a projective and log smooth scheme X_{O_K} over the integer ring O_K extending $(X_K, X_K - D_K)$ such that the log structure of X_{O_K} is defined by $U_K = X_K - D_K \to X_{O_K}$.

(2) There exists a finite tamely ramified extension L of K and a stable curve $(f_{O_L} : X_{O_L} \to O_L, D_{O_L})$ over O_L extending the base change $(f_K : X_K \to K, D_K) \otimes_K L$.

(3) The finite covering D_K and the finite group scheme $J_{N,K}$ of N-torsion points of the Jacobian J_K of X_K are tamely ramified over K.

Then we have (1)⇔(2)⇒(3). We have (3)⇒(1) if $N \geq 3$.

The case $D_K = \emptyset$ is proved essentially in [12]

Proof of Theorem 4.2. The implication (3)⇒(1) for $N \geq 3$ is a special case of Lemma 4.1 where $Y = \text{Spec } O_K$. In fact, if 2 is not invertible in O_K, the condition (b') is satisfied. The implications (2)⇒(3) and (3)⇒(2) for $N \geq 3$ follow from the counterparts in Proposition 1.14. We show (1)⇒(3). If the condition (1) is satisfied, by Theorem 1.8 and Proposition 1.14 (1)⇒(3), there exists an integer $e \geq 1$ such that, if L is a finite separable extension and the ramification index $e_{L/K}$ is divisible by e, the base change $D_K \otimes_K L$ is unramified over L and the action of $I_L \subset I_K$ on $J_{N,K} \otimes_K L$ is unipotent. Hence we have (1)⇒(3). Another proof is found in [10]. By taking an integer $N \geq 3$ invertible in O_K, we have (1)⇔(2).

Proof of Theorem 1.3. Clearly, we have (1)⇒(2). The implications (2)⇒(3) and (3)⇒(2) for $N \geq 3$ follows from Theorem 4.2 (1)⇒(3) and (3)⇒(1) for $N \geq 3$ respectively. We show (2)⇒(1) assuming (a) or (b). If 2 is invertible, the condition (2) implies (3) for $N = 4$ and hence (1) by Lemma 4.1. The condition (2) together with (b) implies (3) for an integer $N \geq 3$ invertible on Y. and (b') in Lemma 4.1 and hence (1) by Lemma 4.1.

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