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Weight spectral sequences and independence of ℓ

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Summary: For a variety over a local field, we show that the alternating sum of the trace of the composition of the actions of an element of the Weil group and an algebraic correspondence on the ℓ-adic etale cohomology is independent of ℓ. We prove the independence by establishing basic properties of weight spectral sequences [14]. We derive them from a new formal construction of weight spectral sequences using the machinary of perverse sheaves.

Let K be a complete discrete valuation field with finite residue field F of order q. We call such a field a local field. The geometric Frobenius Fr_F is the inverse of the map $a \mapsto a^q$ in the absolute Galois group $G_F = \mathrm{Gal}(\bar{F}/F)$. The Weil group W_K is defined as the inverse image of the subgroup $\langle Fr_F \rangle \subset G_F$ by the canonical map $G_K = \mathrm{Gal}(\bar{K}/K) \to G_F$.

Theorem 0.1 *Let X_K be a proper smooth scheme of dimension n over a local field K, $\sigma \in W_K$ be an element of the Weil group and $\Gamma \in CH^n(X_K \times_K X_K)$ be an algebraic correspondence on X_K. Then, for a prime ℓ different from the characteristic of F, the alternating sum*

$$\mathrm{Tr}(\Gamma^* \circ \sigma_* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell)) = \sum_{r=0}^{2n} (-1)^r \mathrm{Tr}(\Gamma^* \circ \sigma_* : H^r(X_{\bar{K}}, \mathbf{Q}_\ell))$$

is in $\mathbf{Z}[1/q]$ and is independent of ℓ.

In the case Γ is the diagonal Δ, it is proved in [15]. If σ is in the wild ramification group $P \subset W_K$, it is proved in [12]. If X_K is an abelian variety, individual $\mathrm{Tr}(\Gamma^* \circ \sigma_* : H^r(X_{\bar{K}}, \mathbf{Q}_\ell))$ is known to be independent of ℓ [7].

Corollary 0.2 *Assume further $n = \dim X_K \leq 2$. Then, for $0 \leq r \leq 2n$, the trace*

$$\mathrm{Tr}(\Gamma^* \circ \sigma_* : H^r(X_{\bar{K}}, \mathbf{Q}_\ell))$$

is in $\mathbf{Z}[1/q]$ and is independent of ℓ.

As an application of Theorem 0.1, we prove some cases of the following conjecture. For $\sigma \in W_K$, let $n(\sigma)$ be the integer such that the image of σ by $G_K \to G_F$ is $Fr_F^{n(\sigma)}$.

Conjecture 0.3 (cf. [17] C_4 and C_5) *Let X_K be a proper smooth scheme over a local field K, $r \geq 0$ be an integer and ℓ be a prime number invertible in F.*

1. For $\sigma \in W_K$ with $n(\sigma) \geq 0$, the eigenpolynomial

$$\det(1 - \sigma_* T : H^r(X_{\bar{K}}, \mathbf{Q}_\ell))$$

is in $\mathbf{Z}[T]$ and independent of ℓ.

2. The eigenpolynomial

$$\det(1 - Fr_* T : H^r(X_{\bar{K}}, \mathbf{Q}_\ell)^I)$$

is in $\mathbf{Z}[T]$ and independent of ℓ.

If $r \leq 1$, it is proved in [7]. If char $K > 0$, it is proved in [18]. As an application of Theorem 0.1, we prove the following.

Corollary 0.4 *If $\dim X_K \leq 2$, Conjecture 0.3 is true.*

To state more results on Conjecture 0.3, we recall the monodromy filtration and the weight monodromy conjecture. Let $I \subset G_K$ be the inertia subgroup and $t_\ell : I \to \mathbf{Z}_\ell(1)$ be the canonical map sending $\sigma \in I$ to $(\sigma(\pi^{1/\ell^n})/\pi^{1/\ell^n})_n \in \mathbf{Z}_\ell(1)$. By the monodromy theorem of Grothendieck [16] Appendix, there exist a nilpotent endomorphism $N \in \mathrm{End}(H^r(X_{\bar{K}}, \mathbf{Q}_\ell))(-1)$ and an open subgroup $J \subset I$ such that, for $\sigma \in J$, the action of σ on $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ is given by $\exp(t_\ell(\sigma)N)$. Let $M_\bullet$ be the increasing filtration on $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ characterized by the conditions

(1) $M_s = H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ and $M_{-s-1} = 0$ for a sufficiently large integer s.

(2) For $s \in \mathbf{Z}$, the map N sends M_s to $M_{s-2}(1)$.

(3) For $s \geq 0$, the induced map $N^s : Gr_s^M \to Gr_{-s}^M(s)$ is an isomorphism.

The filtration $M_\bullet$ is called the monodromy filtration. The following weight monodromy conjecture asserts that the monodromy filtration gives the weight filtration.

Conjecture 0.5 [3] *Let X_K be a proper smooth scheme over a local field K and let $r \geq 0$ and s be integers. Let $\sigma \in W_K$ be an element of the Weil group and $n(\sigma)$ be the integer such that the image of σ by $G_K \to G_F$ is $Fr_F^{n(\sigma)}$. Then, the eigenvalues of the action of σ on $Gr_s^M H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ are algebraic numbers and the complex absolute values of their conjugates are $q^{(r+s)n(\sigma)/2}$.*

Conjecture 0.5 is proved if $r \leq 2$, [14] Satz 2.13, or char $K > 0$, [5], [11] [18].

Corollary 0.6 *Let $r \geq 0$ be an integer.*

1. Assume there exists an algebraic correspondence $\Gamma_r \in CH^n(X_K \times X_K)_{\mathbf{Q}}$ such that Γ_r^* on $H^s(X_{\bar{K}}, \mathbf{Q}_\ell)$ is the identity if $s = r$ and 0 if $s \neq r$. Then, Conjecture 0.3.1 is true.

2. Assume further that $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ satisfies the weight monodromy conjecture, Conjecture 0.5. Then Conjecture 0.3.2 is true.

Theorem 1 is derived rathar directly from basic properties of weight spectral sequences [14] for a semi-stable scheme over the integer ring using alteration [2]. In the first part of this paper, we establish the necessary basic properties, Propositions 1.14, 1.16, 1.18, 1.19 etc. of weight spectral sequences such as functoriality, compatibility with the duality, push-forward and chern classes etc. We prove them using a new construction of weight spectral sequences. The construction is a consequence of the identification, Proposition 1.11, of the graded pieces of the monodromy filtration on the perverse sheaf of nearby cycles. We deduce the identification from the facts Proposition 1.5 that the sheaves of nearby cycles are tame and that the $Rf^!\Lambda$ is a dualizing sheaf for a semi-stable scheme, using some elementary linear algebra.

We expect that the same argument using the weight spectral sequence of Mokrane [13] shows the equality

$$\mathrm{Tr}(\Gamma^* \circ \sigma_* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell)) = \mathrm{Tr}(\Gamma^* \circ \sigma_* : D_{pst} H^*(X_{\bar{K}}, \mathbf{Q}_p)).$$

We plan to work this out in a forthcoming paper.

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1. Weight spectral sequence.

1.1 Monodromy filtrations.

Let C be an abelian category, A be an object of C and $n \leq 0$ be an integer. Let $F_\bullet$ be an increasing filtration on A satisfying $F_{-1}A = 0$ and $F_nA = A$ and $G^\bullet$ be a decreasing filtration on A satisfying $G^0A = A$ and $G^{n+1}A = 0$. We put $M_rA = \sum_{p-q=r} F_pA \cap G^qA$. The increasing filtration $M_\bullet$ satisfies $M_nA = A$ and $M_{-n-1}A = 0$. We consider the filtrations on $Gr_r^M A = M_rA/M_{r-1}A$ induced by F and G. They are defined by $F_pGr_r^MA = \mathrm{Im}(F_pA \cap M_rA \to Gr_r^MA)$ and $G^qGr_r^MA = \mathrm{Im}(G^qA \cap M_rA \to Gr_r^MA)$. Similarly, we consider the filtration $G^qGr_p^FA = \mathrm{Im}(G^qA \cap F_pA \to Gr_p^FA)$ on $Gr_p^FA = F_pA/F_{p-1}A$ induced by G.

Lemma 1.1 *1. For integers p, q and r satisfying $p - q = r$, there is a canonical isomorphism $Gr_p^FGr_r^MA \to Gr_G^qGr_p^FA$ induced by the natural maps $F_pA \cap G^qA \to Gr_p^FGr_r^MA$ and $F_pA \cap G^qA \to Gr_G^qGr_p^FA$.*

2. The filtrations F and G on Gr_r^MA are r-opposite to each other. Namely, $Gr_G^qGr_p^FGr_r^MA = 0$ for $p - q \neq r$. There is a canonical isomorphism $Gr_r^MA \to \bigoplus_{p-q=r} Gr_G^qGr_p^FA$.

Proof. 1. The kernel of the surjection $F_pA \cap G^qA \to Gr_q^GGr_F^pA$ is $(F_{p-1}A \cap G^qA) + (F_pA \cap G^{q+1}A)$. We show that the map $F_pA \cap G^qA \to Gr_p^FGr_r^MA$ is surjective and its kernel K is equal to $(F_{p-1}A \cap G^qA) + (F_pA \cap G^{q+1}A)$. We show $F_pA \cap M_rA =$

$\sum_{p'-q'=r,p'\leq p} F_{p'}A \cap G^{q'}A$. Since $M_rA \subset \sum_{p'-q'=r,p'<p}(F_{p'}A \cap G^{q'}A) + G^qA$, we have $F_pA \cap M_rA \subset \sum_{p'-q'=r,p'<p}(F_{p'}A \cap G^{q'}A) + (F_pA \cap G^qA)$. The other inclusion is obvious. It follows immediately from this equality that the map $F_pA \cap G^qA \to Gr_p^F Gr_r^M A$ is a surjection. We show that the kernel $K = (F_pA \cap G^qA \cap M_{r-1}A) + (F_pA \cap G^qA \cap F_{p-1}A \cap M_rA)$ is equal to $(F_{p-1}A \cap G^qA) + (F_pA \cap G^{q+1}A)$. Since $M_{r-1}A \subset F_{p-1}A + G^{q+1}A$, we have $K \subset (F_pA \cap G^qA) \cap (F_{p-1}A + G^{q+1}A) = F_pA \cap ((F_{p-1}A \cap G^qA) + G^{q+1}A) = (F_{p-1}A \cap G^qA) + (F_pA \cap G^{q+1}A)$. The other inclusion is obvious. Thus the required isomorphism $Gr_p^F Gr_r^M A \to Gr_G^q Gr_p^F A$ is defined.

2. By 1, we have $Gr_G^{q'} Gr_p^F Gr_r^M A = 0$ for $p - q' \neq r$ and the filtrations F and G on $Gr_r^M A = M_rA/M_{r-1}A$ are r-opposite to each other. By [4] Proposition (1.2.5), we have a canonical isomorphism $Gr_r^M A \to \bigoplus_{p-q=r} Gr_G^q Gr_p^F Gr_r^M A$. Hence we have a canonical isomorphism $Gr_r^M A \to \bigoplus_{p-q=r} Gr_G^q Gr_p^F A$.

Let N be a nilpotent endomorphism of A and $n \geq 0$ be an integer satisfying $N^{n+1} = 0$. Then there exists a unique increasing filtration filtration $M_\bullet$ on A characterized by the following properties (1)-(3).

(1) $M_nA = A$ and $M_{-n-1}A = 0$.

(2) $N : A \to A$ sends M_rA into $M_{r-2}A$ for $r \in \mathbf{Z}$.

(3) If $r \geq 0$, the induced map $N^r : Gr_r^M A \to Gr_{-r}^M A$ is an isomorphism.

The filtration $M_\bullet$ is called the monodromy filtration defined by N. We define an increasing filtration $F_\bullet$ by $F_pA = \mathrm{Ker}(N^{p+1} : A \to A)$ and a decreasing filtration $G^\bullet$ by $G^qA = \mathrm{Im}(N^q : A \to A)$. We have $F_0A = \mathrm{Ker}(N : A \to A)$ and $G^0A = A$. We call $F_\bullet$ the kernel filtration and $G^\bullet$ the image filtration. The monodromy filtration is described by the kernel filtration and the image filtration as follows.

Proposition 1.2 *Let N be a nilpotent endomorphism of A and let $M_\bullet, F_\bullet$ and $G^\bullet$ be the monodromy, kernel and the image filtrations on A. Then we have $M_rA = \sum_{p-q=r}(F_pA \cap G^qA)$.*

Proof. We put $M'_rA = \sum_{p-q=r}(F_pA \cap G^qA)$ and show that the filtration $M'_\bullet$ satisfies the conditions (1)-(3) above. Let $n \geq 0$ be an integer satisfying $N^{n+1} = 0$. Since $F_nA = A, F_{-1}A = 0$ and $G^0A = A, G_{n+1}A = 0$, we have $M'_nA = A$ and $M'_{-n-1}A = 0$ and the condition (1) is satisfied. Since $NF_pA \subset F_{p-1}A$ and $NG^qA \subset G^{q+1}A$, we have $NM'_rA \subset M'_{r-2}A$ and the condition (2) is satisfied.

We show that the induced map $N^r : Gr_r^{M'} A \to Gr_{-r}^{M'} A$ is an isomorphism to complete the proof. As in Lemma 1.1, we identify $Gr_r^{M'} A = \bigoplus_{p-q=r,p\geq 0,q\geq 0} Gr_G^q Gr_p^F A$ and $Gr_{-r}^{M'} A = \bigoplus_{p-q=r,p-r\geq 0,q+r\geq 0} Gr_G^{q+r} Gr_{p-r}^F A$. It is sufficient to show that the map $N^r : Gr_G^q Gr_p^F A \to Gr_G^{q+r} Gr_{p-r}^F A$ is an isomorphism for $p \geq r \geq 0$ and $q \geq 0$. We deduce it from the following Lemma.

Lemma 1.3 *For $p, q \geq 0$, the induced map $N^q : Gr_{p+q}^F A \to Gr_p^F A$ is an injection. Its image is equal to $G^q Gr_p^F A$.*

We complete the proof of Proposition 1.2 admitting Lemma 1.3. By Lemma 1.3, the maps $N^q : Gr^F_{p+q}A/NGr^F_{p+q+1}A \to Gr^q_G Gr^F_p A$ and $N^{q+r} : Gr^F_{p+q}A/NGr^F_{p+q+1}A \to Gr^{q+r}_G Gr^F_{p-r}A$ are isomorphisms. Hence the assertion follows.

Proof of Lemma. By the definition of the kernel filtration, we have $(N^q)^{-1}F_{p+1}A = F_{p+q+1}A$ for $p, q \geq 0$. Hence, the induced map $N^q : Gr^F_{p+q}A \to Gr^F_p A$ is injective. Similarly, we have $G_q A \cap F_p A = \mathrm{Im}(N^q : F_{p+q}A \to F_p A)$. Hence we obtain $G^q Gr^F_p A = \mathrm{Im}(G^q A \cap F_p A \to Gr^F_p A) = \mathrm{Im}(N^q : Gr^F_{p+q}A \to Gr^F_p A)$.

We give a characterization of the kernel filtration under a certain hypothesis.

Lemma 1.4 *Let A be an object of an abelian category C, N be an endomorphism of A and $n \geq 0$ be an integer. Let $F'_\bullet$ be an increasing filtration of A satisfying $F'_{-1}A = 0$ and $F'_n A = A$.*

1. If the following condition (1) is satisfied, we have $N^{n+1} = 0$.

(1) For $p \geq 0$, the map N sends $F'_p A$ to $F'_p A$ and the induced endomorphism $N : Gr^{F'}_p A \to Gr^{F'}_p A$ is the 0-map.

2. Assume the condition (1) is satisfied and let $F_p A = \mathrm{Ker}(N^{p+1} : A \to A)$ be the kernel filtration. Further if the following conditions (2) and (3) are satisfied, we have $F_p A = F'_p A$ for $p \geq 0$.

(2) $F_0 A = F'_0 A$.

(3) For $p \geq 0$, there is no non-0 map from a subobject of $\mathrm{Coker}(N : Gr^{F'}_{p+1}A \to Gr^{F'}_p A)$ *to $Gr^{F'}_{p'}A$ for $p' > p+1$.*

Proof. 1. The condition (1) means $NF'_p A \subset F'_{p-1}A$. Hence $N^{p+1}F'_p A = 0$ and the assertion follows.

2. Since $N^{p+1}F'_p A = 0$, we have $F_p A \supset F'_p A$. We show $F_p A = F'_p A$ by induction on $p \geq 0$. By (2), it holds for $p = 0$. We assume $F'_{p'}A = F_{p'}A$ for $p' \leq p$ and show $F'_{p+1}A = F_{p+1}A$. We show $F_{p+1}A/F'_{p+1}A = 0$. By the induction hypothesis, we have $Gr^F_p A = Gr^{F'}_p A$. Since the map $N : Gr^F_{p+1}A \to Gr^F_p A$ is injective, it induces an injection $F_{p+1}A/F'_{p+1}A = Gr^F_{p+1}A/Gr^{F'}_{p+1}A \to \mathrm{Coker}(N : Gr^{F'}_{p+1}A \to Gr^{F'}_p A)$. By the assumption (3), there is no non-zero map $F_{p+1}A/F'_{p+1}A \to Gr^{F'}_{p'}A$ for $p' > p+1$. Hence the injection $F_{p+1}A/F'_{p+1}A \to A/F'_{p+1}A = F'_n A/F'_{p+1}A$ is the 0-map and the assertion follows.

1.2 Nearby cycles on semi-stable schemes.

Let K be a henselian discrete valuation field with residue field F. The spectrum of the integer ring O_K will be denoted by S. We say a scheme X locally of finite presentation over S is strictly semi-stable of relative dimension n if it is, Zariski locally on X, etale over $\mathrm{Spec}\, O_K[T_0, \ldots, T_n]/(T_0 \cdots T_n - \pi)$ for a prime element π of K. If the residue field F is perfect, a scheme X locally of finite presentation over S is strictly semi-stable if and only if the following conditions (1)-(3) are satisfied.

(1) X is regular and flat over S.

(2) The generic fiber X_K is smooth.

(3) The closed fiber X_F is a divisor of X with simple normal crossings.

In the rest of the paper, X denotes a strictly semi-stable scheme over O_K of relative dimension n and $Y = X_F$ denotes the closed fiber of X. Let $\bar{K}$ be a separable closure of K and K^{ur} be the maximum unramified extension of K in $\bar{K}$. Let $\bar{F}$ be the residue field of K^{ur}. It is a separable closure of F. Let $I = \mathrm{Gal}(\bar{K}/K^{ur}) \subset G_K = \mathrm{Gal}(\bar{K}/K)$ be the inertia subgroup. It is the kernel of the canonical surjection $G_K \to G_F = \mathrm{Gal}(\bar{F}/F)$. For a prime number ℓ invertible in F, let $t_\ell : I \to \mathbf{Z}_\ell(1)$ be the canonical surjection defined by $\sigma \mapsto (\sigma(\pi^{1/\ell^n})/\pi^{1/\ell^n})_n$ for a prime element π of K. Let S^{ur} denote the spectrum of the integer ring $O_{K^{ur}}$. Let $i : Y = X_F \to X, j : X_K \to X, \bar{i} : Y_{\bar{F}} \to X_{S^{ur}}$ and $\bar{j} : X_{\bar{K}} \to X_{S^{ur}}$ be the canonical maps.

Let ℓ be a prime number invertible in O_K and let Λ denote either of $\mathbf{Z}/\ell^m\mathbf{Z}, \mathbf{Z}_\ell$ and $\mathbf{Q}_\ell$. For $p \geq 0$, let $R^p\psi\Lambda = \bar{i}^*R^p\bar{j}_*\Lambda$ denote the sheaf of nearby cycles. It is a sheaf on $Y_{\bar{F}}$ with an action of G_K compatible with the action of the quotient $G_F = G_K/I$ on $Y_{\bar{F}}$. Similarly $R\psi\Lambda = \bar{i}^*R\bar{j}_*\Lambda$ is defined as a object of the derived category on $Y_{\bar{F}}$ with an action of G_K.

In this subsection, we recall an explicit computation, Corollary 1.6 below, of the sheaves $R^p\psi\Lambda$ (cf. [14]). We deduce it from the following Proposition. We identify a sheaf on Y with its pull-back on $Y_{\bar{F}}$ with the G_F-action.

Proposition 1.5 *Let $f : X \to S = \mathrm{Spec}\ O_K$ be a strictly semi-stable scheme of relative dimension n. Then*

1. ([10] Theorem 1.2, [14] Korollar 2.25) The action of the inertia subgroup I on $R^p\psi\Lambda$ is trivial for $p \geq 0$.

2. ([10] Theorem 1.4) The map $\Lambda \to Rf^!\Lambda(-n)[-2n]$ sending 1 to the canonical class $[X]$ is an isomorphism.

We introduce some notation to state the computation of $R^p\psi\Lambda$. Let $D_1, \ldots, D_m$ be the irreducible components of $Y = X_F$. For a non-empty subset $I \subset \{1, \ldots, m\}$, we put $Y_I = \bigcap_{i\in I} D_i$ and let $a_I : Y_I \to Y$ be the immersion. The scheme Y_I is smooth of dimension $n - p$ over F if $\mathrm{Card}\ I = p + 1$. For an integer $p \geq 0$, we put $Y^{(p)} = \coprod_{I\subset\{1,\ldots,m\},\mathrm{Card}\ I=p+1} Y_I$ and let $a_p : Y^{(p)} \to Y$ be the natural map. We put $a_{-1} = id_Y$. We identify the exterior power $\Lambda^p a_{0*}\Lambda$ with $a_{(p-1)*}\Lambda$. Let $\theta : \Lambda \to i^*R^1j_*\Lambda(1)$ be the map defined by the class $[\pi] \in H^1(K, \Lambda(1))$ of a prime element π of K. It is independent of the choice of π. Let θ also denote the map $i^*R^{p-1}j_*\Lambda \to i^*R^pj_*\Lambda(1)$ induced by θ by the cup-product.

Corollary 1.6 *1. Let $p \geq 0$ be an integer. The canonical map $i^*R^pj_*\Lambda \to R^p\psi\Lambda$ is surjective. The map $\theta : i^*R^{p-1}j_*\Lambda \to i^*R^pj_*\Lambda(1)$ induces a map $\bar{\theta} : R^{p-1}\psi\Lambda \to i^*R^pj_*\Lambda(1)$. The sequences*

$$0 \longrightarrow R^{p-1}\psi\Lambda \xrightarrow{\bar{\theta}} i^*R^pj_*\Lambda(1) \longrightarrow R^p\psi\Lambda(1) \longrightarrow 0,$$

$$0 \longrightarrow R^p\psi\Lambda \xrightarrow{\bar{\theta}} i^*R^{p+1}j_*\Lambda(1) \xrightarrow{\theta} \ldots \xrightarrow{\theta} i^*R^nj_*\Lambda(n-p) \longrightarrow 0$$

are exact.

2. (cf. [10] Remarks 1.5 (c), [14] Satz 2.8) *The map* $a_{0*}\Lambda \to i^*R^1j_*\Lambda(1)$ *sending 1 on an irreducible component* D_i *to the class of a uniformizer of* D_i *is an isomorphism. For* $p \geq 0$*, the cup-product induces an isomorphism* $a_{(p-1)*}\Lambda \to i^*R^pj_*\Lambda(p)$*.*

To deduce Corollary 1.6 from Proposition 1.5, we apply the following Lemma.

Lemma 1.7 *Let* T *be an element in the inertia* I *such that* $t_\ell(T)$ *is a generator of* $\mathbf{Z}_\ell(1)$*. Then,*

1. The isomorphism $i^*Rj_*\Lambda \to R\Gamma(I, R\psi\Lambda)$ *induces a quasi-isomorphism with the mapping fiber*

$$i^*Rj_*\Lambda \longrightarrow \text{Fiber}[R\psi\Lambda \xrightarrow{T-1} R\psi\Lambda].$$

2. ([14] Lemma 1.2) *Let* $\theta : i^*Rj_*\Lambda \to i^*Rj_*\Lambda(1)[1]$ *be the map defined by the class* $[\pi] \in H^1(K, \Lambda(1)) = Hom(\Lambda, \Lambda(1)[1])$ *of a prime element* π *of* K*. Then the diagram*

$$\begin{array}{ccccccc}
i^*Rj_*\Lambda & \longrightarrow & [0 & \longrightarrow & R\psi\Lambda & \xrightarrow{T-1} & R\psi\Lambda] \\
{\scriptstyle\theta}\downarrow & & & & \downarrow{\scriptstyle 1\otimes t_\ell(T)} & & \\
i^*Rj_*\Lambda(1)[1] & \longrightarrow & [R\psi\Lambda(1) & \xrightarrow{(T-1)\otimes 1} & R\psi\Lambda(1) & \longrightarrow & 0]
\end{array}$$

is commutative, where the horizontal maps are the isomorphism in 1.

Proof of Lemma. 1. By Proposition 1.5.1, the natural map $R\psi\Lambda^P \to R\psi\Lambda$ is an isomorphism. Hence, the isomorphism $i^*Rj_*\Lambda \to R\Gamma(I, R\psi\Lambda)$ induces an isomorphism $i^*Rj_*\Lambda \to R\Gamma(I/P, R\psi\Lambda)$. Since I/P is a cyclic group generated by T, the assertion follows.

Proof of Corollary 1.6. 1. By Proposition 1.5.1 and Lemma 1.7.1, we obtain an exact sequence $0 \to R^{p-1}\psi\Lambda \xrightarrow{\partial} i^*R^pj_*\Lambda \to R^p\psi\Lambda \to 0$ where $\partial : R^{p-1}\psi\Lambda \to i^*R^pj_*\Lambda$ is the boundary map. We show that the composition $i^*R^{p-1}j_*\Lambda \to R^{p-1}\psi\Lambda \xrightarrow{\partial} i^*R^pj_*\Lambda$ tensored with the map $\Lambda \to \Lambda(1)$ sending 1 to $t_\ell(T)$ is equal to the cup-product with $[\pi] \in H^1(K, \Lambda(1))$. We may assume that the residue field is separably closed. Then the class $[\pi] \in H^1(K, \Lambda(1))$ is identified with the class of $t_\ell \in Hom(I, \Lambda(1)) = H^1(K, \Lambda(1))$. By Lemma 1.7.2, the twisted composition $i^*R^{p-1}j_*\Lambda \to i^*R^pj_*\Lambda(1)$ is the same as the map $\theta : i^*R^{p-1}j_*\Lambda \to i^*R^pj_*\Lambda(1)$. Thus the assertions except the last exact sequence are proved. The last exact sequence is deduced from the first one inductively.

2. More precisely, we prove the following Lemma. Let $i^{(p)} = i \circ a_p : Y^{(p)} \to X$ for $p \geq 0$ be the canonical map. The map $i^{(p)} : Y^{(p)} \to X$ is the disjoint sum of regular immersions of codimension $p+1$.

Lemma 1.8 *Let* $p \geq 0$ *and consider the diagram*

$$\begin{array}{ccc}
i^*R^{p+1}j_*\Lambda(p+1) & \longleftarrow & a_{p*}\Lambda \\
\downarrow & & \downarrow \\
R^{p+2}i^!\Lambda(p+1) & \longrightarrow & a_{p*}R^{2p+2}i^{(p)!}\Lambda(p+1)
\end{array}$$

with the arrows defined as follows. The top horizontal is defined by the cup product of the map $a_{0*}\Lambda \to i^*R^1j_*\Lambda(1)$ in Corollary 1.6.2. The left vertical is the boudary map. The bottom horizontal is induced by the dual of the quasi-isomorphism $\Lambda_Y \to [a_{0*}\Lambda \to \cdots \to a_{p*}\Lambda \to \cdots \to a_{n*}\Lambda]$. The right vertical is the map sending 1 to the canonical class $[Y^{(p)}]$. Then the arrows are isomorphisms and the diagram is commutative.

Proof. First we show that the three arrows except the top horizontal one are isomorphisms. It is clear for the left vertical arrow. We show that the right vertical arrow is an isomorphism. Let $i_0 : s \to S$ be the immersion of the closed point and, for $p \geq 0$, let $f^{(p)} : Y^{(p)} \to \mathrm{Spec}\ F$ be the canonical map. The map $f^{(p)}$ is smooth of relative dimension $n - p$. Identifying $Ri^{(p)!}Rf^!\Lambda = Rf^{(p)!}Ri_0^!\Lambda$, we consider a commutative diagram

$$\begin{array}{ccc}
\Lambda(n-p)[2(n-p)] & \longrightarrow & i^{(p)*}Rf^!\Lambda(-p)[-2p] \\
\downarrow & & \downarrow \\
Rf^{(p)!}\Lambda & \longrightarrow & Rf^{(p)!}Ri_0^!\Lambda(1)[2] = Ri^{(p)!}Rf^!\Lambda(1)[2].
\end{array} \tag{1}$$

The maps are induced by those sending 1 to the canonical classes. The top horizontal arrow is an isomorphism by Proposition 1.5.2, the left vertical is an isomorphism since $f^{(p)}$ is smooth of relative dimension $n-p$ and the bottom horizontal is an isomorphism since O_K is a discrete valuation ring. Hence the right vertical arrow is an isomorphism. By Proposition 1.5.2, it implies that the right vertical arrow in the diagram of Lemma 1.8 is an isomorphism.

We show that the bottom horizontal arrow is an isomorphism. We have $Ri^!\Lambda = RHom_{\Lambda_X}(i_*\Lambda, \Lambda_X)$ and $Ri^{(p)!}\Lambda = RHom_{\Lambda_X}(i_*^{(p)}\Lambda, \Lambda_X)$. By the quasi-isomorphism $\Lambda_Y \to [a_{0*}\Lambda \to \cdots \to a_{p*}\Lambda \to \cdots \to a_{n*}\Lambda]$, we obtain a spectral sequence $E_1^{p,q} = R^qi^{(-p)!}\Lambda \Rightarrow R^{p+q}i^!\Lambda$. By the isomorphism $\Lambda \to Ri^{(p)!}\Lambda(p+1)[2(p+2)]$, we have $E_1^{p,2p} = \Lambda_{Y^{(p)}}(-(p+1))$ and $E_1^{p,q} = 0$ for $q \neq 2p$. Thus the spectral sequence degenerates at E_1 and we obtain an isomorphism $R^pi^!\Lambda \to R^{2p}i^{(p)!}\Lambda$. The bottom horizontal arrow is induced by this isomorphism with p replaced by $p+1$ and is an isomorphism.

To complete the proof, we show that the diagram is commutative. Shrinking X, we may assume $Y = D_0 \cup \cdots \cup D_p$ and $Y^{(p)} = D_0 \cap \cdots \cap D_p$. We have $Rj_*\Lambda = RHom_{\Lambda_X}(j_!\Lambda, \Lambda_X)$. From the exact sequence $0 \to j_!\Lambda \to \Lambda_X \to \Lambda_Y \to 0$, we obtain a quasi-isomorphism $j_!\Lambda \to \mathrm{Fiber}(\Lambda_X \to \Lambda_Y)$. The map $i^*Rj_*\Lambda \to Ri^!\Lambda[1]$ is the dual of the map $\Lambda_Y[-1] \to j_!\Lambda$ defined by the commutative diagram

$$\begin{array}{ccc}
\Lambda_Y[-1] & \longrightarrow & \mathrm{Fiber}(\Lambda_X \to \Lambda_Y) \\
\downarrow & & \| \\
j_!\Lambda & \longrightarrow & \mathrm{Fiber}(\Lambda_X \to \Lambda_Y).
\end{array}$$

Hence the composition $i^*Rj_*\Lambda \to Ri^!\Lambda[1] \to Ri^{(p)!}\Lambda[p+1]$ is the dual of the composition of the natural map $\Lambda_{Y^{(p)}}[-(p+1)] \to [\Lambda_X \to i_*^{(0)}\Lambda \to \cdots \to i_*^{(p)}\Lambda]$ with the inverse of the quasi-isomorphism $j_!\Lambda \to [\Lambda_X \to i_*^{(0)}\Lambda \to \cdots \to i_*^{(p)}\Lambda]$. For $0 \leq i \leq p$, let $i_i : D_i \to X, i'_i : D_i \to Y$ and $j_i : X - D_i \to X$ be the immersions. By the commutative diagram

$$\begin{array}{ccc} \bigotimes_{i=0}^{p} j_{i!}\Lambda & \longrightarrow & \bigotimes^{Lp}_{i=0}[\Lambda_X \to \Lambda_{D_i}] \\ \downarrow & & \downarrow \\ j_!\Lambda & \longrightarrow & [\Lambda_X \to i_*^{(0)}\Lambda \to \cdots \to i_*^{(p)}\Lambda], \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccccc} \bigotimes_{i=0}^{p} i'_{i*}\Lambda & \longrightarrow & \bigotimes_{i=0}^{p} R^1 j_{i*}\Lambda(1) & \longrightarrow & \bigotimes_{i=0}^{p} R^2 i_i^!\Lambda(1) \\ \downarrow & & \downarrow & & \downarrow \\ a_{p*}\Lambda & \longrightarrow & R^{p+1}j_*\Lambda(p+1) & \longrightarrow & R^{2(p+1)}i^!\Lambda(p+1). \end{array}$$

Since the canonical class is defined as the product, the assertion follows.

1.3 Construction of weight spectral sequences.

Let X be a strictly semi-stable scheme of relative dimension n over $S = \text{Spec } O_K$ as in the previous section. Let C be the abelian category of perverse Λ-sheaves on $Y = X_F$ [1]. In the terminology of perverse sheaves, Corollary 1.6 implies the following.

Lemma 1.9 *1. The complex $A = R\psi\Lambda[n]$ is a perverse sheaf.*

2. The shifted canonical filtration $F'_pA = \tau_{\leq p-n}A = (\tau_{\leq p}R\psi\Lambda)[n]$ is a filtration of a perverse sheaf $A = R\psi\Lambda[n]$ by sub perverse sheaves.

3. Let $p \geq 0$ be an integer. Then, the graded piece $Gr_p^{F'}A = R^p\psi\Lambda[n-p]$ is quasi-isomorphic to the complex

$$[i^*R^{p+1}j_*\Lambda(1) \xrightarrow{\theta} \cdots \xrightarrow{\theta} i^*R^nj_*\Lambda(n-p)]$$

where $i^*R^nj_*\Lambda(n-p)$ is put on degree 0. The truncation

$$[i^*R^{p+q+1}j_*\Lambda(q+1) \xrightarrow{\theta} \cdots \xrightarrow{\theta} i^*R^nj_*\Lambda(n-p)]$$

defines a filtration $G'^qGr_p^{F'}A$ of $Gr_p^{F'}A$ by sub perverse sheaves.

Proof. Since $Y^{(p)}$ is smooth of dimension $n-p$ and $a_p : Y^{(p)} \to Y$ is finite, the complex $a_{p*}\Lambda[n-p]$ is a perverse sheaf. By the isomorphism $a_{(p-1)*}\Lambda \to i^*R^pj_*\Lambda(p)$ in Corollary 1.6.2, the complex $i^*R^pj_*\Lambda(p)[n-(p+1)]$ is a perverse sheaf. The assertions follow from this and Corollary 1.6.1 immediately.

Corollary 1.10 *There is a canonical isomorphism*

$$Gr_G'^{q}Gr_p^{F'}A \longrightarrow a_{(p+q)*}\Lambda(-p)[n-(p+q)]$$

of perverse sheaves.

Proof. By definition of the filtration G', we have a canonical isomorphism $Gr'^{q}_{G} Gr^{F'}_{p} A \to i^* R^{p+q+1} j_* \Lambda(q+1)[n-(p+q)]$. It is sufficient to compose it with the inverse of the shifted and twisted of the isomorphism $a_{(p+q)*}\Lambda \to R^{p+q+1} j_* \Lambda(p+q+1)$ in Corollary 1.6.2.

In the case $p = 0$, the canonical isomorphism $Gr'^{q}_{G} Gr^{F'}_{0} A \to a_{q*}\Lambda[n-q]$ is defined by the canonical quasi-isomorphism $\Lambda \to [a_{0*}\Lambda \to \cdots \to a_{p*}\Lambda \to \cdots \to a_{n*}\Lambda]$. In fact, since the map $\theta : \Lambda \to i^* R^1 j_* \Lambda(1)$ is equal to the composition $\Lambda \to a_{0*}\Lambda \to i^* R^1 j_* \Lambda(1)$ of the canonical maps, the diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[d] & a_{0*}\Lambda \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{p*}\Lambda \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{n*}\Lambda \arrow[d] \\
R^0\psi\Lambda \arrow[r, "\bar{\theta}"'] & i^* R^1 j_* \Lambda(1) \arrow[r, "\theta"'] & \cdots \arrow[r, "\theta"'] & i^* R^{p+1} j_* \Lambda(p+1) \arrow[r, "\theta"'] & \cdots \arrow[r, "\theta"'] & i^* R^{n+1} j_* \Lambda(n+1)
\end{tikzcd}$$

is commutative (cf. [14] Satz 2.9). Hence the assertion follows.

Proposition 1.11 *Let X be a strictly semi-stable scheme of relative dimension n over $S = \operatorname{Spec} O_K$. Let T be an element of the inertia group I such that $t_\ell(T)$ is a generator of $\mathbf{Z}_\ell(1)$.*

1. The operator $\nu = T - 1$ on $A = R\psi\Lambda[n]$ satisfies ν^n.

2. Let $M_\bullet$ be the monodromy filtration on A defined by ν. Then there exists a canonical isomorphism

$$Gr^M_r A \to \bigoplus_{p-q=r} a_{(p+q)*}\Lambda(-p)[n-(p+q)].$$

The filtration M and the canonical isomorphism are independent of the choice of T.

If $\Lambda = \mathbf{Q}_\ell$, the nilpotent monodromy operator N defines a monodromy filtration on A. It is the same as the filtration in Proposition 1.11.2. We also call the induced filtration $M_\bullet$ on $R\psi\Lambda = A[-n]$ the monodromy filtration.

Corollary 1.12 ([14] Satz 2.10) *Assume further X is proper over O_K. Then the monodromy filtration $M_\bullet$ on $R\psi\Lambda$ induces a spectral sequence*

$$E_1^{p,q} = \bigoplus_{i \geq \max(0,-p)} H^{q-2i}(Y^{(p+2i)}_{\bar{F}}, R\psi\Lambda(-i)) \Rightarrow H^{p+q}(X_{\bar{K}}, \Lambda).$$

The spectral sequence in Corollary 1.12 is called the weight spectral sequence. The induced increasing filtration $W_\bullet$ on the limit $H^r(X_{\bar{K}}, \Lambda)$ is called the weight filtration.
Proof of Corollary 1.12. The monodromy filtration $M_\bullet$ on $R\psi\Lambda$ induces a spectral sequence $E_1^{p,q} = H^{p+q}(Y_{\bar{F}}, Gr^M_{-p} R\psi\Lambda) \Rightarrow H^{p+q}(Y_{\bar{F}}, R\psi\Lambda) = H^{p+q}(X_{\bar{K}}, \Lambda)$. By Proposition 1.11, the E_1-term $E_1^{p,q}$ is canonically isomorphic to

$$H^{p+q}(Y_{\bar{F}}, \bigoplus_{i \geq \max(0,-p)} a_{(p+2i)*}\Lambda(-i)[-(p+2i)]) = \bigoplus_{i \geq \max(0,-p)} H^{q-2i}(Y^{(p+2i)}_{\bar{F}}, \Lambda(-i)).$$

We deduce Proposition 1.11 from the following Lemma.

Lemma 1.13 *1.* $\nu^{n+1} = 0$ *and the kernel filtration* $F_pA = \mathrm{Ker}(\nu^{p+1} : A \to A)$ *is equal to the shifted canonical filtration* $F'_pA = \tau_{\le p-n}A = (\tau_{\le p}R\psi\Lambda)[n]$.

2. For $p, q \ge 0$, *the image* $\mathrm{Im}(\nu^q : Gr^{F'}_{p+q}A \to Gr^{F'}_pA)$ *is equal to the filtration* $G'^qGr^{F'}_pA$ *defined in Lemma 1.9.3.*

3. Let $p, q \ge 0$. *For the canonical isomorphism in Corollary 1.10, the diagram*

$$\begin{array}{ccc} Gr'^q_G Gr^{F'}_{p+1}A & \longrightarrow & a_{(p+1+q)*}\Lambda(-(p+1))[n-(p+1+q)] \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle 1\otimes t_\ell(T)} \\ Gr'^{q+1}_G Gr^{F'}_pA & \longrightarrow & a_{(p+q+1)*}\Lambda(-p)[n-(p+q+1)] \end{array}$$

is commutative.

Proof of Proposition 1.11. We prove Proposition 1.11 admitting Lemma 1.13. We have a canonical isomorphism $Gr^M_rA \to \bigoplus_{p-q=r} Gr^q_GGr^F_rA$ by Proposition 1.2 and Lemma 1.1. By Lemmas 1.3 and 1.13.2, we have $Gr^q_GGr^F_rA = Gr^q_{G'}Gr^{F'}_rA$. By Corollary 1.10, we have a canonical isomorphism $Gr^q_{G'}Gr^{F'}_rA \to a_{(p+q)*}\Lambda(-p)[n-(p+q)]$.

We show that the monodromy filtration $M_\bullet$ is independent of T. Let T' be another element of I such that $t_l(T') = ut_l(T)$ for $u \in \mathbf{Z}_\ell^\times$. Then the induced map $Gr^M_rA \to Gr^M_{r-2}A$ by $T'-1$ is $u\nu$. Hence $M_\bullet$ is independent of T. The choice of T does not appear in the definition of the canonical isomorphism.

Proof of Lemma 1.13. 1. We show that ν and $F'_\bullet$ satisfy the conditions (1)-(3) in Lemma 1.4. It is clear that $\nu = T-1$ maps F'_pA to F'_pA. By Proposition 1.5.1, the induced map ν on $Gr^{F'}_pA = R^p\psi\Lambda[n-p]$ is the 0-map. Hence the condition (1) is satisfied and we have $\nu^{n+1} = 0$.

We show the condition (2) is satisfied. Namely we show that $F'_0A = \Lambda[n] = R^0\psi\Lambda[n]$ is equal to $F_0A = \mathrm{Ker}(T-1 : A \to A)$. By Lemma 1.9.1, the complex $F'_0A = \Lambda[n]$ is a perverse sheaf on Y. By Lemma 1.8, we have an isomorphism $R^{p+2}i^!\Lambda \to a_{p*}\Lambda(-(p+1))$. Hence the complex $Ri^!\Lambda[n+2]$ is a perverse sheaf. By the distinguished triangle $Ri^!\Lambda \to i^*\Lambda \to i^*Rj_*\Lambda \to$, we have ${}^p\mathcal{H}^0(i^*Rj_*\Lambda[n]) = \Lambda[n], {}^p\mathcal{H}^1(i^*Rj_*\Lambda[n]) = Ri^!\Lambda[n+2]$ and ${}^p\mathcal{H}^q(i^*Rj_*\Lambda[n]) = 0$ for $q \neq 0, 1$. Hence by Lemma 1.7.1, we obtain an exact sequence

$$0 \longrightarrow \Lambda[n] \longrightarrow A \xrightarrow{T-1} A \longrightarrow Ri^!\Lambda[n+2] \longrightarrow 0$$

of perverse sheaves. Therefore the canonical map $\Lambda[n] = R^0\psi\Lambda[n] \to \mathrm{Ker}(T-1 : A \to A)$ is an isomorphism and the condition (2) is satisfied.

We show the condition (3) is satisfied. First, we show that the cokernel of the map $\nu : Gr^{F'}_{p+1}A = R^{p+1}\psi\Lambda[n-(p+1)] \to Gr^{F'}_pA = R^p\psi\Lambda[n-p]$ is isomorphic to $i_*R^{p+1}j_*\Lambda[n-p]$. By Lemma 1.7.1, the bottom arrow of the commutative diagram

$$\begin{array}{ccc} R\psi\Lambda & = & \mathrm{Cone}[0 \to R\psi\Lambda] \\ {\scriptstyle T-1}\downarrow & & \downarrow{\scriptstyle \mathrm{Cone(0,id)}} \\ R\psi\Lambda & \xleftarrow[T-1]{} & \mathrm{Cone}[i_*Rj_*\Lambda \to R\psi\Lambda] \end{array}$$

is a quasi-isomorphism. Hence we have a commutative diagram

$$\begin{array}{ccc} R^{p+1}\psi\Lambda[n-(p+1)] & = & [0 \quad \to R^{p+1}\psi\Lambda] \\ {\scriptstyle\nu}\downarrow & & \downarrow \\ R^p\psi\Lambda[n-p] & \longrightarrow & [i_*R^{p+1}j_*\Lambda \to R^{p+1}\psi\Lambda] \end{array} \tag{2}$$

where the bottom horizontal arrow is a quasi-isomorphism and $R^{p+1}\psi\Lambda$ in the right column are put on degree $p+1-n$. Hence the cokernel is isomorphic to $i_*R^{p+1}j_*\Lambda[n-p] = a_{p*}\Lambda(-(p+1))[n-p]$.

We show that there is no non-0 map from a sub perverse sheaf of Coker($\nu : R^{p+1}\psi\Lambda[n-(p+1)] \to R^p\psi\Lambda[n-p]) \simeq a_{p*}\Lambda(-(p+1))[n-p]$ to $Gr^{F'}_{p'}A = R^{p'}\psi\Lambda[n-p']$ for $p' > p+1$. By Corollary 1.10, the perverse sheaf $R^{p'}\psi\Lambda[n-p']$ is isomorphic to a successive extension of $a_{(p'+q)*}\Lambda(-(p+q+1))[n-(p'+q)]$ for $q \geq 0$. Hence, there is no non-0 map from a sub perverse sheaf of Coker($\nu : R^{p+1}\psi\Lambda[n-(p+1)] \to R^p\psi\Lambda[n-p]) \simeq a_{p*}\Lambda(-(p+1))[n-p]$ to $Gr^{F'}_{p'}A$ if $p' > p$. Thus the condition (3) is also satisfied. By applying Lemma 1.4, we obtain $F_pA = F'_pA$ for $p \geq 0$.

2. By induction on q, we deduce a commutative diagram

$$\begin{array}{ccc} R^{p+q}\psi\Lambda[n-(p+q)] \xrightarrow{\bar\theta} & [0 \to & i_*R^{p+q+1}j_*\Lambda(1) \xrightarrow{\theta} \cdots \xrightarrow{\theta} i_*R^nj_*\Lambda(n-(p+q))] \\ {\scriptstyle\nu^q}\downarrow & & \downarrow{\scriptstyle t_\ell(T)^{\otimes q}} \\ R^p\psi\Lambda[n-p] \xrightarrow[\bar\theta]{} & [i_*R^{p+1}j_*\Lambda(1) \xrightarrow[\theta]{} \cdots \xrightarrow[\theta]{} & i_*R^{p+q+1}j_*\Lambda(q+1) \xrightarrow[\theta]{} \cdots \xrightarrow[\theta]{} i_*R^nj_*\Lambda(n-p)] \end{array}$$

from the commutative diagram (2) and the proof of Corollary 1.6. The horizontal arrows are quasi-isomorphisms and $i_*R^nj_*\Lambda(n-p)$ and $i_*R^nj_*\Lambda(n-(p+q))$ in the right column are put on degree 0. Hence the assertion follows.

3. It follows from the commutative diagram above and the definition of the canonical isomorphism.

1.4 Properties of weight spectral sequences I, functorialty and duality.

In this subsection, we establish the functoriality and the compatiblity with the Poincaré duality for weight spectral sequences. In the next subsection, using these properties, we establish the compatiblity with push-forward, chern classes etc.

We begin with the functoriality. Let X and X' be strictly semi-stable schemes over S and $f : X \to X'$ be a morphism over S. To formulate the functoriality of weight spectral sequences, we introduce some notations. Let $D_1, \ldots, D_m$ be the irreducible components of $Y = X_F$ and $D'_1, \ldots, D'_{m'}$ be the irreducible components of $Y' = X'_F$. We define $Y'^{(p)} = \bigcup_{I\subset\{1,\ldots,m'\},\text{Card } I=p+1} Y'_I$ and $a'_p : Y'^{(p)} \to Y' = X'_F$ for $p \geq 0$ similarly as for X. For $p \geq 0$, we define maps $f^{(p)*} : f^*a'_{p*}\Lambda \to a_{p*}\Lambda$ and $f^{(p)*} : H^q(Y'^{(p)}_{\bar F}, \Lambda) \to H^q(Y^{(p)}_{\bar F}, \Lambda)$. Since $\sum_{i'=1}^{m'} f^*D'_{i'} = \sum_{i=1}^m D_i$ as divisors, there exists a unique $i' \in \{1, \ldots, m'\}$ such that $f(D_i) \subset D'_{i'}$ for each $i \in \{1, \ldots, m\}$. We define a map $\varphi : \{1, \ldots, m\} \to \{1, \ldots, m'\}$ by requiring $f(D_i) \subset D'_{\varphi(i)}$. Renumbering if necessary, we assume that the map φ is increasing. For a subset $I \subset \{1, \ldots, m'\}$ and an integer $p' \geq p = \text{Card } I - 1$, we put $\mathcal{I}_{I',p'} = \{I \subset \{1,\dots,m\} |\ \mathrm{Card}\ I = p'+1$ and φ induces a surjection $I \to I'\}$. For $I \in \mathcal{I}_{I',p}$, the map f induces a map $f_{I'I} : Y_I = \bigcap_{i\in I} D_i \to Y'_{I'} = \bigcap_{i'\in I'} D'_{i'}$ and the maps $f^*_{I'I} : f^*a'_{I'*}\Lambda \to a_{I*}\Lambda$ and $f^*_{I'I} : H^q(Y_{I',\bar F},\Lambda) \to H^q(Y_{I,\bar F},\Lambda)$ are defined. They are independent of the choice of numbering. We define maps $f^{(p)*} : f^*a'_{p*}\Lambda \to a_{p*}\Lambda$ and $f^{(p)*} : H^q(Y'^{(p)}_{\bar F},\Lambda) \to H^q(Y^{(p)}_{\bar F},\Lambda)$ to be the sum $\sum_{I'\subset\{1,\dots,m'\},\mathrm{Card}\ I'=p+1}\sum_{I\in\mathcal{I}_{I',p}} f^*_{I'I}$.

Proposition 1.14 *Let X and X' be strictly semi-stable schemes over S of relative dimension n and n' respectively and $f : X \to X'$ be a morphism over S. Then,*

1. The inverse image $f^*A'[n-n']$ and $f^*M_rA'[n-n']$ are perverse sheaves on $Y = X_F$. The inverse image $f^*M_rA'[n-n']$ gives the monodromy filtration on $f^*A'[n-n']$.

2. The natural map $f^* : f^*A'[n-n'] \to A$ sends the filtration $f^*M_rA'[n-n']$ into M_rA. We have a commutative diagram

$$\begin{array}{ccc} f^*Gr^M_r A'[n-n'] & \longrightarrow & \bigoplus_{p-q=r} f^*a'_{(p+q)*}\Lambda(-p)[n-(p+q)] \\ {\scriptstyle Gr^M_r f^*}\downarrow & & \downarrow{\scriptstyle \oplus f^{(p+q)*}} \\ Gr^M_r A & \longrightarrow & \bigoplus_{p-q=r} a_{(p+q)*}\Lambda(-p)[n-(p+q)] \end{array}$$

where the horizontal arrows are the canonical map in Proposition 1.11. The left vertical arrow is the map induced by $f^* : f^*A'[n-n'] \to A$ and the right vertical arrow is the direct sum of $f^{(p+q)*}$.

Corollary 1.15 *Assume further X and X' are proper over S. Then we have a map of weight spectral sequences*

$$\begin{array}{ccc} E'^{p,q}_1 = \bigoplus_{i\geq\max(0,-p)} H^{q-2i}(Y'^{(p+2i)}_{\bar F},\Lambda(-i)) & \Rightarrow & H^{p+q}(X'_{\bar K},\Lambda) \\ {\scriptstyle \oplus f^{(p+2i)*}}\downarrow & & \downarrow{\scriptstyle f^*_{\bar K}} \\ E^{p,q}_1 = \bigoplus_{i\geq\max(0,-p)} H^{q-2i}(Y^{(p+2i)}_{\bar F},\Lambda(-i)) & \Rightarrow & H^{p+q}(X_{\bar K},\Lambda). \end{array}$$

Proof of Proposition 1.14. 1. By the isomorphism $Gr^M_r A' \to \bigoplus_{p-q=r} a'_{(p+q)*}\Lambda(-p)[n'-(p+q)]$ in Proposition 1.11, it is sufficient to show that $f^*a'_{p*}\Lambda[n-p]$ is a perverse sheaf. Hence, it is sufficient to show that $f^*a_{I*}\Lambda[n-p]$ is a perverse sheaf for $I' \subset \{1,\dots,m'\}$ and Card $I' = p+1$. We put $\mathcal{I}_{I',p'} = \{I \subset \{1,\dots,m\} |\ \mathrm{Card}\ I = p'+1$ and φ induces a surjection $I \to I'\}$ as above. Then the inverse image $f^{-1}(Y'_{I'})$ is equal to the union $\bigcup_{I\in\mathcal{I}_{I',p}} Y_I$. Hence, we have an exact sequence $f^*a'_{I'*}\Lambda \to \bigoplus_{I\in\mathcal{I}_{I',p}} a_{I*}\Lambda \to \cdots \to \bigoplus_{I\in\mathcal{I}_{I',n'}} a_{I*}\Lambda$. Thus $f^*a'_{I'*}\Lambda[n-p]$ is a perverse sheaf and the assertion is proved.

2. By the definition of the canonical isomorphism $Gr^q_G Gr^F_p A \to a_{(p+q)*}\Lambda(-p)[n-$

$(p+q)]$, it is sufficient to show that the diagrams

$$\begin{array}{ccc} f^*Gr^q_GGr^F_pA'[n-n'] & \longrightarrow & f^*i'^*R^{p+q+1}j'_*\Lambda(q+1)[n-(p+q)] \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ Gr^q_GGr^F_pA & \longrightarrow & i^*R^{p+q+1}j_*\Lambda(q+1)[n-(p+q)], \end{array}$$

$$\begin{array}{ccc} f^*i'^*R^pj'_*\Lambda(p) & \longrightarrow & f^*a'_{(p-1)*}\Lambda \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^{(p)*}} \\ i^*R^pj_*\Lambda(p) & \longrightarrow & a_{(p-1)*}\Lambda \end{array}$$

are commutative. The commutativity of the first square is clear from the construction. We show the commutativity of the second diagram. By Proposition 1.5.2, it is reduced to the case where $p=1$. In the case $p=1$, it follows from $f^*[D'_{i'}]=\sum_{i\in\varphi^{-1}(i')}[D_i]$.

Next, we establish compatibility with the Poincaré duality. Let $f:X\to S$ be a strictly semi-stable scheme over S of relative dimension n. We define a perverse sheaf A' by $A'=R\psi Rf^!\Lambda[-n]=A\otimes Rf^!\Lambda[-2n]$. Let $f_Y:Y\to \mathrm{Spec}\ F$ be the closed fiber of f and let $f^{(p)}=f_Y\circ a_p:Y^{(p)}\to \mathrm{Spec}\ F$ be the canonical map for $p\geq 0$. The map $f^{(p)}$ is smooth of relative dimension $n-p$. Let D_Y and $D_{Y^{(p)}}$ denote the functors $R\mathcal{H}om(\quad, Rf^!_Y\Lambda)$ and $R\mathcal{H}om(\quad, Rf^{(p)!}\Lambda)$. We define a commutative diagram of isomorphisms

$$\begin{array}{ccc} Gr^M_rA' & \longrightarrow & \bigoplus_{q-p=r}a_{(p+q)*}D_{Y^{(p+q)}}\Lambda(p)[-(n-(p+q))] \\ \big\downarrow & & \big\downarrow \\ D_YGr^M_{-r}A & \longrightarrow & \bigoplus_{p-q=-r}D_Y(a_{(p+q)*}\Lambda)(p)[-(n-(p+q))]. \end{array} \tag{3}$$

The left verticall arrow is induced by the canonical isomorphism $A'\to D_YA=R\mathcal{H}om(A,Rf^!_Y\Lambda)$ [9] 4.3. We recall the definition of $A'\to D_YA$. The product defines a pairing $A\times A'\to R\psi Rf^!\Lambda$. Let $R\psi_0$ denote the nearby cycle functor for S itself. We have a base change map $R\psi Rf^!\Lambda\to Rf^!_YR\psi_0\Lambda=Rf^!_Y\Lambda$ loc.cit 4.3.b). The composite pairing $A\times A'\to Rf^!_Y\Lambda$ induces the required map. By [9] Théorème 4.2, the canonical map $A'\to D_YA$ is an isomorphism. The bottom horizontal arrow in the diagram (3) is the dual of the canonical isomorphism in Proposition 1.11. The right vertical arrow is defined by the canonical isomorphism $a_{(p+q)*}D_{Y^{(p+q)}}\Lambda\to D_Y(a_{(p+q)*}\Lambda)$. The top horizontal arrow is defined by the commutativity of the diagram.

Proposition 1.16 *Let $f:X\to S$ be a strictly semi-stable scheme over S of relative dimension n. Then we have a commutative diagram of isomorphisms*

$$\begin{array}{ccc} Gr^M_rA(n) & \longrightarrow & \bigoplus_{p-q=r}a_{(p+q)*}\Lambda(n-p)[n-(p+q)] \\ \big\downarrow & & \big\downarrow \\ Gr^M_rA' & \longrightarrow & \bigoplus_{q-p=r}a_{(p+q)*}D_{Y^{(p+q)}}\Lambda(p)[-(n-(p+q))] \end{array}$$

where the top horizontal arrow is the twist of the canonical map in Proposition 1.11 and the bottom horizontal arrow is the top horizontal arrow in the diagram (3) above. The left vertical arrow is induced by the canonical map $\Lambda(n)[2n] \to Rf^!\Lambda$. The right vertical arrow is the direct sum of $(-1)^q$-times that induced by the canonical isomorphisms $\Lambda(n-(p+q))[2(n-(p+q))] \to Rf^{(p+q)!}\Lambda \to D_{Y^{(p+q)}}\Lambda$ sending the (p,q)-components to the (q,p)-components.

Corollary 1.17 *Assume further X is proper and $\Lambda = \mathbf{F}_\ell$ or $\mathbf{Q}_\ell$. Then, we have an isomorphism of the weight spectral sequence with its dual*

$$\begin{array}{ccc}
E_1^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{q-2i}(Y_{\bar F}^{(p+2i)}, \Lambda(n-i)) & \Rightarrow & H^{p+q}(X_{\bar K}, \Lambda(n)) \\
\downarrow & & \downarrow \\
E_1'^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{2(n-(p+2i))-(q-2i)}(Y_{\bar F}^{(p+2i)}, \Lambda(-(p+i))^* & \Rightarrow & H^{2n-(p+q)}(X_{\bar K}, \Lambda)^*.
\end{array}$$

*The superscript * denotes the linear dual. The right vertical arrow is induced by the pairing $H^{p+q}(X_{\bar K}, \Lambda(n)) \times H^{2n-(p+q)}(X_{\bar K}, \Lambda) \to \Lambda$. The left vertical arrow is the direct sum of the $(-1)^{p+i}$-times that induced by the pairing $H^{q-2i}(Y_{\bar F}^{(p+2i)}, \Lambda(n-i)) \times H^{2(n-(p+2i))-(q-2i)}(Y_{\bar F}^{(p+2i)}, \Lambda(-(p+i)) \to \Lambda$.*

Proof of Corollary. The canonical isomorphism $\Lambda(n) \to Rf^!\Lambda[-2n]$ induces an isomorphism of spectral sequences

$$\begin{array}{ccc}
E_1^{p,q} = H^{p+q}(Y_{\bar F}, Gr^M_{-p}A(n)[-n]) & \Rightarrow & H^{p+q}(X_{\bar K}, \Lambda(n)) \\
\downarrow & & \downarrow \\
E_1^{p,q} = H^{p+q}(Y_{\bar F}, Gr^M_{-p}A'[-n]) & \Rightarrow & H^{p+q-2n}(X_{\bar K}, D_X\Lambda).
\end{array}$$

The assertion follows form Proposition 1.16 and the Poincaré duality $H^q(V, D_V\Lambda) \simeq H^{-q}(V,\Lambda)^*$ for a proper smooth scheme V over a separably closed field.

Proof of Proposition. By the definition of the canonical isomorphisms, it is sufficient to show that the diagram

$$\begin{array}{ccc}
Gr^q_G Gr^F_p A(n) & \longrightarrow & a_{(p+q)*}\Lambda(n-p)[n-(p+q)] \\
\downarrow & & \downarrow \\
D_Y Gr^p_G Gr^F_q A & \longrightarrow & D_Y(a_{(p+q)*}\Lambda)(q)[-(n-(p+q))]
\end{array} \tag{4}$$

is commutative. The horizontal arrows are the twist and the dual of the canonical isomorphism in Corollary 1.10. The left vertical arrow is induced by the composition $A \to A'(n) \to D_Y A$. The right vertical arrow is induced by $(-1)^q$-times the canonical isomorphism $\Lambda(n-(p+q))[2(n-(p+q))] \to Rf^{(p+q)!}\Lambda$.

First we prove the case $q = 0$. In this case, we have $Gr^F_0 A = \Lambda[n]$ and the composition via the lower left is the same as the composition of isomorphisms

$$Gr^0_G Gr^F_p A(n) \longrightarrow Gr^F_p Gr^0_G A' \longrightarrow Gr^F_p D_Y(\Lambda[n]) \longrightarrow D_Y(a_{p*}\Lambda[n-p]).$$

The middle arrow is induced by the base change map $A' = R\psi Rf^!\Lambda[-n] \to D_Y(\Lambda[n]) = Rf^!R\psi\Lambda[-n]$. The last arrow is induced by the dual of the quasi-isomorphism $\Lambda \to [a_{0*}\Lambda \to \cdots \to a_{p*}\Lambda \to \cdots \to a_{n*}\Lambda]$ since the diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[d] & a_{0*}\Lambda \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{p*}\Lambda \arrow[r] \arrow[d] & \cdots \arrow[r] & a_{n*}\Lambda \arrow[d] \\
R^0\psi\Lambda \arrow[r, "\bar\theta"'] & i^*R^1j_*\Lambda(1) \arrow[r, "\theta"'] & \cdots \arrow[r, "\theta"'] & i^*R^{p+1}j_*\Lambda(p+1) \arrow[r, "\theta"'] & \cdots \arrow[r, "\theta"'] & i^*R^{n+1}j_*\Lambda(n+1)
\end{tikzcd}$$

is commutative.

We compute the right vertical arrow in the diagram (4) in the case $q = 0$. We identify $D_Y(a_{p*}\Lambda) = a_{p*}Ra_p^!Rf_Y^!\Lambda$. Let $i_0 : \mathrm{Spec}\ F \to S, j_0 : \mathrm{Spec}\ K \to S$ and $i^{(p)} = i \circ a_p : Y^{(p)} \to X$ for $p \geq 0$ be the canonical maps. Then by the commutative diagram (1) in the proof of Lemma 1.8, the right vertical arrow in the diagram (4) is induced by the composition of the isomorphisms

$$\Lambda(n-p)[2(n-p)] \to i^{(p)*}Rf^!\Lambda(-p)[-2p] \to Ri^{(p)!}Rf^!\Lambda(1)[2] \leftarrow Ra_p^!Rf_Y^!\Lambda.$$

The maps are induced by the canonical isomorphisms $\Lambda(n)[2n] \to Rf^!\Lambda, \Lambda(-(p+1))[-2(p+1)] \to Ri^{(p)!}\Lambda$ and $\Lambda(-1)[-2] \to Ri_0^!\Lambda$. Thus the case $q = 0$ is reduced to the commutativity of the diagram of isomorphisms

$$\begin{tikzcd}
Gr_G^0 Gr_p^F R\psi Rf^!\Lambda \arrow[r] \arrow[d] & a_{p*}i^{(p)*}Rf^!\Lambda(-p)[-p] \arrow[r] & a_{p*}Ri^{(p)!}Rf^!\Lambda(1)[p+2] \arrow[d] \\
Gr_p^F D_Y\Lambda \arrow[r] & D_Y(a_{p*}\Lambda[-p]) \arrow[r] & a_{p*}Ra_p^!Rf_Y^!Ri_0^!\Lambda(1)[p+2].
\end{tikzcd}$$

The top left arrow is induced by the canonical map in Corollary 1.10, the top right is induced by the canonical map $\Lambda(-(p+1))[-2(p+1)] \to Ri^{(p)!}\Lambda$, the right vertical is induced by the isomorphism $Ri^{(p)!}Rf^! \to Ra_p^!Rf_Y^!Ri_0^!$, the left vertical is induced by $R\psi Rf^!\Lambda \to Rf^!\Lambda = D_Y\Lambda$, the bottom left is induced by the quasi-isomorphism $\Lambda \to [a_{0*}\Lambda \to \cdots \to a_{p*}\Lambda \to \cdots \to a_{n*}\Lambda]$ and the bottom right is induced by the canonical isomorphism $\Lambda \to Ri_0^!\Lambda(1)[2]$ with the identifycation $D_Y(a_{p*}\Lambda) = a_{p*}Ra_p^!Rf_Y^!\Lambda$.

Since the diagram

$$\begin{tikzcd}
R\psi Rf^!\Lambda \arrow[r] \arrow[d] & i^*Rj_*Rf^!\Lambda(1)[1] \arrow[r] \arrow[d] & Ri^!Rf^!\Lambda(1)[2] \arrow[d, "\simeq"] \\
Rf_Y^!R\psi\Lambda \arrow[r] & Rf_Y^!i_0^*Rj_{0*}\Lambda(1)[1] \arrow[r] & Rf_Y^!Ri_0^!\Lambda(1)[2]
\end{tikzcd}$$

is commutative, we obtain a commutative diagram of isomorphisms

$$\begin{tikzcd}
Gr_G^0R^p\psi R^{-2n}f^!\Lambda \arrow[r] \arrow[d] & i^*R^{p+1}j_*R^{-2n}f^!\Lambda(1) \arrow[d] & \\
R^{p-2n}f_Y^!R^0\psi_0\Lambda \arrow[d, equal] & R^{p+2}i^!R^{-2n}f^!\Lambda(1) \arrow[r] \arrow[d] & a_{p*}R^{2p+2}i^{(p)!}R^{-2n}f^!\Lambda(1) \arrow[d] \\
R^{p-2n}f_Y^!\Lambda \arrow[r] & R^{p-2n}f_Y^!R^2i_0^!\Lambda(1) \arrow[r] & a_{p*}R^{2p-2n}f^{(p)!}R^2i_0^!\Lambda(1).
\end{tikzcd}$$

The left bottom arrow is induced by $\Lambda \to R^2 i_0^! \Lambda(1)$ sending 1 to the canonical class $[s]$ of the closed point $s \in S$. The right horizontal arrows are induced by the dual of the quasi-isomorphism $\Lambda \to [a_{0*}\Lambda \to \cdots \to a_{p*}\Lambda \to \cdots \to a_{n*}\Lambda]$. Hence, if $q = 0$, the commutativity of the diagram (4) follows from Lemma 1.8.

We reduce the general case to the case $q = 0$. Since the adjoint of T on A' is T^{-1} on A and $T^{-1} - 1 = -T^{-1}(T-1)$ induces $-\nu$ on $Gr^q_G Gr^F_p A$, the diagram

$$\begin{array}{ccc} Gr^0_G Gr^F_{p+q} A(n) & \xrightarrow{\nu^q} & Gr^q_G Gr^F_p A(n) \\ \downarrow & & \downarrow \\ D_Y Gr^{p+q}_G Gr^F_0 A & \xrightarrow[(-1)^q \nu^q]{} & D_Y Gr^p_G Gr^F_q A \end{array}$$

is commutative. Hence the assertion follows from Lemma 1.13.3.

1.5 Properties of weight spectral sequences II, push-forward, chern classes etc.

Let X and X' be proper and strictly semi-stable schemes over S of relative dimension n and $n-r$ and let $f : X \to X'$ be a morphism over S. For $p \geq 0$, we define the push-forward map $f_*^{(p)} : H^q(Y^{(p)}_{\bar F}, \Lambda) \to H^{q-2r}(Y'^{(p)}_{\bar F}, \Lambda(-r))$ to be the sum $\sum_{I' \subset \{1,\dots,m'\}, \mathrm{Card}\ I' = p+1} \sum_{I \in \mathcal{I}_{I',p}} f_{I'I*}$ similarly as $f^{(p)*}$.

Proposition 1.18 *Let X and X' be proper and strictly semi-stable schemes over S of relative dimension n and $n' = n - r$ and let $f : X \to X'$ be a morphism over S. Let $\Lambda = \mathbf{F}_\ell$ or $\mathbf{Q}_\ell$. Then, we have a map of spectral sequence*

$$\begin{array}{ccc} E_1^{p,q} = \bigoplus_{i \geq \max(0,-p)} H^{q-2i}(Y^{(p+2i)}_{\bar F}, \Lambda(-i)) & \Rightarrow & H^{p+q}(X_{\bar K}, \Lambda) \\ \Big\downarrow {\scriptstyle \oplus f_*^{(p+2i)}} & & \Big\downarrow {\scriptstyle f_*} \\ E_1'^{p,q-2r} = \bigoplus_{i \geq \max(0,-p)} H^{q-2r-2i}(Y'^{(p+2i)}_{\bar F}, \Lambda(-i-r)) & \Rightarrow & H^{p+q-2r}(X'_{\bar K}, \Lambda(-r)). \end{array}$$

The right vertical map f_* is the push-forward map and the left vertical map is the direct sum of $f_*^{(p+2i)}$.

Proof. For a morphism $f : V \to V'$ of proper smooth schemes of $\dim V = d, \dim V' = d' - r$ over a separably closed field, we have a commutative diagram

$$\begin{array}{ccc} H^q(V, \Lambda) & \longrightarrow & H^{2d-q}(V, \Lambda(d))^* \\ {\scriptstyle f_*} \Big\downarrow & & \Big\downarrow {\scriptstyle (f^*)^*} \\ H^{q-2r}(V, \Lambda(-r)) & \longrightarrow & H^{2d-q}(V, \Lambda(d))^*. \end{array}$$

The horizontal arrows are the isomorphisms of Poincaré duality and the right vertical arrow is the dual of the pull back f^*. Hence the required map is obtained as the composition of the dual of the map in Proposition 1.14 with the maps in Proposition 1.16 for X and for X'.

Proposition 1.19 *Let X be a proper and strictly semi-stable scheme over S of relative dimension n and $\mathcal{E}$ be a locally free O_X-module of rank r. Then, for $0 \leq k \leq r$, the chern class $c_k(\mathcal{E})$ induces a map of spectral sequences*

$$\begin{array}{ccc} E_1^{p,q} = \bigoplus_{i \geq \max(0,-p)} H^{q-2i}(Y_{\bar{F}}^{(p+2i)}, \Lambda(-i)) & \Rightarrow & H^{p+q}(X_{\bar{K}}, \Lambda) \\ \Big\downarrow {\scriptstyle c_k(\mathcal{E})\cup} & & \Big\downarrow {\scriptstyle c_k(\mathcal{E})\cup} \\ E_1^{p,q} = \bigoplus_{i \geq \max(0,-p)} H^{q-2i+2k}(Y_{\bar{F}}^{(p+2i)}, \Lambda(-i+k)) & \Rightarrow & H^{p+q+2k}(X_{\bar{K}}, \Lambda(k)). \end{array}$$

Proof. First we prove the case where $k = r$. Let $\pi : E = Spec_{O_X} S^\bullet \mathcal{E} \to X$ be the vector bundle over X associated to $\mathcal{E}$ and $i : X \to E$ be the 0-section. We consider the isomorphism $\Lambda_X \to Ri^! \Lambda_E(r)[2r]$ sending 1 to the canonical class $[X]$. The induced map $\Lambda \to \Lambda(r)[2r]$ is the top chern class $c_r(\mathcal{E})$. Since the monodromy filtration $M_\bullet A_E$ of $A_E = R\psi\Lambda[n+r]$ is equal to $\pi_s^* M_\bullet A_X[r]$, it induces an isomorphism of filtered complexes $R\psi\Lambda_X \to Ri^! R\psi\Lambda_E(r)[2r] = R\psi\Lambda_X \otimes Ri^! \Lambda_E(r)[2r]$. By adjunction $i_* Ri^! \to \mathrm{id}$ and by pull-back i^*, it induces a map of filtered complexes $R\psi\Lambda_X \to R\psi\Lambda_X(r)[2r]$. The induced maps on the graded pieces are induced by $\Lambda_{Y^{(p)}} \to Ri^!_{Y^{(p)}} \Lambda_{E \times_X Y^{(p)}}(r)[2r]$ where $i^!_{Y^{(p)}} : Y^{(p)} \to E \times_X Y^{(p)}$ denotes the immersion. Hence the assertion follows.

There is an alternative proof when $\Lambda = \mathbf{F}_\ell$ or $\mathbf{Q}_\ell$. Let $P = \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ be the $\mathbf{P}^r \supset E$-bundle associated to $\mathcal{E} \oplus \mathcal{O}$ and $i : X \to P$ be the 0-section. Then the map $c_r(\mathcal{E})$ is equal to the composition $i^* \circ i_*$. Hence the assertion in this case follows form Propositions 1.14 and 1.18.

Since $c_1(\mathcal{E}) = c_1(\mathcal{L})$ for $\mathcal{L} = \Lambda^r \mathcal{E}$, the case $k = 1$ is proved.

We prove the general case. Let $P = \mathbf{P}(\mathcal{E})$ be the $\mathbf{P}^{r-1}$-bundle associated to $\mathcal{E}$ and $\pi : P \to X$ be the projection. Let $h = c_1(O(1))$ be the first chern class of the tautological invertible sheaf. Then we have a commutative diagram

$$\begin{array}{ccc} H^p(X_{\bar{K}}, \Lambda) & \xrightarrow{\oplus c_k(\mathcal{E})} & \bigoplus_{k=0}^{r-1} H^{p+2k}(X_{\bar{K}}, \Lambda(k)) \\ \Big\| & & \Big\downarrow {\scriptstyle \oplus(-1)^k h^{(r-k)} \circ \pi^*} \\ H^p(X_{\bar{K}}, \Lambda) & \xrightarrow[h^r \circ \pi^*]{} & H^{p+2r}(P_{\bar{K}}, \Lambda(r)) \end{array}$$

and the right vertical arrow is an isomorphism. Thus the assertion follows from Proposition 1.14 and the case $k = 1$.

We generalize Proposition 1.19 for an element in K-groups. We briefly recall the terminology. For a scheme X, let $K(X)$ be the Grothendieck group of the category of locally free O_X-modules of finite rank. It is the quotient of the free abelian group generated by the isomorphism classes $[\mathcal{E}]$ of locally free O_X-modules of finite rank by the relations $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ for exact sequences $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$. The γ-filtration $F^n K(X)$ on $K(X)$ is defined as follows. Let $\lambda_t : K(X) \to 1 + tK(X)[[t]] \subset K(X)[[t]]^\times$ be the canonical map sending the class $[\mathcal{E}]$ of a locally free O_X-module $\mathcal{E}$ to $\sum_q [\Lambda^q \mathcal{E}] t^q$. For $x \in K(X)$, we put $\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x) = 1 + \sum_{n>0} \gamma^n(x) t^n$. For $n = 1$, $F^1 K(X)$ is defined to be the kernel of the map $K(X) \to \mathbf{Z}^{\pi_0(X)}$ sending $[\mathcal{E}]$ to rank $\mathcal{E}$. For

$n \geq 1$, $F^n K(X)$ is defined as the subgroup generated by the elements of the form $\gamma^{n_1}(x_1)\cdots\gamma^{n_r}(x_r)$ where $x_i \in F^1K(X)$ and $\sum_i n_i \geq n$. We put $F^0K(X) = K(X)$.

Let X be a strictly semi-stable scheme over S. The map $ch : K(X) \to CH^*(X_K)_{\mathbf{Q}}$ sending the class $[\mathcal{E}]$ of a locally free O_X-module $\mathcal{E}$ to its chern character $(ch_i(\mathcal{E}))_i$ is a ring homomorphism. It is compatible with the γ-filtration and induces a homomorphism $ch : Gr^*_F K(X) \to CH^*(X_K)_{\mathbf{Q}}$ of graded rings.

Lemma 1.20 *The map $ch : Gr^*_F K(X)_{\mathbf{Q}} \to CH^*(X_K)_{\mathbf{Q}}$ is surjective.*

Proof. Since X is regular, the map $K(X) \to K(X_K)$ is surjective by [8] Corollary 2.2.7.1. Hence the map $Gr^*_F K(X) \to Gr^*_F K(X_K)$ is surjective. Let $CH^*(X_K)_{\mathbf{Q}} \to Gr^*_F K(X_K)_{\mathbf{Q}}$ be the natural map sending the class $[F]$ of an integeral closed subscheme F to the class of $[O_F]$. Then, by Riemann-Roch [6] Corollary 18.3.2, it is the inverse of $ch : Gr^*_F K(X_K)_{\mathbf{Q}} \to CH^*(X_K)_{\mathbf{Q}}$.

Proposition 1.21 *Let X be a proper and strictly semi-stable scheme over S and $k \geq 0$ be an integer. Let $a \in F^kK(X)$ be an element of the k-th γ-filtration. Let $\Lambda = \mathbf{Q}_\ell$. Then, the chern character $ch_k(a)$ induces a map of spectral sequences*

$$\begin{array}{ccc}
E_1^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{q-2i}(Y_{\bar F}^{(p+2i)}, \Lambda(-i)) & \Rightarrow & H^{p+q}(X_{\bar K}, \Lambda) \\
\Big\downarrow {\scriptstyle ch_k(a)\cup} & & \Big\downarrow {\scriptstyle ch_k(a)\cup} \\
E_1^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{q-2i+2k}(Y_{\bar F}^{(p+2i)}, \Lambda(-i+k)) & \Rightarrow & H^{p+q+2k}(X_{\bar K}, \Lambda(k)).
\end{array}$$

Proof. By the definition of the γ-filtration, we may assume $a = \gamma^k([\mathcal{E}] - \text{rank } \mathcal{E})$ for a locally free O_X-module $\mathcal{E}$. Then $ch_k(a) = c_k(\mathcal{E})$ and the assertion follows from Proposition 1.19.

Finally, we establish the functoriality for an algebraic correspondence. As a preliminary, we construct a resolution of the product of strictly semi-stable schemes.

Lemma 1.22 *Let X and X' be strictly semi-stable schemes over S. Let $D_1, \ldots, D_m$ be the irreducible components of $Y = X_F$ and $\mathcal{I}_i = O(-D_i) \subset O_X$ be the ideal defining D_i for $i = 1, \ldots, m$. Similarly, let $D'_1, \ldots, D'_{m'}$ be the irreducible components of $Y' = X'_F$ and $\mathcal{I}'_i = O(-D'_i) \subset O_{X'}$ be the ideal defining D'_i for $i = 1, \ldots, m'$. We put $\Delta'' = \Delta \times \Delta'$ where $\Delta = \{1, \ldots, m\}, \Delta' = \{1, \ldots, m'\}$ and regard Δ'' as a partially ordered set with the product order. Then,*

1. The blow-up X'' of $X \times_S X'$ by the ideal $\prod_{(i,i')\in\Delta''}(\prod_{j=1}^{i} pr_1^*\mathcal{I}_j + \prod_{j'=1}^{i'} pr_2^*\mathcal{I}'_{j'})$ is strictly semi-stable over S.

2. The closed fiber $Y'' = X''_F$ is the sum of the proper transforms $D''_{ii'}$ of $D_i \times_F D'_{i'}$ for $(i,i') \in \Delta''$. For $(i,i'), (j,j') \in \Delta''$, if the intersection $D''_{i,i'} \cap D''_{j,j'}$ is not empty, we have either $(i,i') \leq (j,j')$ or $(j,j') \leq (i,i')$.

Proof. The question is local on X and on X'. We may assume X and X' are etale over $X_0 = \text{Spec } O_K[T_0, \dots, T_n]/(T_0 \cdots T_n - \pi)$ and $X'_0 = \text{Spec } O_K[T'_0, \dots, T'_{n'}]/(T'_0 \cdots T'_{n'} - \pi)$ respectively. Further, we may assume $X = X_0, X' = X'_0, m = n+1$ and $m' = n'+1$. We describe the blow-up $X'' \to X \times_S X'$ using a cone decomposition. Let Σ be the set of totally ordered subsets of Δ'' and call an element $\sigma \in \Sigma$ a face of Δ''. Let $M = \mathbf{N}^m +_{\mathbf{N}} \mathbf{N}^{m'}$ be the amalgamate sum of $\mathbf{N} \to \mathbf{N}^m : 1 \mapsto (1, \dots, 1)$ and $\mathbf{N} \to \mathbf{N}^{m'} : 1 \mapsto (1, \dots, 1)$. For $i \in \Delta$ and $i' \in \Delta'$, let e_i and $e_{i'} \in M$ be the images of the standard bases. Let $N = Hom(M, \mathbf{N})$ be the dual monoid. For $(i, i') \in \Delta''$, we define $f_{ii'} \in N$ by $f_{ii'}(e_i) = f_{ii'}(e'_{i'}) = 1$ and $f_{ii'}(e_j) = f_{ii'}(e'_{j'}) = 0$ for $j \neq i, j' \neq i'$. By identifying (i, i') with $f_{ii'}$, we regard Δ'' as a subset of N. The monoid N is generated by Δ''.

Lemma 1.23 *For a face $\sigma \in \Sigma$, let $N_\sigma \subset N$ be the submonoid generated by $f_{i,i'}$ for $(i, i') \in \sigma$ and M_σ be the submonoid $\{x \in M^{\text{gp}} | f(x) \geq 0$ for $f \in N_\sigma\}$ of the associated group $M^{\text{gp}} = \{xx'^{-1} | x, x' \in M\} \simeq \mathbf{Z}^{m+m'-1}$. Then the family $(N_\sigma)_{\sigma \in \Sigma}$ is a regular proper subdivision of N. Namely the following conditions (1)-(6) are satisfied.*

(1) For $\sigma \in \Sigma$, $N_\sigma \cap \Delta'' = \sigma$.

(2) For $v \in \Delta''$, $\{v\} \in \Sigma$.

(3) For $\sigma \in \Sigma$ and $x \in M_\sigma$, the subset $\sigma_x = \{f \in \sigma | f(x) = 0\} \subset \sigma$ is in Σ.

(4) For $\sigma, \tau \in \Sigma$, there exists $x \in M_\sigma$ such that $N_\sigma \cap N_\tau = N_{\sigma_x}$.

(5) $N = \bigcup_{\sigma \in \Sigma} N_\sigma$.

(6) For $\sigma \in \Sigma$, the monoid N_σ is isomorphic to $\mathbf{N}^r$ for some $r \geq 0$.

Proof of Lemma. We use multiplicative notation to denote the operation in M. The condition (2) is clear from the definition. Let $s : N \to \mathbf{N}$ be the map $f \mapsto f(\prod_{i=1}^m e_i) = f(\prod_{i'=1}^{m'} e'_{i'})$. Then $\Delta'' = \{f \in N | s(f) = 1\}$ and the condition (1) follows. It also follows from this that, for $\sigma \in \Sigma$, the map $\mathbf{N}^\sigma \to N_\sigma$ sending the standard basis $e_{(i,i')}$ to $f_{ii'}$ is an isomorphism. Hence the condition (6) follows. The condition (3) follows easily from (6). To show the conditions (4) and (5) are satisfied, we define a map

$$\Sigma \to \bigcup_{r=1}^{m+m'-1} \left\{ \begin{array}{c} g : \Delta \times \{1\} \amalg \Delta' \times \{2\} \\ \to \{0, 1, \dots, r\} \end{array} \middle| \begin{array}{c} g|_{\Delta \times \{1\}} \text{ and } g|_{\Delta' \times \{2\}} \text{ are increasing,} \\ \text{the image of } g \text{ contains } \{1, \dots, r\} \text{ as} \\ \text{a subset and } g(m, 1) = g(m', 2) = r \end{array} \right\}$$

by sending $\sigma \in \Sigma$ to the map $g_\sigma : \Delta \times \{1\} \amalg \Delta' \times \{2\} \to \{0, 1, \dots, \text{Card } \sigma\}$ defined by $g_\sigma(i, 1) = \text{Card } \{(i', j) \in \sigma | i' \leq i\}$ and $g_\sigma(i', 2) = \text{Card } \{(i, j') \in \sigma | j' \leq i'\}$. It is a bijection since the inverse is given by sending a map $g : \Delta \times \{1\} \amalg \Delta' \times \{2\} \to \{0, 1, \dots, r\}$ to the face $\sigma_g \subset \Delta''$ defined by $\sigma_g = \{(\min_{g(i,1) \geq j} i, \min_{g(i',2) \geq j} i') | 1 \leq j \leq r\}$. The dual monoid N is identified with the monoid $\{(a_1, \dots, a_m, b_1, \dots, b_{m'}) \in \mathbf{N}^{m+m'} | \sum_{i=1}^m a_i = \sum_{i'=1}^{m'} b_{i'}\}$ by the map $f \mapsto (f(e_1), \dots, f(e_m), f(e'_1), \dots, f(e'_{m'}))$. We have

$$N_\sigma = \left\{ f \in N \middle| \begin{array}{c} \sum_{j=1}^i f(e_j) \leq \sum_{j'=1}^{i'} f(e'_{j'}) \text{ if } g_\sigma(i, 1) \leq g_\sigma(i', 2) \text{ and} \\ \sum_{j'=1}^{i'} f(e'_{j'}) \leq \sum_{j=1}^i f(e_j) \text{ if } g_\sigma(i', 2) \leq g_\sigma(i, 1) \end{array} \right\}.$$

For $f \in N$, we put $\Phi = \{\sum_{j=1}^{i} f(e_j), \sum_{j'=1}^{i'} f(e'_{j'}) | 1 \leq i \leq m, 1 \leq i' \leq m'\}$ and define a map $g : \Delta \times \{1\} \amalg \Delta' \times \{2\} \to \{0, 1, \ldots, r\}$ by $g(i, 1) = \text{Card } \{c \in \Phi | c \leq \sum_{j=1}^{i} f(e_j)\}$ and $g(i', 2) = \text{Card } \{c \in \Phi | c \leq \sum_{j'=1}^{i'} f(e'_{j'})\}$. Then we have $f \in N_{\sigma_g}$ and the condition (5) is satified. It is easy to deduce $N_\sigma \cap N_\tau = N_{\sigma \cap \tau}$ from the description of N_σ above. The condition (4) follows from this and (6).

We return to the proof of Lemma 1.22. Let $P_\Sigma \to \text{Spec } \mathbf{Z}[M]$ be the blow-up of the spectrum of the monoid algebra $\mathbf{Z}[M]$ by the ideal $\prod_{(i,i') \in \Delta''}(\prod_{j=1}^{i} e_j, \prod_{j'=1}^{i'} e'_{j'})$. We show that the scheme P_Σ is obtained by naturally patching $\mathbf{Z}[M_\sigma]$ for $\sigma \in \Sigma$. It is obtained by patching $U_\epsilon = \text{Spec } \mathbf{Z}[M][(\prod_{j \leq i} e_j / \prod_{j' \leq i'} e'_{j'})^{\epsilon(i,i')}, (i, i') \in \Delta'']$ for $\epsilon : \Delta'' \to \{\pm 1\}$. For a map $\epsilon : \Delta'' \to \{\pm 1\}$, we have $U_\epsilon = \text{Spec } \mathbf{Z}[M_\epsilon]$ where M_ϵ is the submonoid $M + \langle (\prod_{j \leq i} e_j / \prod_{j' \leq i'} e'_{j'})^{\epsilon(i,i')}, (i, i') \in \Delta'' \rangle \subset M^{\text{gp}}$. For a face $\sigma \in \Sigma$, we have $M_\sigma = M + \langle \prod_{j \leq i} e_j / \prod_{j' \leq i'} e'_{j'}, g(i, 1) \geq g(i', 2); \prod_{j' \leq i'} e'_{j'} / \prod_{j \leq i} e_j, g(i, 1) \leq g(i', 2) \rangle$. Hence, if we put $\sigma_\epsilon = \{(i, i') \in \Delta'' | f_{ii'}(x) \geq 0 \text{ if } x \in M_\epsilon\}$ for $\epsilon : \Delta'' \to \{\pm 1\}$, we have $M_\epsilon = M_{\sigma_\epsilon}$. Therefore P_Σ is obtained by patching $\mathbf{Z}[M_\sigma]$ for $\sigma \in \Sigma$.

We show that X'' is identified with $(X \times_S X') \times_{\text{Spec } \mathbf{Z}[M]} P_\Sigma$. We define a map $X \times_S X' \to \text{Spec } \mathbf{Z}[M]$ by $e_i \mapsto T_{i-1}$ and $e'_{i'} \to T'_{i'-1}$. Since the blow-up X'' is obtained by patching $X \times_S X'[(\prod_{j \leq i} t_j / \prod_{j' \leq i'} t'_{j'})^{\epsilon(i,j)}, (i, j) \in \Delta''] = (X \times_S X') \times_{\text{Spec } \mathbf{Z}[M]} U_\epsilon$ for $\epsilon : \bar{\Delta}'' \to \{\pm 1\}$, it is identified with the fiber product $(X \times_S X') \times_{\text{Spec } \mathbf{Z}[M]} P_\Sigma$.

We show 1. It is sufficient to show that $(X \times_S X') \times_{\text{Spec } \mathbf{Z}[M]} \text{Spec } \mathbf{Z}[M_\sigma] = \text{Spec } O_K[M_\sigma]/(e_1 \cdots e_m - \pi)$ is strictly semi-stable for $\sigma \in \Sigma$. For $r = \text{Card } \sigma$, the monoid M_σ is isomorphic to $\mathbf{N}^r \times \mathbf{Z}^{m+m'-1-r}$ and the composition of $\mathbf{N} \to M \to M_\sigma$ with the projection $M_\sigma \to \mathbf{N}^r$ sends 1 to $(1, \ldots, 1)$. Hence $O_K[M_\sigma]/(e_1 \cdots e_m - \pi)$ is isomorphic to $O_K[S_1, \ldots, S_r, U_1^{\pm 1}, \ldots, U_{m+m'-1-r}^{\pm 1}]/(S_1 \cdots S_r - \pi)$. Thus the assertion follows.

We show 2. For $(i, i') \in \Delta''$, the proper transform $D''_{ii'}$ is the closed subscheme of X'' corresponding to the face $\{(i, i')\} \in \Sigma$. Hence $D_{ii'}$ is a divisor of X'' and we have $X''_F = \sum_{(i,i') \in \Delta''} D_{ii'}$. For $(i, i'), (j, j') \in \Delta''$, if either of $(i, i') \leq (j, j')$ and $(j, j') \leq (i, i')$ is not satisfied, there is no face $\sigma \in \Sigma$ such that $(i, i'), (j, j') \in \sigma$. Hence for such $(i, i'), (j, j')$, the intersection $D_{ii'} \cap D_{jj'}$ is empty.

For integers $p, k \geq 0$, we define a map $ch_k^{(p)} : F^k K(X'') \to CH^{k-p}(Y'^{(p)} \times_F Y^{(p)})_{\mathbf{Q}}$. Let $p_1 : X'' \to X$ and $p_2 : X'' \to X'$ be the compositions of $X'' \to X \times_S X'$ with the projections. For subsets $I \subset \Delta$ and $I' \subset \Delta'$ with $\text{Card } I = \text{Card } I'$, let $I \wedge I' \subset \Delta''$ be the graph of the increasing bijection $I \to I'$. It is a face of Δ''. Let $f_{I \wedge I'} : Y''_{I \wedge I'} = \bigcap_{(i,i') \in I \wedge I'} D''_{ii'} \to Y_I \times_F Y'_{I'}$ denote the restriction of f. If $\text{Card } I = \text{Card } I' = p + 1$, we have $\dim(Y_I \times_F Y'_{I'}) = n + n' - 2p$ and $\dim Y''_{I \wedge I'} = n + n' - p$. We define a map $ch_k^{(p)} : F^k K(X'') \to CH^{k-p}(Y^{(p)} \times_F Y'^{(p)})_{\mathbf{Q}}$ by

$$ch_k^{(p)}(a) = \sum_{I' \subset \Delta, I' \subset \Delta', \text{Card } I = \text{Card } I' = p+1} f_{I \wedge I' *}(ch_k(a|_{Y''_{I \wedge I'}})).$$

Proposition 1.24 *Let X and X' be proper and strictly semi-stable schemes over S of relative dimension n and n'. Let $X'' \to X \times_S X'$ be the normalization of the blow-up as in Lemma 1.22. Let $a \in F^k K(X'')$ be an element in the k-th γ-filtration. Let*

$\Gamma = ch_k(a)$ *be the image of a by* $ch_k : F^kK(X'') \to CH^k(X_K \times_K X'_K)_{\mathbf{Q}}$ *and, for* $p \geq 0$, *let* $\bar\Gamma^{(p)} = ch_k^{(p)}(a) \in CH^{k-p}(Y^{(p)} \times_F Y'^{(p)})_{\mathbf{Q}}$ *be the element defined above. Let* $\Lambda = \mathbf{Q}_\ell$. *Then, we have a map of weight spectral sequences*

$$\begin{array}{ccc} E_1'^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{q-2i}(Y'^{(p+2i)}_{\bar F}, \Lambda(-i)) & \Rightarrow & H^{p+q}(X'_{\bar K}, \Lambda) \\ \Big\downarrow {\scriptstyle \oplus\bar\Gamma^{(p+2i)*}} & & \Big\downarrow {\scriptstyle \Gamma^*} \\ E_1^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{q-2i+2(k-n)}(Y^{(p+2i)}_{\bar F}, \Lambda(-i+k-n)) & \Rightarrow & H^{p+q+2(k-n)}(X_{\bar K}, \Lambda(k-n)). \end{array}$$

Proof. The map $\Gamma^* : H^r(X'_{\bar K}, \Lambda) \to H^{r+2(k-n)}(X_{\bar K}, \Lambda(k-n))$ is the composition of

$$H^r(X'_{\bar K}, \Lambda) \xrightarrow{pr_2^*} H^r((X \times_K X')_{\bar K}, \Lambda) \xrightarrow{ch_k(a)\cup} H^{r+2k}((X \times_K X')_{\bar K}, \Lambda(k)) \xrightarrow{pr_{1*}} H^{r+2(k-n)}(X_{\bar K}, \Lambda(k-n))$$

and, for $p \geq 0$, the map $\bar\Gamma^{(p)*} : H^r(Y'^{(p)}_{\bar F}, \Lambda) \to H^{r+2(k-n)}(Y^{(p)}_{\bar F}, \Lambda(k-n))$ is the composition of

$$H^r(Y'^{(p)}_{\bar F}, \Lambda) \xrightarrow{p_2^*} H^r(Y''^{(p)}_{\bar F}, \Lambda) \xrightarrow{ch_k(a|_{Y''^{(p)}})\cup} H^{r+2k}(Y''^{(p)}_{\bar F}, \Lambda(k)) \xrightarrow{p_{1*}} H^{r+2(k-n)}(Y^{(p)}_{\bar F}, \Lambda(k-n)).$$

Hence the assertion follows from Propositions 1.14, 1.18 and 1.21.

2. Independence of ℓ.

Let X_K be a proper smooth scheme of dimension n over a field K, $\sigma \in G_K$, $\Gamma \in CH^n(X_K \times_K X_K)$ and ℓ be a prime number different from the characterstic of K. Recall that the map $\Gamma^* : H^r(X_{\bar K}, \mathbf{Q}_\ell) \to H^r(X_{\bar K}, \mathbf{Q}_\ell)$ is defined as the composition

$$H^r(X_{\bar K}, \mathbf{Q}_\ell) \overset{pr_2^*}{\to} H^r((X \times_K X)_{\bar K}, \mathbf{Q}_\ell) \overset{[\Gamma]\cup}{\to} H^{r+2n}((X \times_K X)_{\bar K}, \mathbf{Q}_\ell(n)) \overset{pr_{1*}}{\to} H^r(X_{\bar K}, \mathbf{Q}_\ell)$$

where $[\Gamma] \in H^{2n}((X_K \times_K X_K)_{\bar K}, \mathbf{Q}_\ell(n))$ denotes the image by the cycle map. We compute the alternating sum Tr $(\Gamma^* \circ \sigma_* : H^*(X_{\bar K}, \mathbf{Q}_\ell))$ using an alteration. Let L be a finite normal extension K, W_L be a proper and smooth scheme over L and $f : W_L \to X_K$ be a proper, surjective and generically finite morphism over K. We fix an embedding $L_0 \to \bar K$ of the separable closure L_0 of K in L and let σ also denote the extension to L of the restriction of σ to L_0. Let $pr_2 : W_{\sigma,L} = W_L \times_L L \to L$ be the base change by σ and f_σ denote the composition $f \circ pr_1 : W_{\sigma,L} \to X$. Let $\Gamma_\sigma \in CH^n(W_L \times_L W_{\sigma,L})$ be the pull-back $(f \times f_\sigma)^*\Gamma$ of Γ by $f \times f_\sigma : W_L \times_L W_{\sigma,L} \to X_K \times_K X_K$. It induces a homomorphism $\Gamma_\sigma^* : H^*(W_{\sigma,\bar L}, \mathbf{Q}_\ell) \to H^*(W_{\bar L}, \mathbf{Q}_\ell)$. The isomorphism $\sigma^* = 1 \times \sigma^* : W_{\sigma,\bar L} \to W_{\bar L} = W \times_L \bar L$ also induces an isomorphism $\sigma_* = (\sigma^*)^* : H^*(W_{\bar L}, \mathbf{Q}_\ell) \to H^*(W_{\sigma,\bar L}, \mathbf{Q}_\ell)$.

Lemma 2.1 *We have an equality*

$$[W_L : X_L]\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : H^*(X_{\bar K}, \mathbf{Q}_\ell)) = \mathrm{Tr}\ (\Gamma_\sigma^* \circ \sigma_* : H^*(W_{\bar L}, \mathbf{Q}_\ell)).$$

Proof. We prove the equality by showing the commutativity of the diagram.

$$\begin{array}{ccccc} H^*(W_{\bar L},\mathbf{Q}_\ell) & \xrightarrow{\sigma_*} & H^*(W_{\sigma,\bar L},\mathbf{Q}_\ell) & \xrightarrow{\Gamma_\sigma^*} & H^*(W_{\bar L},\mathbf{Q}_\ell) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_{\sigma *}}\downarrow & & {\scriptstyle f^*}\uparrow \\ H^*(X_{\bar K},\mathbf{Q}_\ell) & \xrightarrow[\sigma_*]{} & H^*(X_{\bar K},\mathbf{Q}_\ell) & \xrightarrow[\Gamma^*]{} & H^*(X_{\bar K},\mathbf{Q}_\ell). \end{array}$$

We show the commutativity. By the functoriality of the cycle map, the pull-back of the class $[\Gamma] \in H^{2d}(X_{\bar K} \times X_{\bar K}, \mathbf{Q}_\ell(d))$ by $f \times f_\sigma$ is $[\Gamma_\sigma] \in H^{2d}(W_{\bar L} \times W_{\sigma,\bar L}, \mathbf{Q}_\ell(d))$. Therefore, by the definition of the map $\Gamma^* : H^*(X_{\bar K},\mathbf{Q}_\ell) \to H^*(X_{\bar K},\mathbf{Q}_\ell)$ and of $\Gamma_\sigma^* : H^*(W_{\sigma,\bar L},\mathbf{Q}_\ell) \to H^*(W_{\bar L},\mathbf{Q}_\ell)$, the right square is commutative. The left square is commutative by the transport of structure.

We show the equality. Since $f_* \circ f^*$ is the multiplication by the degree $[W_L : X_L]$, the map $f^* : H^*(X_{\bar K},\mathbf{Q}_\ell) \to H^*(W_{\bar L},\mathbf{Q}_\ell)$ is injective. By the commutative diagram above, the image $\mathrm{Im}\ \Gamma_\sigma^* \circ \sigma_*$ is a subspace of $\mathrm{Im}\ f^*$. Hence the trace $\mathrm{Tr}\ (\Gamma_\sigma^* \circ \sigma_* : H^*(W_{\bar L},\mathbf{Q}_\ell))$ is equal to the trace of the restriction on $\mathrm{Im}\ f^*$. Thus we obtain

$$\begin{aligned}\mathrm{Tr}\ (\Gamma_\sigma^* \circ \sigma_* : H^*(W_{\bar L},\mathbf{Q}_\ell)) &= \mathrm{Tr}\ (\Gamma^* \circ \sigma_* \circ f_* \circ f^* : H^*(X_{\bar K},\mathbf{Q}_\ell)) \\ &= [W_L : X_L]\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : H^*(X_{\bar K},\mathbf{Q}_\ell)).\end{aligned}$$

To prove Theorem 0.1, we apply the following theorem on alteration.

Lemma 2.2 ([2] Theorem 5.9) *Let K be a local field and X_K be a proper scheme over K. Then there exist a finite normal extension L of K, a strictly semi-stable and projective scheme W over the integer ring O_L and a proper, surjective and generically finite morphism $f : W_L \to X_K$ over K.*

Proof of Theorem 0.1. Since $\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : H^*(X_{\bar K},\mathbf{Q}_\ell))$ is in $\mathbf{Z}_\ell$, it is sufficient to show that $\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : H^*(X_{\bar K},\mathbf{Q}_\ell))$ is a rational number independent of ℓ. First we prove the case $n(\sigma) \geq 0$, where $n(\sigma)$ is the integer such that the image of σ by $G_K \to G_F$ is $Fr_F^{n(\sigma)}$.

Let $L, W \to \mathrm{Spec}\ O_L$ and $f : W_L \to X_K$ be as in Lemma 2.2. Let $W_\sigma = W \times_{O_L} O_L$, $f_\sigma : W_{\sigma,L} \to X$ and $\Gamma_\sigma = (f \times f_\sigma)^*\Gamma \in CH^n(W_L \times_L W_{\sigma,L})$ be as in Lemma 2.1. By Lemma 2.1, it is sufficient to show that $\mathrm{Tr}\ (\Gamma_\sigma^* \circ \sigma_* : H^*(W_{\bar L},\mathbf{Q}_\ell))$ is a rational number independent of ℓ. Let E be the residue field of L and $V = W_E$ be the closed fiber. We identify the closed fiber $V_{\bar\sigma} = W_{\sigma,E}$ with the base change $V \times_E E$ by the map $\bar\sigma : E \to E$ induced by σ. By numbering the irreducible components of V and $V_{\bar\sigma}$, we define the blow-up $W'' \to W \times_{O_L} W_\sigma$ as in Lemma 1.22. By Lemma 1.20, there exists an element $a \in F^nK(W'')_{\mathbf{Q}}$ such that $ch_n(a) = \Gamma_\sigma \in CH^n(W_L \times_L W_{\sigma,L})_{\mathbf{Q}}$. For $p \geq 0$, we define $\bar\Gamma_\sigma^{(p)} = ch_n^{(p)}(a) \in CH^{n-p}(V^{(p)} \times_E V_{\bar\sigma}^{(p)})$ as in Proposition 1.24. Then by Proposition 1.24, we have a map of weight spectral sequences

$$\begin{array}{ccc} E_1'^{p,q} = \bigoplus_{i \geq \max(0,-p)} H^{q-2i}(V_{\bar\sigma\bar E}^{(p+2i)}, \mathbf{Q}_\ell(-i)) & \Rightarrow & H^{p+q}(W_{\sigma,\bar L},\mathbf{Q}_\ell) \\ {\scriptstyle \oplus\bar\Gamma_\sigma^{(p+2i)*}}\downarrow & & \downarrow{\scriptstyle \Gamma_\sigma^*} \\ E_1^{p,q} = \bigoplus_{i \geq \max(0,-p)} H^{q-2i}(V_{\bar E}^{(p+2i)}, \mathbf{Q}_\ell(-i)) & \Rightarrow & H^{p+q}(W_{\bar L},\mathbf{Q}_\ell). \end{array}$$

By transport of structure, we have an isomorphism of spectral sequences

$$\begin{array}{ccc} E_1^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{q-2i}(V_{\bar{E}}^{(p+2i)}, \mathbf{Q}_\ell(-i)) & \Rightarrow & H^{p+q}(W_{\bar{L}}, \mathbf{Q}_\ell) \\ \Big\downarrow {\scriptstyle \oplus\bar{\sigma}_*^{(p+2i)}} & & \Big\downarrow {\scriptstyle \sigma_*} \\ E_1'^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{q-2i}(V_{\bar{\sigma},\bar{E}}^{(p+2i)}, \mathbf{Q}_\ell(-i)) & \Rightarrow & H^{p+q}(W_{\sigma,\bar{L}}, \mathbf{Q}_\ell). \end{array}$$

For $p \geq 0$, let $\sigma^{(p)}_{\mathrm{geom}} : V^{(p)}_{\bar{\sigma},\bar{E}} \to V^{(p)}_{\bar{E}}$ denote the composition $\varphi^{f\cdot n(\sigma)} \circ \sigma^{*(p)}$ where φ denote the absolute Frobenius and $q = p^f$ is the order of the residue field F. It is a morphism of schemes over $\bar{E}$. Since the action of the absolute Frobenius on $H^q(V^{(p)}_{\bar{E}}, \mathbf{Q}_\ell)$ is the identity, we obtain an endomorphism of spectral sequences

$$\begin{array}{ccc} E_1^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{q-2i}(V_{\bar{E}}^{(p+2i)}, \mathbf{Q}_\ell(-i)) & \Rightarrow & H^{p+q}(W_{\bar{L}}, \mathbf{Q}_\ell) \\ \Big\downarrow {\scriptstyle \oplus\bar{\Gamma}_{\bar{\sigma}}^{(p+2i)*}\circ\sigma_{\mathrm{geom}*}^{(p+2i)}} & & \Big\downarrow {\scriptstyle \Gamma_\sigma^*\circ\sigma_*} \\ E_1^{p,q} = \bigoplus_{i\geq \max(0,-p)} H^{q-2i}(V_{\bar{E}}^{(p+2i)}, \mathbf{Q}_\ell(-i)) & \Rightarrow & H^{p+q}(W_{\bar{L}}, \mathbf{Q}_\ell). \end{array}$$

Hence we have an equality

$$\begin{aligned} \mathrm{Tr}(\Gamma_\sigma^* \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)) &= \sum_{i\geq \max(0,-p)} (-1)^p \mathrm{Tr}(\Gamma_{\bar{\sigma}}^{(p+2i)*} \circ \sigma_{\mathrm{geom}*}^{(p+2i)} : H^*(V_{\bar{E}}^{(p+2i)}, \mathbf{Q}_\ell(-i))) \\ &= \sum_{p=0}^{n} (-1)^p (p+1) \mathrm{Tr}(\Gamma_{\bar{\sigma}}^{(p)*} \circ \sigma_{\mathrm{geom}*}^{(p)} : H^*(V_{\bar{E}}^{(p)}, \mathbf{Q}_\ell)). \end{aligned}$$

Let $\Gamma_{\sigma^{(p)}_{\mathrm{geom}*}} \in CH^{n-p}(V^{(p)}_{\bar{\sigma},\bar{E}} \times V^{(p)}_{\bar{E}})$ be the class of the graph of $\sigma^{(p)}_{\mathrm{geom}}$ and let ${}^t\Gamma_{\sigma^{(p)}_{\mathrm{geom}}} \in CH^{n-p}(V^{(p)}_{\bar{E}} \times V^{(p)}_{\bar{\sigma},\bar{E}})$ be its transpose. By Lefschetz Trace formula, we have

$$\mathrm{Tr}\ (\Gamma_{\bar{\sigma}}^{(p)*} \circ \sigma_{\mathrm{geom}*}^{(p)} : H^*(V_{\bar{E}}^{(p)}, \mathbf{Q}_\ell)) = (\Gamma_{\bar{\sigma}}^{(p)}, {}^t\Gamma_{\sigma^{(p)}_{\mathrm{geom}}}).$$

The right hand side is the intersection number in $V_{\bar{E}}^{(p+2i)} \times V_{\bar{\sigma},\bar{E}}^{(p+2i)}$ and is a rational integer independent of ℓ. Thus the proof of the case $n(\sigma) \geq 0$ is completed.

The case $n(\sigma) < 0$ is reduced to the case $n(\sigma) \geq 0$ by the following Lemma.

Lemma 2.3 *Let L and L' be a field of characteristic 0. Let A_0 and B_0 (resp. A_1 and B_1) be endomorphisms of L-vector spaces V_0 (resp. of V_1) of finite dimensions commutative to each other and let A'_0 and B'_0 (resp. A'_1 and B'_1) be endomorphisms of L'-vector spaces V'_0 (resp. of V'_1) of finite dimensions commutative to each other. Assume that* $\mathrm{Tr}(A_0^m B_0^n) - \mathrm{Tr}(A_1^m B_1^n)$ *and* $\mathrm{Tr}(A_0'^m B_0'^n) - \mathrm{Tr}(A_1'^m B_1'^n)$ *are rational numbers and*

$$\mathrm{Tr}(A_0^m B_0^n) - \mathrm{Tr}(A_1^m B_1^n) = \mathrm{Tr}(A_0'^m B_0'^n) - \mathrm{Tr}(A_1'^m B_1'^n)$$

for integers $n, m \geq 0$. If B_0, B_1, B'_0 and B'_1 are invertible, then $\mathrm{Tr}(A_0 B_0^{-1}) - \mathrm{Tr}(A_1 B_1^{-1})$ *and* $\mathrm{Tr}(A'_0 B_0'^{-1}) - \mathrm{Tr}(A'_1 B_1'^{-1})$ *are rational numbers and*

$$\mathrm{Tr}(A_0 B_0^{-1}) - \mathrm{Tr}(A_1 B_1^{-1}) = \mathrm{Tr}(A'_0 B_0'^{-1}) - \mathrm{Tr}(A'_1 B_1'^{-1}).$$

Proof. Since

$$\frac{\det(1-(B_1-A_1T)S)}{\det(1-(B_0-A_0T)S)} = \exp\sum_{n=1}^{\infty}\frac{(\mathrm{Tr}(B_0-A_0T)^n - \mathrm{Tr}(B_1-A_1T)^n)}{n}S^n$$

we have

$$\frac{\det(1-(B_1-A_1T)S)}{\det(1-(B_0-A_0T)S)} = \frac{\det(1-(B'_1-A'_1T)S)}{\det(1-(B'_0-A'_0T)S)}$$

in $\mathbf{Q}(T,S)^\times$. If we put

$$\frac{\det(1-(B_1-A_1T)S)}{\det(1-(B_0-A_0T)S)} = (-S^{-1})^{n_0}\sum_{n=0}^{\infty} f_n S^{-n}$$

in $\mathbf{Q}(T)((S^{-1}))^\times$ where $n_0 = \dim V_0 - \dim V_1$, we have $f_0 = \det(B_1 - A_1T)/\det(B_0 - A_0T) \in \mathbf{Q}(T)^\times$. Hence we have

$$\frac{\det(B_1-A_1T)}{\det(B_0-A_0T)} = \frac{\det(B'_1-A'_1T)}{\det(B'_0-A'_0T)}$$

in $\mathbf{Q}(T)$. If we put $\det(B_1-A_1T)/\det(B_0-A_0T) = \sum_{n=0}^{\infty} c_n T^n \in \mathbf{Q}[[T]]^\times$, we have $\mathrm{Tr}(A_0B_0^{-1}) - \mathrm{Tr}(A_1B_1^{-1}) = c_1/c_0$. Hence the assertion follows.

Proof of Corollary 0.2. Since H^3 and H^4 are the duals of H^1 and H^0 if $n=2$, it is a consequence of Theorem 0.1 and Lemma 2.4(1) below.

Lemma 2.4 *Let $K, X_K, n, r, \sigma, \Gamma$ and ℓ be as in Theorem 0.1. If one of the following conditions (1)-(3) is satisfied, the trace $\mathrm{Tr}(\Gamma^*\circ\sigma_* : H^r(X_{\bar K}, \mathbf{Q}_\ell))$ is in $\mathbf{Z}[1/q]$ and is independent of ℓ.*

(1) $r \leq 1$.

(2) X_K is an abelian variety.

(3) There exists an algebraic correspondence $\Gamma_r \in CH^n(X_K\times X_K)_{\mathbf{Q}}$ such that Γ_r^* on $H^s(X_{\bar K}, \mathbf{Q}_\ell)$ is the identity if $s = r$ and 0 if $s \neq r$.

Proof. Since the trace $\mathrm{Tr}(\Gamma^*\circ\sigma_* : H^r(X_{\bar K}, \mathbf{Q}_\ell))$ is in $\mathbf{Z}_\ell$, it is sufficient to show that it is in $\mathbf{Q}$ and is independent of ℓ. If the condition (3) is satisfied, it is sufficient to apply Theorem 0.1 to the composition $\Gamma_r\Gamma$. If X_K is an abelian variety, the condition (3) is satisfied and the case (2) is reduced to the case (3). We reduce the case (1) to the case (2). Since the case $r = 0$ is clear, we may assume $r = 1$.

We assume $r = 1$ and reduced it to the case where X_K is an abelian variety. We may assume X_K is connected. Let $L = \Gamma(X_K, O)$ be the constant field of X_K. It is a finite separable extension of K. Let A_L be the Albanese variety of the variety X_K over L and let $A'_K = \mathrm{Res}_{L/K}A_L$ be the Weil restriction. We identify $H^1(X_{\bar L}, \mathbf{Q}_\ell)$ with $H^1(A_{\bar L}, \mathbf{Q}_\ell)$ and $H^1(X_{\bar K}, \mathbf{Q}_\ell) = \mathrm{Ind}_{G_L}^{G_K} H^1(X_{\bar L}, \mathbf{Q}_\ell)$ with $H^1(A'_{\bar K}, \mathbf{Q}_\ell) = \mathrm{Ind}_{G_L}^{G_K} H^1(A_{\bar L}, \mathbf{Q}_\ell)$. We will define an endomorphism f of an abelian variety A_K such that the endomorphism Γ^* of $H^1(X_{\bar K}, \mathbf{Q}_\ell)$ is identified with the endomorphism f^* on $H^1(A_{\bar K}, \mathbf{Q}_\ell)$. Applying Lemma

2.2 to a closed subscheme of $X_K \times_K X_K$, we may assume that there exists a proper smooth scheme W_K of dimension n and morphisms $p_1, p_2 : W_K \to X_K$ such that $\Gamma^* = p_{1*} \circ p_2^*$. Let $M \supset L$ be the constant field of W_K. Replacing K by a purely inseparable extension of K, we may assume M is a separable extension of K. Let B_M be the Albanese variety of the variety W_K over M and $B'_K = \mathrm{Res}_{M/K} B_M$ be the Weil restriction. By the functoriality, the map $p_2 : W_K \to X_K$ induces $p_{2*} : B_M \to A_L \otimes_L M$. We define $p_{2*} : B'_K \to A'_K$ to be the composition $B'_K = \mathrm{Res}_{M/K} B_M \to \mathrm{Res}_{M/K}(A_L \otimes_L M) \overset{N_{M/L}}{\to} \mathrm{Res}_{L/K} A_L = A'_K$. The map $p_2^* : H^1(X_{\bar{K}}, \mathbf{Q}_\ell) \to H^1(W_{\bar{K}}, \mathbf{Q}_\ell)$ is identified with $p_2^* : H^1(A'_{\bar{K}}, \mathbf{Q}_\ell) \to H^1(B'_{\bar{K}}, \mathbf{Q}_\ell)$.

Let A_L^* and B_M^* be the Picard varieties of X_K over L and of W_K over M respectively. They are identified with the dual abelian varieties of A_L and of B_M. The Weil restrictions $A'^*_K = \mathrm{Res}_{L/K} A^*_L$ and $B'^*_K = \mathrm{Res}_{M/K} B^*_M$ are identified with the dual of A'_K and of B'_K respectively. The push-forward of invertible sheaves defines a map $p_{1*} : B^*_M \to A^*_L \otimes_L M$ of abelian varieties over M. We identify $H^1(X_{\bar{L}}, \mathbf{Q}_\ell(1))$ and $H^1(W_{\bar{M}}, \mathbf{Q}_\ell(1))$ with the Tate modules $T_\ell A^*_L \otimes \mathbf{Q}_\ell$ and $T_\ell B^*_M \otimes \mathbf{Q}_\ell$. Then the map $p_{1*} : H^1(W_{\bar{M}}, \mathbf{Q}_\ell(1)) \to H^1(X_{\bar{L}}, \mathbf{Q}_\ell(1))$ is identified with the map $(p_{1*})_* : T_\ell B^*_M \otimes \mathbf{Q}_\ell \to T_\ell A^*_L \otimes \mathbf{Q}_\ell$. Let $(p_{1*})^* : A'_K \to B'_K$ be the dual of the composition $B'^*_K = \mathrm{Res}_{M/K} B^*_M \to \mathrm{Res}_{M/K}(A^*_L \otimes_L M) \overset{N_{M/L}}{\to} \mathrm{Res}_{L/K} A^*_L = A'^*_K$. Then the map $p_{1*} : H^1(W_{\bar{M}}, \mathbf{Q}_\ell(1)) \to H^1(X_{\bar{L}}, \mathbf{Q}_\ell(1))$ is the same as the pull-back by the map $(p_{1*})^* : A'_K \to B'_K$. Therefore the composition $\Gamma^* = p_{1*} \circ p_2^*$ on $H^1(X_{\bar{K}}, \mathbf{Q}_\ell)$ is the pull-back f^* by the endomorphism $f = p_{2*} \circ (p_{1*})^* : A'_K \to A'_K$. Hence the case $r = 1$ is reduced to the case where X_K is an abelian variety. Thus Lemma is proved.

We prove Corollary 0.6 by comparing the monodromy filtration with the weight filtration.

Lemma 2.5 *Let X be a proper and strictly semi-stable scheme over the integer ring of a local field K and let $r \geq 0$ be an integer. Let $W_\bullet$ be the weight filtration on $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$.*

1. We have $W_r H^r(X_{\bar{K}}, \mathbf{Q}_\ell) = H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ and $W_{-r-1} H^r(X_{\bar{K}}, \mathbf{Q}_\ell) = 0$. For $s \in \mathbf{Z}$, the nilpotent monodromy operator $N \in \mathrm{End}(H^r(X_{\bar{K}}, \mathbf{Q}_\ell))(1)$ maps W_s to W_{s-2}.

2. Let $\sigma \in W_K$ be an element of the Weil group with $n(\sigma) \geq 0$ and s be an integer. Let α be an eigenvalue of the action of σ on $Gr^W_s H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$. Then α is an algebraic integer and the complex absolute values of their conjugates are $q^{(r+s)n(\sigma)/2}$.

Proof. 1. Clear from the definition of the weight filtration.

2. A consequence of the Weil conjecture [5].

Corollary 2.6 *Let X be a proper and strictly semi-stable scheme over the integer ring of a local field K and let $r \geq 0$ be an integer. Then Conjecture 0.5 for $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ is true if and only if the monodromy filtration and the weight filtration on $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ are the same.*

Lemma 2.7 *Let X_K be a proper smooth scheme over a local field K. Then Conjecture 0.5 is true for $r \leq 2$.*

Proof. In [14] Satz 2.13, Conjecture 0.5 is proved for a proper and strictly semi-stable scheme X over the integer ring of a local field K assuming $\dim X_K \leq 2$. We reduce it to this case. By Lemma 2.2, there is a finite extension L of K, a projective smooth scheme W_L over L and a proper surjective and generically finite morphism $f : W_L \to X_K$. Since the restricition to $G_L \subset G_K$ of the representation $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ is a direct summand of $H^r(W_{\bar{L}}, \mathbf{Q}_\ell)$, it is sufficient to show the assertion for W_L over L. By replacing X_K by W_L, we may assume X_K is projective. Further replacing X_K by its hyperplane section, we may assume $\dim X_K = r$. Applying Lemma 2.2, we may assume that X_K is the generic fiber of a projective and strictly semi-stable scheme X over the integer ring O_K. Thus the assertion is proved.

Proof of Corollary 0.6. 1. By Lemma 2.4.(3),

$$\det(1 - \sigma_* T : H^r(X_{\bar{K}}, \mathbf{Q}_\ell)) = \exp(-\sum_{n=1}^{\infty} \frac{\mathrm{Tr}(\sigma_*^n : H^r(X_{\bar{K}}, \mathbf{Q}_\ell))}{n} T^n)$$

is in $\mathbf{Q}[T]$ and independent of ℓ. It is sufficient to show that the eigenvalues of σ_* acting on $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ is an algebraic integer. We may replace σ by some power. Hence it follows from Lemmas 2.2 and 2.5.2.

2. Similarly as above, it is sufficient to show that the trace $\mathrm{Tr}(\sigma_* : H^r(X_{\bar{K}}, \mathbf{Q}_\ell)^I)$ is a rational number independent of ℓ. Let $M_\bullet, F_\bullet$ and $G^\bullet$ denote the monodromy, the kernel and the image filtrations on $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$. By 1 and by the assumption that $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ satisfies Conjecture 0.5, the trace $\mathrm{Tr}(\sigma_* : Gr^M_s H^r(X_{\bar{K}}, \mathbf{Q}_\ell))$ is a rational number independent of ℓ. By Proposition 1.2 and Lemmas 1.1, 1.3, we have an isomorphism $Gr^M_s H^r(X_{\bar{K}}, \mathbf{Q}_\ell) = \bigoplus_{t-u=s} Gr^u_G Gr^F_t H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$. Further the map $N^t : Gr^u_G Gr^F_t H^r(X_{\bar{K}}, \mathbf{Q}_\ell) \to Gr^{u+t}_G F_0 H^r(X_{\bar{K}}, \mathbf{Q}_\ell)(-t)$ is an isomorphism. Hence we have an equality $\mathrm{Tr}(\sigma_* : Gr^M_s H^r(X_{\bar{K}}, \mathbf{Q}_\ell)) = \sum_{t-u=s} q^t \cdot \mathrm{Tr}(\sigma_* : Gr^{u+t}_G F_0 H^r(X_{\bar{K}}, \mathbf{Q}_\ell))$. From this we deduce that $\mathrm{Tr}(\sigma_* : Gr^t_G F_0 H^r(X_{\bar{K}}, \mathbf{Q}_\ell))$ and hence $\mathrm{Tr}(\sigma_* : F_0 H^r(X_{\bar{K}}, \mathbf{Q}_\ell)) = \sum_t \mathrm{Tr}(\sigma_* : Gr^t_G F_0 H^r(X_{\bar{K}}, \mathbf{Q}_\ell))$ are rational numbers independent of ℓ. The action of the inertia I on $F_0 H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ factors through a finite quotient I/J and the I-fixed part $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)^I$ is equal to $(F_0 H^r(X_{\bar{K}}, \mathbf{Q}_\ell))^{I/J}$. Hence, for a complete set $T \subset I$ of representatives, we have $[I : J] \cdot \mathrm{Tr}(\sigma_* : H^r(X_{\bar{K}}, \mathbf{Q}_\ell)^I) = \sum_{\tau \in T} \mathrm{Tr}((\sigma \circ \tau)_* : F_0 H^r(X_{\bar{K}}, \mathbf{Q}_\ell))$ and the assertion follows.

Proof of Corollary 0.4. The assertion 1 follows from Corollary 0.2 by the same argument as in the proof of Corollary 0.6.1. The assertion 2 follows from Corollary 0.6.1 and Lemma 2.7.

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