UTMS 2001–20 July 11, 2001

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UNIQUE CONTINUATION ALONG AN ANALYTIC CURVE FOR THE ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider an elliptic partial differential operator $P(x, \partial)$ with analytic coefficients and discuss the unique continuation along an analytic curve. That is, let $P(x, \partial)u = 0$ in a simply connected domain $\Omega \subset \mathcal{R}^n$, $\gamma \subset \Omega$ be an analytic curve and let $\{x^j\}_{j \in \mathcal{N}} \subset \gamma$ have an accumulation point. Our main result asserts that if $u(x^j) = 0$, $j \in \mathcal{N}$, then $u(x) = 0$ for any $x \in \gamma$. Furthermore we apply such uniqueness to an isotropic Lamé system with constant Lamé coefficients and the Kirchhoff plate equation with analytic coefficients.

1. INTRODUCTION

Let $\Omega \subset \mathcal{R}^2$ be a bounded simply connected domain and L be a straight line which intersects Ω. Assume that L_1 is a interval on L such that $\overline{L_1} \subset L \cap \Omega$.

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\Delta u + k^2 u = 0 \qquad in \quad \Omega. \tag{1.1}$$

Date: July 10, 2001.

1991 *Mathematics Subject Classification.* 35B60, 30B40.

Key words and phrases. Unique continuation, analytic curve, discrete set, elliptic equation.

The first named author is partly supported by NSF of China (19971016). The second named author is partly supported by Sanwa Systems Development Company Limited(Tokyo). This work has been finished during the stay of the first named author at the Graduate School of Mathematical Sciences of the University of Tokyo and he thanks the university for the support.

Then, in Bruckner, Cheng and Yamamoto [2], it is proved that if $u(x^j) = 0$ for $x^j \in L_1$, $j \in \mathcal{N}$ which are mutually distinct, then $u = 0$ on $\overline{\Omega \cap L_1}$. This is a unique continuation property along a line from a discrete set. This unique continuation is restricted to the straight line L and we have no information of u outside L. In fact, $u = u(x_1, x_2) = x_2 e^{ikx_1}$ satisfies (1.1) and $u(x_1, 0) = 0$, $x_1 \in \mathcal{R}$, while $u(x_1, x_2) \neq 0$ if $x_2 \neq 0$.

The main purpose of this paper is to extend such unique continuation along a straight line to an elliptic partial differential operator with analytic coefficient of the form:

$$(\Delta^m u)(x) + \sum_{|\alpha| \le 2m-1} a_\alpha(x) \partial^\alpha u(x) = 0 \tag{1.2}$$

in a simply connected domain $\Omega \subset \mathcal{R}^n$, where a_α, $|\alpha| \le 2m - 1$, satisfy conditions on analyticity.

This paper is composed of five sections.

- Section 2. Formulation and the main result
- Section 3. Holomorphic extension of the fundamental solution
- Section 4. Proof of the main result
- Section 5. Applications to the equations of elasticity.

2. Formulation and the main result

Let $x = (x_1, \cdots, x_n) \in \mathcal{R}^n$ and $z = (z_1, \cdots, z_n) \in \mathcal{C}^n$. We set

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathcal{N} \cup \{0\})^n, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$\partial_j = \frac{\partial}{\partial x_j}, \ 1 \le j \le n, \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad \Delta = \sum_{j=1}^n \partial_j^2.$$

Throughout this paper, we assume that $\Omega \subset \mathcal{R}^n$ is a simply connected domain. By $I(z^0)$, we denote the isotropic cone with vertex at $z^0 = (z_1^0, \cdots, z_n^0) \in \mathcal{C}^n$:

$$I(z^0) = \left\{ z \in \mathcal{C}^n \mid \sum_{j=1}^{n} (z_j - z_j^0)^2 = 0 \right\}. \tag{2.1}$$

We define the kernel of harmonicity hull $N(\Omega)$ of Ω by

$$N(\Omega) = \{ z \in \mathcal{C}^n \mid CH\,(\mathcal{R}^n \cap I(z)) \subset \Omega \} \tag{2.2}$$

(Ebenfelt [6]). Here $CH\,(A)$ denotes the convex hull of a set $A \subset \mathcal{R}^n$.

We consider an elliptic partial differential operator:

$$P(x, \partial)u(x) = \Delta^m u(x) + \sum_{|\alpha| \le 2m-1} a_\alpha(x) \partial^\alpha u(x), \quad x \in \Omega. \tag{2.3}$$

Throughout this paper, we assume

(2.4) $a_\alpha, \quad |\alpha| \le 2m-1,$ can be extended as holomorphic functions in $N(\Omega)$.

We are ready to state our main result:

Theorem 2.1. *Suppose that $u \in C^{2m}(\Omega)$ satisfies*

$$P(x, \partial)u(x) = 0, \qquad x \in \Omega. \tag{2.5}$$

Let γ be an analytic curve such that $\overline{\gamma} \subset \Omega$ and let the discrete set $\{x^j\}_{j \in \mathcal{N}}$ be on γ. If

$$u(x^j) = 0, \qquad j \in \mathcal{N},$$

then $u = 0$ on $\overline{\gamma}$.

In Theorem 2.1, by the analytic curve γ, we mean that, for any $x^* \in \gamma$, there exist small $\delta > 0$, $\mu > 0$ and an interval $I = (0, l)$ such that $\gamma \cap O_{x^*}(\delta)$ can be represented by

$$x(\xi) = (x_1(\xi), \cdots, x_n(\xi)), \quad \xi \in I$$

where $O_{x^*}(\delta) = \{x \,|\, |x - x^*| < \delta\}$ and $x(\cdot)$ can be extended as an analytic function in

$$\{\xi + i\eta \in \mathcal{C} \,|\, \xi \in I, \, -\mu < \eta < \mu\}. \tag{2.6}$$

See Bukhgeim [3] for other unique continuation from a discrete set. As for unique continuation along a line which has character similar to our main result, we refer to Alessandrini and Favaron [1], Cheng, Hon and Yamamoto [4], Cheng and Yamamoto [5].

In the case of $n = 2$ (a planar domain) and $m = 1$ (a second order elliptic operator), we have

Corollary 2.2. *We consider the case: $n = 2$ and $m = 1$. In* (2.3)*, suppose that $a_0 \le 0$ and* (2.4) *holds. If $\gamma \subset \Omega$ be a closed curve which is analytic in the sense of Theorem 2.1 and assume that $\overline{\gamma} \subset \Omega$. Let $\{x^j\}_{j \in \mathcal{N}} \subset \gamma$. Then $u = 0$ on $\overline{\Omega}$ follows from that $u(x^j) = 0$, $j \in \mathcal{N}$.*

In fact, by Theorem 2.1, we have $u|_\gamma = 0$. Therefore since $a_0 \le 0$ on $\overline{\Omega}$, the uniqueness of the Dirichlet boundary value problem yields $u = 0$ in the domain bounded by γ. Thus the classical unique continuation (e.g. Hörmander [7], Isakov [8]) implies that $u = 0$ on $\overline{\Omega}$.

It is interesting to compare this unique continuation with the following unique continuation of a harmonic function from the boundary: let a bounded domain $\Omega \subset \mathcal{R}^2$ have C^1-boundary $\partial\Omega$, and $\Gamma_0 \subset \partial\Omega$ be closed and have positive measure.

If $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_0)$ satisfies $\Delta u = 0$ in Ω and $u = |\nabla u| = 0$ on Γ_0, then $u = 0$ in Ω (e.g. [8]).

In contrast with this unique continuation, the corollary means that, for some set $\omega \subset \Omega$ of measure 0 (i.e. $\omega = \{x_j\}_{j \in \mathcal{N}}$), $u = 0$ in ω (without $|\nabla u| = 0$ in ω) may yield that $u = 0$ over $\overline{\Omega}$.

3. Holomorphic extension of the fundamental solution

Let $E(x,t)$ be the Green function of $P(x,\partial)$:

$$P(x,\partial)E(\cdot,t) = \delta_t, \qquad x \in \Omega, \tag{3.1}$$

where δ_t is the Dirac delta function at the point t. Under the assumption for P, it is known (e.g. John [9]) that there exists a fundamental solution E. We define the solid isotropic cone with vertex at $t \in \mathcal{R}^n$ by

$$S(t) = \{z = x + iy \in \mathcal{C}^n \mid \; < x - t, y >= 0, \quad |x - t| \le |y|\} \tag{3.2}$$

where $< \cdot, \cdot >$ denotes the scalar product in $\mathcal{R}^n$ and we set $|x| = \sqrt{< x, x >}$ for $x \in \mathcal{R}^n$. Then we have

Theorem 3.1. *(Ebenfelt* [6]*) Under the assumption* (2.4)*, the fundamental solution $E(\cdot,t)$ extends as a holomorphis function in $\mathcal{C}^n \setminus S(t)$.*

4. Proof of the main result

First Step: Assume that x^* is a accumulation point of $\{x^j\}_{j \in \mathcal{N}}$ and $l_1 \subset \gamma$ is a small connected part, which contains x^*, such that l_1 can be represented by $x(\xi) = (x_1(\xi), \cdots, x_n(\xi)), \quad \xi \in I_1$. Here I_1 is an interval. For simplicity, we denote I_1 by $(0,\ell)$ and $x^* = (x_1(\frac{\ell}{2}), \cdots, x_n(\frac{\ell}{2}))$.

It is sufficient to prove that $u = 0$ on a part l_1, because we can repeat a similar argument until the whole γ is covered.

Since $\overline{\gamma} \subset \Omega$, we can take a C^2-closed curve $\Gamma \subset \Omega$ and a sufficiently small $\delta > 0$ such that

$$\mathrm{dist}\,(x(\xi), \Gamma) \ge \delta, \qquad \xi \in (0, \ell). \tag{4.1}$$

By Ω_Γ, we denote the domain bounded by Γ. Then we note that $\overline{\gamma} \subset \Omega_\Gamma$.

Furthermore, for $\mu_1 > 0$, we set

$$D_{\mu_1} = \{z \in \mathcal{C} \mid 0 < \mathrm{Re}\, z < \ell,\ |\mathrm{Im}\, z| < \mu_1\}. \tag{4.2}$$

In this step, we will prove

Lemma 4.1. *(Complex extension). Suppose that u satisfies*

$$P(x, \partial)u = 0 \qquad \text{in } \Omega. \tag{4.3}$$

Then there exist $\mu_1 > 0$ and a complex-valued function $G(z)$ which is holomorphic in D_{μ_1} such that

$$G(\xi) = u(x(\xi)), \qquad 0 < \xi < \ell. \tag{4.4}$$

Proof of Lemma 4.1: Let $E(\cdot, t)$ be a fundamental solution to the operator $P(x, \partial)^*$ in Ω:

$$P(x, \partial)^* u = \Delta^m u + \sum_{|\alpha| \le 2m-1} (-1)^{|\alpha|} \partial^\alpha (a_\alpha u),$$

which is the formal adjoint of $P(x, \partial)$. Then, in view of Theorem 3.1([6]), $E(\cdot, t)$ can be extended as a holomorphic function $E(z, t)$ for $z \in \mathcal{C}^n \setminus S(t)$. On the other hand, by the Green formula, we have

$$\begin{aligned} u(x) &= \int_{\Omega_\Gamma} P(\zeta,\partial)^* E(x,\zeta)u(\zeta)d\zeta - \int_{\Omega_\Gamma} P(\zeta,\partial)u(\zeta)E(x,\zeta)d\zeta \\ &= \int_\Gamma \{(M(\zeta,\partial)E)(x,\zeta)u(\zeta) - (\widetilde{M}(\zeta,\partial)u)(\zeta)E(x,\zeta)\}d\sigma_\zeta, \quad x \in \Omega_\Gamma, \end{aligned}$$

where M and $\widetilde{M}$ are differential operators involving derivatives of orders at most $2m-1$. In particular, on l_1, we have

$$\begin{aligned} u(x(\xi)) &= \int_\Gamma (M(\zeta,\partial)E)(x(\xi),\zeta)u(\zeta)d\sigma_\zeta \\ &\quad - \int_\Gamma (\widetilde{M}(\zeta,\partial)u)(\zeta)E(x(\xi),\zeta)d\sigma_\zeta, \qquad 0 < \xi < l. \end{aligned} \tag{4.5}$$

Next we will prove that there exists a positive constant $\mu_1 > 0$, which depends on μ, Γ, δ, Ω_Γ and γ, such that

$$\{x(z) \,|\, z \in D_{\mu_1}\} \subset \mathbb{C}^n \setminus \bigcup_{\zeta\in\Gamma} S(\zeta). \tag{4.6}$$

Here $\mu > 0$ is given in (2.6).

Proof of (4.6)**:** Let us denote the analytic extension of $x(\xi)$ by $x(\xi + i\eta) = a(\xi + i\eta) + ib(\xi + i\eta)$, $\xi, \eta \in \mathcal{R}$, where a, b are $\mathcal{R}^n$-valued, and $x(\xi) = a(\xi)$ for $0 < \xi < \ell$. Moreover by any $\epsilon > 0$, there exists $\nu = \nu(\epsilon) > 0$ such that

$$if \quad |\eta| < \nu, \quad then \quad |b(\xi + i\eta)| < \epsilon \quad for \quad 0 < \xi < \ell. \tag{4.7}$$

We set $z = \xi + i\eta$ with $\xi, \eta \in \mathcal{R}$. By (4.1), we have

$$\inf_{0\le\xi\le\ell, \zeta\in\Gamma} |x(\xi) - \zeta| \ge \delta.$$

Choosing $\mu_2 > 0$ sufficiently small for δ, ℓ, Ω_Γ and γ, we obtain

$$\inf_{0\le\xi\le\ell, |\eta|\le\mu_2,\, \zeta\in\Gamma} |a(\xi + i\eta) - \zeta| \ge \frac{\delta}{2}. \tag{4.8}$$

In view of (4.7), we can take $\mu_3 = \nu(\delta/2)$ such that

$$\sup_{0\le\xi\le\ell,\,|\eta|\le\mu_3} |b(\xi+i\eta)| < \frac{\delta}{2}.$$

Setting $\mu_1 = \min\{\mu, \mu_2, \mu_3\}$, we see that, if $z = \xi + i\eta \in D_{\mu_1}$ and $\zeta \in \Gamma$, then $|a(\xi+i\eta)-\zeta| > |b(\xi+i\eta)|$, that is, $x(z) \notin S(\zeta)$ for any $\zeta \in \Gamma$. Thus the proof of (4.6) is completed.

We define a complex-valued function

$$\begin{aligned} G(z) \quad = \quad & \int_\Gamma (M(\zeta,\partial)E)(x(z),\zeta)u(\zeta)d\sigma_\zeta \\ & - \int_\Gamma (\widetilde{M}(\zeta,\partial)u)(\zeta)E(x(z),\zeta)d\sigma_\zeta, \qquad z \in D_{\mu_1}, \end{aligned} \tag{4.9}$$

where $E(x(z),\zeta)$ is a holomorphic extension of the fundamental solution $E(x(\xi),\zeta)$. By (4.6), (4.8) and (4.9), Theorem 3.1 implies that the function $G(z)$ is holomorphic in D_{μ_1}. By (4.5) and (4.9), we see (4.4). Thus the proof of Lemma 4.1 is complete.

Second Step: In this step, we will complete the proof of Theorem 2.1. Since $x^j \in \{x(\xi); 0 < \xi < \ell\}$, we take $\xi^j \in (0,\ell)$ such that $x^j = x(\xi^j)$, $j \in \mathcal{N}$. By $u(x^j) = 0$, $j \in \mathcal{N}$, we have $G(\xi^j) = 0$, $j \in \mathcal{N}$. In view of Lemma 4.1, the function G is holomorphic in D_{μ_1} and $\xi^j \in (0,\ell) \subset D_{\mu_1}$, so that the unicity theorem for analytic functions yields that $G = 0$ in D_{μ_1}. Again by (4.4) in Lemma 4.1, the proof of the Theorem 2.1 is complete.

5. Applications to equations of elasticity

Let γ and $\{x^j\}_{j\in\mathcal{N}}$ be taken as in Theorem 2.1.

5.1. **Isotropic Lamé equation with constant Lamé coefficients.** We consider

$$\mu\Delta U + (\lambda+\mu)\nabla(\operatorname{div} U) = 0 \qquad \text{in } \Omega \subset \mathcal{R}^n \tag{5.1}$$

where $U = (u_1, \cdots, u_n)$ denotes displacement and λ, μ are constants such that $\lambda + 2\mu > 0$ and $\mu > 0$. Then we show

Theorem 5.1. *Let $U = (u_1, \cdots, u_n) \in C^4(\Omega)$ satisfy (5.1). We fix $k \in \{1, \cdots, n\}$. If $u_k(x^j) = 0$, $j \in \mathcal{N}$, then $u_k = 0$ on $\overline{\gamma}$.*

Remark 5.2. The uniqueness holds for the respective component of U.

Proof. The reduction of (5.1) to the biharmonic equation $\Delta^2 u_k = 0$, is well-known (e.g. John [10]) and for completeness we show it. That is, we write (5.1) as

$$\mu \Delta u_k + (\lambda + \mu)\partial_k(\operatorname{div} U) = 0, \qquad 1 \le k \le n. \tag{5.2}$$

Therefore taking the divergence, we have $(\lambda + 2\mu)\Delta(\operatorname{div} U) = 0$, that is,

$$\Delta(\operatorname{div} U) = 0. \tag{5.3}$$

Next applying ∂_m^2 to (5.2) and summing over $m = 1, \cdots, n$, we obtain $\mu\Delta^2 u_k + (\lambda + \mu)\partial_k \Delta(\operatorname{div} U) = 0$, which implies $\Delta^2 u_k = 0$, $1 \le k \le n$ by (5.3). Therefore the application of Theorem 2.1 completes the proof of Theorem 5.1. □

5.2. **The Kirchhoff plate equation.** Let $\Omega \subset \mathcal{R}^2$ and let $u = u(x_1, x_2)$ denote the displacement in describing transformation of an isotropic elastic plate from the fixed plane position. Then we can state the classical Kirchhoff plate equation without force terms:

$$\begin{aligned} &(\lambda + \mu)\Delta^2 u + 2\nabla(\lambda + \mu) \cdot \nabla(\Delta u) + \Delta(\lambda + \mu)\Delta u \\ &+2(\partial_1\partial_2\mu)\partial_1\partial_2 u - (\partial_2^2\mu)\partial_1^2 u - (\partial_1^2\mu)\partial_2^2 u = 0 \qquad \text{in } \Omega. \end{aligned} \tag{5.4}$$

Here we assume that λ and μ satisfy (2.4). Therefore the equation (5.4) falls within the form of (2.3), so that Theorem 2.1 yields

Theorem 5.3. *Let $u \in C^4(\Omega)$ satisfy* (5.4). *If $u(x^j) = 0$, $j \in \mathcal{N}$, then $u = 0$ on $\overline{\gamma}$.*

References

1. Alessandrini, G. and Favaron, A., Sharp stability estimates of harmonic continuation along lines. Math. Methods in Appl. Sci. 23 (2000), 1037–1056.
2. Bruckner, G., Cheng, J. and Yamamoto, M., Uniqueness in determining a periodic structure from discrete far field observations. Preprint No.605, Weierstrass Institut für Angewandte Analysis und Stochastik, Berlin (2000).
3. Bukhgeim, A.L., Extension of solutions of elliptic equations from discrete sets. J. Inverse Ill-posed Problems 1 (1993), 17–32.
4. Cheng, J., Hon, Y.C. and Yamamoto, M., Stability in line unique continuation of harmonic functions: general dimensions. J. Inverse Ill-posed Problems 6 (1998), 319–326.
5. Cheng, J. and Yamamoto, M., Unique continuation on a line for harmonic functions. Inverse Problems 14 (1998), 869–882.
6. Ebenfelt, P., Holomorphic extension of solutions of elliptic partial differential equations and a complex Huygens' principle. J. London Math. Soc. 55 (1997), 87–104.
7. Hörmander, L., Linear Partial Differential Operators. Springer-Verlag, Berlin, 1963.
8. Isakov, V., Inverse Problems for Partial Differential Equations. Springer-Verlag, Berlin, 1998.
9. John, F. Plane Waves and Spherical Means Applied to Partial Differential Equations. Interscience Publications, New York, 1955.
10. John, F., Partial Differential Equations. Springer-Verlag, Berlin, 1971.

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