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by

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Abstract

In the present paper we formulate and investigate a one-dimensional inverse acoustic problem in the form of nonlinear system of Volterra integral equations. We prove the conditional stability of the inverse problem with an explicitly given Lipschitz constant depending only on the depth l and a bound of norms of an unknown coefficient and inverse problem data.
The proposed technique can be applied to prove local well-posedness of the inverse problem in $L_2(0,l)$.

List of notations.

$v(z,t)$: the pressure,

$c(z) > 0$: the velocity,

$\rho(z) > 0$: the density,

$v_{z|z=+0} = \beta\delta(t)$: excitation, β : a non-vanishing constant,

$\delta(t)$: the Dirac delta function,

$v(+0,t) = f(t)$: the additional information (inverse problem data),

$c_0 = c(+0), \quad \rho_0 = \rho(+0)$: the *a priori* given constants,

$x = \varphi(z) = \int\limits_0^z \frac{d\xi}{c(\xi)}$: the travelling time variable,

$$u(x,t) = v(z,t) = v(\varphi^{-1}(x),t),$$

$$h(x) = \rho(z) = \rho(\varphi^{-1}(x)), \quad g(x) = c(z) = c(\varphi^{-1}(x)),$$

$\sigma(x) = g(x)h(x)$: the acoustical impedance, $0 < \sigma_* \le \sigma(x)$ for $x > 0$,

$\sigma_0 = \sigma(+0)$: Thus we note that $\sigma_0 = c_0\rho_0$,

$S(x) = -\alpha\sqrt{\sigma(x)/\sigma_0}$: the jump of $u(x,t)$ at $t = x$,

$a(x) = 2S'(x)/S(x)$: the function for which the inverse problem is formulated in integral form,

$\Delta(l) = \{(x,t);\ 0 < x < t < 2l - x\}$: the domain of influence for the forward and the inverse problems,

l: the depth on which the inverse problem is solved,

$M = \|\sigma\|_{H_1(0,l)}$: *a priori* given constant,

$Q = \|f\|_{H_1(0,2l)}$,

S': the differentiation in the argument under consideration,

$u_x = \frac{\partial u}{\partial x}$, $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$,

$$\theta(t) = \begin{cases} 1\ , & t \geq 0, \\ 0\ , & t < 0. \end{cases}$$: the Heaviside function.

1 Introduction

In the present paper, we investigate an inverse problem for a one-dimensional acoustic equation

$$\frac{1}{\rho c^2} v_{tt}(z,t) = \left(\frac{1}{\rho} v_z\right)_z (z,t), \quad z>0, \; t>0. \tag{1.1}$$

Here:

$c(z) > 0$ is the sound speed of the medium,

$\rho(z) > 0$ is the density,

$v(z,t)$ is the pressure.
The equation (1.1) governs the propagation of a small amplitude acoustic wave in the half-space. Moreover the initial and boundary conditions are given:

$$v|_{t<0} = 0, \tag{1.2}$$

$$v_z(+0,t) = \beta\delta(t), \quad t>0, \tag{1.3}$$

$$v(+0,t) = f(t), \quad t>0. \tag{1.4}$$

Here $\delta(t)$ is the Dirac delta function and β is the non vanishing constant.
Inverse Problem 1.
Determine $c(z)$ and $\rho(z)$ (or some their combination), given the information (1.4) concerning a solution $v(z,t)$ to the initial boundary value problem (1.1)-(1.3).

Let us introduce the travelling time variable and new functions

$$\begin{aligned} x &= \varphi(z) = \int_0^z \frac{d\xi}{c(\xi)}, \quad z = \varphi^{-1}(x), \\ u(x,t) &= v(z,t) = v(\varphi^{-1}(x),t), \end{aligned}$$

$$h(x) = \rho(z) = \rho(\varphi^{-1}(x)), \quad g(x) = c(z) = c(\varphi^{-1}(x)).$$

Then (1.1)-(1.4) can be written in the form

$$u_{tt}(x,t) = u_{xx}(x,t) - \frac{\sigma'(x)}{\sigma(x)} u_x(x,t), \quad x > 0, \ t > 0, \tag{1.5}$$

$$u_{|t<0} = 0, \tag{1.6}$$

$$u_x(+0,t) = \alpha\delta(t), \quad t > 0, \tag{1.7}$$

$$u(+0,t) = f(t), \quad t > 0. \tag{1.8}$$

Here

$$\sigma(x) = g(x)h(x) \tag{1.9}$$

is the acoustical impedance and $\alpha = \beta\, c(+0)$.

Inverse Problem 2. Given $\alpha \neq 0$, $f(t)$, find $\sigma(x)$ from (1.5)-(1.8).

This kind of inverse problems have been studied by many authors and we mention only several related results. Blagoveschenskii [8] applied the method by Gelfand and Levitan to prove the unique solvability of (1.5)-(1.8). Romanov [17] proved an analogous theorem for the equation

$$w_{tt}(x,t) = w_{xx}(x,t) - q(x)w(x,t), \tag{1.10}$$

which can be transformed to (1.5) by introducing a new functions:

$$w(x,t) = \frac{u(x,t)}{\sqrt{\sigma(x)}},$$
$$q(x) = -\frac{1}{2}\frac{\sigma''(x)}{\sigma(x)} + \frac{3}{4}\frac{\sigma'(x)^2}{\sigma(x)^2}.$$

See also [16]. Romanov and Yamamoto [18] derived an L_2 conditional stability estimate for a multidimensional analogue of the inverse problem for (1.10) with a concentrated source term. However the existing results do not specify constants appearing in stability estimates and so cannot effectively support error analysis in the numerical experiments. As for numerical algorithms of the inverse problem (1.5) - (1.8), we can refer to Kabanikhin $[12-15]$ and the references therein.

An early numerical result for the acoustic inverse problem was obtained in a discrete form by Baranov and Kunetz [7]. Bamberger *et al* [5, 6] used the conjugate gradient technique for reconstructing acoustical impedance. Reviews and lists of references can be found in Bube [9] and Bube and Burridge [10]. Symes [21] investigated an inverse problem similar to (1.1) -(1.4), and Sacks and Symes [19] considered a multidimesional inverse problem (see also [20, 22] and the references therein). He and Kabanikhin [11] used an optimization technique for a three dimensional inverse acoustic problem. Alekseev [1] and Alekseev and Dobrinskii [2], Baev [4] considered practical applications of the above inverse problems. Azamatov and Kabanikhin [3] proved the well-posednes around one solution in L_2 of a Volterra operator equation. Their result can be combined with the results of our paper to prove the local well-posedness and well-posedness in the neighborhood of the exact solution with the explicit constants.

The purpose of this paper is to offer stability estimation with constants explicitly given in terms of $\beta, c_0, \rho_0, \sigma_*$, the depth l, and the H_1-norms M and Q of σ and f, respectively.

Here we set
$c_0 = c(+0),\ \rho_0 = \rho(+0)$ and $\sigma_* \equiv \inf_{x>0} \sigma(x) > 0.$

Then, on the assumption that $\sigma^{(1)}(+0) = \sigma^{(2)}(+0)$, we will prove the estimate (see Theorem 5.1),

$$\left\|\sigma^{(1)} - \sigma^{(2)}\right\|_{H_1(0,l)} \leq C \left\|f^{(1)} - f^{(2)}\right\|_{H_1(0,2l)},$$

where the constant

$$C(l, M, Q, \beta, c_0, \rho_0, \sigma_*) > 0$$

is explicitly given as a function of parameters $l, M, Q, \beta, c_0, \rho_0, \sigma_*$.

2 Formulation of the inverse problem in integral form

We start with the equations:

$$u_{tt}(x,t) = u_{xx}(x,t) - \frac{\sigma'(x)}{\sigma(x)} u_x(x,t), \quad x > 0,\ t > 0, \tag{2.1}$$

$$u_{|t<0} = 0, \tag{2.2}$$

$$u_x(+0,t) = \alpha\, \delta(t), \tag{2.3}$$

$$u(+0,t) = f(t), \quad t > 0. \tag{2.4}$$

Throughout the paper, we assume that $\sigma \in C^1[0,\infty)$ and

$$0 < \sigma_* \le \sigma(x),\ x \ge 0.$$

It is known that the solution to the forward problem (2.1) - (2.3) has the following form (Romanov [17], Kabanikhin $[12-13]$):

$$u(x,t) = S(x)\theta(t-x) + \tilde{u}(x,t), \tag{2.5}$$

where $\tilde{u}$ is continuous for $x > 0$ and sufficiently smooth for $t > x > 0$. We will derive an equation for the function $S(x)$, using the assumption that $S(x)$ and $\tilde{u}(x,t)$ are sufficiently smooth.

We have from (2.5)

$$u_x(x,t) = S'(x)\theta(t-x) - S(x)\delta(t-x) + \tilde{u}_x(x,t) \tag{2.6}$$

and therefore from (2.3)

$$\Big[S'(x)\theta(t-x) - S(x)\delta(t-x) + \tilde{u}_x(x,t)\Big]_{x=+0} = \alpha\delta(t).$$

Hence

$$S(+0) = -\alpha. \tag{2.7}$$

We differentiate (2.5) twice with respect to x:

$$u_{xx}(x,t) = S''(x)\theta(t-x) - 2S'(x)\delta(t-x) + S(x)\delta'(t-x) + \tilde{u}_{xx}(x,t) \quad (2.8)$$

and then do the same with respect to t

$$u_{tt}(x,t) = S(x)\delta'(t-x) + \tilde{u}_{tt}(x,t). \quad (2.9)$$

Using (2.6), (2.8) and (2.9), we equate in (2.1) all the factors corresponding to the singular term $\delta(t-x)$:

$$2S'(x) - \frac{\sigma'(x)}{\sigma(x)}S(x) = 0. \quad (2.10)$$

It follows from (2.10) that

$$S(x) = S(+0)\sqrt{\frac{\sigma(x)}{\sigma_0}}. \quad (2.11)$$

and it follows from (2.4) and (2.7) that

$$S(+0) = f(+0) = -\alpha. \quad (2.12)$$

Therefore

$$S(x) = -\alpha\sqrt{\frac{\sigma(x)}{\sigma_0}}. \quad (2.13)$$

Using (2.5), (2.10) and (2.13), we can change the inverse problem (2.1)

- (2.4) by (2.14) - (2.17):

$$u_{tt}(x,t) = u_{xx}(x,t) - 2\frac{S'(x)}{S(x)}u_x(x,t), \quad t > x > 0, \tag{2.14}$$

$$u_x(0,t) = 0, \quad t > 0, \tag{2.15}$$

$$u(x, x+0) = S(x), \quad x \geq 0, \tag{2.16}$$

$$u(+0,t) = f(t), \quad t > 0. \tag{2.17}$$

In fact, (2.16) is seen as follows. By (2.1) and (2.2), we conclude that

$$u(x,t) = 0, \qquad 0 < t \leq x,$$

in terms of the uniqueness of solution to the forward problem.

Then, by (2.5), we have

$$\tilde{u}(x,x) = 0, \quad x > 0.$$

Therefore (2.16) follows.

The inverse problem (2.1)-(2.4) is equivalent to

Reduced Inverse Problem. Find $u(x,t)$ for $t > x > 0$ and $S(x)$ for $x > 0$ in (2.14)-(2.17).

In this section, we transform the reduced inverse problem to a system of nonlinear Volterra integral equations.

Using the D'Alembert formula for representing the solution to the Cauchy problem (2.14), (2.15) and (2.17), we obtain

$$u(x,t) = F(x,t) + A\Big[\, 2\frac{S'}{S}u_x \,\Big](x,t), \quad t > x > 0. \tag{2.18}$$

Here

$$\begin{cases} F(x,t) = \frac{1}{2}\Big[f(t-x) + f(t+x)\Big], \\ A[v](x,t) = \frac{1}{2}\int_0^x \left(\int_{t-x+\xi}^{t+x-\xi} v(\xi,\tau)d\tau\right) d\xi. \end{cases} \tag{2.19}$$

We let $t \to x+0$ in (2.18) and take (2.5) into account to obtain

$$S(x) = F(x, x+0) + A\Big[\, 2\frac{S'}{S}u_x\Big](x, x+0), \quad x > 0. \tag{2.20}$$

We introduce a new function

$$a(x) = 2\,\frac{S'(x)}{S(x)}, \tag{2.21}$$

that is,

$$S(x) = -\alpha \exp\left\{\frac{1}{2}\int_0^x a(\xi)d\xi\right\}.$$

Let us differentiate (2.18) and (2.20) with respect to x

$$u_x(x,t) = F_x(x,t) + \frac{1}{2}\int_0^x a(\xi)[u_x(\xi, t+x-\xi) + u_x(\xi, t-x+\xi)]d\xi, \quad t > x > 0. \tag{2.22}$$

$$S'(x) = \frac{d}{dx}F(x, x+0) + \int_0^x a(\xi)u_x(\xi, 2x-\xi)d\xi. \tag{2.23}$$

In terms of (2.21), we can rewrite (2.23)

$$\frac{1}{2}\,a(x)S(x) = f'(2x) + \int_0^x a(\xi)u_x(\xi, 2x-\xi)d\xi. \tag{2.24}$$

The function

$$p(x) = -\frac{1}{\alpha}\exp\left\{-\frac{1}{2}\int_0^x a(\xi)d\xi\right\} \tag{2.25}$$

satisfies

$$p(x) = -\frac{1}{\alpha} - \frac{1}{2}\int_0^x a(\xi)p(\xi)d\xi. \tag{2.26}$$

Therefore, multiplying (2.24) by (2.26) and noting that $p(x)S(x) = 1$, we can obtain

$$\begin{aligned} a(x) &= -2\Big[\frac{1}{\alpha} + \frac{1}{2}\int_0^x a(\xi)p(\xi)d\xi\Big] \\ &\times \Big[f'(2x) + \int_0^x a(\xi)u_x(\xi, 2x-\xi)d\xi\Big]. \end{aligned} \tag{2.27}$$

Hence we reduce the inverse problem to the system of three nonlinear integral equations (2.22), (2.26) and (2.27) with respect to u_x, p and a. For convenience, we can rewrite (2.22), (2.26) and (2.27) in a vector form

$$\Phi(x,t) = G(x,t) + \mathcal{B}(\Phi). \tag{2.28}$$

Here we set:

$$\begin{aligned} \Phi(x,t) &= \Big(\Phi_1(x,t), \Phi_2(x), \Phi_3(x)\Big)^\top, \ G(x,t) = \Big(G_1(x,t), G_2, G_3(x)\Big)^\top, \\ \mathcal{B}(\Phi) &= \Big(\mathcal{B}_1(\Phi), \mathcal{B}_2(\Phi), \mathcal{B}_3(\Phi)\Big)^\top, \end{aligned}$$

$$\begin{aligned} \Phi_1(x,t) &= u_x(x,t), \ \Phi_2(x) = p(x), \ \Phi_3(x) = a(x), \\ G_1(x,t) &= F_x(x,t), \ G_2 = -\frac{1}{\alpha}, \ G_3(x) = -\frac{2}{\alpha}f'(2x), \end{aligned}$$

$$\begin{cases} \mathcal{B}_1(\Phi) = \frac{1}{2}\int_0^x \Phi_3(\xi)\Big[\Phi_1(\xi, t+x-\xi) + \Phi_1(\xi, t-x+\xi)\Big]d\xi, \\ \mathcal{B}_2(\Phi) = -\frac{1}{2}\int_0^x \Phi_3(\xi)\Phi_2(\xi)d\xi, \\ \mathcal{B}_3(\Phi) = -\Big(f'(2x) + \int_0^x \Phi_3(\xi)\Phi_1(\xi, 2x-\xi)d\xi\Big) \\ \qquad \times \int_0^x \Phi_3(\xi)\Phi_2(\xi)d\xi - \frac{2}{\alpha}\int_0^x \Phi_3(\xi)\Phi_1(\xi, 2x-\xi)d\xi. \end{cases} \tag{2.29}$$

For simplicity, by $\Phi \in L_2(l)$, we mean that

$$\Phi_1(x,t) \in L_2(\Delta(l)),$$

$$\Phi_j(x) \in L_2(0,l), \quad j = 2,3,$$

and set

$$\|\Phi\|^2(l) = \|\Phi_1\|^2_{L_2(\Delta(l))} + \sum_{j=2}^{3} \|\Phi_j\|^2_{L_2(0,l)}.$$

We recall that that

$$\Delta(l) = \{(x,t): \ 0 < x < t < 2l - x\}.$$

3 Conditional stability in L_2

In this section, we will prove conditional stability for solution to the system of the nonlinear Volterra equations (2.29) and will give dependency of the constant of the stability estimate.

We introduce $\Sigma(l, M, c_0, \rho_0, \sigma_*)$, the class of possible solutions of the inverse problem, namely, $\sigma(x) \in \Sigma(l, M, c_0, \rho_0, \sigma_*)$ if $\sigma(x)$ satisfies the following conditions:

1. $\sigma(x) \in H_1(0, l) \cap C^1[0, l)$,

2. $\|\sigma\|_{H_1(0,l)} \le M$,

3. $0 < \sigma_* \le \sigma(x), \quad x \in (0, l)$,

4. $\sigma_0 = c_0 \rho_0$.

We define $\mathcal{F}(l, Q, \beta, c_0)$, the class of possible data, namely, $f \in \mathcal{F}(l, Q, \beta, c_0)$ if f satisfies the following conditions:

1. $f \in H_1(0, 2l)$,

2. $\|f\|_{H_1(0,2l)} \le Q$,

3. $f(+0) = -\beta c_0$.

We note that the third condition must follow from (1.9) and (2.12). Suppose that for $f^{(j)} \in \mathcal{F}(l, Q, \beta, c_0)$, there exists $\sigma^{(j)}$ from $\Sigma(l, M, c_0, \rho_0, \sigma_*)$ which solves the inverse problem

$$u^{(j)}_{tt}(x,t) = u^{(j)}_{xx}(x,t) - \frac{(\sigma^{(j)})'(x)}{\sigma^{(j)}(x)} u^{(j)}_x(x,t), \; x > 0, \; t > 0, \quad (3.1)$$

$$u^{(j)}_{|t<0} = 0, \quad (3.2)$$

$$u^{(j)}_x(+0,t) = \alpha \, \delta(t), \quad t > 0, \quad (3.3)$$

$$u^{(j)}(+0,t) = f^{(j)}(t), \quad t > 0, \quad (3.4)$$

for $j = 1, 2$.

We define

$$\begin{aligned} \Phi^{(j)} &= (\Phi_1^{(j)}, \Phi_2^{(j)}, \Phi_3^{(j)})^\top, \\ \Phi_1^{(j)}(x,t) &= u_x^{(j)}(x,t),\ \Phi_2^{(j)}(x) = p^{(j)}(x),\ \Phi_3^{(j)}(x) = a^{(j)}(x); \\ G_1^{(j)}(x,t) &= F_x^{(j)}(x,t) = \frac{1}{2}\Big[(f^{(j)})'(t+x) - (f^{(j)})'(t-x)\Big], \\ G_2^{(j)} &= -\frac{1}{\alpha},\ G_3^{(j)}(x) = -\frac{2}{\alpha}(f^{(j)})'(2x),\ j = 1, 2. \end{aligned}$$

Then we show the stability in our inverse problem where the constant is given by $\|\Phi^{(1)}\|(l)$ and $\|\Phi^{(2)}\|(l)$.

Theorem 3.1. *Suppose that for $G^{(j)} \in L_2(l)$, there exists $\Phi^{(j)} \in L_2(l)$, which solves the inverse problem*

$$\Phi^{(j)}(x,t) = G^{(j)}(x,t) + \mathcal{B}(\Phi^{(j)}), \quad j = 1, 2, \quad (x,t) \in \Delta(l), \tag{3.5}$$

for $j = 1, 2$.
Then

$$\left\|\Phi^{(1)} - \Phi^{(2)}\right\|^2(l) \ \leq\ C_1 \left\|f^{(1)} - f^{(2)}\right\|^2_{H_1(0,2l)}. \tag{3.6}$$

Here we have

$$\begin{aligned} C_1 &= [3l + 3(4\frac{1}{\beta^2 c_0^2} + \Phi_*^4)(1 + 6\Phi_*^2 l)] \\ &\times\ \exp\left\{\Phi_*^2[25l + 6(4\frac{1}{\beta^2 c_0^2} + \Phi_*^4)(1 + 24\Phi_*^2 l) + 12Q^2 + 12\Phi_*^4]\right\}, \end{aligned} \tag{3.7}$$

$$\Phi_* = \max\{\left\|\Phi^{(1)}\right\|(l),\ \left\|\Phi^{(2)}\right\|(l)\}, \quad Q = \left\|f^{(1)}\right\|_{H_1(0,2l)}.$$

Remark 3.1. *Notice that if $\|G\|(l)$ is small, then $\|\Phi\|(l)$ is small as well, which implies that $\|\sigma'/\sigma\|_{L_2(0,l)}$ is small. In partcular, when $\sigma \equiv const$, then $u(x,t) = -\alpha\theta(t-x)$.*

Proof.

We set

$$\begin{aligned}
\Phi^{(j)}(x,t) &= \left(\Phi_1^{(j)}(x,t), \Phi_2^{(j)}(x), \Phi_3^{(j)}(x)\right)^\top, \ j=1,2,\\
\tilde\Phi(x,t) &= (\tilde\Phi_1(x,t), \tilde\Phi_2(x), \tilde\Phi_3(x))^\top = \Phi^{(1)}(x,t) - \Phi^{(2)}(x,t),\\
\tilde G(x,t) &= (\tilde G_1(x,t), \tilde G_2(x), \tilde G_3(x))^\top = G^{(1)}(x,t) - G^{(2)}(x,t),\\
\tilde f(x) &= f^{(1)}(x) - f^{(2)}(x), \ \tilde F(x,t) = F^{(1)}(x,t) - F^{(2)}(x,t).
\end{aligned}$$

Then it follows from (3.5) that

$$\tilde\Phi(x,t) = \tilde G(x,t) + \mathcal{B}(\Phi^{(1)}) - \mathcal{B}(\Phi^{(2)}), \quad (x,t)\in\Delta(l). \tag{3.8}$$

Recalling the definition (2.29), we will estimate the first component $\tilde\Phi_1$ by the right-hand side of (3.8):

$$\begin{aligned}
&|\tilde\Phi_1(x,t)| \le |\tilde G_1(x,t)| + \\
&+ \frac12\sqrt{\int_0^x |\tilde\Phi_3(\xi)|^2 d\xi}\left[\sqrt{\int_0^x |\Phi_1^{(1)}(\xi,t+x-\xi)|^2 d\xi} + \sqrt{\int_0^x |\Phi_1^{(1)}(\xi,t-x+\xi)|^2 d\xi}\,\right]\\
&+ \frac12\sqrt{\int_0^x |\Phi_3^{(2)}(\xi)|^2 d\xi}\left[\sqrt{\int_0^x |\tilde\Phi_1(\xi,t+x-\xi)|^2 d\xi} + \sqrt{\int_0^x |\tilde\Phi_1(\xi,t-x+\xi)|^2 d\xi}\,\right].
\end{aligned}$$

Therefore, noting that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ and $(\sqrt{a}+\sqrt{b})^2 \le 2a+2b$ for $a\ge 0$ and $b \ge 0$, we obtain

$$\begin{aligned}
|\tilde\Phi_1(x,t)|^2 \le\ & 3|\tilde G_1(x,t)|^2\\
&+ \frac32\int_0^x |\tilde\Phi_3(\xi)|^2 d\xi \int_0^x \left[|\Phi_1^{(1)}(\xi,t+x-\xi)|^2 + |\Phi_1^{(1)}(\xi,t-x+\xi)|^2\right] d\xi\\
&+ \frac32\int_0^x |\Phi_3^{(2)}(\xi)|^2 d\xi \int_0^x \left[|\tilde\Phi_1(\xi,t+x-\xi)|^2 + |\tilde\Phi_1^2(\xi,t-x+\xi)|^2\right] d\xi.
\end{aligned} \tag{3.9}$$

Henceforth, for $\Phi_1 \in L_2(\Delta(l))$, we set

$$\|\Phi_1\|^2_{L_2(\Delta(l,x))} = \int_0^x \left(\int_\xi^{2l-\xi} |\Phi_1(\xi,\tau)|^2 d\tau \right) d\xi$$

and we note that $\|\Phi_1\|_{L_2(\Delta(l,l))} = \|\Phi_1\|_{L_2(\Delta(l))}$.

From (3.9), it follows that

$$\begin{aligned}
&\left\|\tilde{\Phi}_1\right\|^2_{L_2(\Delta(l,x))} \le 3\left\|\tilde{G}_1\right\|^2_{L_2(\Delta(l,x))} \\
+ &\ \frac{3}{2}\int_0^x \int_\xi^{2l-\xi} \left\{ \int_0^\xi |\tilde{\Phi}_3(\xi')|^2 d\xi' \int_0^\xi \left[|\Phi_1^{(1)}(\xi',\tau+\xi-\xi')|^2 + |\Phi_1^{(1)}(\xi',\tau-\xi+\xi')|^2 \right] d\xi' \right. \\
+ &\ \left. \int_0^\xi |\Phi_3^{(2)}(\xi')|^2 d\xi' \int_0^\xi \left[|\tilde{\Phi}_1(\xi',\tau+\xi-\xi')|^2 + |\tilde{\Phi}_1^2(\xi',\tau-\xi+\xi')|^2 \right] d\xi' \right\} d\tau d\xi \\
\le &\ 3\left\|\tilde{G}_1\right\|^2_{L_2(\Delta(l,x))} + 12\Phi_*^2 \int_0^x \int_0^\xi |\tilde{\Phi}_3(\xi')|^2 d\xi' d\xi + 12\Phi_*^2 \int_0^x \left\|\tilde{\Phi}_1\right\|^2_{L_2(\Delta(l,\xi))} d\xi,
\end{aligned} \tag{3.10}$$

where

$$\Phi_* = \max\{\left\|\Phi^{(1)}\right\|(l),\ \left\|\Phi^{(2)}\right\|(l)\}.$$

We estimate the second component of the left-hand side of (3.8):

$$\begin{aligned}
|\tilde{\Phi}_2(x)| &\le \frac{1}{2}\int_0^x |\Phi_3^{(1)}(\xi)\tilde{\Phi}_2(\xi)| d\xi + \frac{1}{2}\int_0^x |\Phi_2^{(2)}(\xi)\tilde{\Phi}_3(\xi)| d\xi \\
&\le \frac{1}{2}\sqrt{\int_0^x |\Phi_3^{(1)}(\xi)|^2 d\xi}\sqrt{\int_0^x |\tilde{\Phi}_2(\xi)|^2 d\xi} + \frac{1}{2}\sqrt{\int_0^x |\Phi_2^{(2)}(\xi)|^2 d\xi}\sqrt{\int_0^x |\tilde{\Phi}_3(\xi)|^2 d\xi}.
\end{aligned}$$

Therefore

$$|\tilde{\Phi}_2(\xi)|^2 \le \frac{1}{2}\Phi_*^2 \int_0^\xi \left[|\tilde{\Phi}_2(\xi')|^2 + |\tilde{\Phi}_3(\xi')|^2 \right] d\xi' \tag{3.11}$$

and

$$\left\|\tilde{\Phi}_2\right\|^2_{L_2(0,x)} \le \frac{1}{2}\Phi_*^2 \int_0^x \left[\left\|\tilde{\Phi}_2\right\|^2_{L_2(0,\xi)} + \left\|\tilde{\Phi}_3\right\|^2_{L_2(0,\xi)} \right] d\xi. \tag{3.12}$$

Let us estimate $\tilde{\Phi}_3$, using the third equation from the system (3.8)

$$|\tilde{\Phi}_3(x)| \ \le \ |\tilde{G}_3(x)| + \sum_{j=1}^{5} \omega_j, \tag{3.13}$$

where

$$\begin{aligned}
\mathcal{B}_4(\Phi) &= \int_0^x \Phi_3(\xi)\Phi_1(\xi, 2x-\xi)d\xi, \\
\omega_1(x) &= 2|(f^{(1)})'(2x)|\Big|\mathcal{B}_2(\Phi^{(1)}) - \mathcal{B}_2(\Phi^{(2)})\Big|, \\
\omega_2(x) &= \frac{2}{|\alpha|}\Big|\mathcal{B}_4(\Phi^{(1)}) - \mathcal{B}_4(\Phi^{(2)})\Big|, \\
\omega_3(x) &= \Big|\mathcal{B}_2(\Phi^{(1)})\Big||\alpha|\omega_2(x), \\
\omega_4(x) &= 2\Big|\mathcal{B}_4(\Phi^{(2)})\Big|\Big|\mathcal{B}_2(\Phi^{(1)}) - \mathcal{B}_2(\Phi^{(2)})\Big|, \\
\omega_5(x) &= 2\Big|(f^{(1)})'(2x) - (f^{(2)})'(2x)\Big|\Big|\mathcal{B}_2(\Phi^{(2)})\Big|.
\end{aligned}$$

First we have

$$\omega_1(x) \ \le \ |(f^{(1)})'(2x)|\Phi_* \left(\sqrt{\int_0^x |\tilde{\Phi}_2(\xi)|^2 d\xi} + \sqrt{\int_0^x |\tilde{\Phi}_3(\xi)|^2 d\xi}\right),$$

that is,

$$\omega_1^2(x) \ \le \ 2[(f^{(1)})'(2x)]^2\Phi_*^2 \int_0^x \Big[|\tilde{\Phi}_2(\xi)|^2 + |\tilde{\Phi}_3(\xi)|^2\Big]d\xi. \tag{3.14}$$

Second,

$$\begin{aligned}
\omega_2(x) \ &\le \ \frac{2}{|\alpha|}\int_0^x \Big|\tilde{\Phi}_3(\xi)\Phi_1^{(1)}(\xi, 2x-\xi) + \Phi_3^{(2)}(\xi)\tilde{\Phi}_1(\xi, 2x-\xi)\Big|d\xi \\
&\le \ \frac{2}{|\alpha|}\Bigg[\ \sqrt{\int_0^x |\tilde{\Phi}_3(\xi)|^2 d\xi}\sqrt{\int_0^x |\Phi_1^{(1)}(\xi, 2x-\xi)|^2 d\xi} + \\
&+ \ \sqrt{\int_0^x |\Phi_3^{(2)}(\xi)|^2 d\xi}\ \sqrt{\int_0^x |\tilde{\Phi}_1(\xi, 2x-\xi)|^2 d\xi}\ \Bigg],
\end{aligned}$$

so that

$$\omega_2^2(x) \;\le\; \frac{8}{\alpha^2}\int_0^x |\tilde{\Phi}_3(\xi)|^2 d\xi \int_0^x |\Phi_1^{(1)}(\xi,2x-\xi)|^2 d\xi + \frac{8}{\alpha^2}\Phi_*^2 \int_0^x |\tilde{\Phi}_1(\xi,2x-\xi)|^2 d\xi. \tag{3.15}$$

Third,

$$\begin{aligned}\omega_3(x) \;&\le\; \Big|\mathcal{B}_2(\Phi^{(1)})\Big||\alpha|\omega_2(x) \le \frac{|\alpha|}{2}\omega_2(x)\int_0^x \Big|\Phi_3^{(1)}(\xi)\Phi_2^{(1)}(\xi)\Big| d\xi \\ &\le\; \frac{|\alpha|}{2}\omega_2(x)\sqrt{\int_0^x |\Phi_3^{(1)}(\xi)|^2 d\xi}\sqrt{\int_0^x |\Phi_2^{(1)}(\xi)|^2 d\xi} \le \frac{|\alpha|}{2}\omega_2(x)\Phi_*^2,\end{aligned}$$

and hence

$$\omega_3^2(x) \;\le\; \frac{\alpha^2}{4}\Phi_*^4\omega_2^2(x). \tag{3.16}$$

Furthermore

$$\begin{aligned}\omega_4(x) \;&\le\; \sqrt{\int_0^x |\Phi_3^{(2)}(\xi)|^2 d\xi}\sqrt{\int_0^x |\Phi_1^{(2)}(\xi,2x-\xi)|^2 d\xi} \\ &\times\; \Phi_*\left(\sqrt{\int_0^x |\tilde{\Phi}_2(\xi)|^2 d\xi} + \sqrt{\int_0^x |\tilde{\Phi}_3(\xi)|^2 d\xi}\right),\end{aligned}$$

and

$$\omega_4^2(x) \;\le\; 2\Phi_*^4 \int_0^x |\Phi_1^{(2)}(\xi,2x-\xi)|^2 d\xi \int_0^x \Big[|\tilde{\Phi}_2(\xi)|^2 + |\tilde{\Phi}_3(\xi)|^2\Big] d\xi. \tag{3.17}$$

Finally,

$$\omega_5(x) \;\le\; \Phi_*^2\Big|(f^{(1)})'(2x) - (f^{(2)})'(2x)\Big|,$$

so that

$$\omega_5^2(x) \;\le\; \Phi_*^4\Big|(f^{(1)})'(2x) - (f^{(2)})'(2x)\Big|^2. \tag{3.18}$$

Taking into account (3.14)-(3.18) and $\Big(\sum\limits_{k=1}^{6}|b_k|\Big)^2 \le 6\sum\limits_{k=1}^{6}|b_k|^2$, we obtain from (3.13)

$$
\begin{aligned}
|\tilde{\Phi}_3(x)|^2 &\le \frac{24}{\alpha^2}\Big[(f^{(1)})'(2x)-(f^{(2)})'(2x)\Big]^2 + 6\sum_{j=1}^{5}\omega_j^2\\
&\le \frac{24}{\alpha^2}\Big[(f^{(1)})'(2x)-(f^{(2)})'(2x)\Big]^2\\
&+ 12\Phi_*^2\Big[(f^{(1)})'(2x)\Big]^2\int_0^x\Big[|\tilde{\Phi}_2(\xi)|^2+|\tilde{\Phi}_3(\xi)|^2\Big]d\xi\\
&+ 12\Big(4\frac{1}{\alpha^2}+\Phi_*^4\Big)\\
&\times \Bigg[\int_0^x|\tilde{\Phi}_3(\xi)|^2d\xi\int_0^x|\Phi_1^{(1)}(\xi,2x-\xi)|^2d\xi+\Phi_*^2\int_0^x|\tilde{\Phi}_1(\xi,2x-\xi)|^2d\xi\Bigg]\\
&+ 12\Phi_*^4\int_0^x|\Phi_1^{(2)}(\xi,2x-\xi)|^2d\xi\int_0^x\Big[|\tilde{\Phi}_2(\xi)|^2+|\tilde{\Phi}_3(\xi)|^2\Big]d\xi\\
&+ 6\Phi_*^4\Big[(f^{(1)})'(2x)-(f^{(2)})'(2x)\Big]^2.
\end{aligned}
$$

That is,

$$
\begin{aligned}
|\tilde{\Phi}_3(x)|^2 &\le \mu_0\Big[(f^{(1)})'(2x)-(f^{(2)})'(2x)\Big]^2+\mu_1\int_0^x|\tilde{\Phi}_1(\xi,2x-\xi)|^2d\xi\\
&+ \mu_2(x)\int_0^x|\tilde{\Phi}_2(\xi)|^2d\xi+\mu_3(x)\int_0^x|\tilde{\Phi}_3(\xi)|^2d\xi, \qquad (3.19)
\end{aligned}
$$

where we set

$$
\begin{aligned}
\mu_0 &= 6\Big(4\frac{1}{\alpha^2}+\Phi_*^4\Big),\\
\mu_1 &= 12(4\frac{1}{\alpha^2}+\Phi_*^4)\Phi_*^2 = 2\Phi_*^2\mu_0,\\
\mu_2(x) &= 12\Phi_*^2\Big[(f^{(1)})'(2x)\Big]^2+12\Phi_*^4\int_0^x|\Phi_1^{(2)}(\xi,2x-\xi)|^2d\xi,\\
\mu_3(x) &= 12\Phi_*^2\Big[(f^{(1)})'(2x)\Big]^2+12(4\frac{1}{\alpha^2}+\Phi_*^4)\int_0^x|\Phi_1^{(1)}(\xi,2x-\xi)|^2d\xi
\end{aligned}
$$

$$+ \quad 12\Phi_*^4 \int_0^x |\Phi_1^{(2)}(\xi, 2x-\xi)|^2 d\xi.$$

Therefore

$$\begin{aligned} \left\|\tilde{\Phi}_3\right\|_{L_2(0,x)} &\leq \mu_0 \int_0^x [\tilde{f}'(2\xi)]^2 d\xi + \frac{1}{2}\mu_1 \left\|\tilde{\Phi}_1\right\|^2_{L_2(\Delta(l,x))} \\ &+ \int_0^x \mu_2(\xi) \left\|\tilde{\Phi}_2\right\|_{L_2(0,\xi)} d\xi + \int_0^x \mu_3(\xi) \left\|\tilde{\Phi}_3\right\|_{L_2(0,\xi)} d\xi. \end{aligned}$$

Next we consider a system of inequalities (3.10), (3.11) and (3.19). By the Cauchy-Bunyakovskii inequality and change of variables, we obtain that

$$\tilde{G}_1(x,t) = \tilde{F}_x(x,t) = \frac{1}{2}\Big[-\tilde{f}'(t-x) + \tilde{f}'(t+x)\Big],$$

$$\begin{aligned} \left\|\tilde{G}_1\right\|^2_{L_2(\Delta(l,x))} &= \int_0^x \int_\xi^{2l-\xi} \Big[\tilde{F}_x(\xi,\tau)\Big]^2 d\tau d\xi \\ &= \frac{1}{4}\int_0^x \int_\xi^{2l-\xi} \Big[|\tilde{f}'(\tau-\xi)|^2 - 2|\tilde{f}'(\tau-\xi)||\tilde{f}'(\tau+\xi)| \\ &+ |\tilde{f}'(\tau+\xi)|^2\Big] d\tau d\xi \leq x \left\|\tilde{f}'\right\|^2_{L_2(0,2l)}. \end{aligned} \tag{3.20}$$

For simplicity, we set

$$\begin{cases} \nu = \left\|\tilde{f}'\right\|_{L_2(0,2l)}, \\ \Psi_1(x) = \left\|\tilde{\Phi}_1\right\|^2_{L_2(\Delta(l,x))}, \quad x \in (0,l), \\ \Psi_j(x) = \int_0^x \tilde{\Phi}_j^2(\xi) d\xi, \quad j = 2,3. \end{cases} \tag{3.21}$$

Then it follows from (3.10), (3.12), (3.19) and (3.20) that

$$\Psi_1(x) \leq 3\nu^2 x + 12\Phi_*^2 \int_0^x \Psi_3(\xi) d\xi + 12\Phi_*^2 \int_0^x \Psi_1(\xi) d\xi, \tag{3.22}$$

$$\Psi_2(x) \leq \frac{1}{2}\Phi_*^2 \int_0^x [\Psi_2(\xi) + \Psi_3(\xi)] d\xi, \tag{3.23}$$

$$\Psi_3(x) \leq \frac{\mu_0}{2}\nu^2 + \frac{1}{2}\mu_1 \Psi_1(x) + \int_0^x \Big[\mu_2(\xi)\Psi_2(\xi) + \mu_3(\xi)\Psi_3(\xi)\Big] d\xi. \tag{3.24}$$

Substituting (3.22) into (3.24) and noting that

$$\begin{aligned} \int_0^x \int_0^\xi \tilde{\Phi}_1^2(\xi', 2\xi - \xi') d\xi' d\xi &= \frac{1}{2} \int_0^x \left(\int_{\xi'}^{2x-\xi'} \tilde{\Phi}_1^2(\xi', \zeta) d\zeta \right) d\xi' \\ &= \frac{1}{2} \left\| \tilde{\Phi}_1 \right\|^2_{L_2(\Delta(l,x))}, \end{aligned} \tag{3.25}$$

for $\Psi = \Psi_1 + \Psi_2 + \Psi_3$, we obtain

$$\Psi(x) \leq \nu^2 \Big(3x + \frac{\mu_0}{2} + \frac{3}{2}\mu_1 x \Big) + \int_0^x \sum_{j=1}^3 \gamma_j(\xi)\Psi_j(\xi) d\xi, \tag{3.26}$$

where

$$\begin{aligned} \gamma_1 &= 12\Phi_*^2 \Big(1 + \frac{1}{2}\mu_1\Big), \quad \gamma_2(x) = \frac{1}{2}\Phi_*^2 + \mu_2(x), \\ \gamma_3(x) &= 6\Phi_*^2(2 + \mu_1) + \frac{1}{2}\Phi_*^2 + \mu_3(x). \end{aligned}$$

Let us introduce a new function

$$V(x) = \nu^2 \Big(3l + \frac{\mu_0}{2} + \frac{3}{2}\mu_1 l \Big) + \int_0^x \sum_{j=1}^3 \gamma_j(\xi)\Psi_j(\xi) d\xi.$$

Then $\Psi(x) \leq V(x)$ and

$$V'(x) = \sum_{j=1}^3 \gamma_j(x)\Psi_j(x) \leq V(x) \sum_{j=1}^3 \gamma_j(x).$$

Therefore

$$\frac{V'(x)}{V(x)} \leq \sum_{j=1}^3 \gamma_j(x)$$

and the Gronwall inequality yields

$$\Psi(x) \leq V(x) \leq V(0) \exp\left\{ \int_0^x \sum_{j=1}^3 \gamma_j(\xi) d\xi \right\}. \tag{3.27}$$

On the other hand, we have

$$\begin{aligned}\int_0^x \gamma_2(\xi)d\xi &\leq \frac{x}{2}\Phi_*^2 + \int_0^x \mu_2(\xi)d\xi \leq \frac{x}{2}\Phi_*^2 + 12\Phi_*^2\int_0^x \Big[(f^{(1)})'(2\xi)\Big]^2 d\xi \\ &+ 12\Phi_*^4\int_0^x\int_0^\xi |\Phi_1^{(2)}(\xi', 2\xi-\xi')|^2 d\xi' d\xi \\ &\leq \frac{x}{2}\Phi_*^2 + 6\Phi_*^2 \left\|(f^{(1)})'\right\|^2_{L_2(0,2l)} + 6\Phi_*^6 \qquad (3.28)\end{aligned}$$

and

$$\begin{aligned}\int_0^x \gamma_3(\xi)d\xi &= \int_0^x [6\Phi_*^2(2+\mu_1) + \frac{1}{2}\Phi_*^2 + \mu_3(\xi)]d\xi \\ &\leq \frac{1}{2}\Phi_*^2 x + 6\Phi_*^2(2+\mu_1)x + 6\Phi_*^2\left\|(f^{(1)})'\right\|^2_{L_2(0,2l)} \\ &+ 6(4\frac{1}{\alpha^2} + \Phi_*^4)\Phi_*^2 + 6\Phi_*^6. \qquad (3.29)\end{aligned}$$

At the last inequality, we have applied (3.25). Hence by (3.28) and (3.29), we obtain

$$\begin{aligned}\int_0^x \sum_{j=1}^3 \gamma_j(\xi)d\xi &\leq 25\Phi_*^2 x + 12\Phi_*^2\left\|(f^{(1)})'\right\|^2_{L_2(0,2l)} + 12\Phi_*^6 \\ &+ 6\Phi_*^2(4\frac{1}{\alpha^2} + \Phi_*^2)(1+24\Phi_*^2 x) \qquad (3.30)\end{aligned}$$

We substitute (3.30) into (3.27), so that

$$\begin{aligned}\left\|\Phi^{(1)} - \Phi^{(2)}\right\|^2(l) &\leq C_1\left\|(f^{(1)})' - (f^{(2)})'\right\|^2_{L_2(0,2l)} \\ &\leq C_1\left\|f^{(1)} - f^{(2)}\right\|^2_{H_1(0,2l)}, \qquad (3.31)\end{aligned}$$

where the constants $C_1 > 0$ is given by (3.7).

Thus we have proved the conditional stability where the dependency of the constants C_1 is given by (3.7).

In the next section, we will sharpen (3.6), that is, we will estimate Φ_* in terms of l and M.

4 Sharp estimation of Φ^* by $\|\sigma\|_{H_1(0,l)}$.

For sharpening Theorem 3.1, we will first estimate L_2-norms of Φ_1 and Φ_2 by $\|\Phi_3\|_{L_2(0,l)}$. Let the vector-function $\Phi = (\Phi_1, \Phi_2\, \Phi_3)^\top$ solve the system

$$\Phi(x,t) \;=\; G(x,t) + \mathcal{B}(\Phi), \quad (x,t) \in \Delta(l). \tag{4.1}$$

We begin with the equation for Φ_2:

$$\Phi_2(x) \;=\; -\frac{1}{\alpha} - \frac{1}{2}\int_0^x \Phi_3(\xi)\Phi_2(\xi)d\xi, \quad x \in (0,l). \tag{4.2}$$

Therefore

$$|\Phi_2(x)|^2 \;\le\; \frac{2}{\alpha^2} + \frac{1}{2}\int_0^x |\Phi_3(\xi)|^2 d\xi \int_0^x |\Phi_2(\xi)|^2 d\xi,$$

and

$$\|\Phi_2\|^2_{L_2(0,x)} \;\le\; \frac{2l}{\alpha^2} + \frac{1}{2}\,\|\Phi_3\|^2_{L_2(0,x)} \int_0^x \|\Phi_2\|^2_{L_2(0,\xi)}\, d\xi.$$

Hence the Gronwall inequality yields

$$\|\Phi_2\|^2_{L_2(0,l)} \;\le\; \frac{2l}{\alpha^2} \exp\left\{\frac{l}{2}\,\|\Phi_3\|^2_{L_2(0,l)}\right\}. \tag{4.3}$$

Let us estimate the L_2- norm of Φ_1 from the first equation of the system (4.1):

$$\Phi_1(x,t) \;=\; F_x(x,t) + \frac{1}{2}\int_0^x \Phi_3(\xi)\Big[\Phi_1(\xi, t+x-\xi) + \Phi_1(\xi, t-x+\xi)\Big]d\xi.$$

Since, for $(x,t) \in \Delta(l)$, we have

$$|\Phi_1(x,t)|^2 \;\le\; 2|F_x(x,t)|^2$$

$$+\; \int_0^x |\Phi_3(\xi)|^2 d\xi \int_0^x \Big[|\Phi_1(\xi, t+x-\xi)|^2 + |\Phi_1(\xi, t-x+\xi)|^2\Big] d\xi,$$

then (3.20) yields

$$\|\Phi_1\|^2_{L_2(\Delta(l,x))} \;\le\; 2x\,\|f'\|^2_{L_2(0,2l)} + 2\,\|\Phi_3\|^2_{L_2(0,x)} \int\limits_0^x \|\Phi_1\|^2_{L_2(\Delta(l,\xi))}\,d\xi, \quad x \in (0,l),$$

so that we obtain

$$\|\Phi_1\|^2_{L_2(\Delta(l))} \;\le\; 2l\exp\left\{2l\,\|\Phi_3\|^2_{L_2(0,l)}\right\}\|f'\|^2_{L_2(0,2l)}\,. \tag{4.4}$$

Next, since

$$\Phi_3(x) = a(x) = \frac{\sigma'(x)}{\sigma(x)},$$

we have

$$\|\Phi_3\|^2_{L_2(0,l)} \;\le\; \frac{1}{\sigma_*^2}\,\|\sigma\|^2_{H_1(0,l)} = \frac{1}{\sigma_*^2}M^2. \tag{4.5}$$

Consequently we obtain from (4.4) and (4.3) that

$$\|\Phi_1\|^2_{L_2(\Delta(l))} \;\le\; 2l\exp\left\{2l\frac{1}{\sigma_0^2}M^2\right\}Q^2,$$

$$\|\Phi_2\|^2_{L_2(0,l)} \;\le\; \frac{2l}{\alpha^2}\exp\left\{\frac{1}{2\sigma_*^2}lM^2\right\},$$

and therefore

$$\|\Phi\|^2(l) \le 2l\exp\left\{2l\frac{1}{\sigma_0^2}M^2\right\}Q^2 + \frac{2l}{\beta^2c_0^2}\exp\left\{\frac{1}{2\sigma_*^2}lM^2\right\} + \frac{1}{\sigma_*^2}M^2$$

and

$$\Phi_*^2 \le 2l\exp\left\{2l\frac{1}{\sigma_0^2}M^2\right\}Q^2 + \frac{2l}{\beta^2c_0^2}\exp\left\{\frac{1}{2\sigma_*^2}lM^2\right\} + \frac{1}{\sigma_*^2}M^2. \tag{4.6}$$

5 Estimation of $\left\|\sigma^{(1)}-\sigma^{(2)}\right\|_{H_1(0,l)}$.

We recall that $(\sigma^{(j)})'(x)=a^{(j)}(x)\sigma^{(j)}(x)=\Phi_3^{(j)}(x)\sigma^{(j)}(x), \quad j=1,2$ and denote

$$\tilde{\sigma}(x)=\sigma^{(1)}(x)-\sigma^{(2)}(x), \quad \tilde{\Phi_3}(x)=\Phi_3^{(1)}(x)-\Phi_3^{(2)}(x).$$

Therefore

$$\begin{aligned}(\sigma^{(1)})'(x)-(\sigma^{(2)})'(x) &= \Phi_3^{(1)}(x)[\sigma^{(1)}(x)-\sigma^{(2)})(x)] \\ &+ [\Phi_3^{(1)}(x)-\Phi_3^{(2)}(x)]\sigma^{(2)}(x) \\ &= \Phi_3^{(1)}(x)\tilde{\sigma}(x)+\tilde{\Phi_3}(x)\sigma^{(2)}(x).\end{aligned}$$

Since $\tilde{\sigma}(0)=0$ by the assumption, we have

$$\tilde{\sigma}(x) = \int_0^x \tilde{\sigma}'(\xi)d\xi, \tag{5.1}$$

so that we obtain

$$\tilde{\sigma}'(\xi) = \Phi_3^{(1)}(\xi)\int_0^\xi \tilde{\sigma}'(\xi')d\xi' + \tilde{\Phi_3}(\xi)\Big[\sigma^{(2)}(+0)+\int_0^\xi (\sigma^{(2)})'(\xi')d\xi'\Big].$$

Hence

$$\|\tilde{\sigma}'\|^2_{L_2(0,x)} \leq 4[|\sigma^{(2)}(+0)|^2+xM^2]\left\|\tilde{\Phi_3}\right\|^2_{L_2(0,x)}+2l\int_0^x |\Phi_3^{(1)}(\xi)|^2\,\|\tilde{\sigma}'\|^2_{L_2(0,\xi)}\,d\xi.$$

By the Gronwall inequality, we have

$$\|\tilde{\sigma}'\|^2_{L_2(0,l)} \leq 4[|\sigma^{(2)}(+0)|^2+lM^2]\left\|\tilde{\Phi_3}\right\|^2_{L_2(0,l)}\exp\left\{2l\left\|\Phi_3^{(1)}\right\|^2_{L_2(0,l)}\right\}.$$

Using (3.6) and (4.5), we obtain

$$\begin{aligned}\|\tilde{\sigma}'\|^2_{L_2(0,l)} &\leq 4[|\sigma^{(2)}(+0)|^2+lM^2]\exp\left\{2l\frac{1}{\sigma_*^2}M^2\right\}C_1\left\|f^{(1)}-f^{(2)}\right\|^2_{H_1(0,2l)} \\ &\leq C_2\left\|f^{(1)}-f^{(2)}\right\|^2_{H_1(0,2l)},\end{aligned} \tag{5.2}$$

where we set

$$C_2 = 4[c_0^2\rho_0^2 + lM^2]\exp\left\{2l\frac{1}{\sigma_*^2}M^2\right\}C_1. \tag{5.3}$$

By (5.1), we obtain

$$\|\tilde{\sigma}\|^2_{L_2(0,l)} \le \frac{l^2}{2}\|\tilde{\sigma}'\|^2_{L_2(0,l)} \le \frac{l^2}{2}C_2\left\|f^{(1)} - f^{(2)}\right\|^2_{H_1(0,2l)}. \tag{5.4}$$

Therefore it follows from (5.2) and (5.4) that

$$\|\tilde{\sigma}\|^2_{H_1(0,l)} \ \le \ C\left\|f^{(1)} - f^{(2)}\right\|^2_{H_1(0,2l)}.$$

Here we set

$$C = C_2\left(\frac{l^2}{2} + 1\right). \tag{5.5}$$

Therefore, in terms of (3.7), (4.6), (5.3) and (5.5), we have proved our main result:

Theorem 5.1. *Suppose that for $f^{(1)}, f^{(2)}$ from $\mathcal{F}(l, Q, \beta, c_0)$, there exist $\sigma^{(1)}, \sigma^{(2)} \in \Sigma(l, M, c_0, \rho_0, \sigma_*)$ which solve the inverse problem* (2.1)-(2.4) *with $f^{(1)}$ and $f^{(2)}$, respectively. Then*

$$\left\|\sigma^{(1)} - \sigma^{(2)}\right\|^2_{H_1(0,l)} \ \le \ C\left\|f^{(1)} - f^{(2)}\right\|^2_{H_1(0,2l)},$$

where

$$\begin{aligned} C(l, M, Q, \beta, c_0, \rho_0, \sigma_*) &= 4(l+1)[c_0^2\rho_0^2 + lM^2]\exp\left\{2l\frac{1}{\sigma_*^2}M^2\right\} \\ &\times\ [3l + 3(4\frac{1}{\beta^2 c_0^2} + \eta^2)(1 + 6\eta l)] \\ &\times\ \exp\left\{\eta[25l + 6(4\frac{1}{\beta^2 c_0^2} + \eta^2)(1 + 24\eta l) + 12Q^2 + 12\eta^2]\right\}, \\ \eta &= 2l\exp\left\{2l\frac{1}{\sigma_0^2}M^2\right\}Q^2 + \frac{2l}{\beta^2 c_0^2}\exp\left\{\frac{1}{2\sigma_*^2}lM^2\right\} + \frac{1}{\sigma_*^2}M^2. \end{aligned}$$

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