UTMS 2001–18 June 4, 2001

Global uniqueness and stability for a class of multidimesional inverse hyperbolic problems with two unknowns

by

Masahiro YAMAMOTO and Xu ZHANG

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

GLOBAL UNIQUENESS AND STABILITY FOR A CLASS OF MULTIDIMESIONAL INVERSE HYPERBOLIC PROBLEMS WITH TWO UNKNOWNS

Masahiro Yamamoto* and Xu Zhang†

June 1, 2001

Abstract

In this paper, we obtain the global uniqueness and stablity estimate for a class of multidimensional inverse hyperbolic problems of determining source terms and an initial value from a single measurement of boundary values or interior values. By means of a suitable transformation, we reduce the problem to the observability inequalities for nonconservative hyperbolic equations with memory. Then, using the compactness/uniqueness argument, we can prove the uniqueness and the stability by a new kind of unique continuation property of a non-local hyperbolic equation.

*Department of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba Meguro Tokyo 153 Japan. E-mail: myama@ms.u-tokyo.ac.jp.

†School of Mathematics, Sichuan University, Chengdu 610064, China; and Departamento de Matemática Aplicada, Universidad Complutense, 28040 Madrid Spain. This author is partially supported by NSF of China under grant 19901024 and the Postdoctoral Fellowship of the Spanish Education and Culture Ministry. E-mail: xuzhang@fudan.edu.

1 Introduction

We consider the following hyperbolic equation:

$$\begin{cases} y_{tt}(t,x) - \Delta y(t,x) = \lambda(t,x)y(t,x) + \mu(t,x)f(x), & (t,x) \in Q, \\ y(t,x) = 0, & (t,x) \in \Sigma, \\ y(0,x) = 0, \quad y_t(0,x) = g(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $x = (x_1, ..., x_n)$, $Q \triangleq (0,T) \times \Omega$, $\Sigma = (0,T) \times \partial\Omega$, $T > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain which is either convex or of class $C^{1,1}$. In (1.1), both $f \in L^2(\Omega)$ and $g \in H_0^1(\Omega)$ are unknown and we fix $\lambda \in W^{1,\infty}(0,T; L^\infty(\Omega))$ and $\mu \in C^3(\overline{Q})$. The source term $\mu(t,x)f(x)$ is assumed to cause the vibration and we are required to determine $f = f(x)$ and $g = g(x)$ from the boundary observation $\frac{\partial y}{\partial \nu}|_\Sigma$ or the interior observation $y|_{(0,T)\times G}$, where $\nu = \nu(x)$ stands for the unit outward normal vector to $\partial\Omega$ at x and G is a suitable subdomain of Ω. The solution of (1.1) is denoted by $y = y(f,g)(t,x)$. As for the unique existence of the solution $y = y(f,g) \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)) \cap C^2([0,T]; L^2(\Omega))$ of (1.1), we refer to [9], [16].

The main purpose of this paper is to study the following two problems.

Uniqueness: *Does*

$$\left.\frac{\partial y(f,g)}{\partial \nu}\right|_\Sigma = 0 \quad \text{or} \quad y(f,g)\big|_{(0,T)\times G} = 0 \tag{1.2}$$

imply $f \equiv 0$ and $g \equiv 0$?

Stability: *Is the map*

$$\left.\frac{\partial y(f,g)}{\partial \nu}\right|_\Sigma \to (f,g) \quad \text{or} \quad y(f,g)\big|_{(0,T)\times G} \to (f,g) \tag{1.3}$$

continuous (in some suitable Hilbert spaces)?

Concerning the above problems, if g is known, i.e., there is only one unknown in (1.1), then one can find an extensive references ([4], [5], [8], [13], [18], [19], [20], [21] and the references cited therein), and great progress has been made there. The main tool to solve this problem is a kind of weighted energy estimate, which is usually referred as Carleman estimate.

As for the inverse problem for two unknown functions, however, to our best knowledge, there are no any results available in the literature even if $\lambda(t,x) \equiv 0$ and $\mu(t,x) \equiv \mu(t)$

except in the case when $\mu(t)$ has a special form such as $\mu(t) = C_1 e^{C_2 t}$ for some constants C_1 and C_2. The main difficulty is that a usual Carleman estimate seems not to work for this problem.

We note that for the case $\lambda(t,x) \equiv \lambda(x)$ and $\mu(t,x) \equiv C_1 e^{C_2 t}$ for some constants $C_1 \neq 0$ and C_2, if we put

$$Z = \frac{\partial}{\partial t}\Big(\frac{y}{\mu}\Big), \tag{1.4}$$

then the equation (1.1) is reduced to

$$\begin{cases} Z_{tt} - \Delta Z = [\lambda(x) - C_2^2]Z - 2C_2 Z_t & \text{in } Q, \\ Z = 0 & \text{on } \Sigma, \\ Z(0,x) = \frac{g(x)}{C_1}, \quad Z_t(0,x) = f(x) - \frac{2C_2 g(x)}{C_1} & x \in \Omega. \end{cases} \tag{1.5}$$

Therefore the global uniqueness and stability estimate for our problem follows from the known observability inequalities for the wave equation (1.5) ([17], [23]).

For the general λ and μ, when we use the transform (1.4), we find that the equation that Z satisfies is a hyperbolic equation with memory and with various lower order terms (*see* (4.5)) and our inverse problem is reduced to the observability estimate for such a sort of hyperbolic equation. We remark that, as far as we know, unique continuation property for this sort of hyperbolic equation (with large memory) is not published in the literature. This is another difficulty of our problem. Thus we need some technical conditions (*see* (2.1)). Fortunately, our technical conditions admit several interesting cases such as $\lambda(t,x) =$ constant and $\mu(t,x) \equiv \mu(t)$.

The rest of this paper is organized as follows. In Section 2, we state our main results. In order to give proofs of these results, in Section 3, we derive two observability inequalities for the hyperbolic equation with memory, which have their independent interest. The final section, Section 4, is devoted to the proof of our main results.

2 Main Results

In this paper, C denotes a generic constant which may be different from line to line but is independent of f and g. In what follows, for a set $S \subset \mathbb{R}^n$ and $\varepsilon > 0$, put

$$\mathcal{O}_\varepsilon(S) \stackrel{\triangle}{=} \{y \in \mathbb{R}^n \;\big|\; |y - x| < \varepsilon \text{ for some } x \in S\}.$$

We need the following assumptions:

$$
\begin{cases}
\lambda \in W^{1,\infty}(0,T;L^\infty(\Omega)), \quad \mu \in C^3(\overline{Q}),\\
\frac{\partial \lambda}{\partial x_{j_0}} = \frac{\partial \mu}{\partial x_{j_0}} = 0 \text{ for some } j_0 \in \{0,1,\cdots,n\},\\
\min_{(t,x)\in\overline{Q}} |\mu(t,x)| > 0,
\end{cases} \tag{2.1}
$$

where $x_0 \triangleq t$.

We recall that $y(f,g) \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T];H_0^1(\Omega))$ is the solution of (1.1).

Our main results in this paper are stated as follows.

Theorem 2.1 *Let (2.1) hold and $T > \operatorname{diam}\Omega$. Then there exists a positive constant C such that*

$$
\begin{array}{ll}
C^{-1}(|f|_{L^2(\Omega)} + |g|_{H_0^1(\Omega)}) \le \left|\frac{\partial y(f,g)}{\partial \nu}\right|_{H^1(0,T;L^2(\partial\Omega))} \le C(|f|_{L^2(\Omega)} + |g|_{H_0^1(\Omega)}),\\
\qquad \forall\, (f,g) \in L^2(\Omega)\times H_0^1(\Omega).
\end{array} \tag{2.2}
$$

Theorem 2.2 *Let (2.1) hold. Let $G = \Omega \cap \mathcal{O}_\delta(\partial\Omega)$ for some $\delta > 0$. Let $T > \sup_{x\in\Omega\setminus G} |x - x_0|$. Then there exists a positive constant C such that*

$$
\begin{array}{ll}
C^{-1}(|f|_{L^2(\Omega)} + |g|_{H_0^1(\Omega)}) \le |y(f,g)|_{H^2(0,T;L^2(G))} \le C(|f|_{L^2(\Omega)} + |g|_{H_0^1(\Omega)}),\\
\qquad \forall\, (f,g) \in L^2(\Omega)\times H_0^1(\Omega).
\end{array} \tag{2.3}
$$

The proofs of Theorems 2.1–2.2 will be given in Section 4. Now several remarks are in order.

Remark 2.1 *One can easily check that $\lambda(t,x) = \text{constant}$ and $\mu(t,x) \equiv \mu(t) \in C^3[0,T]$ with $\mu(t) \neq 0$ satisfies Assumption (2.1).*

Remark 2.2 *If $(f,g) \in H^{-1}(\Omega)\times L^2(\Omega)$, then one can obtain a similar global uniqueness and stability result.*

Remark 2.3 *Using the same method, one can consider more general hyperbolic equations (with various lower order terms).*

Remark 2.4 *The condition in the second line of (2.1) is technical. It is open how to drop this condition.*

3 Observability estimate for hyperbolic equation with memory

3.1 Statement of the results

Let us consider the following hyperbolic equation with memory:

$$\begin{cases} y_{tt} - \Delta y = \sum\limits_{|\alpha|\le 1} \Big[A_\alpha(t,x)D^\alpha y(t,x) + \int_0^t B_\alpha(t,s,x)D^\alpha y(s,x)ds\Big] & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x) & x \in \Omega. \end{cases} \tag{3.1}$$

In (3.1), $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$ is a multi-index with nonnegative integer components and $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_n$, $D^\alpha = \left(\frac{\partial}{\partial t}\right)^{\alpha_0}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.

In order to prove Theorems 2.1–2.2, we shall derive observability inequalities for (3.1) which estimate the initial energy $|(y_0, y_1)|_{H_0^1(\Omega)\times L^2(\Omega)}$ of (3.1) by means of boundary data or interior data.

If $A_\alpha = 0$ ($|\alpha| = 1$) and $B_\alpha = 0$ ($|\alpha| \le 1$), then the multiplier method yields the observability inequalities (e.g. [9], [15]). Furthermore, for the case $B_\alpha \equiv 0$ ($|\alpha| \le 1$), the corresponding observability problem for (3.1) is now well-understood ([1], [3], [12], [17], [23]); and the main tool is a Carleman estimate or microlocal analysis. However, as long as the first equation in (3.1) with integral terms is concerned, it is quite difficult to apply those tools. Therefore we will use another approach developed in [6]–[7] to establish the desired observability inequalities.

We need the following assumptions:

$$\begin{cases} A_0 \in L^\infty(Q), \quad B_0 \in L^\infty((0,T)\times Q); \\ A_\alpha \in C^1(\overline{Q}) \text{ and } B_\alpha \in C^1([0,T]\times\overline{Q}) \text{ for any } |\alpha| = 1; \\ \frac{\partial A_\alpha}{\partial x_{j_0}} = \frac{\partial B_\alpha}{\partial x_{j_0}} \equiv 0 \text{ for some } j_0 \in \{1,2,\cdots,n\} \text{ and for all } |\alpha| \le 1. \end{cases} \tag{3.2}$$

Our results on observability inequalities are the following:

Theorem 3.1 *Let (3.2) hold and $T > \operatorname{diam}\Omega$. Then there exists a positive constant C such that*

$$\begin{aligned} &C^{-1}(|y_1|_{L^2(\Omega)} + |y_0|_{H_0^1(\Omega)}) \le \left|\frac{\partial y}{\partial \nu}\right|_{L^2(\Sigma)} \le C(|y_1|_{L^2(\Omega)} + |y_0|_{H_0^1(\Omega)}), \\ &\qquad \forall\, (y_1, y_0) \in L^2(\Omega)\times H_0^1(\Omega), \end{aligned} \tag{3.3}$$

where $y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ *is the weak solution of (3.1).*

Theorem 3.2 *Let (3.2) hold. Let* $G = \Omega \cap \mathcal{O}_\delta(\partial\Omega)$ *for some* $\delta > 0$ *and* $T > \sup_{x\in\Omega\setminus G} |x - x_0|$. *Then there exists a positive constant* C *such that*

$$\begin{aligned} &C^{-1}(|y_1|_{L^2(\Omega)} + |y_0|_{H_0^1(\Omega)}) \le |y|_{H^1(0,T;L^2(G))} \le C(|y_1|_{L^2(\Omega)} + |y_0|_{H_0^1(\Omega)}), \\ &\qquad \forall\ (y_1, y_0) \in L^2(\Omega) \times H_0^1(\Omega), \end{aligned} \tag{3.4}$$

where $y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ *is the weak solution of (3.1).*

The proof of Theorems 3.1–3.2 will be given in subsections 3.3–3.4.

Remark 3.1 *As for observability inequalities (or equivalently, exact controllability) for equations with memory terms, we refer to [6], [7], [10], [14], [22]. Compared with the results in these references, our observability estimates (Theorems 3.1–3.2) are more suitable to our inverse problems and are generalization in some cases.*

3.2 Some preliminaries

In order to prove Theorems 3.1–3.2, we need some preliminaries. For any fixed

$$\begin{cases} a_0 \in L^\infty(Q), \quad b_0 \in L^\infty((0,T)\times Q); \\ a_\alpha \in C^1(\overline{Q}) \text{ and } b_\alpha \in C^1([0,T]\times\overline{Q}) \text{ for any } |\alpha| = 1, \end{cases} \tag{3.5}$$

let us consider the following hyperbolic equation with memory

$$\begin{cases} u_{tt} - \Delta u = \displaystyle\sum_{|\alpha|\le 1} \Big[a_\alpha(t,x) D^\alpha u(t,x) + \int_0^t b_\alpha(t,s,x) D^\alpha u(s,x)ds\Big] + h(t,x) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & x \in \Omega. \end{cases} \tag{3.6}$$

First of all, by means of Galerkin's method and an energy estimate, one has the following result.

Lemma 3.1 *Let (3.5) hold,* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* $h \in L^1(0,T; L^2(\Omega))$. *Then (3.6) admits a unique weak solution* $u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. *Furthermore it holds*

$$|u|_{C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))} \le C(|u_0|_{H_0^1(\Omega)} + |u_1|_{L^2(\Omega)} + |h|_{L^1(0,T;L^2(\Omega))}). \tag{3.7}$$

By [11] and Lemma 3.1, one easily sees that

Lemma 3.2 *Let (3.5) hold, $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ and $h\in L^1(0,T;L^2(\Omega))$. Then the weak solution u of (3.6) satisfies*

$$\frac{\partial u}{\partial\nu}\in L^2(\Sigma).$$

Furthermore

$$\Big|\frac{\partial u}{\partial\nu}\Big|_{L^2(\Sigma)}\le C(|u_0|_{H_0^1(\Omega)}+|u_1|_{L^2(\Omega)}+|h|_{L^1(0,T;L^2(\Omega))}). \tag{3.8}$$

Next, by the transposition method and Lemma 3.1, we obtain

Lemma 3.3 *Let (3.5) hold, $(u_0,u_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ and $h\in L^1(0,T;H^{-1}(\Omega))$. Then (3.6) admits a unique weak solution $u\in C([0,T];L^2(\Omega))\cap C^1([0,T];H^{-1}(\Omega))$. Furthermore*

$$|u|_{C([0,T];L^2(\Omega))\cap C^1([0,T];H^{-1}(\Omega))}\le C(|u_0|_{L^2(\Omega)}+|u_1|_{H^{-1}(\Omega)}+|h|_{L^1(0,T;H^{-1}(\Omega))}). \tag{3.9}$$

Moreover, we need the following result.

Lemma 3.4 *Let (3.5) hold, $(u_0,u_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ and $h=0$. Denote*

$$w=w(t,x)\overset{\triangle}{=}\int_0^t u(s,x)ds,\quad (t,x)\in Q, \tag{3.10}$$

where u is the weak solution of (3.6). Then $w\in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$. Furthermore

$$|w|_{C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))}\le C(|u_0|_{L^2(\Omega)}+|u_1|_{H^{-1}(\Omega)}). \tag{3.11}$$

Proof. Denote

$$\psi(t,x)\overset{\triangle}{=}w(t,x)+\xi(x),\quad (t,x)\in Q, \tag{3.12}$$

where $\xi\in H_0^1(\Omega)$ solves

$$\begin{cases}\Delta\xi=u_1 & \text{in }\Omega,\\ \xi=0 & \text{on }\partial\Omega.\end{cases} \tag{3.13}$$

Integrating the first equation of (3.6) from 0 to t, noting that $u=w_t=\psi_t$ and using integration by parts, we see that

$$\begin{aligned}&\psi_{tt}-\Delta\psi\\ &=\int_0^t\Big[a_0(s,x)u(s,x)+\int_0^s b_0(s,\tau,x)u(\tau,x)d\tau\Big]ds\\ &\quad+\sum_{|\alpha|=1}\int_0^t\Big[a_\alpha(s,x)D^\alpha\psi_s(s,x)+\int_0^s b_\alpha(s,\tau,x)D^\alpha\psi_\tau(\tau,x)d\tau\Big]ds\\ &=\mathcal{F}(\psi)+H,\end{aligned} \tag{3.14}$$

where

$$\mathcal{F}(\psi)=\mathcal{F}(\psi)(t,x)\triangleq \sum_{|\alpha|=1}\Big\{a_\alpha(t,x)D^\alpha\psi(t,x)-\int_0^t\Big[a_{\alpha,s}(s,x)+\int_s^t b_{\alpha,s}(\tau,s,x)d\tau \\ -b_\alpha(s,s,x)\Big]D^\alpha\psi(s,x)ds\Big\} \tag{3.15}$$

and

$$\begin{aligned} H=H(t,x)\triangleq &\int_0^t\Big[a_0(s,x)+\int_s^t b_0(\tau,s,x)d\tau\Big]u(s,x)ds\\ &-\sum_{|\alpha|=1,\alpha_0=0}\Big[a_\alpha(0,x)+\int_0^t b_\alpha(\tau,0,x)d\tau\Big]D^\alpha\xi(x)\\ &-\Big\{a_{(1,0,\cdots,0)}(0,x)+\int_0^t b_{(1,0,\cdots,0)}(\tau,0,x)d\tau\Big\}u_0(x).\end{aligned} \tag{3.16}$$

Thus one sees that ψ solves

$$\begin{cases}\psi_{tt}-\Delta\psi=\mathcal{F}(\psi)+H & \text{in } Q,\\ \psi=0 & \text{on } \Sigma,\\ \psi(0,x)=\xi(x),\quad \psi_t(0,x)=u_0(x) & x\in\Omega.\end{cases} \tag{3.17}$$

By (3.13), (3.16) and Lemma 3.3, we have

$$|H|_{C([0,T];L^2(\Omega))}\le C(|u_0|_{L^2(\Omega)}+|u_1|_{H^{-1}(\Omega)}). \tag{3.18}$$

Thus, by (3.17)–(3.18) and Lemma 3.1, we obtain

$$\begin{aligned}&|\psi|_{C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))}\\ &\le C(|\xi|_{H_0^1(\Omega)}+|u_0|_{L^2(\Omega)}+|H|_{L^1(0,T;L^2(\Omega))})\le C(|u_0|_{L^2(\Omega)}+|u_1|_{H^{-1}(\Omega)}).\end{aligned} \tag{3.19}$$

Now the desired result follows from (3.19) and (3.12) immediately. $\square$

The following result is known, which can be found, for example, in [23].

Lemma 3.5 *Let (3.5) hold, $b_\alpha=0$ for all $|\alpha|\le 1$, $h=0$ and $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$. Let $T>\operatorname{diam}\Omega$. Then there is a constant $C>0$ such that the weak solution $u\in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ of (3.6) satisfies*

$$|u_0|_{H_0^1(\Omega)}+|u_1|_{L^2(\Omega)}\le C\Big|\frac{\partial u}{\partial\nu}\Big|_{L^2(\Sigma)},\quad \forall\,(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega). \tag{3.20}$$

Finally, we show

Lemma 3.6 *Let (3.5) hold, $T>\operatorname{diam}\Omega$, $(u_0,u_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ and $h=0$. Suppose that the weak solution $u\in C([0,T];L^2(\Omega))\cap C^1([0,T];H^{-1}(\Omega))$ of (3.6) satisfies $u=0$ in $(0,T)\times G$, where $G\triangleq\Omega\cap\mathcal{O}_\delta(\partial\Omega)$ for some $\delta>0$. Then $u\in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$.*

Proof. We borrow an idea from [6]–[7]. Choose an open subset Ω_0 such that $\overline{\Omega_0} \subset \Omega$ and $\Omega \setminus \Omega_0 \subset G$, and let

$$u^\varepsilon(t,x) = (u * \rho_\varepsilon)(t,x) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} u(t, x-\eta)\rho_\varepsilon(\eta)d\eta,$$

where $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ is the Friedrichs mollifier. We take $\varepsilon > 0$ so small that $\operatorname{supp} u^\varepsilon \subset [0,T] \times \Omega_0$. It is easy to see that u^ε satisfies

$$u_{tt}^\varepsilon - \Delta u^\varepsilon = \sum_{|\alpha|\le 1} \Big[a_\alpha(t,x)D^\alpha u^\varepsilon(t,x) + \int_0^t b_\alpha(t,s,x)D^\alpha u^\varepsilon(s,x)ds\Big] + H^\varepsilon(t,x), \tag{3.21}$$

where

$$\begin{aligned} H^\varepsilon(t,x) \stackrel{\triangle}{=} & \sum_{|\alpha|\le 1} \Big[\Big((a_\alpha D^\alpha u) * \rho_\varepsilon - a_\alpha(t,x)D^\alpha u^\varepsilon(t,x)\Big) \\ & + \int_0^t \Big((b_\alpha(t,s,\cdot)D^\alpha u(s,\cdot)) * \rho_\varepsilon - b_\alpha(t,s,x)D^\alpha u^\varepsilon(s,x)\Big)ds\Big]. \end{aligned} \tag{3.22}$$

By means of the Friedrichs lemma (e.g. Vol. III (p. 9) in [2]), we have

$$|(a_0 u) * \rho_\varepsilon - a_0 u^\varepsilon|_{L^2(\Omega)} + |(b_0 u) * \rho_\varepsilon - b_0 u^\varepsilon|_{L^2(\Omega)} \le C|u|_{L^2(\Omega)} \tag{3.23}$$

and

$$\begin{aligned} & \sum_{|\alpha|=1} \Big[|(a_\alpha D^\alpha u) * \rho_\varepsilon - a_\alpha D^\alpha u^\varepsilon|_{L^2(\Omega)} + |(b_\alpha D^\alpha u) * \rho_\varepsilon - b_\alpha D^\alpha u^\varepsilon|_{L^2(\Omega)}\Big] \\ & \le C \sum_{|\alpha|=1} |D^\alpha u|_{H^{-1}(\Omega)}. \end{aligned} \tag{3.24}$$

Here and henceforth $C > 0$ denotes a generic constant which is independent of $\varepsilon > 0$.

By (3.23)–(3.24) and Lemma 3.3, for all small $\varepsilon > 0$, we obtain

$$|H^\varepsilon|_{C[0,T];L^2(\Omega))} \le C. \tag{3.25}$$

We decompose u^ε as

$$u^\varepsilon \stackrel{\triangle}{=} p^\varepsilon + q^\varepsilon, \tag{3.26}$$

where p^ε and q^ε are the solutions of

$$\begin{cases} p_{tt}^\varepsilon - \Delta p^\varepsilon = \sum\limits_{|\alpha|\le 1} a_\alpha D^\alpha p^\varepsilon + H^\varepsilon + \sum\limits_{|\alpha|\le 1} \int_0^t b_\alpha(t,s,x)D^\alpha u^\varepsilon(s,x)ds & \text{in } Q, \\ p^\varepsilon = 0 & \text{on } \Sigma, \\ p^\varepsilon(0,x) = p_t^\varepsilon(0,x) = 0 & x \in \Omega \end{cases} \tag{3.27}$$

and

$$\begin{cases} q^\varepsilon_{tt} - \Delta q^\varepsilon = \sum_{|\alpha|\le 1} a_\alpha D^\alpha q^\varepsilon & \text{in } Q, \\ q^\varepsilon = 0 & \text{on } \Sigma, \\ q^\varepsilon(0,x) = u^\varepsilon(0,x), \quad q^\varepsilon_t(0,x) = u^\varepsilon_t(0,x) & x \in \Omega \end{cases} \tag{3.28}$$

respectively.

Denote

$$\begin{aligned} R^\varepsilon &= R^\varepsilon(t,x) \triangleq \sum_{|\alpha|\le 1} \int_0^t b_\alpha(t,s,x) D^\alpha u^\varepsilon(s,x)ds, \\ w^\varepsilon &= w^\varepsilon(t,x) \triangleq \int_0^t u^\varepsilon(s,x)ds. \end{aligned} \tag{3.29}$$

Then

$$\begin{aligned} R^\varepsilon &= \int_0^t b_0(t,s,x)u^\varepsilon(s,x)ds + \sum_{|\alpha|=1} \int_0^t b_\alpha(t,s,x) D^\alpha w^\varepsilon_s(s,x)ds \\ &= \int_0^t b_0(t,s,x)u^\varepsilon(s,x)ds + \sum_{|\alpha|=1} b_\alpha(t,t,x) D^\alpha w^\varepsilon(t,x) \\ &\quad -b_{(1,0,\cdots,0)}(t,0,x)u^\varepsilon(0,x) - \sum_{|\alpha|=1} \int_0^t b_{\alpha,s}(t,s,x) D^\alpha w^\varepsilon(s,x)ds. \end{aligned} \tag{3.30}$$

Now, by (3.21), (3.25), (3.29)–(3.30) and Lemma 3.4, we have

$$|H^\varepsilon + R^\varepsilon|_{L^1(0,T;L^2(\Omega))} \le C. \tag{3.31}$$

Thus, applying Lemmas 3.1–3.2 to (3.27), for all small $\varepsilon > 0$, we obtain

$$|p^\varepsilon|_{C([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))} + \Big|\frac{\partial p^\varepsilon}{\partial \nu}\Big|_{L^2(\Sigma)} \le C. \tag{3.32}$$

On the other hand, applying Lemma 3.5 to (3.28), noting that $p^\varepsilon = -q^\varepsilon$ in $(0,T)\times(\Omega \setminus \overline{\Omega_0})$, and using (3.32), we conclude that

$$|q^\varepsilon|_{C([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))} \le C\Big|\frac{\partial q^\varepsilon}{\partial \nu}\Big|_{L^2(\Sigma)} = C\Big|\frac{\partial p^\varepsilon}{\partial \nu}\Big|_{L^2(\Sigma)} \le C \tag{3.33}$$

for all small $\varepsilon > 0$. Hence it follows from (3.26) and (3.32)– (3.33) that for any small $\varepsilon > 0$ one has

$$|u^\varepsilon|_{C([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))} \le C, \tag{3.34}$$

which yields the desired result. $\square$

3.3 Proof of Theorem 3.1

Step 1. Let us decompose the solution y of (3.1) as

$$y = p + q, \tag{3.35}$$

where p and q are the solutions of

$$\begin{cases} p_{tt} - \Delta p = \sum_{|\alpha|\le 1} \Big[A_\alpha(t,x) D^\alpha p(t,x) + \int_0^t B_\alpha(t,s,x) D^\alpha y(s,x) ds \Big] & \text{in } Q \\ p = 0 & \text{on } \Sigma \\ p(0,x) = p_t(0,x) = 0 & x \in \Omega \end{cases} \tag{3.36}$$

and

$$\begin{cases} q_{tt} - \Delta q = \sum_{|\alpha|\le 1} A_\alpha(t,x) D^\alpha q(t,x) & \text{in } Q \\ q = 0 & \text{on } \Sigma \\ q(0,x) = y_0(x), \quad q_t(0,x) = y_1(x) & x \in \Omega \end{cases} \tag{3.37}$$

respectively.

By (3.37) and Lemma 3.5, one has

$$|y_0|_{H_0^1(\Omega)} + |y_1|_{L^2(\Omega)} \le C \Big| \frac{\partial q}{\partial \nu} \Big|_{L^2(\Sigma)}. \tag{3.38}$$

Thus, by (3.35), we obtain

$$|y_0|_{H_0^1(\Omega)} + |y_1|_{L^2(\Omega)} \le C \Big[\Big| \frac{\partial y}{\partial \nu} \Big|_{L^2(\Sigma)} + \Big| \frac{\partial p}{\partial \nu} \Big|_{L^2(\Sigma)} \Big]. \tag{3.39}$$

However, by (3.36) and Lemma 3.2, we have

$$\Big| \frac{\partial p}{\partial \nu} \Big|_{L^2(\Sigma)} \le C |\mathcal{H}|_{L^1(0,T;L^2(\Omega))}, \tag{3.40}$$

where

$$\mathcal{H} = \mathcal{H}(t,x) \triangleq \sum_{|\alpha|\le 1} \int_0^t B_\alpha(t,s,x) D^\alpha y(s,x) ds. \tag{3.41}$$

Denote

$$w = w(t,x) \triangleq \int_0^t y(s,x) ds. \tag{3.42}$$

Then

$$\begin{aligned}\mathcal{H} &= \int_0^t B_0(t,s,x)y(s,x)ds + \sum_{|\alpha|=1}\int_0^t B_\alpha(t,s,x)D^\alpha w_s(s,x)ds\\ &= \int_0^t B_0(t,s,x)y(s,x)ds + \sum_{|\alpha|=1}\Big[B_\alpha(t,t,x)D^\alpha w(t,x)\\ &\quad -B_{(1,0,\cdots,0)}(t,0,x)y_0(x) - \int_0^t B_{\alpha,s}(t,s,x)D^\alpha w(s,x)ds.\end{aligned} \tag{3.43}$$

Thus, by (3.43) and Lemma 3.4, one sees that

$$|\mathcal{H}|_{L^1(0,T;L^2(\Omega))} \le C(|y_0|_{L^2(\Omega)} + |y_1|_{H^{-1}(\Omega)}). \tag{3.44}$$

Now, combining (3.39)–(3.40) and (3.44), we obtain

$$|y_0|_{H_0^1(\Omega)} + |y_1|_{L^2(\Omega)} \le C\Big[\Big|\frac{\partial y}{\partial \nu}\Big|_{L^2(\Sigma)} + |y_0|_{L^2(\Omega)} + |y_1|_{H^{-1}(\Omega)}\Big]. \tag{3.45}$$

Step 2. Let us prove that

$$|y_0|_{L^2(\Omega)} + |y_1|_{H^{-1}(\Omega)} \le C\Big|\frac{\partial y}{\partial \nu}\Big|_{L^2(\Sigma)}. \tag{3.46}$$

We will use the compactness/uniqueness argument. Assume that (3.46) is false. Then, by (3.45), there is a sequence $\{y_0^m, y_1^m\}_{m=1}^\infty \subset H_0^1(\Omega)\times L^2(\Omega)$ such that

$$|y_0^m|_{L^2(\Omega)} + |y_1^m|_{H^{-1}(\Omega)} = 1 \qquad \text{for all } m, \tag{3.47}$$

$$\Big|\frac{\partial y^m}{\partial \nu}\Big|_{L^2(\Sigma)} \to 0 \qquad \text{as } m\to\infty, \tag{3.48}$$

$$(y_0^m, y_1^m) \to (y_0^\infty, y_1^\infty) \text{ weakly in } H_0^1(\Omega)\times L^2(\Omega) \qquad \text{as } m\to\infty, \tag{3.49}$$

$$(y^m, y_t^m) \to (y^\infty, y_t^\infty) \text{ weakly}^* \text{ in } L^\infty(0,T;H_0^1(\Omega)\times L^2(\Omega)) \qquad \text{as } m\to\infty, \tag{3.50}$$

where y^m is the weak solution of (3.1) with initial data (y_0^m, y_1^m), and y^∞ is the weak solution of (3.1) with initial data (y_0^∞, y_1^∞). By (3.47)–(3.50) and (3.45), we obtain

$$\begin{cases} \frac{\partial y^\infty}{\partial \nu} = 0,\\ |y_0^\infty|_{L^2(\Omega)} + |y_1^\infty|_{H^{-1}(\Omega)} = 1. \end{cases} \tag{3.51}$$

Define

$$\mathcal{G} \triangleq \Big\{y \in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))\Big|\ \begin{array}{l} y \text{ is the weak solution of (3.1)}\\ \text{and } \frac{\partial y}{\partial \nu} = 0 \text{ on } \Sigma\Big\}.\end{array} \tag{3.52}$$

It is easy to see that $\mathcal{G}$ is a Banach space equipped with the norm of $C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$. By (3.45), one sees easily that $\mathcal{G}$ is finite dimensional. Let us adopt an argument in [1] to show that $\mathcal{G}=\{0\}$. By (3.2) and Lemma 3.6, we see that $\frac{\partial}{\partial x_{j_0}}$ is a linear operator from $\mathcal{G}$ into $\mathcal{G}$. Choose any element v of $\mathcal{G}$. Then $\left(\frac{\partial}{\partial x_{j_0}}\right)^k v \in \mathcal{G}$ for any $k \in \mathbb{N}$. Since $\mathcal{G}$ is finite dimensional, there is an integer $N \geq 1$ such that

$$\Big(\frac{\partial}{\partial x_{j_0}}\Big)^N v + c_1\Big(\frac{\partial}{\partial x_{j_0}}\Big)^{N-1} v + \cdots + c_N v = 0 \quad \text{ in } Q \tag{3.53}$$

for some constants $c_1, \cdots, c_N$. Noting that $\frac{\partial v}{\partial \nu} = 0$ on Σ, similarly to [7], one sees that $v = 0$ in Q. Consequently we obtain

$$\mathcal{G} = \{0\}. \tag{3.54}$$

However (3.51) contradicts (3.54). Thus the proof of (3.46) is complete. Combining (3.45) and (3.46), we conclude the desired estimates. □

From the proof of Theorem 3.1, it is easy to see that we have actually proved the following new unique continuation property for a class of hyperbolic equations with memory.

Theorem 3.3 *Assume that (3.2)holds, $T > \operatorname{diam}\Omega$ and $(y_0, y_1) \in L^2(\Omega)\times H^{-1}(\Omega)$. Suppose that the weak solution $y \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$ of (3.1) satisfies $y = 0$ in $(0,T)\times G$, where $G \stackrel{\triangle}{=} \Omega \cap \mathcal{O}_\delta(\partial\Omega)$ for some $\delta > 0$. Then $y \equiv 0$ in Q.*

3.4 Proof of Theorem 3.2

We need the following observability inequality for a class of hyperbolic equations without memory, which can be proved by Carleman estimates (e.g., [17]).

Lemma 3.7 *Let (3.5) hold, $b_\alpha = 0$ for all $|\alpha| \leq 1$, $h = 0$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Let $G = \Omega \cap \mathcal{O}_\delta(\partial\Omega)$ for some $\delta > 0$ and $T > \sup_{x\in\Omega\setminus G}|x - x_0|$. Then there is a constant $C > 0$ such that the weak solution $u \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ of (3.6) satisfies*

$$|u_0|_{H_0^1(\Omega)} + |u_1|_{L^2(\Omega)} \leq C|u|_{H^1(0,T;L^2(G))}, \quad \forall\, (u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega). \tag{3.55}$$

Now, by means of Lemma 3.7, similarly to Theorem 3.1, one can prove Theorem 3.2.

□

4 Proof of the main results

Proof of Theorem 2.1. Denote

$$z = z(t,x) \stackrel{\triangle}{=} \frac{y(t,x)}{\mu(t,x)}. \tag{4.1}$$

Then, by (1.1), we see that z solves

$$\begin{cases} z_{tt} - \Delta z = \ell_1 z + \ell_2 z_t + \langle\, \ell_3, \nabla z \,\rangle + f & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0,x) = 0, \quad z_t(0,x) = \frac{g(x)}{\mu(0,x)} & x \in \Omega, \end{cases} \tag{4.2}$$

where $\langle\, \cdot, \cdot \,\rangle$ denotes the scalar product in $\mathbb{R}^n$, and

$$\begin{cases} \ell_1 = \ell_1(t,x) \stackrel{\triangle}{=} \lambda(t,x) + \frac{\Delta\mu(t,x) - \mu_{tt}(t,x)}{\mu(t,x)}, \\ \ell_2 = \ell_2(t,x) \stackrel{\triangle}{=} -\frac{2\mu_t(t,x)}{\mu(t,x)}, \\ \ell_3 = \ell_3(t,x) \stackrel{\triangle}{=} \frac{2\nabla\mu(t,x)}{\mu(t,x)}. \end{cases} \tag{4.3}$$

Now denote

$$Z = Z(t,x) \stackrel{\triangle}{=} z_t(t,x). \tag{4.4}$$

Then, by (4.2), it is easy to see that Z solves

$$\begin{cases} Z_{tt} - \Delta Z = \mathcal{Q}(Z) & \text{in } Q, \\ Z = 0 & \text{on } \Sigma, \\ Z(0,x) = Z_0(x) \stackrel{\triangle}{=} \frac{g(x)}{\mu(0,x)}, \quad Z_t(0,x) = Z_1(x) \stackrel{\triangle}{=} f(x) + \frac{\ell_2(0,x)}{\mu(0,x)} g(x) & x \in \Omega, \end{cases} \tag{4.5}$$

where

$$\mathcal{Q}(Z) \stackrel{\triangle}{=} (\ell_1 + \ell_{2,t})Z + \ell_2 Z_t + \langle\, \ell_3, \nabla Z \,\rangle + \ell_{1,t} \int_0^t Z(s,x)ds + \langle\, \ell_{3,t}, \nabla \int_0^t Z(s,x)ds \,\rangle .$$

However, applying Theorem 3.1 to (4.5), we obtain

$$\begin{aligned} & C^{-1}(|Z_1|_{L^2(\Omega)} + |Z_0|_{H_0^1(\Omega)}) \le \left|\frac{\partial Z}{\partial \nu}\right|_{L^2(\Sigma)} \le C(|Z_1|_{L^2(\Omega)} + |Z_0|_{H_0^1(\Omega)}), \\ & \forall\ (Z_1, Z_0) \in L^2(\Omega) \times H_0^1(\Omega), \end{aligned} \tag{4.6}$$

which implies the desired result immediately. □

Proof of Theorem 2.2. By means of Theorem 3.2, similarly to Theorem 2.1, one can prove Theorem 2.2. We omit the details. □

References

[1] C. Bardos, G. Lebeau & J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Control & Optim.*, **30** (1992), 1024–1065.

[2] L. Hörmander, *The Analysis of Linear Partial Differential Operators (I–IV)*, Springer-Verlag, Berlin, 1983–1985.

[3] O. Yu. Imanuvilov, *On Carleman estimates for hyperbolic equations,* Report Number 00-06, Iowa State University, Department of Mathematics, 2000.

[4] V. Isakov, *Inverse Soure Problems,* Amer. Math. Soc., Providence, Rhode Island, 1990.

[5] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin, 1998.

[6] J. Kim, *Control of a plate equation with large memory, Differential and Integral Equations,* **5** (1992), 261–279.

[7] J. Kim, *Control of a second-order integro-differential equation, SIAM J. Control & Optim.*, **31** (1993), 101–110.

[8] M. V. Klibanov, *Carleman estimates and inverse problems in the last two decades,* in *Surveys on Solutions Methods for Inverse Problems* (2000), Springer-Verlag, Wien, 119–146.

[9] V. Komornik, *Exact Controllability and Stabilization, the Multiplier Method*, Masson, Paris, 1994.

[10] I. Lasiecka, *Controllability of a viscoelastic Kirchhoff plate,* in *Control and Estimation of Distributed Parameter Systems,* Internat. Ser. Numer. Math. **91**, Birkhäuser-Verlag, Basel, 1989, 237–247.

[11] I. Lasiecka, J. L. Lions & R. Triggiani, *Nonhomogeneous boundary value problems for second order hyperbolic operators, J. Math. Pures Appl.*, **69** (1986), 149–192.

[12] I. Lasiecka, R. Triggiani & X. Zhang, *Nonconservative wave equations with purely Neumann B.C.: global uniqueness and observability in one shot,* in *Differential Geometric Methods in the Control of Partial Differential Equations, Contemp. Math. (Amer. Math. Soc.),* **268** (2000), 227–325.

[13] M. M. Lavrentév, V. G. Romanov & S. P. Shishataskii, *Ill-Posed Problems of Mathematics Physics and Analysis*, American Mathematical Society, Vol. **64**, (1986).

[14] G. Leugering, *Exact boundary controllability of an integrodifferential equation, Appl. Math. Optim.,* **15** (1987), 223–250.

[15] J. L. Lions, *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués.* Tomes 1 & 2, Masson, Paris, 1988.

[16] J. L. Lions & E. Magenes, *Non-homogeneous Boundary Value Problems and Applications (Vol. I)*, Springer-Verlag, New York, 1972.

[17] K. Liu, M. Yamamoto & X. Zhang, *Observability inequality by internal observation and its applications*, preprint.

[18] J.-P. Puel & M. Yamamoto, *Om a global estimate in a linear inverse hyperbolic problem, Inverse Problems,* **12** (1996), 995–1002.

[19] M. Yamamoto, *Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method, Inverse Problems,* **11** (1995), 481–496.

[20] M. Yamamoto, *On ill-posedness and a Tikhonov regularization for a multidimentional inverse hyperbolic problem, J. Math. Kyoto Univ.,* **36** (1996), 825–856.

[21] M. Yamamoto, *Uniqueness and stability in multidimentional hyperbolic inverse problems, J. Math. Pures Appl.,* **78** (1999), 65–98.

[22] J. Yan, *Contrôlabilité exacte pour des systèmes à mémoire, Rev. Mat. Univ. Complut. Madrid,* **5** (1992), 319–327.

[23] X. Zhang, *Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities, SIAM J. Control & Optim,* **39** (2000), 812–834.

UTMS

2001–7 Shigeo Kusuoka: *Nonlinear transformation containing rotation and Gaussian measure.*

2001–8 Kazuya Kato and Takeshi Saito: *Conductor formula of Bloch.*

2001–9 Hiroshige Kajiura, Yutaka Matsuo and Tadashi Takayanagi: *Exact Tachyon condensation on noncommutative torus.*

2001–10 Mikio Furuta and Yukio Kametani: *The Seiberg-Witten equations and equivariant e-invariants.*

2001–11 Igor Trooshin and Masahiro Yamamoto: *Spectral properties of non-symmetric systems of ordinary differential operators.*

2001–12 Hiroshige Kajiura: *T-duality group for open string theory.*

2001–13 Nariya Kawazumi and Shigeyuki Morita: *The primary approximation to the cohomology of the moduli space of curves and cocycles for the Mumford-Morita-Miller classes.*

2001–14 Xu Bin: *The degree of symmetry of compact smooth manifolds.*

2001–15 K. Habiro, S. Kamada, Y. Matsumoto and K. Yoshikawa: *Algebraic formulae for the q-inverse in a free group.*

2001–16 Maurizio Grasselli, Masaru Ikehata and Masahiro Yamamoto: *Direct and inverse inequalities for the isotropic Lamé system with variable coefficients and applications to an inverse source problem.*

2001–17 Kangsheng Liu, Masahiro Yamamoto and Xu Zhang: *Observability inequalities by internal observation and its applications.*

2001–18 Masahiro Yamamoto and Xu Zhang: *Global uniqueness and stability for a class of multidimesional inverse hyperbolic problems with two unknowns.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001   FAX +81-3-5465-7012