THE PRIMARY APPROXIMATION TO THE COHOMOLOGY OF THE MODULI SPACE OF CURVES AND COCYCLES FOR THE MUMFORD-MORITA-MILLER CLASSES

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ABSTRACT. We prove that the image of the natural homomorphism from the Sp-invariant part of the cohomology of the abelianization of the Torelli group, which is expressed as a quotient of the polynomial algebra generated by connected trivalent graphs, to the cohomology of the moduli space of smooth projective curves coincides exactly with the tautological algebra generated by the Mumford-Morita-Miller classes. Furthermore, we give an explicit algorithm to determine the cohomology class corresponding to any given trivalent graph. This is based on some contraction formula concerning the cohomology of the mapping class group with symplectic coefficients together with a simple formula relating the IH moves of trivalent graphs to certain operation in the tautological algebra.

Proofs are given twofold. The first proof is given in the framework of group cohomology with twisted coefficients while the second one is given in the context of symplectic representation theory. Thereby we show an exact correspondence between the two different approaches to the tautological algebra.

CONTENTS

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1. Statement of the main results

Let $\mathbf{M}_g$ be the moduli space of smooth projective curves of genus g and let $\mathcal{M}_g$ be the mapping class group of a closed oriented surface Σ_g of genus g. As is well known, there exists a close connection between these two objects. More precisely, $\mathcal{M}_g$ acts on the Teichmüller space $\mathcal{T}_g$ of genus g properly discontinuously and the quotient space $\mathcal{T}_g/\mathcal{M}_g$ can be naturally identified with $\mathbf{M}_g$. As was essentially established by Teichmüller, the space $\mathcal{T}_g$ is contractible (in fact homeomorphic to $\mathbb{R}^{6g-6}$ for $g \geq 2$) so that we have a canonical isomorphism

$$H^*(\mathbf{M}_g;\mathbb{Q}) \cong H^*(\mathcal{M}_g;\mathbb{Q}).$$

Let $\pi : \mathbf{C}_g \to \mathbf{M}_g$ be the universal family of curves over the moduli space. The corresponding orbifold fundamental groups are given by the group extension $\pi_1\Sigma_g \to \mathcal{M}_{g,*} \to \mathcal{M}_g$ where $\mathcal{M}_{g,*}$ denotes the mapping class group of Σ_g relative to a base point. We have also a canonical isomorphism $H^*(\mathbf{C}_g;\mathbb{Q}) \cong H^*(\mathcal{M}_{g,*};\mathbb{Q})$. Let $e \in H^2(\mathcal{M}_{g,*};\mathbb{Q})$ be the universal Euler class of the tangent bundle along the fiber of the universal Σ_g-bundle and let $e_i \in H^{2i}(\mathcal{M}_g;\mathbb{Q})$ be the i-th Mumford-Morita-Miller classes (see [58][44][43]). Actually Mumford defined these classes at the level of the rational Chow algebras of the moduli spaces. Namely he defined classes $K_{\mathbf{C}_g/\mathbf{M}_g} \in \mathcal{A}^1(\mathbf{C}_g)$, $\kappa_i \in \mathcal{A}^i(\mathbf{M}_g)$ and called them tautological classes. The cohomology classes e, e_i are nothing but their images in the rational cohomology (up to signs). The tautological algebras $\mathcal{R}^*(\mathbf{C}_g)$ and $\mathcal{R}^*(\mathbf{M}_g)$ of the moduli spaces are defined to be the subalgebras of the rational Chow algebras generated by the above tautological classes (see [41][7][18][25]). Similarly we define the tautological algebras $\mathcal{R}^*(\mathcal{M}_{g,*})$ and $\mathcal{R}^*(\mathcal{M}_g)$ of the mapping class groups to be the subalgebras of $H^*(\mathcal{M}_{g,*};\mathbb{Q})$ and $H^*(\mathcal{M}_g;\mathbb{Q})$ generated by the classes $e, e_1, e_2, \cdots$. These are simply the projected images of the original tautological algebras in the rational cohomology.

Let H denote the first integral homology $H_1(\Sigma_g;\mathbb{Z})$ of Σ_g and let $H_\mathbb{Q} = H \otimes \mathbb{Q}$. The intersection number induces a non-degenerate skew symmetric bilinear form on $H_\mathbb{Q}$ and the automorphism group of $H_\mathbb{Q}$ preserving this form is the symplectic group $Sp(H_\mathbb{Q})$. Let $\Lambda^3 H_\mathbb{Q}$ denote the third exterior power of $H_\mathbb{Q}$. Then $H_\mathbb{Q}$ can be considered as a natural submodule of $\Lambda^3 H_\mathbb{Q}$ by the embedding $H_\mathbb{Q} \ni u \mapsto u \wedge \omega_0 \in \Lambda^3 H_\mathbb{Q}$ where $\omega_0 \in \Lambda^2 H_\mathbb{Q}$ is the symplectic class. We denote the quotient $\Lambda^3 H_\mathbb{Q}/H_\mathbb{Q}$ simply by $U_\mathbb{Q}$. $U_\mathbb{Q}$ is an irreducible representation of the algebraic group $Sp(H_\mathbb{Q})$ corresponding to the Young diagram $[1^3]$ (see [16][55][56] for details of symplectic representation theory related to the mapping class group and [8] for generalities).

Extending earlier results in [52][53], the second author constructed in [56] a morphism

$$\begin{array}{ccc}
\pi_1\Sigma_g & \longrightarrow & [1^2]\widetilde{\times} H_\mathbb{Q} \\
\downarrow & & \downarrow \\
\mathcal{M}_{g,*} & \xrightarrow{\rho_2} & \Big(\big([1^2]\oplus[2^2]\big)\widetilde{\times}_{\text{torelli}}\Lambda^3 H_\mathbb{Q}\Big) \rtimes Sp(H_\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathcal{M}_g & \xrightarrow[\rho_2]{} & \big([2^2]\widetilde{\times} U_\mathbb{Q}\big) \rtimes Sp(H_\mathbb{Q})
\end{array} \tag{1}$$

of group extensions where we use the symbol $\widetilde{\times}$ to indicate central extensions. The top horizontal homomorphism in the above diagram is the second term in the Malcev completion of $\pi_1\Sigma_g$ and the targets of the other two homomorphisms ρ_2 are semi-direct products of $Sp(H_{\mathbb{Q}})$ with two step nilpotent groups whose extension classes are given by certain Sp-submodules $[2^2] \subset H^2(U_{\mathbb{Q}}) = \Lambda^2 U_{\mathbb{Q}}^*$ and $[1^2]^{\mathrm{torelli}} \oplus [2^2] \subset H^2(\Lambda^3 H_{\mathbb{Q}}) = \Lambda^2(\Lambda^3 H_{\mathbb{Q}}^*)$ (see the above cited papers for details). By a general property of cohomology of semi-direct products (see §3), the diagram (1) induces the following commutative diagram

$$\begin{array}{ccc} \left(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*)\right)^{Sp} & \xrightarrow{\rho_2^*} & H^*(\mathcal{M}_{g,*};\mathbb{Q}) \\ \uparrow & & \uparrow \\ \left(\Lambda^* U_{\mathbb{Q}}^*\right)^{Sp} & \xrightarrow[\rho_2^*]{} & H^*(\mathcal{M}_g;\mathbb{Q}), \end{array} \tag{2}$$

where the superscript Sp means the Sp-invariant part of the corresponding $Sp(H_{\mathbb{Q}})$-module. Now let $\left([1^2]^{\mathrm{torelli}} \oplus [2^2]\right)$ and $([2^2])$ be the ideals of $\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*)$ and $\Lambda^* U_{\mathbb{Q}}^*$ generated by $[1^2]^{\mathrm{torelli}} \oplus [2^2]$ and $[2^2]$ respectively. Then it can be shown that the images under the homomorphisms ρ_2^* in (2) of both subspaces

$$\left([1^2]^{\mathrm{torelli}} \oplus [2^2]\right)^{Sp} \subset \left(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*)\right)^{Sp}$$
$$([2^2])^{Sp} \subset \left(\Lambda^* U_{\mathbb{Q}}^*\right)^{Sp}$$

are trivial (see Proposition 3.2). Since there exist canonical isomorphisms

$$\left(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*)\right)^{Sp} / \left([1^2]^{\mathrm{torelli}} \oplus [2^2]\right)^{Sp} \cong \Big(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*) / \left([1^2]^{\mathrm{torelli}} \oplus [2^2]\right)\Big)^{Sp}$$
$$\left(\Lambda^* U_{\mathbb{Q}}^*\right)^{Sp} / ([2^2])^{Sp} \cong \left(\Lambda^* U_{\mathbb{Q}}^* / ([2^2])\right)^{Sp},$$

we obtain the following commutative diagram

$$\begin{array}{ccc} \Big(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*) / \left([1^2]^{\mathrm{torelli}} \oplus [2^2]\right)\Big)^{Sp} & \xrightarrow{\rho_2^*} & H^*(\mathcal{M}_{g,*};\mathbb{Q}) \\ \uparrow & & \uparrow \\ \left(\Lambda^* U_{\mathbb{Q}}^* / ([2^2])\right)^{Sp} & \xrightarrow[\rho_2^*]{} & H^*(\mathcal{M}_g;\mathbb{Q}). \end{array} \tag{3}$$

Now our first main result is the following.

Theorem 1.1. *The images of the homomorphisms ρ_2^* in (3) are exactly the tautological algebras $\mathcal{R}^*(\mathcal{M}_{g,*})$ and $\mathcal{R}^*(\mathcal{M}_g)$ so that we have the following commutative diagram*

$$\begin{array}{ccc} \Big(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}^*) / \left([1^2]^{torelli} \oplus [2^2]\right)\Big)^{Sp} & \xrightarrow{\rho_2^*} & \mathcal{R}^*(\mathcal{M}_{g,*}) \\ \uparrow & & \uparrow \\ \left(\Lambda^* U_{\mathbb{Q}}^* / ([2^2])\right)^{Sp} & \xrightarrow[\rho_2^*]{} & \mathcal{R}^*(\mathcal{M}_g). \end{array} \tag{4}$$

Moreover, in the stable range (namely in degrees $\leq \frac{2}{3}g$), both homomorphisms ρ_2^* are isomorphisms.

The spaces of Sp-invariants appearing in (2) can be described explicitly by applying Weyl's classical representaion theory. Let $\mathcal{G}_{2k}$ denote the set of all the isomorphism classes of *connected* trivalent graphs with $2k$ vertices and let $\mathcal{G}^0_{2k}$ be the subset of $\mathcal{G}_{2k}$ consisting of those trivalent graphs *without loops* where a loop means an edge both of whose endpoints are the same vertex. We set $\mathcal{G}$ (resp. $\mathcal{G}^0$) to be the disjoint union of $\mathcal{G}_{2k}$ (resp. $\mathcal{G}^0_{2k}$) for all $k \geq 1$. Then we can construct canonical surjections from the polynomial algebras generated by trivalent graphs to the Sp-invariant subspaces making the following diagram commutative.

$$
\begin{array}{ccc}
\mathbb{Q}[\Gamma;\Gamma\in\mathcal{G}] & \xrightarrow{\Phi_\alpha} & \left(\Lambda^*(\Lambda^3 H^*_{\mathbb{Q}})\right)^{Sp} \\
\uparrow & & \uparrow \\
\mathbb{Q}[\Gamma;\Gamma\in\mathcal{G}^0] & \xrightarrow[\Phi_\beta]{} & \left(\Lambda^* U^*_{\mathbb{Q}}\right)^{Sp}
\end{array}
\tag{5}
$$

(see [53] and §2, §3 below for details). By combining (2)(4)(5), we obtain our second main theorem.

Theorem 1.2. *The homomorphisms ρ_2^* in* (4)*, where we let the genus g run over all values ≥ 2, can be realized at the cocycle level by the following commutative diagram*

$$
\begin{array}{ccc}
\mathbb{Q}[\Gamma;\Gamma\in\mathcal{G}] & \xrightarrow{\Phi_\alpha} & \mathbb{Q}[g][e,e_1,e_2,\cdots] \\
\uparrow & & \uparrow \\
\mathbb{Q}[\Gamma;\Gamma\in\mathcal{G}^0] & \xrightarrow[\Phi_\beta]{} & \mathbb{Q}[g][e_1,e_2,\cdots].
\end{array}
\tag{6}
$$

Here the targets express polynomial algebras generated by the classes $e, e_1, e_2, \cdots$ with coefficients in the polynomial algebra $\mathbb{Q}[g]$ on the genus g. Moreover, for any trivalent graph Γ, there is an explicit recursive algorithm to determine the cohomology classes $\alpha_\Gamma \in \mathbb{Q}[g][e,e_1,e_2,\cdots]$ and $\beta_\Gamma \in \mathbb{Q}[g][e_1,e_2,\cdots]$.

Here and henceforth, for each connected trivalent graph Γ we denote by α_Γ and β_Γ the images of Γ by the maps Φ_α and Φ_β respectively. (However, in §2 and §3, we will use the same letters $\alpha_\Gamma, \beta_\Gamma$ also for the corresponding group cocycles). For the disjoint union $\Gamma \amalg \Gamma'$ of any two trivalent graphs Γ and Γ', we set

$$\alpha_{\Gamma\amalg\Gamma'} = \alpha_\Gamma \alpha_{\Gamma'}, \quad \beta_{\Gamma\amalg\Gamma'} = \beta_\Gamma\beta_{\Gamma'}$$

(cf. Lemma 2.1). It should be mentioned that the cohomology classes α_Γ and β_Γ do depend on the genus g. In fact, their coefficients are non-constant elements of $\mathbb{Q}[g]$. However, if we expand the coefficients in "$(-2g)$-adic" way, then the resultant coefficients turn out to be independent of the genus and have certain geometrical meaning in the context of trivalent graphs (see §8 for details). These results are based on the following explicit formulas.

Theorem 1.3. (i) *For any connected trivalent graph Γ with $2k$ vertices, we have*

$$\alpha_\Gamma = (-1)^k e_k + e(\text{lower terms in } \ e, e_1, e_2, \cdots).$$

(ii) *Let $\Gamma(k)$ be the connected trivalent graph with $2k$ vertices depicted in Figure 1.1. Then we have*

$$\sum\nolimits_{k=1}^\infty t^k \alpha_{\Gamma(k)} = e_0 - \left(\left(\frac{1+2te}{1+3te}\right)^2 + \frac{te}{1+3te} e_* \left(\frac{-t}{1+2te}\right)\right)^{-1} e_* \left(\frac{-t}{1+2te}\right),$$

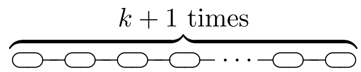

FIGURE 1.1. Trivalent graphs $\Gamma(k)$

where $e_0 = 2 - 2g$ *and*

$$e_*\left(\frac{-t}{1+2te}\right) = \sum\nolimits_{k=0}^{\infty} e_k \left(\frac{-1}{1+2te}\right)^k.$$

(iii) *Let Γ_1, Γ_2 be two trivalent graphs with $2k$ vertices and suppose that Γ_2 can be obtained from Γ_1 by applying an IH (or equivalently a Whitehead) move. Namely there exists a subgraph shaped like the letter I in Γ_1 such that Γ_2 is obtained by replacing the subgraph $I \subset \Gamma_1$ by a subgraph shaped like the letter H. Let τ_1 and τ_2 be the corresponding edges of Γ_1 and Γ_2 respectively. Then we have*

$$\alpha_{\Gamma_1} - \alpha_{\Gamma_2} = e(\alpha_{\Gamma_2 \setminus \tau_2} - \alpha_{\Gamma_1 \setminus \tau_1})$$

where $\Gamma_i \setminus \tau_i$ $(i = 1, 2)$ denotes the trivalent graph with $(2k-2)$ vertices obtained from Γ_i by removing the edge τ_i. Here if a disjoint circle, denoted by Γ_0, appears in the new graphs, then we set $\alpha_{\Gamma_0} = -2g$.

We can see the effect of the formula (iii) in Theorem 1.3 above in the following simple example.

Example 1.4. Let Γ_1 be a trivalent graph with two vertices which has two loops and let Γ_2 be a trivalent graph with two vertices without loop, namely a *theta* graph. Then it was proved in [47][50] that

$$\alpha_{\Gamma_1} = -e_1 - 4g(g-1)e, \quad \alpha_{\Gamma_2} = -e_1 + 6ge$$

(see also [19] for a proof in the context of algebraic geometry). Now there is an embedding $I \subset \Gamma_1$ such that Γ_2 is obtained from Γ_1 by replacing I by H. Then $\Gamma_2 \setminus \tau_2$ is a circle while $\Gamma_1 \setminus \tau_1$ is the disjoint union of two circles. On the other hand, we have

$$\alpha_{\Gamma_1} - \alpha_{\Gamma_2} = e\big(-2g - (-2g)^2\big) = e(\alpha_{\Gamma_2 \setminus \tau_2} - \alpha_{\Gamma_1 \setminus \tau_1})$$

which checks our formula in this case.

Main part of the results of §2 – §7 and §13 of the present paper has been announced in [35] while those of §9 – §12 appear here for the first time.

2. DESCRIPTIONS OF $\big(\Lambda^*(\Lambda^3 H^*_{\mathbb{Q}})\big)^{Sp}$ AND $(\Lambda^* U^*_{\mathbb{Q}})^{Sp}$ IN TERMS OF GRAPHS

Let $\mathcal{I}_{g,*}$ and $\mathcal{I}_g$ be the Torelli groups corresponding to $\mathcal{M}_{g,*}$ and $\mathcal{M}_g$ respectively. Namely they are subgroups consisting of elements which act on H trivially. Then it is one of the fundamental results of Johnson that

$$H_1(\mathcal{I}_{g,*}; \mathbb{Q}) \cong \Lambda^3 H_{\mathbb{Q}}, \quad H_1(\mathcal{I}_g; \mathbb{Q}) \cong U_{\mathbb{Q}}$$

for any $g \geq 3$ (see [28][30]). In this section, we describe the $Sp(2g, \mathbb{Q})$-invariant part of the rational cohomology algebras $H^*(\Lambda^3 H; \mathbb{Q}) = H^*(\Lambda^3 H_{\mathbb{Q}}) \cong \Lambda^*(\Lambda^3 H^*_{\mathbb{Q}})$ and $H^*(U; \mathbb{Q}) = H^*(U_{\mathbb{Q}}) \cong \Lambda^* U^*_{\mathbb{Q}}$ of the above Sp-modules explicitly. To do so, we use a fundamental result of Weyl in the classical representation theory. It turned out that this is a specific case of Kontsevich's general framework given in [37][38].

We consider $\Lambda^3 H$ as a natural Sp-submodule of $H^{\otimes 3}$ by the injection $\Lambda^3 H \subset H^{\otimes 3}$ defined by

$$\Lambda^3 H \ni a_1 \wedge a_2 \wedge a_3 \longmapsto \sum_{\sigma \in \mathfrak{S}_3} \operatorname{sgn} \sigma \; a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes a_{\sigma(3)} \in H^{\otimes 3}$$

where $a_i \in H$. Here $\mathfrak{S}_n$ is the n-th symmetric group and $\operatorname{sgn} \sigma$ denotes the sign of the permutation σ. It should be remarked that $\Lambda^3 H$ is a direct summand of $H^{\otimes 3}$ as a $\mathbb{Z}$-module. Similarly we consider $\Lambda^{2k}(\Lambda^3 H)$ as a natural Sp-submodule of $H^{\otimes 6k}$ defined by the injection

$$\Lambda^{2k}(\Lambda^3 H) \ni \xi_1 \wedge \cdots \wedge \xi_{2k} \longmapsto \sum_{\sigma \in \mathfrak{S}_{2k}} \operatorname{sgn} \sigma \; \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(2k)} \in H^{\otimes 6k}$$

where $\xi_i \in \Lambda^3 H$.

Now let V be a set consisting of $2k$ vertices with labels $\{1, 2, \cdots, 2k\}$ and let C be a graph with k edges such that the set of vertices of C is exactly equal to V. In other words, C is nothing but a way of making k parings out of labeled $2k$ vertices so that if we write E_C for the set of edges of C, then we can write

$$E_C = \big\{\{i_s, j_s\}; s = 1, \cdots, k\big\}$$

where

$$\{i_s, j_s; s = 1, \cdots, k\} = \{1, 2, \cdots, 2k\}.$$

We always assume that $i_s < j_s$ for all s. We can imagine such a graph C visually by putting the $2k$ vertices on a straight line in numerical order and join each pair $\{i_s, j_s\}$ by a curved edge. We call such a graph a *linear chord diagram* because if we close the straight line to obtain a circle, then we obtain the usual chord diagram which appears in the theory of Vassiliev's knot invariants (see [2]). For any linear chord diagram C, let

$$a_C \in (H_{\mathbb{Q}}^{\otimes 2k})^{Sp} \tag{7}$$

be the invariant tensor defined by permuting the tensor product $(\omega_0)^{\otimes k}$ in such a way that the s-th part $(\omega_0)_s$ goes to $(H_{\mathbb{Q}})_{i_s} \otimes (H_{\mathbb{Q}})_{j_s}$, where $(H_{\mathbb{Q}})_i$ denotes the i-th component of $H_{\mathbb{Q}}^{\otimes 2k}$, and multiplied by the factor $\operatorname{sgn} C$. Here sgn C is the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 2k-1 & 2k \\ i_1 & j_1 & \cdots & i_k & j_k \end{pmatrix}.$$

Each linear chord diagram C also defines an $Sp(2g; \mathbb{Z})$-invariant homomorphism

$$\alpha_C : H^{\otimes 2k} \longrightarrow \mathbb{Z} \tag{8}$$

by the rule

$$\alpha_C(u_1 \otimes \cdots \otimes u_{2k}) = \operatorname{sgn} C \prod_s u_{i_s} \cdot u_{j_s}.$$

Now let C be a linear chord diagram with $6k$ vertices with labels $\{1, 2, \cdots, 6k\}$. Then we can construct another graph Γ_C by joining three vertices of C with labels $3i+1, 3i+2, 3i+3$ to a single point for all $i = 0, 1, \cdots, 2k-1$. Clearly Γ_C is a trivalent graph with $2k$ vertices. Conversely we can *lift* any trivalent graph Γ with $2k$ vertices to a linear chord diagram C with $6k$ vertices such that $\Gamma_C = \Gamma$.

Now it is easy to see that the restriction of α_C to the submodule $\Lambda^{2k}(\Lambda^3 H) \subset H^{\otimes 6k}$ depends only on the associated trivalent graph Γ_C. Hence, if we are given a trivalent graph Γ with $2k$ vertices, then we can define an $Sp(2g,\mathbb{Z})$-invariant homomorphism

$$\alpha_\Gamma : \Lambda^{2k}(\Lambda^3 H) \longrightarrow \mathbb{Q}$$

by setting

$$\alpha_\Gamma = \frac{1}{(2k)!}\alpha_C$$

where C is any lift of Γ. We put the factor $(2k)!$ for later use (see Lemma 2.1 below). We can consider α_Γ as a $2k$-cocycle of the abelian group $\Lambda^3 H$ because we have a natural embedding

$$\mathrm{Hom}(\Lambda^{2k}(\Lambda^3 H), \mathbb{Q}) \subset Z^{2k}(\Lambda^3 H; \mathbb{Q})$$

given by

$$f(\xi_1, \cdots, \xi_{2k}) = f(\xi_1 \wedge \cdots \wedge \xi_{2k}) \ (f \in \mathrm{Hom}(\Lambda^{2k}(\Lambda^3 H), \mathbb{Q}),\ \xi_i \in \Lambda^3 H)$$

where $Z^{2k}(\Lambda^3 H; \mathbb{Q})$ denotes the set of $\mathbb{Q}$-valued $2k$-cocycles of $\Lambda^3 H$. Throughout this paper, we always use the Alexander-Whitney cup product on the cocycle level.

Lemma 2.1. *Let Γ and Γ' be two trivalent graphs with $2k$ and 2ℓ vertices, respectively. Then the disjoint union $\Gamma \amalg \Gamma'$ is a trivalent graph with $2k+2\ell$ vertices. In such a situation, two cocycles $\alpha_{\Gamma\amalg\Gamma'}$ and $\alpha_\Gamma \cup \alpha_{\Gamma'}$ are cohomologous to each other.*

Proof. Choose linear chord diagrams C, C' which are lifts of Γ, Γ' respectively. Then the sum $C + C'$ which is defined by putting the first vertex of C' right after the last vertex of C is a lift of $\Gamma \amalg \Gamma'$. We have

$$\begin{aligned}
&\alpha_{\Gamma\amalg\Gamma'}(\xi_1, \cdots, \xi_{2k+2\ell}) \\
=&\frac{1}{(2k+2\ell)!} \sum_{\sigma \in \mathfrak{S}_{2k+2\ell}} \operatorname{sgn}\sigma\, \alpha_{C+C'}\big(\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(2k+2\ell)}\big) \\
=&\frac{(2k)!(2\ell)!}{(2k+2\ell)!} \sum_{(2k,2\ell)-\textit{shuffles}} \operatorname{sgn}\sigma\, \alpha_\Gamma\big(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(2k)}\big) \cdot \alpha_{\Gamma'}\big(\xi_{\sigma(2k+1)}, \cdots, \xi_{\sigma(2k+2\ell)}\big)
\end{aligned}$$

where an element $\sigma \in \mathfrak{S}_{2k+2\ell}$ is called a $(2k, 2\ell)$-*shuffle* if

$$\begin{aligned}
&\sigma(1) < \sigma(2) < \cdots < \sigma(2k) \quad \text{and} \\
&\sigma(2k+1) < \cdots < \sigma(2k+2\ell).
\end{aligned}$$

Here we have used the fact that $\operatorname{sgn}(C + C') = \operatorname{sgn} C \operatorname{sgn} C'$. Observe that

$$\alpha_\Gamma(\xi_1, \cdots, \xi_{2k}) \cdot \alpha_{\Gamma'}(\xi_{2k+1}, \cdots, \xi_{2k+2\ell}) = \alpha_\Gamma \cup \alpha_{\Gamma'}(\xi_1, \cdots, \xi_{2k+2\ell})$$

because the degree of the cocycle α_Γ is even for any Γ. It is easy to see that for any $(2k, 2\ell)$-shuffle σ, the correspondence

$$\begin{aligned}
(\Lambda^3 H)^{2k+2\ell} \ni &(\xi_1, \cdots, \xi_{2k+2\ell}) \longmapsto \\
&\operatorname{sgn}\sigma\, \alpha_\Gamma\big(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(2k)}\big) \cdot \alpha_{\Gamma'}\big(\xi_{\sigma(2k+1)}, \cdots, \xi_{\sigma(2k+2\ell)}\big)
\end{aligned}$$

defines a cocycle of the group $\Lambda^3 H$ which is cohomologous to the cup product $\alpha_\Gamma \cup \alpha_{\Gamma'}$. Since the number of $(2k, 2\ell)$-shuffles is exactly equal to $\frac{(2k+2\ell)!}{(2k)!(2\ell)!}$, we can conclude that $\alpha_{\Gamma\amalg\Gamma'}$ is cohomologous to $\alpha_\Gamma \cup \alpha_{\Gamma'}$. □

Let $\mathcal{G}_{2k}$ denote the set of isomorphism classes of *connected* trivalent graphs with $2k$ vertices and let

$$\mathcal{G} = \coprod_{k \geq 1} \mathcal{G}_{2k}$$

be the disjoint union of $\mathcal{G}_{2k}$ for all $k \geq 1$. We consider the polynomial algebra

$$\mathbb{Q}[\varGamma; \varGamma \in \mathcal{G}]$$

generated by the elements of $\mathcal{G}$. In view of Lemma 2.1, we can define an algebra homomorphism

$$\Phi_\alpha : \mathbb{Q}[\varGamma; \varGamma \in \mathcal{G}] \longrightarrow H^*(\Lambda^3 H; \mathbb{Q}) \cong \mathrm{Hom}(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}), \mathbb{Q})$$

by sending each element $\varGamma \in \mathcal{G}_{2k}$ to the cohomology class $[\alpha_\varGamma] \in H^{2k}(\Lambda^3 H; \mathbb{Q})$ which is represented by the cocycle $\alpha_\varGamma$ of the group $\Lambda^3 H$. Since $H_{\mathbb{Q}}$ is the fundamental representation of the group $Sp(2g; \mathbb{Q})$, this group acts on the cohomology group $H^*(\Lambda^3 H; \mathbb{Q})$ naturally. By the definition of the homomorphism Φ_α, it is clear that its image $\mathrm{Im}\,\Phi_\alpha$ lies in the $Sp(2g; \mathbb{Q})$-invariant part of $H^*(\Lambda^3 H; \mathbb{Q})$.

Proposition 2.2. *The homomorphism*

$$\Phi_\alpha : \mathbb{Q}[\varGamma; \varGamma \in \mathcal{G}] \longrightarrow \mathrm{Hom}(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}), \mathbb{Q})^{Sp} \cong \left(\Lambda^*(\Lambda^3 H^*_{\mathbb{Q}})\right)^{Sp}$$

defined above is surjective and is an isomorphism for degrees $\leq \frac{2}{3}g$.

Proof. The surjectivity follows from a classical result of Weyl which shows that any Sp-invariant homomorphism $H_{\mathbb{Q}}^{\otimes 2r} \to \mathbb{Q}$ can be described as a linear combination of various iterated contractions using the intersection pairing $\mu : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$.

We prove the latter statement. If we assume that $2k \leq \frac{2}{3}g$, then we have $6k \leq 2g$. It follows that the number of members of any symplectic basis of H, which is $2g$, is greater than or equal to $6k$. As is well known, the number of ways of $3k$-fold iterated contractions $H^{\otimes 6k} \to \mathbb{Q}$, or equivalently the number of linear chord diagrams with $6k$ vertices, is equal to $(6k-1)!!$. It is now a simple matter to observe that we can choose an appropriate set of permutations of certain $6k$ members out of a symplectic basis of H so that it can serve as the dual basis of the above set of linear chord diagrams under the natural pairing between the two sets. Since $\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$ is an Sp direct summand of $H_{\mathbb{Q}}^{\otimes 6k}$, we have the desired result. The above argument, in fact, proves that

$$\dim(H_{\mathbb{Q}}^{\otimes 2r})^{Sp} = (2r-1)!!$$

for any r and for all $g \geq r$. □

Next we describe $H^*(U; \mathbb{Q})^{Sp} \cong H^*(U_{\mathbb{Q}})^{Sp}$ where $U = \Lambda^3 H / H$. $U_{\mathbb{Q}}$ is also an $Sp(2g; \mathbb{Q})$-module. We define an $Sp(2g; \mathbb{Q})$-equivariant homomorphism

$$q : \Lambda^3 H_{\mathbb{Q}} \longrightarrow \Lambda^3 H_{\mathbb{Q}} \tag{9}$$

by setting

$$q(\xi) = \xi - \frac{1}{2g-2} C\xi \wedge \omega_0 \quad (\xi \in \Lambda^3 H_{\mathbb{Q}}).$$

Here $C : \Lambda^3 H \to H$ is the contraction given by

$$C(u \wedge v \wedge w) = 2\{(u \cdot v)w + (v \cdot w)u + (w \cdot u)v\} \ (u, v, w \in H)$$

and $\omega_0 \in \Lambda^2 H$ is the symplectic class (in homology) defined as

$$\omega_0 = x_1 \wedge y_1 + \cdots + x_g \wedge y_g$$

where $x_1, \cdots, x_g, y_1, \cdots, y_g$ is any symplectic basis of H. It is easy to see that the homomorphism q annihilates any element of $H_{\mathbb{Q}}$ so that it induces an Sp-embedding

$$q : U_{\mathbb{Q}} \longrightarrow \Lambda^3 H_{\mathbb{Q}}.$$

In fact $\operatorname{Im} q = \operatorname{Ker} C \otimes \mathbb{Q}$ and we have a canonical decomposition of the $Sp(2g, \mathbb{Q})$-module

$$\Lambda^3 H_{\mathbb{Q}} \cong U_{\mathbb{Q}} \oplus H_{\mathbb{Q}}$$

into irreducible direct summands.

A *loop* is an edge connecting a single vertex as before. It is straightforward to see that if a trivalent graph $\Gamma \in \mathcal{G}_{2k}$ has a loop, then

$$\alpha_\Gamma(\xi_1, \cdots, \xi_{2k}) = 0 \quad (\xi_i \in \Lambda^3 H_{\mathbb{Q}})$$

whenever at least one ξ_i is contained in $\operatorname{Im} q \subset \Lambda^3 H_{\mathbb{Q}}$.

With these facts in mind, we define $\mathcal{G}^0_{2k} \subset \mathcal{G}_{2k}$ to be the subset consisting of connected trivalent graphs *without loops*. We write

$$\mathcal{G}^0 = \coprod_{k \geq 1} \mathcal{G}^0_{2k}$$

for the disjoint union of $\mathcal{G}^0_{2k}$ for all $k \geq 1$ and consider the polynomial algebra

$$\mathbb{Q}[\Gamma; \Gamma \in \mathcal{G}^0]$$

generated by the elements of $\mathcal{G}^0$. Now we define a homomorphism

$$\Phi_\beta : \mathbb{Q}[\Gamma; \Gamma \in \mathcal{G}^0] \longrightarrow H^*(U_{\mathbb{Q}})^{Sp} \cong \operatorname{Hom}(\Lambda^* U_{\mathbb{Q}}, \mathbb{Q})^{Sp}$$

by sending each element $\Gamma \in \mathcal{G}^0_{2k}$ to the homomorphism

$$\beta_\Gamma : \Lambda^{2k} U_{\mathbb{Q}} \xrightarrow{\Lambda^{2k} q} \Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}) \xrightarrow{\alpha_\Gamma} \mathbb{Q}$$

which is the composition of α_Γ followed by $\Lambda^{2k} q$.

Proposition 2.3. *The above homomorphism*

$$\Phi_\beta : \mathbb{Q}[\Gamma; \Gamma \in \mathcal{G}^0] \longrightarrow \operatorname{Hom}(\Lambda^* U_{\mathbb{Q}}, \mathbb{Q})^{Sp} \cong \left(\Lambda^* U^*_{\mathbb{Q}}\right)^{Sp}$$

is surjective and is an isomorphism for degrees $\leq \frac{2}{3} g$.

Proof. We have a natural Sp-isomorphism

$$H^*(\Lambda^3 H_{\mathbb{Q}}) \cong H^*(U_{\mathbb{Q}}) \otimes H^*(H_{\mathbb{Q}})$$

so that $H^*(U_{\mathbb{Q}})$ is a direct summand of $H^*(\Lambda^3 H_{\mathbb{Q}})$. Hence the result follows from the definition of Φ_β and Proposition 2.2. □

If we idetify $U_{\mathbb{Q}}$ with $\operatorname{Im} q \subset \Lambda^3 H_{\mathbb{Q}}$, then we can describe β_Γ as a certain linear combination

$$\beta_\Gamma = \alpha_\Gamma + \sum_i c_i \alpha_{\Gamma_i} \quad (c_i \in \mathbb{Q})$$

where each Γ_i has at least one loop. The coefficients c_i can be explicitly determined. They depend on the genus g but the dependence is only a matter of form. More precisely, the $(-2g)$-*adic expansion* of c_i is independent of g for any i.

3. Construction of cocycles of the mapping class groups

We begin by recalling the following morphism

$$
\begin{array}{ccc}
\pi_1\Sigma_g & \longrightarrow & H \\
\downarrow & & \downarrow \\
\mathcal{M}_{g,*} & \xrightarrow{\rho_1} & \frac{1}{2}\Lambda^3 H \rtimes Sp(2g,\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathcal{M}_g & \xrightarrow[\rho_1]{} & \frac{1}{2}\Lambda^3 H/H \rtimes Sp(2g,\mathbb{Z})
\end{array} \tag{10}
$$

of group extensions which was constructed in [53] and is a projection of the morphism (1) in §1 (see also Hain's closely related works [14][15] in the context of algebraic geometry). Here the top horizontal homomorphism is the abelianization and the other two homomorphisms ρ_1 are described as follows. Namely certain crossed homomorphisms

$$
\tilde{k} : \mathcal{M}_{g,*} \longrightarrow \frac{1}{2}\Lambda^3 H
$$

$$
\tilde{k} : \mathcal{M}_{g} \longrightarrow \frac{1}{2}\Lambda^3 H/H
$$

are defined and we have

$$
\rho_1(\varphi) = (\tilde{k}(\varphi), \rho_0(\varphi)) \quad (\varphi \in \mathcal{M}_{g,*} \text{ or } \mathcal{M}_g)
$$

where $\rho_0 : \mathcal{M}_g \longrightarrow Sp(2g,\mathbb{Z})$ is the classical homomorphism.

Since the targets of the homomorphisms ρ_1 above are semi-direct products, we discuss a few general facts concerning cohomology of groups which are semi-direct products. For a group G and a (left) G-module M, we denote by $C^*(G;M)$ the standard normalized cochain complex of G with values in M and by $Z^*(G;M)$ the set of cocycles in $C^*(G;M)$. See, for example, [26]. Suppose a group Q acts on a group N, namely there is given a homomorphism of groups $Q \to \mathrm{Aut}(N)$. Then a group law on the product set $N \times Q$ is defined by

$$
(n_1, q_1)(n_2, q_2) = (n_1 q_1(n_2), q_1 q_2), \quad (n_1, n_2 \in N,\ q_1, q_2 \in Q).
$$

We denote by $N \rtimes Q$ the set $N \times Q$ with this group law and call it the semi-direct product of N and Q. We have a natural extension of groups

$$
1 \longrightarrow N \xrightarrow{i} N \rtimes Q \xrightarrow{\pi} Q \longrightarrow 1
$$

where $i(n) = (n,1)$ and $\pi(n,q) = q$ for $n \in N$, $q \in Q$. It admits a canonical splitting homomorphism $s : Q \to N \rtimes Q$ given by $s(q) = (1,q)$.

Let M be an $N \rtimes Q$-module. For an r-cochain $c \in C^r(N;M)$ we define its natural extension $\widetilde{c} \in C^r(N \rtimes Q; M)$ by setting

$$
\begin{aligned}
&\widetilde{c}\big((n_1,q_1),(n_2,q_2),\cdots,(n_r,q_r)\big) \\
=&c\big(n_1, q_1(n_2), q_1q_2(n_3), \ldots, q_1q_2\cdots q_{r-1}(n_r)\big)
\end{aligned} \tag{11}
$$

for $n_i \in N, q_i \in Q$. It is easy to see that if we restrict this extending operation to the Q-invariant cochains, then the resultant map

$$
C^*(N;M)^Q \ni c \mapsto \tilde{c} \in C^*(N \rtimes Q; M)
$$

is a cochain map which is multiplicative with respect to the Alexander-Whitney cup product. For example, when $M = H_1(N;\mathbb{Z})$, the natural map $1_H : N \to N/[N,N] = H_1(N;\mathbb{Z})$ defined by $n \mapsto [n] = n \bmod [N,N]$ induces a 1-cocycle introduced by the second author in [47]

$$k_0 := \widetilde{1_H} \in Z^1(N \rtimes Q; H_1(N;\mathbb{Z})) \tag{12}$$

which is explicitly given by

$$k_0(n,q) = [n] \in H_1(N;\mathbb{Z}), \quad ((n,q) \in N \rtimes Q).$$

Consider the case where N is abelian. Suppose that there is given a homomorphism $\rho : G \to N \rtimes Q$ from a group G to the semi-direct product $N \rtimes Q$. Then ρ can be expressed as $\rho = (f, \bar\rho)$. Here $f : G \to N$ is a 1-cocycle of G with values in the G-module N where the action is given through the homomorphism $\bar\rho : G \to Q$. Then, for the trivial module $\mathbb{Q}$, we have a linear map

$$f^* : \mathrm{Hom}(\Lambda^* N, \mathbb{Q})^Q \hookrightarrow Z^*(N;\mathbb{Q})^Q \longrightarrow Z^*(N \rtimes Q;\mathbb{Q}) \xrightarrow{\rho^*} Z^*(G;\mathbb{Q})$$

This means any Q-invariant linear form on the exterior product $\Lambda^* N$ induces a cocycle of G in a natural way. The cocycle $f^*c \in Z^r(G;\mathbb{Q})$ induced by $c \in \mathrm{Hom}(\Lambda^r N, \mathbb{Q})^Q$ is explicitly given by

$$\begin{aligned}(f^*c)(g_1, g_2, \ldots, g_r) &= c\,(f(g_1), \bar\rho(g_1)f(g_2), \cdots, \bar\rho(g_1 \cdots g_{r-1})f(g_r)) \\ &= c_* f^r(g_1, g_2, \ldots, g_r) \quad (g_i \in G)\end{aligned}$$

where $f^r \in Z^r(G; \Lambda^r N)$ means the r-th power of the 1-cocycle f with respect to the Alexander-Whitney cup product, and $c_* : C^*(G;\Lambda^r N) \to C^*(G;\mathbb{Q})$ denotes the cochain map induced by the G-homomorphism $c : \Lambda^r N \to \mathbb{Q}$. Consequently we obtain

$$[f^*c] = c_*[f]^r \in H^r(G;\mathbb{Q}).$$

Now if we apply the above procedure to the homomorphisms ρ_1 in (10) and combine it with the results of §2, we obtain the following commutative diagram.

$$\begin{array}{ccccc}
\mathbb{Q}[\Gamma; \Gamma \in \mathcal{G}] & \xrightarrow{\Phi_\alpha} & \mathrm{Hom}(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}), \mathbb{Q})^{Sp} & \xrightarrow{\tilde{k}^*} & Z^*(\mathcal{M}_{g,*};\mathbb{Q}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{Q}[\Gamma; \Gamma \in \mathcal{G}^0] & \xrightarrow{\Phi_\beta} & \mathrm{Hom}(\Lambda^* U_{\mathbb{Q}}, \mathbb{Q})^{Sp} & \xrightarrow{\tilde{k}^*} & Z^*(\mathcal{M}_g;\mathbb{Q}).
\end{array}$$

As was already mentioned in §1, for each trivalent graph Γ with $2k$ vertices, we denote simply by α_Γ and β_Γ the cocycles of $\mathcal{M}_{g,*}$ and $\mathcal{M}_g$ constructed above. Then, summing up the above arguments, we obtain the following proposition which gives explicit formulas for these cocycles.

Proposition 3.1. *Let Γ be a trivalent graph with $2k$ vertices. Choose a linear chord diagram C with $6k$ vertices such that the associated trivalent graph Γ_C is equal to Γ. Then we have*

$$\alpha_\Gamma(\varphi_1, \cdots, \varphi_{2k}) = \frac{1}{(2k)!} \sum_{\sigma \in \mathfrak{S}_{2k}} \mathrm{sgn}\,\sigma\ \alpha_C(\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(2k)})$$

where $\varphi_i \in \mathcal{M}_{g,*}$ $(i = 1, \cdots, 2k)$, $\xi_i = \rho_0(\varphi_1 \cdots \varphi_{i-1})\tilde{k}(\varphi_i)$ and $\alpha_C : H_{\mathbb{Q}}^{\otimes 6k} \to \mathbb{Q}$ is the homomorphism given in (8), §2.

Similarly we have

$$\beta_\Gamma(\varphi_1,\cdots,\varphi_{2k}) = \frac{1}{(2k)!}\sum_{\sigma\in\mathfrak{S}_{2k}} \operatorname{sgn}\sigma\ \alpha_C(q(\xi_{\sigma(1)})\otimes\cdots\otimes q(\xi_{\sigma(2k)}))$$

where $\varphi_i \in \mathcal{M}_g$ $(i=1,\cdots,2k)$ *and* $\tilde\varphi_i \in \mathcal{M}_{g,*}$ *is any element such that* $\tilde\varphi_i$ *projects to* φ_i *under the natural projection* $\mathcal{M}_{g,*}\to\mathcal{M}_g$*. Also* $\xi_i = \rho_0(\tilde\varphi_1\cdots\tilde\varphi_{i-1})\tilde{k}(\tilde\varphi_i)$ *and* $q:\Lambda^3H_\mathbb{Q}\to\Lambda^3H_\mathbb{Q}$ *is the homomorphism given in* (9), §2.

Proposition 3.2. *Under the homomorphisms*

$$\left(\Lambda^*(\Lambda^3H^*_\mathbb{Q})\right)^{Sp}\xrightarrow{\rho_2^*}H^*(\mathcal{M}_{g,*};\mathbb{Q}) \text{ and } \left(\Lambda^*U^*_\mathbb{Q}\right)^{Sp}\xrightarrow{\rho_2^*}H^*(\mathcal{M}_g;\mathbb{Q})$$

in (2)*, the Sp-invariant parts of the ideals* $\left([1^2]^{torelli}\oplus[2^2]\right)$ *and* $([2^2])$ *go to zero.*

Proof. The same argument as in §7 of [53] shows that the above homomorphisms ρ_2^* factor as

$$\begin{array}{ccccc}
\left(\Lambda^*(\Lambda^3H^*_\mathbb{Q})\right)^{Sp} & \longrightarrow & H^*(([1^2]\oplus[2^2])\widetilde{\times}_{\text{torelli}}\Lambda^3H_\mathbb{Q})^{Sp} & \longrightarrow & H^*(\mathcal{M}_{g,*};\mathbb{Q}) \\
\uparrow & & \uparrow & & \uparrow \\
\left(\Lambda^*U^*_\mathbb{Q}\right)^{Sp} & \longrightarrow & H^*([2^2]\widetilde{\times}U_\mathbb{Q})^{Sp} & \longrightarrow & H^*(\mathcal{M}_g;\mathbb{Q}).
\end{array}$$

On the other hand, clearly the ideals generated by $[1^2]^{\text{torelli}}\oplus[2^2]$ and $[2^2]$ vanish in $H^*(([1^2]\oplus[2^2])\widetilde{\times}_{\text{torelli}}\Lambda^3H_\mathbb{Q})$ and $H^*([2^2]\widetilde{\times}U_\mathbb{Q})$ respectively. The claim follows from this immediately. □

4. Twisted Mumford-Morita-Miller classes and $H^*(\mathcal{M}_{g,*};\Lambda^*H)$

Let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g fixing an embedded disk $D^2\subset\Sigma_g$ pointwise, and $\overline{\mathcal{M}}_{g,*}$ the fiber product $\mathcal{M}_{g,*}\times_{\mathcal{M}_g}\mathcal{M}_{g,*}$ induced by the forgetful homomorphism $\mathcal{M}_{g,*}\to\mathcal{M}_g$. It is easy to see that the correspondence $\overline{\mathcal{M}}_{g,*}\ni(\varphi,\psi)\mapsto(\psi\varphi^{-1},\varphi)\in\pi_1\Sigma_g\rtimes\mathcal{M}_{g,*}$ defines an isomorphism.

As in [47], let us consider the 1-cocycle

$$k_0:\overline{\mathcal{M}}_{g,*}=\pi_1\Sigma_g\rtimes\mathcal{M}_{g,*}\longrightarrow H_1(\pi_1\Sigma_g)=H$$

defined in (12) which is explicitly given by $k_0(\varphi,\psi)=[\psi\varphi^{-1}]$ for $(\varphi,\psi)\in\overline{\mathcal{M}}_{g,*}$. Let $\pi:\overline{\mathcal{M}}_{g,*}\to\mathcal{M}_{g,*}$ (resp. $\bar\pi:\overline{\mathcal{M}}_{g,*}\to\mathcal{M}_{g,*}$) be the first (resp. the second) projection so that we have the fiber square

$$\begin{array}{ccc}
\overline{\mathcal{M}}_{g,*} & \xrightarrow{\bar\pi} & \mathcal{M}_{g,*} \\
{\scriptstyle\pi}\downarrow & & \downarrow \\
\mathcal{M}_{g,*} & \longrightarrow & \mathcal{M}_g.
\end{array} \tag{13}$$

Now we write simply $\bar e$ for $(\bar\pi)^*(e)\in H^2(\overline{\mathcal{M}}_{g,*};\mathbb{Z})$ and we consider the cohomology class $\bar e^i{k_0}^j\in H^{2i+j}(\overline{\mathcal{M}}_{g,*};\Lambda^jH)$. We define

$$m_{i,j}=\pi_!(\bar e^i{k_0}^j)\in H^{2i+j-2}(\mathcal{M}_{g,*};\Lambda^jH) \tag{14}$$

for $i,j\geq 0$ and $i+j\geq 2$, where $\pi_!:H^k(\overline{\mathcal{M}}_{g,*};M)\to H^{k-2}(\mathcal{M}_{g,*};M)$ is the Gysin homomorphism (or the integration along the fibers of the map π). We call them the *twisted Mumford-Morita-Miller classes*. In fact, when $j=0$ and $i\geq 1$, we have $m_{i+1,0}=e_i$.

On the mapping class group $\mathcal{M}_{g,1}$ these classes are nothing but the cohomology classes $(-1)^j m_{i,j}$ introduced by the first author in [34], where they are called the *generalized* Morita-Mumford classes. In order to verify this, we introduce the mapping class group $\mathcal{M}_g^2$ of Σ_g fixing two distinct points $p_0, p_1 \in \Sigma_g$ pointwise. Choose a simple curve ℓ in Σ_g connecting p_1 to p_0. Define a 1-cocycle $\omega_\ell \in Z^1(\mathcal{M}_g^2; H)$ by

$$\omega_\ell(\varphi) = \varphi(\ell) - \ell \in H$$

for $\varphi \in \mathcal{M}_g^2$. Here $\varphi(\ell) - \ell$ is regarded as a 1-cycle on Σ_g. In [34] the generalized Morita-Mumford class $m_{i,j} \in H^*(\mathcal{M}_{g,1}; \Lambda^j H)$ was defined to be the Gysin image of the cohomology class $e^i {\omega_\ell}^j \in H^*(\mathcal{M}_{g,1}^1, \mathcal{M}_{g,1} \times \mathbb{Z}; \Lambda^j H)$. Here $\mathcal{M}_{g,1}^1$ is the mapping class group of Σ_g relative to an embedded disk and a fixed point outside of it. The product $\mathcal{M}_g \times \mathbb{Z}$ is embedded into $\mathcal{M}_{g,1}^1$ as the "boundary" of $\mathcal{M}_{g,1}^1$ (p.140, [34]).

On the other hand, let $\pi : \mathcal{M}_g^2 \to \mathcal{M}_{g,*}$ (resp. $\bar{\pi} : \mathcal{M}_g^2 \to \mathcal{M}_{g,*}$) be the homomorphism defined by forgetting the point p_1 (resp. p_0). Consider a diffeomorphism $\psi_\ell : (\Sigma_g, p_1) \to (\Sigma_g, p_0)$ given by sliding the point p_1 along the curve ℓ. We introduce a homomorphism

$$\alpha_\ell : \mathcal{M}_g^2 \longrightarrow \overline{\mathcal{M}}_{g,*} \tag{15}$$

by the correspondence $\varphi \mapsto (\pi(\varphi), \psi_\ell \bar{\pi}(\varphi) {\psi_\ell}^{-1})$. Then we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_{g,1}^1 & \longrightarrow & \mathcal{M}_g^2 & \xrightarrow{\alpha_\ell} & \overline{\mathcal{M}}_{g,*} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \mathcal{M}_{g,1} & \longrightarrow & \mathcal{M}_{g,*} & = & \mathcal{M}_{g,*}. \end{array} \tag{16}$$

Consequently the following lemma implies that the cohomology class $m_{i,j}$ defined in the present paper essentially coincides with the former one.

Lemma 4.1. *We have*

$$\alpha_\ell^*(k_0) = -\omega_\ell \in Z^1(\mathcal{M}_g^2; H).$$

Proof. For any $\varphi \in \mathcal{M}_g^2$, we have

$$\begin{aligned} \alpha_\ell^*(k_0) =& k_0 \left(\pi(\varphi), \psi_\ell \bar{\pi}(\varphi) {\psi_\ell}^{-1}\right) \\ =& \left[\psi_\ell \bar{\pi}(\varphi) {\psi_\ell}^{-1} \pi(\varphi)^{-1}\right] = \left[\psi_\ell {\psi_{\varphi(\ell)}}^{-1}\right] = \left[\ell \varphi(\ell)^{-1}\right] = \left[\ell - \varphi(\ell)\right] \\ =& -\omega_\ell(\varphi) \end{aligned}$$

completing the proof. □

In [42] Looijenga obtained a remarkable result that the rational stable cohomology of $\mathcal{M}_g$ with coefficients in any finite dimensional irreducible representation of the algebraic group $Sp(2g, \mathbb{Q})$ is isomorphic to a free module over the stable rational cohomology of $\mathcal{M}_g$ together with a description of its free basis. His proof as well as construction of cohomology classes are based on geometric considerations on the moduli orbifold of algebraic curves including, in particular, a theorem in Hodge theory. As for $\mathcal{M}_g$, this result cannot be generalized to integral symplectic coefficients. In fact, for example, the second integral cohomology of $\mathcal{M}_g$ with coefficients in H does not admit the stability [46].

On the basis of Looijenga's noteworthy idea [42] that the stable cohomology with symplectic coefficients is computed only from the Harer stability theorem [21] with *trivial* coefficients, the first author has deduced the following result: the stable integral cohomology of $\mathcal{M}_{g,1}$ with coefficients in $H^{\otimes n}$ is a free module over the stable integral cohomology of $\mathcal{M}_{g,1}$, and certain algebraic combinations of the (modified) twisted Mumford-Morita-Miller classes can serve as its (topologically constructed and new) free basis [33]. Here the Lyndon-Hochschild-Serre spectral sequence for a pair of groups introduced in [34] is used instead of geometric considerations including Hodge theory.

Since it has been found out that the twisted Mumford-Morita-Miller classes can be defined also on $\mathcal{M}_{g,*}$, all the results obtained in [33] hold for the mapping class group $\mathcal{M}_{g,*}$. For example, we have

Theorem 4.2. *If the total degree is smaller than $g/2$, then we have*

$$H^*(\mathcal{M}_{g,*};\Lambda^* H_{\mathbb{Q}}) = H^*(\mathcal{M}_{g,*};\mathbb{Q}) \otimes \left(\otimes_{i,j}\mathbb{Q}[m_{i,j}]\right)$$

where the indices run over the set $\{(i,j);\ i \geq 0, j \geq 1 \text{ and } i+j \geq 2\}$.

The above results show that we have obtained explicit basis for these free modules in terms of the twisted Mumford-Morita-Miller classes defined above.

5. Thom isomorphism theorem for $\mathcal{M}_{g,*}$ and splitting of $H^*(\mathcal{M}_{g,*};M)$

Let $E\operatorname{Diff}_+\Sigma_g \to B\operatorname{Diff}_+\Sigma_g$ be the universal Σ_g-bundle over the classifying space $B\operatorname{Diff}_+\Sigma_g$ of oriented Σ_g-bundles. Then the fiber product

$$E\operatorname{Diff}_+\Sigma_g \times_{B\operatorname{Diff}_+\Sigma_g} E\operatorname{Diff}_+\Sigma_g$$

is an Eilenberg-MacLane space $K(\overline{\mathcal{M}}_{g,*},1)$ and its diagonal set is a "submanifold" of codimension 2. Hence we can define the Thom class $\nu \in H^2(\overline{\mathcal{M}}_{g,*};\mathbb{Z})$, namely the Poincaré dual of the diagonal set (see [46]). The cohomology class ν may be constructed also in the following algebraic way.

Let $\mathcal{M}_g^2$ be the mapping class group of Σ_g fixing ordered 2 points p_0 and p_1 as in §4. Choose a simple curve ℓ in Σ_g connecting p_1 to p_0 and consider the homomorphism $\alpha_\ell : \mathcal{M}_g^2 \to \overline{\mathcal{M}}_{g,*}$ introduced in (15), §4. In §§5-7 we write simply π_1 and π_1^0 for $\pi_1(\Sigma_g, p_1)$ and $\pi_1(\Sigma_g - \{p_0\}, p_1)$, respectively. Let R_g denote the kernel of the inclusion homomorphism

$$R_g = \operatorname{Ker}(\pi_1^0 \to \pi_1).$$

The kernel of $\pi : \mathcal{M}_g^2 \to \mathcal{M}_{g,*}$ is naturally identified with π_1^0. Hence we obtain a morphism of group extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R_g & \longrightarrow & \pi_1^0 & \longrightarrow & \pi_1 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \psi_\ell} & & \\ 1 & \longrightarrow & R_g & \longrightarrow & \mathcal{M}_g^2 & \xrightarrow{\alpha_\ell} & \overline{\mathcal{M}}_{g,*} & \longrightarrow & 1. \end{array} \tag{17}$$

Let M be an $\mathcal{M}_{g,*}$-module. We regard it as an $\overline{\mathcal{M}}_{g,*}$-module by the homomorphism $\pi : \overline{\mathcal{M}}_{g,*} \to \mathcal{M}_{g,*}$. Since the group R_g is a subgroup of a free group π_1^0, we have $H^p(R_g;M) = 0$ for any $p \geq 2$. Hence

$$H^p(R_g;M)^{\pi_1} = 0 \tag{18}$$

for any $p \geq 1$, and the second transgression of the Lyndon-Hochschild-Serre spectral sequence of the extension $R_g \to \pi_1^0 \to \pi_1$ is an $\mathcal{M}_g^2$-equivariant isomorphism

$$d_2{}' : H^1(R_g;M)^{\pi_1} \xrightarrow{\cong} H^2(\pi_1;M).$$

Especially if M is a trivial module $\mathbb{Z}$, then any element of $H^2(\pi_1;\mathbb{Z})$ is $\mathcal{M}_g^2$-invariant. Hence we have $H^1(R_g;\mathbb{Z})^{\mathcal{M}_g^2} = H^1(R_g;\mathbb{Z})^{\pi_1^0}$ so that the morphism (17) induces a commutative diagram

$$
\begin{array}{ccc}
H^0(\overline{\mathcal{M}}_{g,*};H^1(R_g;\mathbb{Z})) & \xrightarrow{d_2} & H^2(\overline{\mathcal{M}}_{g,*};\mathbb{Z}) \\
\Big\| & & {\psi_\ell}^*\Big\downarrow \\
H^0(\pi_1;H^1(R_g;\mathbb{Z})) & \xrightarrow{{d_2}'} & H^2(\pi_1;\mathbb{Z}).
\end{array}
\tag{19}
$$

Thus there exists a unique element $\nu_0 \in H^0(\overline{\mathcal{M}}_{g,*};H^1(R_g;\mathbb{Z}))$ satisfying the equality $< {\psi_\ell}^* d_2\nu_0, [\Sigma_g] >= 1$. The cup product

$$\nu_0\cup : M \xrightarrow{\cong} H^1(R_g;M)^{\pi_1} \tag{20}$$

is an $\mathcal{M}_{g,*}$-isomorphism for any $\mathcal{M}_{g,*}$-module M. We define the cohomology class $\nu \in H^2(\overline{\mathcal{M}}_{g,*};\mathbb{Z})$ by

$$\nu = d_2\nu_0 \in H^2(\overline{\mathcal{M}}_{g,*};\mathbb{Z}).$$

It satisfies

$$\pi_!\nu = 1 \tag{21}$$

and plays the role of the Thom class in the context of mapping class groups as follows.

Theorem 5.1 (Thom isomorphism theorem)**.** *Let M be an $\mathcal{M}_{g,*}$-module.*
(i) *We have a split exact sequence*

$$0 \longrightarrow H^p(\mathcal{M}_{g,*};M) \xrightarrow{\nu\cup} H^{p+2}(\overline{\mathcal{M}}_{g,*};M) \xrightarrow{\alpha_\ell^*} H^{p+2}(\mathcal{M}_g^2;M) \longrightarrow 0.$$

(ii) *For any $u \in H^*(\overline{\mathcal{M}}_{g,*};M)$, we have*

$$\nu u = \nu\pi^* s^* u \in H^*(\overline{\mathcal{M}}_{g,*};M)$$

where $s : \mathcal{M}_{g,*} \to \overline{\mathcal{M}}_{g,*}$ denotes the diagonal map defined by $\varphi \mapsto (\varphi,\varphi)$. This means that the cohomology class ν has its "support" in the "diagonal" $s(\mathcal{M}_{g,*}) \subset \overline{\mathcal{M}}_{g,*}$.
(iii)

$$s^*\nu = e \in H^2(\mathcal{M}_{g,*};\mathbb{Z}).$$

Proof. By (18) the Lyndon-Hochschild-Serre spectral sequence of the extension $\pi_1 \to \overline{\mathcal{M}}_{g,*} \xrightarrow{\pi} \mathcal{M}_{g,*}$ with values in $H^1(R_g;M)$ collapses, namely, the homomorphism

$$\pi^* : H^p(\mathcal{M}_{g,*};H^1(R_g;M)^{\pi_1}) \xrightarrow{\cong} H^p(\overline{\mathcal{M}}_{g,*};H^1(R_g;M)) \tag{22}$$

is an isomorphism. This together with (20) implies that the cup product

$$\nu_0\cup : H^*(\mathcal{M}_{g,*};M) \xrightarrow{\cong} H^*(\overline{\mathcal{M}}_{g,*};H^1(R_g;M)), \quad v \mapsto \nu_0\pi^*(v) \tag{23}$$

is an isomorphism.

We first prove (i). Since the group R_g is free, the lower extension in (17) induces the Gysin exact sequence

$$\cdots \longrightarrow H^p(\overline{\mathcal{M}}_{g,*};H^1(R_g;M)) \xrightarrow{d_2} H^{p+2}(\overline{\mathcal{M}}_{g,*};M) \xrightarrow{{\alpha_\ell}^*} H^{p+2}(\mathcal{M}_g^2;M) \longrightarrow \cdots .$$

Substituting the isomorphism (23) into the above sequence, we obtain the exact sequence to be proved. By (21) we have

$$\pi_! d_2(\nu_0 \cup \pi^* v) = \pi_!(\nu \cup \pi^* v) = v$$

for any $v \in H^p(\mathcal{M}_{g,*}; M)$, which gives a natural splitting of the sequence. This proves (i).

Next we consider (ii). It follows from the isomorphism (22) that the map

$$\pi^* : H^*(\mathcal{M}_{g,*}; H^1(R_g; M)) \to H^*(\overline{\mathcal{M}}_{g,*}; H^1(R_g; M))$$

is a surjection. Hence $\pi^* s^*$ is equal to the identity on $H^*(\overline{\mathcal{M}}_{g,*}; H^1(R_g; M))$. Therefore $\nu_0 u = \pi^* s^*(\nu_0 u) = \nu_0 \pi^* s^* u$, which implies

$$\nu u = d_2(\nu_0 u) = d_2(\nu_0 \pi^* s^* u) = \nu \pi^* s^* u$$

thus proving (ii).

Finally we prove (iii). Choose an embedded disk $D^2 \subset \Sigma_g$ containing the simple curve ℓ. Then we have a morphism of extensions

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathcal{M}_{g,1} & \longrightarrow & \mathcal{M}_{g,*} \\ {\scriptstyle\psi}\downarrow & & {\scriptstyle\psi}\downarrow & & {\scriptstyle s}\downarrow \\ R_g & \longrightarrow & \mathcal{M}_g^2 & \xrightarrow{\alpha_\ell} & \overline{\mathcal{M}}_{g,*} \end{array}$$

which induces the following commutative diagram

$$\begin{array}{ccc} H^0(\overline{\mathcal{M}}_{g,*}; H^1(R_g)) & \xrightarrow{d_2} & H^2(\overline{\mathcal{M}}_{g,*}) \\ {\scriptstyle\psi^*}\downarrow & & {\scriptstyle s^*}\downarrow \\ H^0(\mathcal{M}_{g,*}; H^1(\mathbb{Z})) & \xrightarrow{d_2} & H^2(\mathcal{M}_{g,*}). \end{array}$$

We have $\psi^* \nu_0 = \lambda 1_{\mathbb{Z}} \in H^1(\mathbb{Z})$ for some $\lambda \in \mathbb{Z}$. Hence

$$s^* \nu = s^* d_2 \nu_0 = d_2 \psi^*(\nu_0) = \lambda d_2 1 = \lambda e \in H^2(\mathcal{M}_{g,*}; \mathbb{Z}).$$

If we restrict this formula to $\pi_1 \rtimes \pi_1 \subset \pi_1 \rtimes \mathcal{M}_{g,*} = \overline{\mathcal{M}}_{g,*}$ and apply the usual Thom isomorphism theorem on $\Sigma_g \times \Sigma_g$, we obtain $\lambda = 1$. Thus we have $s^* \nu = e \in H^2(\mathcal{M}_{g,*}; \mathbb{Z})$. This completes the proof of (iii) and hence that of the theorem. □

Next we show that there exist certain canonical decompositions of cohomology groups of mapping class groups.

In [45] Proposition 3.1, the second author proved that the cohomology of the total space of any oriented Σ_g-bundle with values in $\mathbb{Z}[1/(2g-2)]$ has a canonical decomposition. Here we refine this result slightly, which would clarify the behavior of the twisted Mumford-Morita-Miller classes.

Let $h : \Pi \to \mathcal{M}_g$ be a homomorphism from a group Π into the mapping class group $\mathcal{M}_g$. The fiber product $\Pi_* = \Pi \times_{\mathcal{M}_g} \mathcal{M}_{g,*}$ admits an extension of groups

$$1 \longrightarrow \pi_1 \xrightarrow{\iota} \Pi_* \xrightarrow{\pi} \Pi \longrightarrow 1. \tag{24}$$

Let A be a commutative ring with a unit. We denote $H_A = H \otimes_{\mathbb{Z}} A = H_1(\Sigma_g; A)$, which has the intersection pairing $\mu : H_A \otimes_A H_A \to A$. For any $A[\Pi]$-module M we identify the cohomology group $H^1(\pi_1; M)$ with $H_A \otimes_A M$ by the $A[\Pi]$-isomorphism

$$\mu' : H_A \otimes_A M \longrightarrow H^1(\pi_1; M) = \mathrm{Hom}_A(H_A, M)$$

which is defined by $\mu'(v)(u) = (\mu\otimes 1_M)_*(u\otimes v)$ for $u \in H_A$ and $v \in H_A \otimes_A M$. For the rest of this section we write simply $H_A \otimes M$ for $H_A \otimes_A M$. As in the previous sections, we use the notation 1_H also for the abelianization map in $H^1(\pi_1; H_A)$ as well as the identity map of H. When $M = H$, we have $\mu'(\omega_0) = -1_H$ where ω_0 is the symplectic form $\omega_0 = \sum_{i=1}^{g} x_i \otimes y_i - y_i \otimes x_i \in H^{\otimes 2}$ as before.

Proposition 5.2. *Suppose that there exists a cohomology class $\theta \in H^2(\Pi_*; A)$ such that*

$$\pi_!(\theta) = \langle \iota^*\theta, [\Sigma_g]\rangle = 1 \in H^0(\Pi; A).$$

We denote

$$\theta' = \theta - \pi^*\pi_!(\theta^2)$$

which also satisfies $\pi_!(\theta') = 1$. Then we have the following.
(i) *For any $A[\Pi]$-module M, the Lyndon-Hochschild-Serre spectral sequence of the extension (24) collapses at the E_2-term, and the cohomology group $H^*(\Pi_*; M)$ naturally decomposes as*

$$H^*(\Pi_*; M) \cong H^{*-2}(\Pi; M) \oplus H^{*-1}(\Pi; H_A \otimes M) \oplus H^*(\Pi; M).$$

(ii) *There exists a unique element $\chi \in H^1(\Pi_*; H_A)$ satisfying*

$$\iota^*\chi = 1_H \in H^1(\pi_1; H_A), \text{ and } \pi_!(\theta\chi) = \pi_!(\theta'\chi) = 0 \in H^1(\Pi; H_A).$$

(iii) *The homomorphism $\epsilon : H^{*-1}(\Pi; H_A \otimes M) \to H^*(\Pi_*; M)$ given by*

$$\epsilon(v) = (\mu \otimes 1_M)_*(\chi \otimes \pi^* v) \quad (v \in H^{*-1}(\Pi; H_A \otimes M)) \tag{25}$$

is a left inverse of the edge homomorphism $\pi_\sharp : \operatorname{Ker}\pi_! \to E_\infty^{*-1,1} = H^{*-1}(\Pi; H_A \otimes M)$.
(iv) *Explicitly, for any $u \in H^*(\Pi_*; M)$, we have*

$$\begin{aligned} u &= \theta'\pi^*\pi_!(u) - (\mu \otimes 1_M)_* (\chi \otimes \pi^*\pi_!(\chi \otimes u)) + \pi^*\pi_!(\theta u) \\ &= \theta\pi^*\pi_!(u) - (\mu \otimes 1_M)_* (\chi \otimes \pi^*\pi_!(\chi \otimes u)) + \pi^*(\pi_!(\theta u) - \pi_!(\theta^2)\pi_!(u)). \end{aligned}$$

Proof. We have

$$\pi_!(\theta\pi^* v) = \pi_!(\theta'\pi^* v) = v \in H^*(\Pi; M)$$

for all $v \in H^*(\Pi; M)$. This means that the map

$$\psi_2 : H^{*-2}(\Pi; M) \to H^*(\Pi_*; M)$$

defined by $v \mapsto \theta'\pi^* v$ is a right inverse of the Gysin map $\pi_!$ and that the map

$$\varphi_0 : H^*(\Pi_*; M) \to H^*(\Pi; M)$$

defined by $u \mapsto \pi_!(\theta u)$ is a left inverse of the map π^*. Especially the map $\pi_!$ is surjective and the map π^* is injective. Hence the Lyndon-Hochschild-Serre spectral sequence of the extension (24) collapses at the E_2-term.

The map ψ_2 induces the decomposition $H^*(\Pi_*; M) = H^{*-2}(\Pi; M) \oplus \operatorname{Ker}\pi_!$, while the short exact sequence

$$0 \longrightarrow H^*(\Pi; M) \xrightarrow{\pi^*} \operatorname{Ker}\pi_! \xrightarrow{\pi_\sharp} E_2^{*-1,1} = H^{*-1}(\Pi; H_A \otimes M) \longrightarrow 0$$

splits by the map φ_0 restricted to $\operatorname{Ker}\pi_!$. Hence

$$\pi_\sharp : \operatorname{Ker}\pi_! \cap \operatorname{Ker}\varphi_0 \xrightarrow{\cong} H^{*-1}(\Pi; H_A \otimes M) \tag{26}$$

is an isomorphism. This proves (i).

Now we consider the case $M = H_A$. The map

$$\iota^* = \pi_! : H^1(\Pi_*; H_A) \longrightarrow H^1(\pi_1; H_A)^\Pi = H^0(\Pi; H_A \otimes H_A)$$

is surjective by (i), so that there exists an element $\tilde{\chi} \in H^1(\Pi_*; H_A)$ satisfying $\iota^*\tilde{\chi} = 1_H$. If we set $\chi = \tilde{\chi} - \pi^*\pi_!(\theta\tilde{\chi})$, then we have $\iota^*\chi = 1_H$ and $\pi_!(\theta\chi) = \pi_!(\theta'\chi) = 0$.

If there is another χ' satisfying the same conditions as χ, then the difference $\chi' - \chi$ is contained in $\operatorname{Ker}\iota^* = \pi^*(H^1(\Pi; H_A))$. It follows that there exists an element $v \in H^1(\Pi; H_A)$ such that $\chi' - \chi = \pi^*(v)$. But we have $v = \pi_!(\theta\pi^*(v)) = \pi_!(\theta\chi') - \pi_!(\theta\chi) = 0$. This proves (ii).

Next we consider (iii). Since $\pi_!(\chi) = \pi_!(\theta\chi) = 0$, we have $\epsilon(H^{*-1}(\Pi; H_A \otimes M)) \subset \operatorname{Ker}\pi_! \cap \operatorname{Ker}\varphi_0$. Moreover we have

$$\pi_\sharp(\chi) = \iota^*(\chi) = (\mu')^{-1}(1_H) = -\omega_0 \in H^0(\Pi; H_A^{\otimes 2}). \tag{27}$$

Hence

$$\pi_\sharp\epsilon(v) = \pi_\sharp(\mu \otimes 1_M)_*(\chi \otimes \pi^* v) = (1_H \otimes \mu \otimes 1_M)_*(\pi_\sharp(\chi) \otimes v) = v$$

for any $v \in H^{*-1}(\Pi; H_A \otimes M)$. This proves (iii).

Finally we prove (iv). We observe

$$\pi_!(\chi^{\otimes 2}) = \omega_0 \in H^0(\Pi; H_A^{\otimes 2}). \tag{28}$$

In fact, we denote by $T' : {H_A}^{\otimes 4} \to {H_A}^{\otimes 4}$ the switch map $v_1 \otimes v_2 \otimes v_3 \otimes v_4 \mapsto -v_1 \otimes v_3 \otimes v_2 \otimes v_4$ $(v_i \in H_A)$. From the multiplicativity of the Lyndon-Hochschild-Serre spectral sequence, we have

$$\begin{aligned}
&\pi_!(\chi^{\otimes 2}) = \langle \iota^*(\chi) \otimes \iota^*(\chi), [\Sigma_g]\rangle = \langle 1_H \otimes 1_H, [\Sigma_g]\rangle \\
=&(\mu \otimes 1_{{H_A}^{\otimes 2}})T'\left((\mu')^{-1}(1_H) \otimes (\mu')^{-1}(1_H)\right) \\
=&(\mu \otimes 1_{{H_A}^{\otimes 2}})T'\left(\omega_0 \otimes \omega_0\right) = \omega_0.
\end{aligned}$$

Let u be any element of $H^*(\Pi_*; M)$. By what we have shown above there exist $u_0 \in H^*(\Pi; M)$, $u_1 \in H^{*-1}(\Pi; H_A \otimes M)$ and $u_2 \in H^{*-2}(\Pi; M)$ such that

$$u = \theta'\pi^*(u_2) + (\mu \otimes 1_M)_*(\chi \otimes \pi^*(u_1)) + \pi^* u_0.$$

Clearly we have $\pi_! u = u_2$. Since $\pi_!(\theta\theta') = \pi_!(\theta\chi) = 0$, $\pi_!(\theta u) = u_0$. Moreover

$$\begin{aligned}
\pi_!(\chi \otimes u) &= \pi_!(\chi \otimes (\mu \otimes 1_M)_*(\chi \otimes \pi^*(u_1))) = (1_H \otimes \mu \otimes 1_M)_*\pi_!(\chi^{\otimes 2} \otimes \pi^*(u_1)) \\
&= (1_H \otimes \mu \otimes 1_M)_*(\omega_0 u_1) = -u_1
\end{aligned}$$

by (28). This proves (iv) completing the proof of the proposition. □

Now we consider the case $\Pi = \mathcal{M}_{g,*}$. Then $\Pi_* = \overline{\mathcal{M}}_{g,*}$ and the Thom class $\nu \in H^2(\overline{\mathcal{M}}_{g,*}; \mathbb{Z})$ satisfies $\pi_!\nu = 1$ by Theorem 5.1.

Theorem 5.3. *For any $\mathcal{M}_{g,*}$-module M the cohomology group $H^*(\overline{\mathcal{M}}_{g,*}; M)$ decomposes as*

$$H^*(\overline{\mathcal{M}}_{g,*}; M) \cong H^*(\mathcal{M}_{g,*}; M) \oplus H^{*-1}(\mathcal{M}_{g,*}; H \otimes M) \oplus H^{*-2}(\mathcal{M}_{g,*}; M).$$

Explicitly, for any $u \in H^*(\overline{\mathcal{M}}_{g,*}; M)$, we have

$$\begin{aligned}
u &= \nu'\pi^*\pi_!(u) - (\mu \otimes 1_M)_*\left(k_0 \otimes \pi^*\pi_!(k_0 \otimes u)\right) + \pi^* s^* u \\
&= \nu\pi^*\pi_!(u) - (\mu \otimes 1_M)_*\left(k_0 \otimes \pi^*\pi_!(k_0 \otimes u)\right) + \pi^*(s^*(u) - e\pi_!(u))
\end{aligned}$$

where $\nu' = \nu - \pi^*(e) \in H^2(\overline{\mathcal{M}}_{g,*}; \mathbb{Z})$.

Proof. The cohomology class $k_0 \in H^1(\overline{\mathcal{M}}_{g,*}; H)$ satisfies $\iota^*(k_0) = 1_H$ and $\pi_!(\nu k_0) = \pi_!(\nu\pi^* s^*(k_0)) = 0$ by Theorem 5.1, (iii). We have

$$\nu - \pi^*\pi_!(\nu^2) = \nu - \pi^*\pi_!(\nu\pi^* s^*(\nu)) = \nu - \pi^*\pi_!(\nu\pi^*(e)) = \nu - \pi^*(e) = \nu'.$$

Now $\pi_!(\nu u) = \pi_!(\nu\pi^* s^* u) = s^* u$ by Theorem 5.1, (ii). Thus the theorem follows from Proposition 5.2 immediately. □

Recall the exact sequence in Theorem 5.1

$$0 \longrightarrow H^{*-2}(\mathcal{M}_{g,*}; M) \xrightarrow{\nu\cup} H^*(\overline{\mathcal{M}}_{g,*}; M) \xrightarrow{{\alpha_\ell}^*} H^*(\mathcal{M}_g^2; M) \longrightarrow 0$$

where $\alpha_\ell : \mathcal{M}_g^2 \to \overline{\mathcal{M}}_{g,*}$ is the homomorphism introduced in (15), §4. The forgetful extension $\pi_1^0 \to \mathcal{M}_g^2 \to \mathcal{M}_{g,*}$ induces the Gysin map $\pi_\sharp : H^*(\mathcal{M}_g^2; M) \to H^{*-1}(\mathcal{M}_{g,*}; H \otimes M)$.

Lemma 5.4. *For any $u \in H^*(\overline{\mathcal{M}}_{g,*}; M)$ we have*

$$\pi_\sharp {\alpha_\ell}^*(u) = -\pi_!(k_0 \otimes u) \in H^{*-1}(\mathcal{M}_{g,*}; H \otimes M).$$

Proof. By Theorem 5.3

$$u = \nu\pi_!(u) - (\mu \otimes 1_M)_* \left(k_0 \otimes \pi^*\pi_!(k_0 \otimes u)\right) + \pi^*(s^*(u) - e\pi_!(u)).$$

Since ${\alpha_\ell}^*(\nu) = 0$ and $\pi_\sharp\pi^* = 0$, we have

$$\begin{aligned} \pi_\sharp {\alpha_\ell}^*(u) &= -\pi_\sharp {\alpha_\ell}^*(\mu \otimes 1_M)_* \left(k_0 \otimes \pi^*\pi_!(k_0 \otimes u)\right) \\ &= -(1_H \otimes \mu \otimes 1_M)_* \pi_\sharp(k_0)\pi_!(k_0 \otimes u) \\ &= (1_H \otimes \mu \otimes 1_M)_*(\omega_0 \otimes \pi_!(k_0 \otimes u)) = -\pi_!(k_0 \otimes u) \end{aligned}$$

as was to be shown. □

As in [45] Proposition 3.1, in the case $\Pi = \mathcal{M}_g$, we have $\pi_!\left((2-2g)^{-1}e\right) = 1 \in H^0(\mathcal{M}_g; \mathbb{Z}[1/(2-2g)])$. We denote

$$\begin{aligned} e'' &= (2-2g)^{-1}e - (2-2g)^{-2}e_1 \in H^2(\mathcal{M}_{g,*}; \mathbb{Z}[1/(2-2g)]) \\ \chi &= (2-2g)^{-1}m_{1,1} \in H^2(\mathcal{M}_{g,*}; H_{\mathbb{Z}[1/(2-2g)]}). \end{aligned}$$

Here $m_{1,1}$ is a generator of $H^1(\mathcal{M}_{g,*}; H) \cong \mathbb{Z}$ (see [46], where $m_{1,1}$ is denoted by k). Then we have $\pi_!(e'') = 1$ and $\iota^*\chi = 1_H$. Since $H^1(\mathcal{M}_g; H) = 0$ (see [45]), we have $\pi_!(e''\chi) = 0$. Consequently we have the following result.

Theorem 5.5. *Let A be the commutative ring $\mathbb{Z}[1/(2-2g)]$ and M be any $A[\mathcal{M}_g]$-module. Then the cohomology group $H^*(\mathcal{M}_{g,*}; M)$ decomposes as*

$$H^*(\mathcal{M}_{g,*}; M) \cong H^*(\mathcal{M}_g; M) \oplus H^{*-1}(\mathcal{M}_g; H_A \otimes M) \oplus H^{*-2}(\mathcal{M}_g; M).$$

Explicitly, for any $u \in H^*(\mathcal{M}_{g,*}; M)$, we have

$$u = e''\pi_! u - (\mu \otimes 1_M)_* \left(\chi \otimes \pi^*\pi_!(\chi \otimes u)\right) + (2-2g)^{-1}\pi^*\pi_!(eu).$$

6. Contraction formula

The purpose of this section is to prove that the twisted Mumford-Morita-Miller classes $m_{i,j}$ on the mapping class group $\mathcal{M}_{g,*}$ are *stable* under the operation induced by any contraction of coefficients which is derived from the intersection pairing $\mu : H \otimes H \to \mathbb{Z}$.

The key to computing contractions of twisted Mumford-Morita-Miller classes $m_{i,j}$'s is the following theorem due to the second author [46]. Here we give an alternative proof of it.

Theorem 6.1 ([46], Theorem 1.3)**.**

$$\mu_*({k_0}^{\otimes 2}) = 2\nu - e - \bar{e} \in H^2(\overline{\mathcal{M}}_{g,*}; \mathbb{Z}).$$

Proof. Let $\mathcal{M}_{g,2}$ be the mapping class group of Σ_g relative to two disjoint embedded disks $D^2 \amalg D^2 \subset \Sigma_g$. The main ingredient of this proof is

$$\mu_*({k_0}^{\otimes 2}) = 0 \in H^2(\mathcal{M}_{g,2}; \mathbb{Z}). \tag{29}$$

In fact, in view of the Gysin sequences and Theorem 5.1, (i), the kernel of the homomorphism

$$H^2(\overline{\mathcal{M}}_{g,*}; \mathbb{Z}) \to H^2(\mathcal{M}_{g,2}; \mathbb{Z})$$

induced by the forgetful homomorphism $\mathcal{M}_{g,2} \to \mathcal{M}_g^2 \overset{\alpha_\ell}{\to} \overline{\mathcal{M}}_{g,*}$ is generated by the classes ν, e and $\bar{e}$. Hence, if (29) is established, we have $\mu_*({k_0}^{\otimes 2}) = a\nu + be + c\bar{e}$ for some $a, b, c \in \mathbb{Z}$. On the other hand we have $s^*\mu_*({k_0}^{\otimes 2}) = 0$ and $\pi_!\mu_*({k_0}^{\otimes 2}) = \bar{\pi}_!\mu_*({k_0}^{\otimes 2}) = 2g$ by (28). Hence we obtain $a = 2$, $b = c = -1$.

To prove (29) we construct a 1-cochain $c \in C^1(\mathcal{M}_{g,2}; \mathbb{Z})$, which cobounds $\mu_*({k_0}^{\otimes 2})$, in a geometrical way.

Choose two parallel simple paths ℓ and ℓ' connecting the two embedded disks. For any mapping class $\varphi \in \mathcal{M}_{g,2}$ we may consider the algebraic intersection number $\varphi(\ell) \cdot \ell'$ of two simple paths $\varphi(\ell)$ and ℓ'. Now we define a 1-cochain

$$c : \mathcal{M}_{g,2} \longrightarrow \mathbb{Z}$$

by setting $\varphi \mapsto \ell \cdot \varphi(\ell')$. Then we have

$$\begin{aligned}\mu_*({k_0}^{\otimes 2})(\varphi_1, \varphi_2) &= k_0(\varphi_1) \cdot \varphi_1 k_0(\varphi_2) = (\ell - \varphi_1(\ell)) \cdot \varphi_1(\ell' - \varphi_2(\ell')) \\ &= \ell \cdot \varphi_2(\ell') - \ell \cdot \varphi_1\varphi_2(\ell') + \ell \cdot \varphi_1(\ell') = dc(\varphi_1, \varphi_2).\end{aligned}$$

This means $\mu_*({k_0}^{\otimes 2}) = 0 \in H^2(\mathcal{M}_{g,2}; \mathbb{Z})$, as was to be shown. □

Thus any contraction of a single twisted Mumford-Morita-Miller class is expressed by an algebraic combination of other such classes. On the other hand, let M_1 and M_2 be two $\mathcal{M}_{g,*}$-modules. Then we consider the contraction map

$$1 \otimes \mu \otimes 1 : (M_1 \otimes H) \otimes (H \otimes M_2) \to M_1 \otimes M_2$$

which is given by $\xi \otimes x \otimes y \otimes \eta \mapsto \mu(x \otimes y)\xi \otimes \eta$.

Theorem 6.2 (Contraction Formula)**.** *Let $u_i \in H^*(\overline{\mathcal{M}}_{g,*}; M_i)$ $(i = 1, 2)$ be any two elements. Then we have the equality*

$$\begin{aligned}&(1 \otimes \mu \otimes 1)_*(\pi_!(u_1 \otimes k_0)\pi_!(k_0 \otimes u_2)) \\ &= -\pi_!(u_1u_2) + s^*(u_1)\pi_!(u_2) + \pi_!(u_1)s^*(u_2) - e\pi_!(u_1)\pi_!(u_2)\end{aligned}$$

as an element of $H^*(\mathcal{M}_{g,*}; M_1 \otimes M_2)$.

Proof. By what was shown in [26], Theorem 3 in Chapter II, p.126, the identification of the Lyndon-Hochschild-Serre spectral sequence

$$\rho : H^{p_i}(\mathcal{M}_{g,*}; H^{q_i}(\pi_1\Sigma_g) \otimes M_i) \longrightarrow E_2^{p_i,q_i}(M_i)$$

of the extension $\pi_1\Sigma_g \to \overline{\mathcal{M}}_{g,*} \to \mathcal{M}_{g,*}$ preserves the cup product up to sign. More precisely, if $\hat{u}_i \in H^{p_i}(\mathcal{M}_{g,*}; H \otimes M_i)$, we have

$$\rho(\hat{u}_1 \cup \hat{u}_2) = (-1)^{p_1}\rho(\hat{u}_1) \cup \rho(\hat{u}_2) \in E_2^{p_1+p_2,2}(M_1 \otimes M_2),$$

namely,

$$\pi_!(\epsilon(\hat{u}_1)\epsilon(\hat{u}_2)) = (-1)^{p_1}(1 \otimes \mu \otimes 1)_* \left((T'_*\hat{u}_1)\hat{u}_2\right)$$

in $H^{p_1+p_2}(\mathcal{M}_{g,*}; M_1 \otimes M_2)$, where $T' : H \otimes M_1 \to M_1 \otimes H$ is the switch map given by $x \otimes \xi \mapsto \xi \otimes x$ and $\epsilon : H^{p_i}(\mathcal{M}_{g,*}; H \otimes M_i) \to H^{p_i+1}(\overline{\mathcal{M}}_{g,*}; M_i)$ is the homomorphism introduced in Proposition 5.2 (iii).

Let u_i be an element of $H^{p_i+1}(\overline{\mathcal{M}}_{g,*}; M_i)$. Clearly we have $\pi_!(k_0 \otimes u_i) \in H^{p_i}(\mathcal{M}_{g,*}; H \otimes M_i)$ and $(-1)^{p_1+1}T'_*\pi_!(k_0 \otimes u_1) = \pi_!(u_1 \otimes k_0)$. By Theorem 5.3 we have

$$-\epsilon\pi_!(k_0 \otimes u_i) = u_i - \nu'\pi^*\pi_!(u_i) - \pi^*s^*(u_i).$$

Consequently we obtain

$$\begin{aligned}
&(1 \otimes \mu \otimes 1)_*(\pi_!(u_1 \otimes k_0)\pi_!(k_0 \otimes u_2)) \\
=& -\pi_!\left(\epsilon(\pi_!(k_0 \otimes u_1))\epsilon(\pi_!(k_0 \otimes u_2))\right) \\
=& -\pi_!\left((u_1 - \nu'\pi^*\pi_!(u_1) - \pi^*s^*(u_1))(u_2 - \nu'\pi^*\pi_!(u_2) - \pi^*s^*(u_2))\right) \\
=& -\pi_!(u_1u_2) + s^*(u_1)\pi_!(u_2) + \pi_!(u_1)s^*(u_2) - e\pi_!(u_1)\pi_!(u_2)
\end{aligned}$$

as was to be shown. □

This implies that any contraction of two twisted Mumford-Morita-Miller classes can be expressed by an algebraic combination of other twisted Mumford-Morita-Miller classes.

7. Representation of the crossed homomorphism $\tilde{k}$ in terms of k_0

In this section we express the crossed homomorphism

$$\tilde{k} : \mathcal{M}_{g,*} \longrightarrow \frac{1}{2}\Lambda^3 H$$

which is the main ingredient of the representation $\rho_1 : \mathcal{M}_{g,*} \to \frac{1}{2}\Lambda^3 H \rtimes Sp(2g, \mathbb{Z})$, in terms of the twisted Mumford-Morita-Miller classes introduced in §4. More precisely, we prove

Proposition 7.1. *We have the equality*

$$m_{0,3} = -6\,\tilde{k} \in H^1(\mathcal{M}_{g,*}; \Lambda^3 H).$$

The $Sp(2g, \mathbb{Z})$-module $\Lambda^3 H$ is embedded into $H \otimes \Lambda^2 H$ by

$$x \wedge y \wedge z \mapsto x \otimes y \wedge z + y \otimes z \wedge x + z \otimes x \wedge y$$

(see [28]). It induces an injective homomorphism

$$H^1(\mathcal{M}_{g,*}; \Lambda^3 H) \to H^1(\mathcal{M}_{g,*}; H \otimes \Lambda^2 H)$$

which sends the element $m_{0,3} = \pi_!({k_0}^3) \in H^1(\mathcal{M}_{g,*}; \Lambda^3 H)$ to

$$3\pi_!(k_0 \otimes {k_0}^2) \in H^1(\mathcal{M}_{g,*}; H \otimes \Lambda^2 H)$$

because of the anti-commutativity of the cup product. On the other hand we have

$$\pi_!(k_0 \otimes {k_0}^2) = -\pi_\sharp \iota^*({k_0}^2) \in H^1(\mathcal{M}_{g,*}; H \otimes \Lambda^2 H)$$

by Lemma 5.4. Here $\pi_\sharp : H^*(\mathcal{M}_g^2 : \Lambda^2 H) \to H^{*-1}(\mathcal{M}_{g,*}; H \otimes \Lambda^2 H)$ denotes the Gysin map associated to the extension $\pi_1(\Sigma_g^0) \to \mathcal{M}_g^2 \to \mathcal{M}_{g,*}$. Furthermore we have a natural isomorphism $H^1(\mathcal{M}_{g,*}; H \otimes \Lambda^2 H) \cong H^1(\mathcal{M}_{g,1}; H \otimes \Lambda^2 H)$ and hence it suffices to prove it on the mapping class group $\mathcal{M}_{g,1}$. Therefore Proposition 7.1 is reduced to the following proposition.

Proposition 7.2. *We have the equality*

$$\pi_\sharp \iota^*({k_0}^2) = 2\,\tilde{k} \in H^1(\mathcal{M}_{g,1}; H \otimes \Lambda^2 H).$$

Proof. Since the fundamental group $\pi_1(\Sigma_g^0)$ is free, we have $H^2(\pi_1(\Sigma_g^0); \Lambda^2 H) = 0$. This implies that there exists a 1-cochain $\theta \in C^1(\pi_1(\Sigma_g^0); \Lambda^2 H)$ which satisfies $(1_H)^2 = -d\theta$, namely

$$\theta(\gamma\gamma') = \theta(\gamma) + \theta(\gamma') + [\gamma] \wedge [\gamma'] \in \Lambda^2 H \tag{30}$$

for any $\gamma, \gamma' \in \pi_1({\Sigma_g}^0)$. Especially we have

$$\theta([\gamma, \gamma']) = \theta(\gamma\gamma'\gamma^{-1}{\gamma'}^{-1}) = 2[\gamma] \wedge [\gamma'].$$

Explicitly the cochain θ can be constructed by making use of the Magnus expansion of the free group $\pi_1(\Sigma_g^0)$ described by the free differential calculus of Fox as follows. Choose a symplectic generating system $\{\alpha_i, \beta_i\}_{i=1}^g$ of $\pi_1(\Sigma_g^0)$. Then the map $\theta : \pi_1(\Sigma_g^0) \to \Lambda^2 H$ is defined by

$$\theta(\gamma) = \sum_{i=1}^{g} \Big[\frac{\partial\gamma}{\partial\alpha_i}\Big] \wedge [\alpha_i] + \Big[\frac{\partial\gamma}{\partial\beta_i}\Big] \wedge [\beta_i] \in \Lambda^2 H$$

for $\gamma \in \pi_1(\Sigma_g^0)$, where $\partial/\partial\alpha_i$ and $\partial/\partial\beta_i$ denote the free differentials and $[\;] : \mathbb{Z}[\pi_1(\Sigma_g^0)] \to H$ denotes the homomorphism induced by the abelianization $\pi_1(\Sigma_g^0) \to H$.

The crossed homomorphism $\tilde{k}$ can be represented in terms of the cochain θ (see also [36]). Here we regard $\tilde{k}$ as a crossed homomorphism of $\mathcal{M}_{g,1}$ into $\mathrm{Hom}(H, \frac{1}{2}\Lambda^2 H)$.

We define the twisted product $\frac{1}{2}\Lambda^2 H \tilde{\times} H$ to be the product set $\frac{1}{2}\Lambda^2 H \times H$ equipped with the group law

$$(\xi, u)(\eta, v) = (\xi + \eta + \frac{1}{2} u \wedge v, u + v) \quad (\xi, \eta \in \frac{1}{2}\Lambda^2 H, u, v \in H).$$

By (30), the correspondence

$$N_2 \ni \gamma \mapsto (\frac{1}{2}\theta(\gamma), [\gamma]) \in \frac{1}{2}\Lambda^2 H \tilde{\times} H$$

is an isomorphism of groups. Here N_2 denotes the quotient group of $\pi_1(\Sigma_g^0)$ by its second commutator subgroup $[\pi_1(\Sigma_g^0), \Gamma_1(\pi_1(\Sigma_g^0))]$ where $\Gamma_1(\pi_1(\Sigma_g^0))$ is the commutator subgroup of $\pi_1(\Sigma_g^0)$. Hence, by the definition given in [52], the crossed homomorphism $\tilde{k}$ is given by

$$(\frac{1}{2}\theta(\varphi(\gamma)), \varphi[\gamma]) = \Big(\tilde{k}(\varphi)(\varphi[\gamma]) + \frac{1}{2}\varphi(\theta(\gamma)), \varphi[\gamma]\Big),$$

namely,

$$\tilde{k}(\varphi)(\varphi[\gamma]) = \frac{1}{2}\theta(\varphi(\gamma)) - \frac{1}{2}\varphi(\theta(\gamma))$$

for any $\varphi \in \mathcal{M}_{g,1}$ and $\gamma \in \pi_1(\Sigma_g^0)$. For example if φ belongs to the Torelli group $\mathcal{I}_{g,1}$, we have

$$\tilde{k}(\varphi)(\varphi[\gamma]) = \frac{1}{2}\theta(\varphi(\gamma)\gamma^{-1}) = \varphi(\gamma)\gamma^{-1} \mod \mathit{\Gamma}_1(\pi_1(\Sigma_g^0)) \in \Lambda^2 H$$

by (30).

Now we consider the mapping class group $\mathcal{M}_{g,1}^1$ of Σ_g relative to an embedded disk and a fixed point outside of it. It is easy to see that the forgetful homomorphism $\mathcal{M}_{g,1}^1 \to \mathcal{M}_{g,1}$ is a split extension so that we may regard $\mathcal{M}_{g,1}$ as a subgroup of $\mathcal{M}_{g,1}^1$ and the group $\mathcal{M}_{g,1}^1$ is decomposed into the semi-direct product

$$\mathcal{M}_{g,1}^1 \cong \pi_1({\Sigma_g}^0) \rtimes \mathcal{M}_{g,1}.$$

This enables us to consider the cochain $\widetilde{\theta} \in C^1(\mathcal{M}_{g,1}^1; \Lambda^2 H)$ given by

$$\widetilde{\theta}(\gamma\varphi) = \theta(\gamma), \quad (\gamma \in \pi_1(\Sigma_g^0),\, \varphi \in \mathcal{M}_{g,1})$$

as in (11). Then we have

$$\begin{aligned} d\widetilde{\theta}(\gamma_1\varphi_1, \gamma_2\varphi_2) &= \varphi_1\theta(\gamma_2) - \theta(\gamma_1\varphi_1(\gamma_2)) + \theta(\gamma_1) \\ =& \varphi_1\theta(\gamma_2) - \theta(\varphi_1(\gamma_2)) - [\gamma_1] \wedge [\varphi_1(\gamma_2)] \\ =& -2\tilde{k}(\varphi_1)(\phi_1[\gamma_2]) + ({k_0}^2)(\gamma_1\varphi_1, \gamma_2\varphi_2) \end{aligned}$$

for any $\gamma_1, \gamma_2 \in \pi_1(\Sigma_g^0)$ and $\varphi_1, \varphi_2 \in \mathcal{M}_{g,1}$. Hence, if we define $f \in C^2(\mathcal{M}_{g,1}^1; \Lambda^2 H)$ by setting

$$f(\gamma_1\varphi_1, \gamma_2\varphi_2) = -2\tilde{k}(\varphi_1)(\varphi_1[\gamma_2]),$$

f is a 2-cocycle and

$$\iota^*({k_0}^2) = [f] \in H^2(\mathcal{M}_{g,1}^1; \Lambda^2 H).$$

By [26], Proposition 3 in Chapter II, the Gysin image $\pi_\sharp \iota^* f$ is given by

$$(\pi_\sharp \iota^* f)(\varphi)(\gamma) = -f(\varphi, \varphi^{-1}(\gamma)) = 2\tilde{k}(\varphi)(\gamma) \quad (\gamma \in \pi_1(\Sigma_g^0),\, \varphi \in \mathcal{M}_{g,1}).$$

Therefore $\pi_\sharp \iota^* {k_0}^2 = \pi_\sharp \iota^*[f] = 2\tilde{k} \in H^1(\mathcal{M}_{g,1}; H \otimes \Lambda^2 H)$, as was to be shown. This completes the proof of Proposition 7.2 and hence that of Proposition 7.1. □

8. Proof of the main results and algorithm for determining $\alpha_{\mathit{\Gamma}}$

In this section we first prove Theorems 1.2 and 1.3 and then give an algorithm for determining the characteristic class $\alpha_{\mathit{\Gamma}}(g) \in \mathbb{Q}[e, e_1, e_2, \cdots]$ for any trivalent graph $\mathit{\Gamma} \in \mathcal{G}$.

In order to prove the results, we construct a *twisted cohomology class* $\alpha(\mathit{\Gamma})$ of the group $\mathcal{M}_{g,*}$ for any finite graph $\mathit{\Gamma}$ *with endpoints*. Here an *endpoint* means a vertex of degree 1. If we denote by $\nu = \nu(\mathit{\Gamma})$ the number of the endpoints, then $\alpha(\mathit{\Gamma})$ is defined as an element of $H^*(\mathcal{M}_{g,*}; {H_{\mathbb{Q}}}^{\otimes \nu})$. In the case $\mathit{\Gamma}$ is a trivalent graph, the cohomology class $\alpha(\mathit{\Gamma})$ coincides with what we have constructed in the previous sections.

Let $\mathit{\Gamma}$ be a finite graph with endpoints, and v_i, $1 \le i \le m$, the vertices of $\mathit{\Gamma}$ of degree ≥ 2. We assign each vertex v_i the twisted Mumford-Morita-Miller class

$$-\pi_!\left({k_0}^{\otimes d_i}\right) = -\frac{1}{d_i!} m_{0,d_i} \in H^{d_i-2}(\mathcal{M}_{g,*}; \Lambda^{d_i} H_{\mathbb{Q}}) \subset H^{d_i-2}(\mathcal{M}_{g,*}; {H_{\mathbb{Q}}}^{\otimes d_i}),$$

where d_i is the degree of v_i. When $n = 3$ it is equal to the cohomology class of the crossed homomorphism $\tilde{k}$. The Euler characteristic $\chi(\Gamma)$ of the graph Γ satisfies

$$-2\chi(\Gamma) + \nu(\Gamma) = \sum_{i=1}^{m} (d_i - 2).$$

So we obtain a twisted cohomology class

$$\prod_{i=1}^{m} \left(-\pi_! \left({k_0}^{\otimes d_i}\right)\right) \in H^{-2\chi(\Gamma)+\nu(\Gamma)}(\mathcal{M}_{g,*}; {H_{\mathbb{Q}}}^{\otimes \Sigma_{i=1}^m d_i}).$$

In a similar way to the construction in §2 we can construct an $Sp(2g;\mathbb{Z})$-invariant homomorphism

$$\alpha_\Gamma : H^{\otimes \Sigma_{i=1}^m d_i} \longrightarrow H^{\otimes \nu(\Gamma)}.$$

It contracts one H in $\Lambda^{d_i} H_{\mathbb{Q}}$ and another H in $\Lambda^{d_j} H_{\mathbb{Q}}$ by the intersection form μ if an edge connects the vertices v_{i_1} and v_{i_2}. Now we define the cohomology class $\alpha(\Gamma)$ by

$$\alpha(\Gamma) = \alpha_{\Gamma *} \left(\prod_{i=1}^{m} \left(-\pi_! \left({k_0}^{\otimes d_i}\right)\right)\right) \in H^{-2\chi(\Gamma)+\nu(\Gamma)}(\mathcal{M}_{g,*}; {H_{\mathbb{Q}}}^{\otimes \nu(\Gamma)}).$$

The sign of the cohomology class depends on the numbering of the vertices.

We have no need to consider vertices of degree 2. In other words, the cohomology class $\pm\alpha(\Gamma)$ depends only on the topological type of Γ. In fact, $-\pi_!({k_0}^{\otimes 2}) \in H^0(\mathcal{M}_{g,*}; H^{\otimes 2})$ corresponds to the identity 1_H of H by the intersection form μ. From what we have shown in §3, if Γ is a trivalent graph (without endpoints), the cohomology class $\alpha(\Gamma)$ coincides with $\alpha_\Gamma \in H^*(\mathcal{M}_{g,*}; \mathbb{Q})$ in the previous sections.

The contraction formulae given in §6 imply that the cohomology class $\alpha(\Gamma)$ is an algebraic combination of the Euler class e and the twisted Mumford-Morita-Miller classes $m_{i,j}$'s. Thus, for any finite graph Γ without endpoints, $\alpha(\Gamma)$ is a polynomial of the Euler class e and the Mumford-Morita-Miller classes $e_i = m_{i+1,0}$'s. This proves Theorem 1.2.

Proof of Theorem 1.3.

In this proof we regard the symplectic form ω_0 as an element of $H^{\otimes 2}$ given by

$$\omega_0 = \sum_{i=1}^{g} (x_i \otimes y_i - y_i \otimes x_i) \in H^{\otimes 2},$$

where $\{x_i, y_i\}$ is a symplectic basis of H. From (28) we have

$$\omega_0 = \pi_! \left({k_0}^{\otimes 2}\right) \in H^0(\mathcal{M}_{g,*}; H^{\otimes 2}) \subset H^{\otimes 2}.$$

Moreover we identify $H^{\otimes 2}$ with $\mathrm{End}(H)$ by the intersection form μ. Then the symplectic form ω_0 corresponds to the negative of the identity $-1_H \in \mathrm{End}(H)$, and the graded algebra $H^*(\mathcal{M}_{g,*}; H^{\otimes 2})$ acts on the graded module $H^*(\mathcal{M}_{g,*}; H)$.

We first prove (ii). Let t be an indeterminate. We define $B \in H^2(\mathcal{M}_{g,*}; H^{\otimes 2})$, $e_*(t) \in H^*(\mathcal{M}_{g,*}; \mathbb{Z})[[t]]$ and $v(t) \in H^*(\mathcal{M}_{g,*}; H)[[t]]$ by

$$B := -\pi_!(\bar{e}{k_0}^{\otimes 2}),$$

$$e_*(t) := \sum_{i=0}^{\infty} e_i t^i = \pi_! \left(\frac{\bar{e}}{1 - t\bar{e}}\right) \quad \text{and}$$

$$v(t) := -\pi_! \left(\frac{t\bar{e}k_0}{1 - t\bar{e}}\right) = -\sum_{i=1}^{\infty} m_{i,1} t^i,$$

respectively. Then, making use of the contraction formula (Theorem 6.2), we obtain

$$
\begin{aligned}
&(1-tB)v(t) = -\left((1-te)^{-1} - tee_*(t)\right) tm_{1,1}. \\
&\mu_*\left(m_{1,1}\otimes v(t)\right) = (1-te)e_*(t) - e_0\left((1-te)^{-1} - tee_*(t)\right).
\end{aligned} \tag{31}
$$

In fact, we have

$$
\begin{aligned}
&(1-tB)v(t) = v(t) - t(1\otimes\mu)_*\left(\pi_!(\bar{e}k_0\otimes k_0)\pi_!\left(k_0\frac{t\bar{e}}{1-t\bar{e}}\right)\right)\\
=&-\pi_!\left(\frac{t\bar{e}}{1-t\bar{e}}k_0\right) + t\pi_!\left(\frac{t\bar{e}^2}{1-t\bar{e}}k_0\right) - t\pi_!(\bar{e}k_0)\frac{te}{1-te} + te\pi_!(\bar{e}k_0)\pi_!\left(\frac{t\bar{e}}{1-t\bar{e}}\right)\\
=&-\pi_!(t\bar{e}k_0) - t\pi_!(\bar{e}k_0)\frac{te}{1-te} + te\pi_!(\bar{e}k_0)\pi_!\left(\frac{t\bar{e}}{1-t\bar{e}}\right)\\
=&-\left(\frac{1}{1-te} - e\pi_!\left(\frac{t\bar{e}}{1-t\bar{e}}\right)\right)t\pi_!(\bar{e}k_0) = -\left((1-te)^{-1} - tee_*(t)\right)tm_{1,1}.
\end{aligned}
$$

Similarly we can deduce the latter equality in (31) as follows.

$$
\begin{aligned}
&\mu_*\left(m_{1,1}\otimes v(t)\right) = -\mu_*\left(\pi_!(\bar{e}k_0)\pi_!\left(k_0\frac{t\bar{e}}{1-t\bar{e}}\right)\right)\\
=&\pi_!\left(\frac{t\bar{e}^2}{1-t\bar{e}}\right) - e\pi_!\left(\frac{t\bar{e}}{1-t\bar{e}}\right) - e_0\frac{te}{1-te} + ee_0\pi_!\left(\frac{t\bar{e}}{1-t\bar{e}}\right)\\
=&e_*(t) - e_0 - tee_*(t) - e_0\frac{1}{1-te} + e_0 + ee_0te_*(t)\\
=&(1-te)e_*(t) - e_0\left((1-te)^{-1} - tee_*(t)\right).
\end{aligned}
$$

The formulae (31) imply

$$
\mu_*\left(m_{1,1}(1-tB_1)^{-1}tm_{1,1}\right) = e_0 - \left((1-te)^{-2} - te(1-te)^{-1}e_*(t)\right)^{-1}e_*(t).
$$

Now the cohomology class in $H^2(\mathcal{M}_{g,*};H^{\otimes 2})$ indicated by the trivalent graph —⬭— is equal to $-B-2e\cdot 1_H$, so that the cohomology class $\alpha_{\Gamma(i)}$ is equal to $\mu_*\left(m_{1,1}(-B-2e\cdot 1_H)^{i-1}m_{1,1}\right)$. Hence we have

$$
\begin{aligned}
&\sum_{i=1}^{\infty} t^i\alpha_{\Gamma(i)} = \sum_{i=1}^{\infty} t^i\mu_*\left(m_{1,1}(-B-2e\cdot 1_H)^{i-1}m_{1,1}\right)\\
=&t\mu_*\left(m_{1,1}(1+2te+tB)^{-1}m_{1,1}\right)\\
=&-\mu_*\left(m_{1,1}\left(1-\left(-\frac{t}{1+2te}\right)B\right)^{-1}\left(-\frac{t}{1+2te}\right)m_{1,1}\right)\\
=&e_0 - \left(\left(\frac{1+2te}{1+3te}\right)^2 + \frac{te}{1+3te}e_*\left(\frac{-t}{1+2te}\right)\right)^{-1}e_*\left(\frac{-t}{1+2te}\right),
\end{aligned}
$$

as was to be shown.

Next we prove (iii). The trivalent graphs H and I define twisted cohomology classes α_H and α_I in $H^2(\mathcal{M}_{g,*};H^{\otimes 4})$, respectively. Let $T: H^{\otimes 4}\to H^{\otimes 4}$ be an Sp-equivariant map given by

$$
T(a_1\otimes a_2\otimes a_3\otimes a_4) = a_1\otimes a_3\otimes a_2\otimes a_4
$$

$(a_i \in H)$. Then $\alpha_I = -T_*(\alpha_H)$ because of the graded commutativity of the cup product. Hence the assertion (iii) is reduced to the following formula

$$\alpha_H + T_*(\alpha_H) = -e\left((\omega_0 \otimes \omega_0) + T_*(\omega_0 \otimes \omega_0)\right) \tag{32}$$

in $H^2(\mathcal{M}_{g,*}; H^{\otimes 4})$. By the contraction formula (Theorem 6.2) we obtain

$$\begin{aligned}\alpha_H =&(1 \otimes \mu \otimes 1)\left(\pi_!({k_0}^{\otimes 2} \otimes k_0)\pi_!(k_0 \otimes {k_0}^{\otimes 2})\right)\\ =&-\pi_!\left({k_0}^{\otimes 4}\right) - e\,\pi_!\left({k_0}^{\otimes 2}\right) \otimes \pi_!\left({k_0}^{\otimes 2}\right)\\ =&-\pi_!\left({k_0}^{\otimes 4}\right) - e\,\omega_0 \otimes \omega_0.\end{aligned}$$

On the other hand, we have $T_*\left({k_0}^{\otimes 4}\right) = -{k_0}^{\otimes 4}$ in $H^4(\overline{\mathcal{M}}_{g,*}; H^{\otimes 4})$. Therefore we have $\alpha_H + T_*(\alpha_H) = -e\,(\omega_0 \otimes \omega_0) - e\,T_*(\omega_0 \otimes \omega_0)$, as was to be shown.

Finally (i) follows from (ii) and (iii) immediately because by (ii), the leading term of $\alpha_{\Gamma(k)}$ is equal to $(-1)^k e_k$ while by (iii), for any connected trivalent graph Γ, the leading term of α_Γ depends only on the number of vertices of Γ. □

Next we consider how the cohomology class $\alpha(\Gamma)$ behaves under a certain modification of the graph Γ. We call an edge *internal* if it connects two vertices of degree ≥ 2 and is not a loop. Let τ be an internal edge of Γ. Then the quotient graph Γ/τ obtained by collapsing the edge τ is homotopy equivalent to the original graph Γ. We denote by $\Gamma \setminus \tau$ the finite graph obtained from Γ by removing the edge τ. Clearly $\nu(\Gamma/\tau) = \nu(\Gamma \setminus \tau) = \nu(\Gamma)$, and $-\chi(\Gamma) = -\chi(\Gamma/\tau) = -\chi(\Gamma \setminus \tau) + 1$. Then we have

Proposition 8.1.

$$\alpha(\Gamma) = \alpha(\Gamma/\tau) + e\alpha(\Gamma \setminus \tau) \in H^{-2\chi(\Gamma)+\nu(\Gamma)}(\mathcal{M}_{g,*}; H^{\otimes \nu(\Gamma)}).$$

Proof. The contraction formula (Theorem 6.2) implies

$$\begin{aligned}&(1 \otimes \mu \otimes 1)_*\left((-\pi_!({k_0}^{\otimes m_1} \otimes k_0))(-\pi_!(k_0 \otimes {k_0}^{\otimes m_2}))\right)\\ &\quad = -\pi_!({k_0}^{\otimes(m_1+m_2)}) - e(-\pi_!({k_0}^{\otimes m_1}))(-\pi_!({k_0}^{\otimes m_2}))\end{aligned}$$

for $m_1, m_2 \geq 1$. It can be visualized as in Figure 8.1. This proves the proposition. □

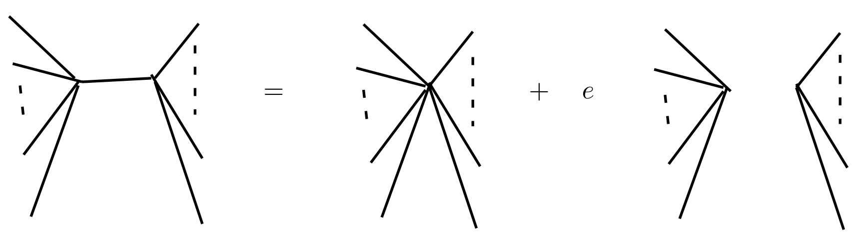

FIGURE 8.1. Collapsing an internal edge.

The bouquet $\check{\Gamma}_m$ of $m+1$ circles may be considered as one of the most degenerate graphs. We define the cohomology class $\check{e}_m \in H^{2m}(\mathcal{M}_{g,*}; \mathbb{Z})$ by

$$\check{e}_m = \alpha(\check{\Gamma}_m) = -\pi_!\left(\mu_*({k_0}^{\otimes 2})^{m+1}\right) = (-1)^m \pi_!\left((\bar{e} + e - 2\nu)^{m+1}\right).$$

We have $\check{e}_0 = -2g$, $\check{e}_1 = -e_1 + 4ge = -e_1 - 2(-2g)e$, and

$$(-1)^m \check{e}_m = e_m + \sum_{k=1}^{m-1} \binom{m+1}{k+1} e^{m-k} e_k + \left((m+1)(2-2g) - 2^{m+1}\right) e^m$$

for $m \geq 1$.

We modify any finite graph without endpoints Γ to a disjoint union of some bouquets with coefficients in the polynomials of the Euler class e by collapsing all the internal edges, so that we obtain the cohomology class $\alpha(\Gamma)$ by substituting the cohomology classes $\check{e}_m$ into the bouquets $\check{\Gamma}_m$.

As an example we consider the trivalent graph Γ_{4-3} of degree 4 in Figure 8.3. The cohomology class $\alpha(\Gamma_{4-3})$ may be computed as in Figure 8.2. This means

$$\alpha_{\Gamma_{4-3}} = \check{e}_2 + 2e\check{e}_1 + e\check{e}_0\check{e}_1 + e(e+1)\check{e}_0^2 = e_2 + ee_1(5-2g) + e^2(2-2g)(-8g-1)$$

(see the third row of Table 8.1).

We now describe the algorithm for determining the cohomology class $\alpha_\Gamma \in \mathbb{Q}[g][e, e_1, e_2, \cdots]$ for any trivalent graph Γ. It is an inductive procedure depending on the number $2k$ of vertices of Γ. If $k = 1$, we already know the answer (see Example 1.4). Suppose that we have known the answer for any Γ whose number of vertices is less than $2k$ ($k > 1$). Then we can determine α_Γ for any Γ with $2k$ vertices as follows. If Γ is not connected, then we know α_Γ because α is multiplicative with respect to the disjoint union of graphs. By Theorem 1.3 (ii) proved above, we know $\alpha_{\Gamma(k)}$. On the other hand, it is easy to see that any two connected trivalent graphs with the same numbers of vertices can be connected to each other by finitely many IH moves. Hence the claim follows by applying Theorem 1.3 (iii) and the induction assumption.

There are 5 and 17 isomorphism classes of connected trivalent graphs with $2k$ vertices for $k = 2$ and $k = 3$ respectively. They are depicted in Figure 8.3 and Figure 8.4.

We can compute the corresponding characteristic classes by applying the above algorithm. They are given in Table 8.1 and Table 8.2.

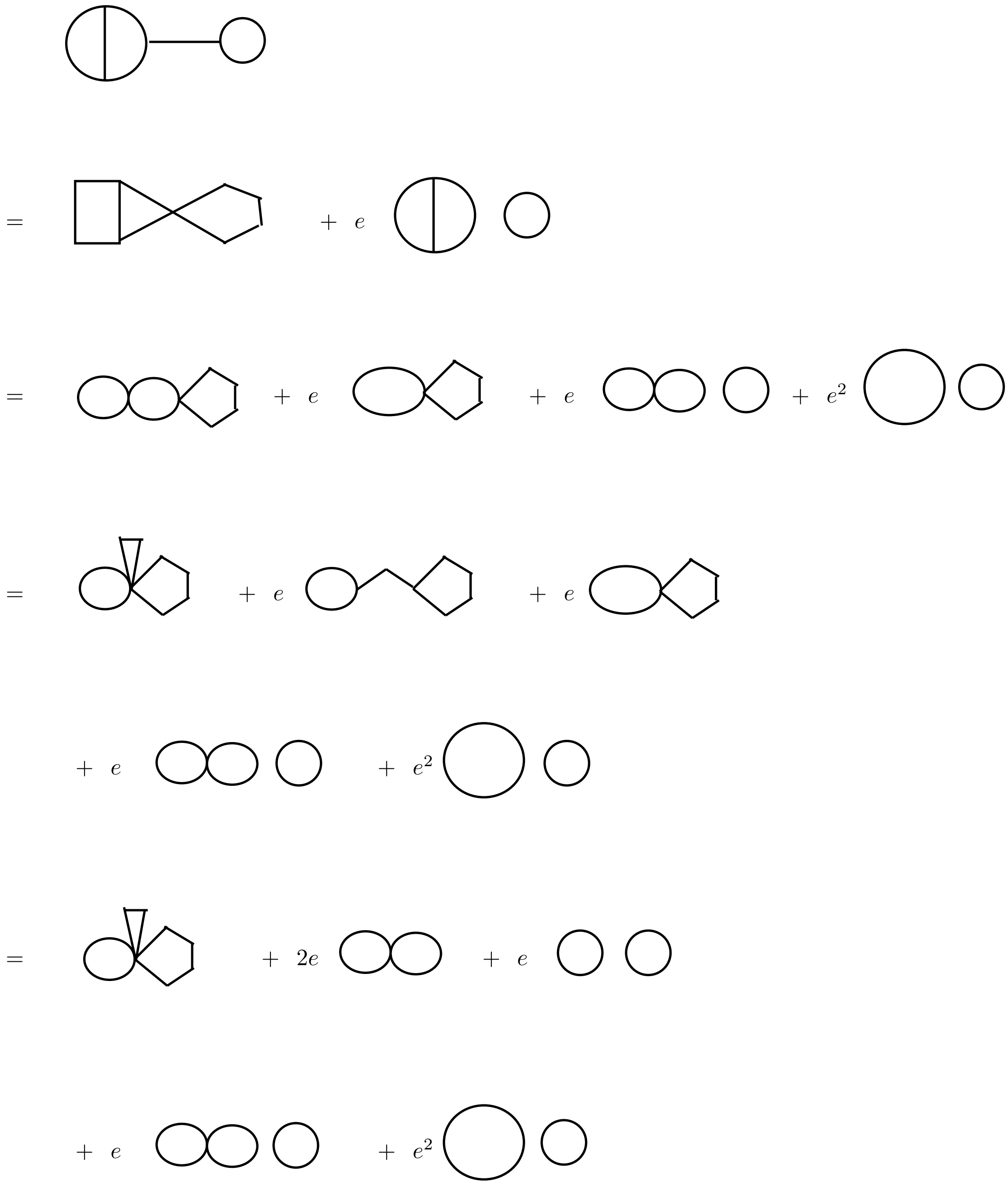

FIGURE 8.2. Computation of $\alpha(\mathit{\Gamma}_{4-3})$.

By using Theorem 1.3, we can show that the coefficients of the polynomials $\alpha_{\mathit{\Gamma}}$ satisfy certain interesting property. For example, we have the following result.

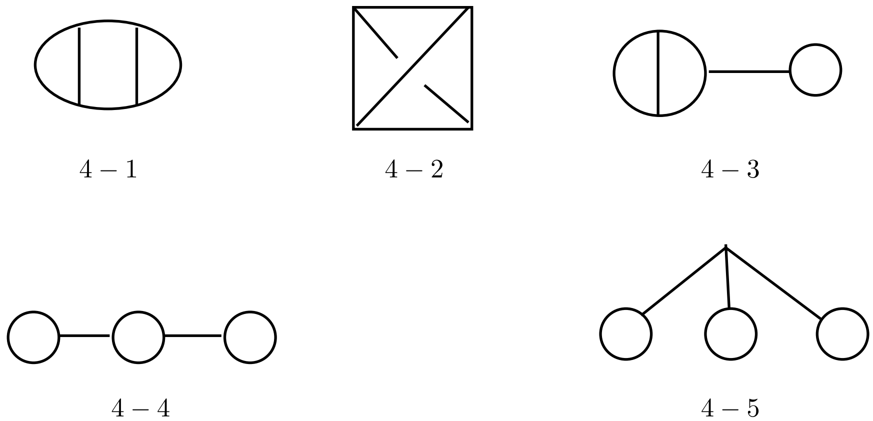

Figure 8.3. Connected trivalent graphs with 4 vertices

Table 8.1. α_Γ for connected trivalent graphs with 4 vertices

	e_2	ee_1	e^2
1	1	6	$4g^2-20g-2$
2	1	6	$-22g-2$
3	1	$-2g+5$	$16g^2-14g-2$
4	1	$-4g+4$	$-8g^3+20g^2-10g-2$
5	1	$-6g+3$	$-16g^3+24g^2-6g-2$

Table 8.2. α_Γ for connected trivalent graphs with 6 vertices

	e_3	ee_2	ee_1^2	e^2e_1	e^3
1	-1	-9	0	$4g-28$	$-32g^2+68g+18$
2	-1	-9	0	$6g-27$	$8g^3-36g^2+64g+18$
3	-1	-8	-1	$16g-22$	$-64g^2+48g+16$
4	-1	-9	0	$2g-29$	$-20g^2+74g+18$
5	-1	-9	0	-30	$-4g^2+82g+18$
6	-1	-9	0	-30	$84g+18$
7	-1	$2g-8$	0	$16g-22$	$8g^3-68g^2+44g+16$
8	-1	$2g-8$	0	$18g-21$	$16g^3-72g^2+40g+16$
9	-1	$2g-8$	0	$14g-23$	$-64g^2+48g+16$
10	-1	$2g-7$	-1	$-4g^2+24g-17$	$32g^3-76g^2+30g+14$
11	-1	$4g-7$	0	$-4g^2+26g-16$	$48g^3-84g^2+22g+14$
12	-1	$4g-7$	0	$-4g^2+24g-17$	$40g^3-80g^2+26g+14$
13	-1	$4g-6$	-1	$-12g^2+28g-13$	$-16g^4+64g^3-76g^2+16g+12$
14	-1	$4g-6$	-1	$-8g^2+32g-12$	$64g^3-88g^2+12g+12$
15	-1	$6g-6$	0	$-12g^2+30g-12$	$-16g^4+80g^3-84g^2+8g+12$
16	-1	$6g-5$	-1	$-20g^2+32g-9$	$-32g^4+96g^3-76g^2+2g+10$
17	-1	$8g-4$	-1	$-32g^2+32g-6$	$-64g^4+128g^3-64g^2-8g+8$

Proposition 8.2. *Let Γ be a connected trivalent graph with $2k$ vertices. Let us write*

$$\alpha_\Gamma = (-1)^k e_k + e\alpha_\Gamma^{(k-1)} + \cdots + e^k\alpha_\Gamma^{(0)}$$

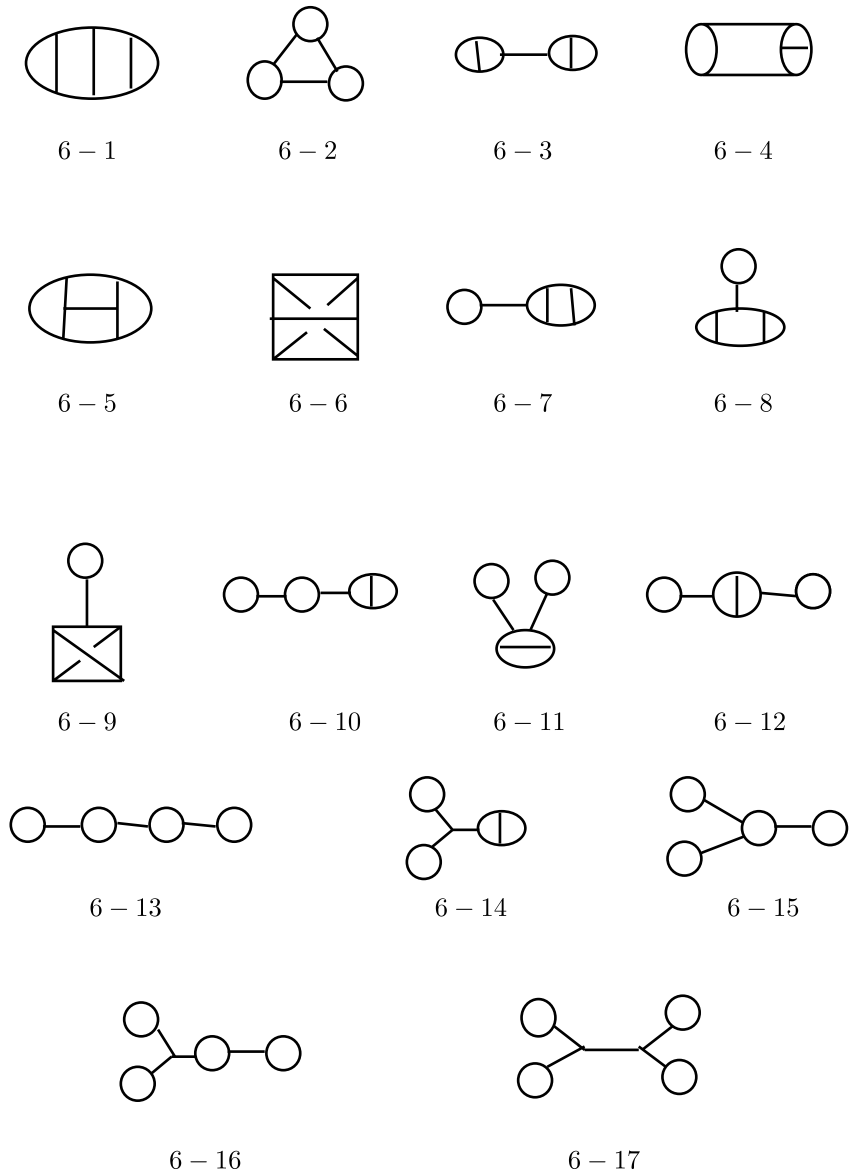

FIGURE 8.4. Connected trivalent graphs with 6 vertices

where $\alpha_\Gamma^{(i)} \in \mathbb{Q}[g][e_1, e_2 \cdot \cdots]$. *Also let us write*

$$\alpha_\Gamma^{(i)} = \sum_J f_J(g) e_1^{j_1} \cdots e_i^{j_i}$$

where $J = (j_1, \cdots, j_i)$ *runs through all multi-indices whose entries are non-negative integers such that* $j_1 + 2j_2 + \cdots + ij_i = i$. *For each* J, *let*

$$f_J(g) = a_0 + a_1(-2g) + \cdots + a_s(-2g)^s + \cdots$$

be the $(-2g)$-adic expansion of $f_J(g)$ and set $|f_J(g)| = \sum_s a_s$. Then for each $i = 0, \cdots, k-1$, the sum

$$\sum_J |f_J(g)|$$

does not depend on the choice of Γ.

Proof. As was already mentioned, any two connected trivalent graphs with the same numbers of vertives can be connected by finitely many IH moves. Hence it suffices to show that the numbers $\sum_J |f_J(g)|$ do not change under any IH move. By a direct computation, we can check that the claim holds for small degrees (see Example 8.3 below). Then the general case follows by induction on the number of vertices using Theorem 1.3 (iii). □

Example 8.3. For trivalent graphs with two vertices as in Exapmle 1.4, we have

$$\alpha_{\Gamma_1} = -e_1 + \{-(-2g)^2 - 2(-2g)\}e, \quad \alpha_{\Gamma_2} = -e_1 - 3(-2g)e.$$

Hence the sums of the coefficients of the $(-2g)$-adic expansions are $-1, -3$ for e_1, e, respectively. The coefficients of the $(-2g)$-adic expansions of polynomials in Table 8.1 are given in Table 8.3.

TABLE 8.3. Coefficients of $(-2g)$-adic expansion

	e_2	ee_1	e^2
1	1	6	$1, 10, -2$
2	1	6	$11, -2$
3	1	$1, 5$	$4, 7, -2$
4	1	$2, 4$	$1, 5, 5, -2$
5	1	$3, 3$	$2, 6, 3, -2$

Hence the sums of these coefficients for e_2, ee_1, e^2 are $1, 6, 9$, respectively, which are independent of the graph Γ. Similar computation for Table 8.2 shows that the sums of the $(-2g)$-adic coefficients for $e_3, e(e_2 + e_1^2), e^2e_1, e^3$ are given by $-1, -9, -30, -24$, respectively, which are also independent of the graph. Further computation for the case of 8 vertices shows that the sums are $1, 12, 58, 140, 45$ for the coefficients of $e^i(\text{polynomials in } e_j)$ $(i = 0, 1, \cdots, 4)$ respectively.

Remark 8.4. We can consider any coefficient of the polynomial α_Γ as an invariant defined for trivalent graphs. However the set of all such coefficients is far from being a complete set of invariants. For example, as mentioned above there are 5 and 17 isomorphism classes of connected trivalent graphs with 4 and 6 vertices, while the numbers of linearly independent coefficients are 4 and 8 respectively. Observe that the sum of the third and the fifth rows in Table 8.3 is equal to twice the fourth row. It might be worthwhile to investigate these coefficients from the viewpoint of classification of trivalent graphs.

9. CHARACTERIZATION OF $e \in H^2(\mathcal{M}_{g,*};\mathbb{Q})$ IN TERMS OF $\left(\Lambda^2(\Lambda^3 H_{\mathbb{Q}})\right)^{Sp}$

In the following three sections §9 ∼ §11, we describe another approach to our main results, namely from the viewpoint of symplectic representation theory. In particular, we prove Theorem 1.1 in this context. It is based on an analysis of the relation between IH moves of trivalent graphs and the ideal $\left([1^2]^{\mathrm{torelli}} \oplus [2^2]\right)$ of $\Lambda^*(\Lambda^3 H^*_{\mathbb{Q}})$.

As the first step, we describe invariant tensors of $\Lambda^2(\Lambda^3 H_{\mathbb{Q}})$ explicitly and by using them we characterize the Euler class $e \in H^2(\mathcal{M}_{g,*};\mathbb{Q})$ (see Corollary 9.6).

Let

$$p : H_{\mathbb{Q}}^{\otimes 6} \longrightarrow \Lambda^2(\Lambda^3 H_{\mathbb{Q}})$$

be the canonical projection. We define two linear chord diagrams Z_1, Z_2 with 6 vertices by

$$Z_1 = \{\{1,2\},\{3,4\},\{5,6\}\}, \quad Z_2 = \{\{1,6\},\{2,5\},\{3,4\}\}. \tag{33}$$

Then the associated graphs Γ_{Z_1} and Γ_{Z_2} are trivalent graphs with 2 vertices: the former has 2 loops while the latter is the *theta* graph which were already introduced in Example 1.4. It is easy to see that two elements $a_{\Gamma_{Z_1}}, a_{\Gamma_{Z_2}}$ form a basis of $\left(\Lambda^2(\Lambda^3 H_{\mathbb{Q}})\right)^{Sp} \cong \mathbb{Q}^2$. Passing to the dual, two elements $\alpha_{\Gamma_{Z_1}}, \alpha_{\Gamma_{Z_2}}$ form a basis of $\mathrm{Hom}(\Lambda^2(\Lambda^3 H_{\mathbb{Q}}),\mathbb{Q})^{Sp}$. Explicitly we have

$$\begin{aligned}
&\alpha_{\Gamma_{Z_1}}\left((a_1\wedge a_2\wedge a_3)\wedge(b_1\wedge b_2\wedge b_3)\right)\\
&\qquad = \sum_{\sigma,\tau\in\mathfrak{S}_3} \operatorname{sgn}\sigma\operatorname{sgn}\tau\,(a_{\sigma(1)}\cdot a_{\sigma(2)})(b_{\tau(1)}\cdot b_{\tau(2)})(a_{\sigma(3)}\cdot b_{\tau(3)})\\
&\alpha_{\Gamma_{Z_1}}\left((a_1\wedge a_2\wedge a_3)\wedge(b_1\wedge b_2\wedge b_3)\right)\\
&\qquad = -\sum_{\sigma,\tau\in\mathfrak{S}_3} \operatorname{sgn}\sigma\operatorname{sgn}\tau\,(a_{\sigma(1)}\cdot b_{\tau(1)})(a_{\sigma(2)}\cdot b_{\tau(2)})(a_{\sigma(3)}\cdot b_{\tau(3)})
\end{aligned}$$

where $a_i, b_i \in H_{\mathbb{Q}}$.

As in [55], we denote by $\mathcal{D}^\ell(2k)$ the set of all linear chord diagrams with $2k$ vertices.

Lemma 9.1. *Let $C, C' \in \mathcal{D}^\ell(2k)$ be two linear chord diagrams with $2k$ vertices. Then*

$$\alpha_C(a_{C'}) = (-1)^{k-r}(2g)^r$$

where $r = r(C,C')$ is the number of connected components of the union $C \cup C'$.

Proof. To prove this, we recall a result of [57], Lemma 3.3. Let i,j be two indices with $1 \le i < j \le 2k$ and let $p_{ij} : H_{\mathbb{Q}}^{\otimes 2k} \to H_{\mathbb{Q}}^{\otimes 2k}$ be the map defined by first taking the contraction of the i-th and the j-th entries by the intersection pairing and then putting ω_0 there. More precisely

$$\begin{aligned}
p_{ij}(a_1\otimes\cdots\otimes a_{2k}) = (a_i\cdot a_j)\sum_{s=1}^{g}\big\{&a_1\otimes\cdots\otimes x_s\otimes\cdots\otimes y_s\otimes\cdots\otimes a_{2k}\\
&- a_1\otimes\cdots\otimes y_s\otimes\cdots\otimes x_s\otimes\cdots\otimes a_{2k}\big\}.
\end{aligned}$$

Let $C \in \mathcal{D}^\ell(2k)$ be any linear chord diagram. Then for any two indices i,j with $1 \le i < j \le 2k$, we have

$$p_{ij}(a_C) = \begin{cases} 2g\,a_C & \{i,j\}\in C\\ -a_{C'} & \{i,j\}\notin C. \end{cases}$$

where C' is the linear chord diagram defined as follows. Let j', i' be indices such that $\{i,j'\},\{i',j\} \in C$. Then

$$C' = C \setminus \{\{i,j'\},\{i',j\}\} \cup \{\{i,j\},\{i',j'\}\}.$$

Now we prove the assertion using the above result. If $r = k$, namely if $C = C'$, then clearly $\alpha_C(a_{C'}) = (2g)^k$. If $r < k$, then by the above formula the factor -1 arises $k - r$ times while the factor $2g$ arises r times. This completes the proof. □

Let the symmetric group $\mathfrak{S}_{6k}$ of degree $6k$ act naturally on $H_{\mathbb{Q}}^{\otimes 6k}$ and hence on $\mathcal{D}^\ell(6k)$. We consider the subgroup $G_k = (\mathfrak{S}_3)^{2k} \rtimes \mathfrak{S}_{2k}$ of $\mathfrak{S}_{6k}$, which is a semi-direct product of $2k$-copies of $\mathfrak{S}_3$ with $\mathfrak{S}_{2k}$, whose i-th $\mathfrak{S}_3$ acts on the $(3i, 3i+1, 3i+2)$ components of $H_{\mathbb{Q}}^{\otimes 6k}$ and $\mathfrak{S}_{2k}$ acts on these $2k$ summands isomorphic to $H_{\mathbb{Q}}^{\otimes 3}$ by permutations.

Lemma 9.2. *Let Γ, Γ' be two trivalent graphs with $2k$ vertices and let $C, C' \in \mathcal{D}^\ell(6k)$ be their arbitrary lifts respectively. Then*

$$\alpha_\Gamma(a_{\Gamma'}) = \frac{1}{(2k)!} \sum_{\sigma \in G_k} \alpha_C(a_{\sigma(C')}) = \frac{1}{(2k)!} \sum_{\sigma \in G_k} \alpha_{\sigma(C)}(a_{C'}).$$

Proof. The first equality follows from

$$\begin{aligned} \alpha_\Gamma(a_{\Gamma'}) =& \alpha_\Gamma\big(p(a_{C'})\big) \\ =& \frac{1}{(2k)!} \alpha_C\big(i \circ p(a_{C'})\big) \\ =& \frac{1}{(2k)!} \sum_{\sigma \in G_k} \alpha_C\big(a_{\sigma(C')}\big). \end{aligned}$$

The second equality is clear. □

Proposition 9.3. *The matrix $\big(\alpha_{\Gamma_{Z_i}}(a_{\Gamma_{Z_j}})\big)$ $(i, j = 1, 2)$ is given by*

$$\begin{pmatrix} 4(2g)^3 - 16(2g)^2 + 16(2g) & -12(2g)^2 + 24(2g) \\ -12(2g)^2 + 24(2g) & 6(2g)^3 - 18(2g)^2 + 12(2g) \end{pmatrix}$$

and its determinant is

$$2^8 3(g-2)(g-1)^2 g^2 (2g+1).$$

Proof. We use Lemma 9.2. By an obvious skew symmetricity of our computation with respect to the former and the latter 3-components of $H_{\mathbb{Q}}^{\otimes 6}$, we have only to compute the values of α_{C_i} on each element in the orbits of a_{C_j} under the action of $(\mathfrak{S}_3)^2$. By explicit graphic computation, we find that the number r of connected components of $C_i \cup \sigma(C_j)$ $(\sigma \in (\mathfrak{S}_3)^2)$ takes the values $3, 2, 1$, on $4, 16, 16$ elements in the orbit, respectively, for $i = j = 1$. Similarly $r = 3, 2, 1$ on $0, 12, 24$ elements in the orbit for $i = 1, j = 2$ and $6, 18, 12$ elements in the orbit for $i = 2, j = 2$. The assertion then follows from Lemma 9.1. □

Remark 9.4. The above value of determinant is consistent with the stable range of the cohomology group $H^2(\Lambda^2(\Lambda^3 H_{\mathbb{Q}})) \cong \mathbb{Q}^2$ which is $g \geq 3$. If $g = 2$, two elements $\alpha_{\Gamma_{Z_1}}, \alpha_{\Gamma_{Z_1}}$ are linearly dependent which reflects the fact that $e_1 = 0$ in this case.

Let $a(g), b(g), c(g), d(g)$ be the entries of the matrix appearing in Proposition 9.3 and also let $D(g)$ be its determinant. Thus

$$a(g) = 4(2g)^3 - 16(2g)^2 + 16(2g), \quad b(g) = c(g) = -12(2g)^2 + 24(2g)$$
$$d(g) = 6(2g)^3 - 18(2g)^2 + 12(2g), \quad D(g) = 2^8 3(g-2)(g-1)^2 g^2 (2g+1).$$

Proposition 9.5. *The projection* $p : \Lambda^2(\Lambda^3 H_{\mathbb{Q}}) \to \big(\Lambda^2(\Lambda^3 H_{\mathbb{Q}})\big)^{Sp}$ *to the* Sp*-invariant part is given by*

$$\begin{aligned} &p(\xi)\\ =&D(g)^{-1}\Big\{\big(d(g)\alpha_{\Gamma_{Z_1}}(\xi) - c(g)\alpha_{\Gamma_{Z_2}}(\xi)\big)a_{\Gamma_{Z_1}} + \big(-b(g)\alpha_{\Gamma_{Z_1}}(\xi) + a(g)\alpha_{\Gamma_{Z_2}}(\xi)\big)a_{\Gamma_{Z_2}}\Big\} \end{aligned}$$

for all $g \geq 3$ *where* $\xi \in \Lambda^2(\Lambda^3 H_{\mathbb{Q}})$.

Proof. This follows directly from Proposition 9.3. □

We define

$$\Gamma_e = \Gamma_{Z_1} - \Gamma_{Z_2}, \quad a_{\Gamma_e} = a_{\Gamma_{Z_1}} - a_{\Gamma_{Z_2}}.$$

Here the subscript e indicates the Euler class $e \in H^2(\mathcal{M}_{g,*};\mathbb{Z})$. This is because it was proved in [50] [53] that

$$\alpha_{\Gamma_e} = -2g(2g+1)e.$$

Corollary 9.6. *If an element* $\xi \in \Lambda^2(\Lambda^3 H_{\mathbb{Q}})$ *satisfies*

$$\alpha_{\Gamma_{Z_1}}(\xi) = 2s, \quad \alpha_{\Gamma_{Z_2}}(\xi) = -3s$$

for some s*, then*

$$p(\xi) = \frac{s}{8(g-1)g(2g+1)} a_{\Gamma_e}.$$

Proof. Computation shows that

$$2d(g) + 3c(g) = 2b(g) + 3a(g) = 2^5 3(g-2)(g-1)g.$$

Then the assertion follows from Proposition 9.5. □

10. IH moves of trivalent graphs and representation theory

In this section, we give a complete description of the linear map

$$f_{IH} : H_{\mathbb{Q}}^{\otimes 4} \longrightarrow \Lambda^2(\Lambda^3 H_{\mathbb{Q}})$$

introduced by Garoufalidis and Nakamura in [11]. It expresses the IH moves of trivalent graphs in the context of symplectic representation theory. Explicitly it is defined as

$$\begin{aligned} f_{IH}(a) =& \sum_{i=1}^{g}\{(a_1 \wedge a_2 \wedge x_i) \wedge (y_i \wedge a_3 \wedge a_4) - (a_1 \wedge a_2 \wedge y_i) \wedge (x_i \wedge a_3 \wedge a_4)\}\\ &- \sum_{i=1}^{g}\{(a_1 \wedge a_3 \wedge x_i) \wedge (y_i \wedge a_4 \wedge a_2) - (a_1 \wedge a_3 \wedge y_i) \wedge (x_i \wedge a_4 \wedge a_2)\} \end{aligned}$$

where $a = a_1 \otimes a_2 \otimes a_3 \otimes a_4 \in H_{\mathbb{Q}}^{\otimes 4}$. For later use (in the computations of Table 11.1 and Table 11.2 in the next section), we denote the i-th term (where $i = 1,2,3,4$) in the above expression by $\tilde{a}_i$ so that $f_{IH}(a) = \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + \tilde{a}_4$. We summarize their result concerning f_{IH} as follows.

Proposition 10.1 ([11])**.** *For any element* $a \in H_{\mathbb{Q}}^{\otimes 4}$, $f_{IH}(a)$ *is invariant under the variable changes* $a_1 \leftrightarrow a_4, a_2 \leftrightarrow a_3$ *and* $a_1, a_4 \leftrightarrow a_2, a_3$ *so that* f_{IH} *factors through* $S^2(S^2 H_{\mathbb{Q}})$ *where* S *denotes the symmetric power. The irreducible decomposition of this* Sp*-module is given by*

$$[4] \oplus [2^2] \oplus [1^2] \oplus [0]$$

(for any $g \geq 4$) and under the composed map

$$p \circ f_{IH} : H_{\mathbb{Q}}^{\otimes 4} \longrightarrow \Lambda^2(\Lambda^3 H_{\mathbb{Q}}) \longrightarrow \Lambda^2 U_{\mathbb{Q}},$$

the first summand goes to 0 *while the other three summands remain nontrivially in* $\Lambda^2 U_{\mathbb{Q}}$.

Now to analize the effect of IH (or equivalently Whitehead) moves on the characteristic classes of moduli space of curves through the homomorphism Φ_α in (6), we have to investigate the map f_{IH} thoroughly. More precisely, as for the summand $[2^2]$ there is nothing to add because the multiplicity of $[2^2]$ in $\Lambda^2(\Lambda^3 H_{\mathbb{Q}})$ is one (see [53]). However there are 3 and 2 copies of $[1^2]$ and $[0]$, respectively, in $\Lambda^2(\Lambda^3 H_{\mathbb{Q}})$ (see the above cited paper) so that we have to determine the exact places of $f_{IH}([1^2])$ and $f_{IH}([0])$.

First we deal with the trivial summand $[0]$. Recall from the previous section that we have the element $a_{\Gamma_e} = a_{\Gamma_{Z_1}} - a_{\Gamma_{Z_2}}$.

Proposition 10.2. *The image of the trivial summand* $[0] \subset S^2(S^2 H_{\mathbb{Q}})$ *under the map* f_{IH} *is the subspace of* $\left(\Lambda^2(\Lambda^3 H_{\mathbb{Q}})\right)^{Sp} \cong \mathbb{Q}^2$ *spanned by the element* a_{Γ_e}.

Proof. Since the image is 1-dimensional, it is enough to compute one particular nontrivial element. We choose $a = x_1 \otimes y_1 \otimes x_1 \otimes y_1 \in H_{\mathbb{Q}}^{\otimes 4}$ for such an element. Then

$$f_{IH}(a) = \sum_{i=1}^{g} \left\{ (x_1 \wedge y_1 \wedge x_i) \wedge (y_i \wedge x_1 \wedge y_1) - (x_1 \wedge y_1 \wedge y_i) \wedge (x_i \wedge x_1 \wedge y_1) \right\}.$$

Hence

$$\alpha_{\Gamma_{Z_1}}(f_{IH}(a)) = 8(g-1), \quad \alpha_{\Gamma_{Z_2}}(f_{IH}(a)) = -12(g-1).$$

In view of Corollary 9.6, this implies that the Sp-invariant component of $f_{IH}(a)$ is a non-zero multiple of the element a_{Γ_e}. This completes the proof. □

Next the effect of the map f_{IH} on the $[1^2]$-component was determined in [56] as follows.

Proposition 10.3 ([56])**.** *The image of the summand* $[1^2] \subset S^2(S^2 H_{\mathbb{Q}})$ *under the map* f_{IH} *coincides exactly with the Torelli summand* $[1^2]^{torelli} \subset \Lambda^2(\Lambda^3 H_{\mathbb{Q}})$.

Here $[1^2]^{torelli} \subset \Lambda^2(\Lambda^3 H_{\mathbb{Q}})$ denotes the *Poincaré dual* of the original Torelli summand $[1^2]^{\mathrm{torelli}} \subset \Lambda^2(\Lambda^3 H_{\mathbb{Q}}^*)$ in cohomology (we use the same symbol).

Thus we find that the graphic operation of IH moves on trivalent graphs fits algebraic structure of the Torelli group (as well as that of the mapping class group) perfectly through symplectic representation theory.

11. Alternative proof of the main results

In this section, we give an alternative proof of our main result in the context of symplectic representation theory. In particular we prove Theorem 1.1 in the context of symplectic representation theory. The main ingredient of our first proof in §6 was the contraction formula (Theorem 6.2) and we obtain Theorem 1.3 by applying it. In contrast with this, the key to the second proof is Proposition 11.1 below which is an enhancement of Theorem 1.3, (iii).

Let Γ_1, Γ_2 be two trivalent graphs with $2k$ vertices. Assume that they are related by an IH-move in the sense of Garoufalidis and Nakamura [11], namely there is an embedding $I \subset \varGamma_1$ such that $\varGamma_2$ is obtained from $\varGamma_1$ by replacing I in $\varGamma_1$ by H. In classical terms, IH-moves are nothing but the Whitehead moves. Let τ_1 and τ_2 be the corresponding edges of $\varGamma_1$ and $\varGamma_2$ respectively. We would like to analize the difference $a_{\varGamma_1} - a_{\varGamma_2} \in (\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}))^{Sp}$.

Proposition 11.1. *Let $\varGamma_i$ $(i = 1, 2)$ be two trivalent graphs with $2k$ vertices which are related by an IH move and let $\tau_i \subset \varGamma_i$ be the corresponding edges. Then the element*

$$\alpha_{\varGamma_1} - \alpha_{\varGamma_2} - \frac{1}{2g(2g+1)}\alpha_{\varGamma_e}\big(\alpha_{\varGamma_1\setminus\tau_1} - \alpha_{\varGamma_2\setminus\tau_2}\big) \ \in (\Lambda^{2k}(\Lambda^3 H^*_{\mathbb{Q}}))^{Sp}$$

belongs to the ideal $\big([1^2]^{torelli} \oplus [2^2]\big)$.

In view of Proposition 3.2 together with the fact that $\alpha_{\varGamma_e} = -2g(2g+1)e$, we obtain another proof of Theorem 1.3, (iii) as an immediate consequence of the above proposition.

Proof of Proposition 11.1.
We first lift $\varGamma_1$ to a linear chord diagram C_1 with $6k$ vertices in such a way that $\{3, 4\} \in C_1$ and the projection of the corresponding chord is equal to the edge τ_1. Let

$$a_{C_1} \in (H_{\mathbb{Q}}^{\otimes 6k})^{Sp}$$

be the invariant tensor associated to C_1. Then we have

$$a_{\varGamma_1} = p(a_{C_1})$$

where $p : H_{\mathbb{Q}}^{\otimes 6k} \rightarrow \Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$ is the canonical projection. Now let

$$f_W : H_{\mathbb{Q}}^{\otimes 4} \longrightarrow H_{\mathbb{Q}}^{\otimes 4}$$

be the Sp-equivariant map defined by

$$f_W(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = a_1 \otimes a_2 \otimes a_3 \otimes a_4 - a_1 \otimes a_3 \otimes a_4 \otimes a_2$$

$(a_i \in H_{\mathbb{Q}})$. It represents the essential part of the map f_{IH}. We consider $H_{\mathbb{Q}}^{\otimes 4}$ as a direct summand of $H_{\mathbb{Q}}^{\otimes 6k}$ by sending the $(1, 2, 3, 4)$ factor of the former space to the $(1, 2, 5, 6)$ factor of the latter space. We set

$$\tilde{f}_W = f_W \otimes \mathrm{id} : H_{\mathbb{Q}}^{\otimes 6k} \longrightarrow H_{\mathbb{Q}}^{\otimes 6k}.$$

Then it is easy to see that

$$\tilde{f}_W(a_{C_1}) = a_{C_1} - a_{C_2}$$

where C_2 is the linear chord diagram obtained by applying the permutation $(2, 5, 6) \mapsto (6, 2, 5)$ to C_1. It follows that $\varGamma_{C_2} = \varGamma_2$ and hence

$$p(\tilde{f}_W(a_{C_1})) = a_{\varGamma_1} - a_{\varGamma_2}.$$

Now consider the *partial projection*

$$p \otimes p : H_{\mathbb{Q}}^{\otimes 6k} = H_{\mathbb{Q}}^{\otimes 6} \otimes H_{\mathbb{Q}}^{\otimes (6k-6)} \longrightarrow \Lambda^2(\Lambda^3 H_{\mathbb{Q}}) \otimes \Lambda^{2k-2}(\Lambda^3 H_{\mathbb{Q}})$$

where we project the first 6 and the remaining $(6k-6)$ factors of $H_{\mathbb{Q}}^{\otimes 6k}$ to $\Lambda^2(\Lambda^3 H_{\mathbb{Q}})$ and $\Lambda^{2k-2}(\Lambda^3 H_{\mathbb{Q}})$ respectively.

Consider the following composition of various Sp-equivariant maps

$$
\begin{array}{ccc}
H_{\mathbb{Q}}^{\otimes 6} \otimes H_{\mathbb{Q}}^{\otimes (6k-6)} & \ni & a_{C_1} \\
\Big\downarrow \tilde{f}_W & & \Big\downarrow \\
H_{\mathbb{Q}}^{\otimes 6} \otimes H_{\mathbb{Q}}^{\otimes (6k-6)} & \ni & a_{C_1} - a_{C_2} \\
\Big\downarrow p\otimes p & & \Big\downarrow \\
\Lambda^2(\Lambda^3 H_{\mathbb{Q}}) \otimes \Lambda^{2k-2}(\Lambda^3 H_{\mathbb{Q}}) & \ni & p \otimes p(a_{C_1} - a_{C_2}) \\
\Big\downarrow \text{wedge product} & & \Big\downarrow \\
\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}) & \ni & a_{\Gamma_1} - a_{\Gamma_2}
\end{array} \tag{34}
$$

under which the element a_{C_1} goes to $a_{\Gamma_1} - a_{\Gamma_2}$. Now let $a = a_1 \otimes a_2 \otimes a_3 \otimes a_4 \in H_{\mathbb{Q}}^{\otimes 4}$ be any element and set $\tilde{a} = a \otimes \omega_0 \in H_{\mathbb{Q}}^{\otimes 4} \otimes' H_{\mathbb{Q}}^{\otimes 2} = H_{\mathbb{Q}}^{\otimes 6}$ where $\otimes'$ means that the $(1,2,3,4,5,6)$ components go to $(1,2,5,6,3,4)$ components. Then we have

$$p \circ (f_W \otimes \mathrm{id})(\tilde{a}) = f_{IH}(a).$$

Hence, by applying Proposition 10.1, Proposition 10.2 and Proposition 10.3, we can conclude that the element $a_{\Gamma_1} - a_{\Gamma_2}$ is divisible by α_{Γ_e} modulo the ideal $\big([1^2]^{\mathrm{torelli}} \oplus [2^2]\big)$. In view of Proposition 10.2, to determine the α_{Γ_e}-factor we have only to compute $\alpha_{\Gamma_{Z_1}}(\tilde{a})$ where a runs through the first 4 components of each monomial in a_{C_1}. Table 11.1 and Table 11.2 indicate all possible cases where this value may not vanish.

Table 11.1. $\alpha_{\Gamma_{Z_1}}(\tilde{a})$

	a_1	a_2	a_3	a_4	$\alpha_{\Gamma_{Z_1}}(\tilde{a}_1)$	$\alpha_{\Gamma_{Z_1}}(\tilde{a}_2)$	$\alpha_{\Gamma_{Z_1}}(\tilde{a}_3)$	$\alpha_{\Gamma_{Z_1}}(\tilde{a}_4)$	$\alpha_{\Gamma_{Z_1}}(\tilde{a})$
1	x_j	y_j	x_k	y_k	$4(g-2)$	$4(g-2)$	0	8	$8(g-1)$
2	y_j	x_j	x_k	y_k	$-4(g-2)$	$-4(g-2)$	-4	-4	$-8(g-1)$
3	x_j	y_j	y_k	x_k	$-4(g-2)$	$-4(g-2)$	-4	-4	$-8(g-1)$
4	y_j	x_j	y_k	x_k	$4(g-2)$	$4(g-2)$	8	0	$8(g-1)$
5	x_j	y_j	x_j	y_j	$4(g-1)$	$4(g-1)$	0	0	$8(g-1)$
6	y_j	x_j	x_j	y_j	$-4(g-1)$	$-4(g-1)$	$-4(g-1)$	$-4(g-1)$	$-16(g-1)$
7	x_j	y_j	y_j	x_j	$-4(g-1)$	$-4(g-1)$	$-4(g-1)$	$-4(g-1)$	$-16(g-1)$
8	y_j	x_j	y_j	x_j	$4(g-1)$	$4(g-1)$	0	0	$8(g-1)$
$1'$	x_j	x_k	y_j	y_k	0	8	$4(g-2)$	$4(g-2)$	$8(g-1)$
$2'$	y_j	x_k	x_j	y_k	-4	-4	$-4(g-2)$	$-4(g-2)$	$-8(g-1)$
$3'$	x_j	y_k	y_j	x_k	-4	-4	$-4(g-2)$	$-4(g-2)$	$-8(g-1)$
$4'$	y_j	y_k	x_j	x_k	8	0	$4(g-2)$	$4(g-2)$	$8(g-1)$
$5'$	x_j	x_j	y_j	y_j	0	0	$4(g-1)$	$4(g-1)$	$8(g-1)$
$8'$	y_j	y_j	x_j	x_j	0	0	$4(g-1)$	$4(g-1)$	$8(g-1)$

In Table 11.1, we assume $j \neq k$ in the cases $1,2,3,4,1',2',3',4'$ and there are no $6',7'$ because they are the same as $6,7$ respectively.

There are other cases where it may not be quite trivial whether the values of $\alpha_{\Gamma_{Z_1}}(\tilde{a})$ are zero or not. We list them in Table 11.2. In the cases $9,10,11,12$, we assume that $j \neq k$.

Table 11.2. Other cases

	a_1	a_2	a_3	a_4	$\alpha_{\Gamma_{Z_1}}(\tilde{a}_1)$	$\alpha_{\Gamma_{Z_1}}(\tilde{a}_2)$	$\alpha_{\Gamma_{Z_1}}(\tilde{a}_3)$	$\alpha_{\Gamma_{Z_1}}(\tilde{a}_4)$	$\alpha_{\Gamma_{Z_1}}(\tilde{a})$
9	x_j	y_k	x_k	y_j	4	4	0	-8	0
10	y_j	x_k	y_k	x_j	4	4	-8	0	0
11	x_j	x_k	y_k	y_j	0	-8	4	4	0
12	y_j	y_k	x_k	x_j	-8	0	4	4	0
$9'$	x_j	y_j	x_j	y_j	0	0	0	0	0
$10'$	y_j	x_j	y_j	x_j	0	0	0	0	0
$11'$	x_j	x_j	y_j	y_j	0	0	0	0	0
$12'$	y_j	y_j	x_j	x_j	0	0	0	0	0

Thus we can conclude that, in all cases other than those listed in Table 11.1, the images of them in $\Lambda^2(\Lambda^3 H_{\mathbb{Q}})$ have no component in the trivial summand [0].

Summing up the above computation, we obtain Table 11.3 which is a complete list of elements of $H_{\mathbb{Q}}^{\otimes 4}$ whose images in $\Lambda^2(\Lambda^3 H_{\mathbb{Q}})$ have non-trivial components in [0]. In Table 11.3, we do not assume that $j \neq k$ and the value $-16(g-1)$ of $\alpha_{\Gamma_{Z_1}}(\tilde{a})$ on the element named number 6 (resp. 7) in Table 11.1 are shared evenly by s_2 and t_2 (resp. s_3 and t_3).

Table 11.3. Non zero terms

	a_1	a_2	a_3	a_4	Table 11.1	$\alpha_{\Gamma_{Z_1}}(\tilde{a})$
s_1	x_j	y_j	x_k	y_k	1, 5	$8(g-1)$
s_2	y_j	x_j	x_k	y_k	2, 6	$-8(g-1)$
s_3	x_j	y_j	y_k	x_k	3, 7	$-8(g-1)$
s_4	y_j	x_j	y_k	x_k	4, 8	$8(g-1)$
t_1	x_j	x_k	y_j	y_k	$1'$, $5'$	$8(g-1)$
t_2	y_j	x_k	x_j	y_k	$2'$, 6	$-8(g-1)$
t_3	x_j	y_k	y_j	x_k	$3'$, 7	$-8(g-1)$
t_4	y_j	y_k	x_j	x_k	$4'$, $8'$	$8(g-1)$

Now it is easy to see that the terms $s_1 \sim s_4$ in Table 11.3 corresponds to $\alpha_{\Gamma_1 \backslash \tau_1}$ while the terms $s_1 \sim s_4$ corresponds to $-\alpha_{\Gamma_2 \backslash \tau_2}$ (the minus sign comes from the sign of the permutation $(1234) \mapsto (1324)$). In both cases, the value $\alpha_{\Gamma_{Z_1}}$ is equal to $8(g-1)$. Hence by Corollary 9.6, the coefficient is equal to $\frac{1}{2g(2g+1)}$. This completes the proof. □

Proof of Theorem 1.1 in the context of symplectic representation theory.

First we prove that the images of the homomorphisms ρ_2^* in (3) coincide exactly with the tautological algebras $\mathcal{R}^*(\mathcal{M}_{g,*}), \mathcal{R}^*(\mathcal{M}_g)$ based on Proposition 11.1 and its corollary (Theorem 1.3, (iii)). We use induction on degrees. The case of degree 2 was already proved in [47][50]. On the other hand, it was proved in [53] that for any genus g and degree $2i$, there exists a degree $2i$ homogeneous element of $\mathbb{Q}[\Gamma; \Gamma \in \mathcal{G}]$ such that the associated characteristic class is equal to a non-zero multiple of e_i. It is easy to see that this element must contain at least one *connected* trivalent graph with $2i$ vertices as a non-trivial term and the sum of the coefficients of such connected graphs is non-zero. Then the induction proceeds again by the fact that any two connected trivalent graphs with the same number of vertices are connected by finitely many IH moves together with Theorem 1.3, (iii).

Next we prove that the homomorphisms ρ_2^* are isomorphisms in the stable range. It is a consequence of the above argument that the dimensions of the degree $2i$ parts of the left hand sides of diagram (4) cannot exceed those of the degree $2i$ parts of the polynomial algebras $\mathbb{Q}[e, e_1, e_2, \cdots]$ and $\mathbb{Q}[e_1, e_2, \cdots]$ respectively. On the other hand, we already know that the tautological algebras have no relations in the stable range (see [43][45]). The claim now follows. The precise value of the stable range $\leq \frac{2}{3}g$ is due to Harer's improved stability theorem in [23]. □

Remark 11.2. Garoufalidis and Nakamura claim in [11] that there exsits a canonical isomorphism

$$\mathbb{Q}[\Gamma; \Gamma \in \mathcal{G}^0]/(IH_0) \cong \Lambda^* U_{\mathbb{Q}}/([2^2])$$

in the stable range where IH_0 denotes the ideal generated by IH_0 moves which are IH moves of certain restricted types. Unfortunately, there seems to be a gap in their proof, because even if we consider only IH_0 moves, the image of the ideal (IH_0) under the canonical map contains the summands $[0]$ and $[1^2]$ other than $[2^2]$. In fact, if their claim were true, it would imply that the leading term of our beta class β_Γ depends only on the number of vertices of Γ and not on its shape. However this is not true (see (35) in §12). Nevertheless we emphasize that our consideration is based on their beautiful idea to relate IH moves of trivalent graphs to symplectic representation theory through their map f_{IH}. (See [12] for a correction of [11]).

12. Unstable relations in the tautological algebras for small g

In this section, we make explicit computations for the case of degree 4 to illustrate our method. Thereby we obtain unstable relations in degree 4 of the tautological algebras $\mathcal{R}^*(\mathcal{M}_g)$ and $\mathcal{R}^*(\mathcal{M}_{g,*})$ for $g = 2, 3, 4, 5$.

First we briefly summarize the case of degree 2. The dimensions of the Sp-invariant part of the module

$$\Lambda^2(\Lambda^3 H_{\mathbb{Q}}) = \Lambda^2 U_{\mathbb{Q}} \oplus \big(U_{\mathbb{Q}} \otimes H_{\mathbb{Q}}\big) \oplus \Lambda^2 H_{\mathbb{Q}}$$

are given in the following table

g	$\Lambda^2 U_{\mathbb{Q}}$	$\big(U_{\mathbb{Q}} \otimes H_{\mathbb{Q}}\big)$	$\Lambda^2 H_{\mathbb{Q}}$	total
2	0	0	1	1
≥ 3	1	0	1	2

A basis of $(\Lambda^2(\Lambda^3 H_{\mathbb{Q}}))^{Sp}$ for $g \geq 3$ can be given as follows. Let Z_1, Z_2 be the linear chord diagrams with 6 vertices defined in (33). Then the associated trivalent graphs are Γ_1, Γ_2 given in Example 1.4. Then of course $\beta_{\Gamma_1} = 0$ because Γ_1 has loops. By applying the results in [57], we obtain

$$\beta_{\Gamma_2} = \alpha_{\Gamma_2} + \frac{6}{2(2g-2)}\alpha_{\Gamma_1} = -\frac{2g+1}{2g-2}\, e_1.$$

If $g = 2$, then $\Lambda^3 H = H$ so that $U = 0$. Hence $\beta_{\Gamma_2} = 0$ in this case. In view of the above formula for the β-class, we obtain the well known fact that $e_1 = 0$ in $H^2(\mathcal{M}_2; \mathbb{Q})$.

Next we consider the case of degree 4. The dimensions of the Sp-invariant part of the module

$$\Lambda^4(\Lambda^3 H_{\mathbb{Q}}) = \Lambda^4 U_{\mathbb{Q}} \oplus \big(\Lambda^3 U_{\mathbb{Q}} \otimes H_{\mathbb{Q}}\big) \oplus \big(\Lambda^2 U_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}}\big) \oplus \big(U_{\mathbb{Q}} \otimes \Lambda^3 H_{\mathbb{Q}}\big) \oplus \Lambda^4 H_{\mathbb{Q}}$$

are given in the following table

g	$\Lambda^4 U_{\mathbb{Q}}$	$(\Lambda^3 U_{\mathbb{Q}} \otimes H_{\mathbb{Q}})$	$(\Lambda^2 U_{\mathbb{Q}} \otimes \Lambda^2 H_{\mathbb{Q}})$	$(U_{\mathbb{Q}} \otimes \Lambda^3 H_{\mathbb{Q}})$	$\Lambda^4 H_{\mathbb{Q}}$	total
2	0	0	0	0	1	1
3	1	0	1	1	1	4
4	2	0	2	1	1	6
5	2	1	2	1	1	7
≥ 6	3	1	2	1	1	8

A basis of $(\Lambda^4(\Lambda^3 H_{\mathbb{Q}}))^{Sp}$ for $g \geq 6$ can be given as follows. Let $\varLambda_i$ $(i = 1, \cdots, 5)$ be the connected trivalent graphs with 4 vertices given in Figure 8.3. Also set

$$\varLambda_6 = Z_2 \amalg Z_2, \quad \varLambda_7 = Z_1 \amalg Z_2, \quad \varLambda_8 = Z_1 \amalg Z_1.$$

Then $\{a_{\varLambda_i}; i = 1, \cdots, 8\}$ form a basis of $(\Lambda^4(\Lambda^3 H_{\mathbb{Q}}))^{Sp}$ $(g \geq 6)$. We know $\alpha_{\varLambda_i}$ by Table 8.1. By the results in [57], the β-class can be evaluated as

$$\begin{aligned} \beta_{\varLambda_1} &= \frac{g(g+1)^2}{(g-1)^3}\, e_2 + \frac{5g^2+2g-1}{4(g-1)^4}\, e_1^2 \\ \beta_{\varLambda_2} &= \frac{g(g+1)(g+2)}{(g-1)^3}\, e_2 + \frac{3(g+1)(4g-1)}{8(g-1)^4}\, e_1^2 \end{aligned} \tag{35}$$

and of course we have

$$\beta_{\varLambda_6} = \left(\beta_{\varGamma_2}\right)^2 = \frac{(2g+1)^2}{4(g-1)^2}\, e_1^2.$$

Thus we have a relation

$$\beta_{\varLambda_6} = 2(2g+1)\big\{ -(g+2)\beta_{\varLambda_1} + (g+1)\beta_{\varLambda_2} \big\}$$

between the β-classes.

To obtain unstable relations in degree 4, we consider the following 8 elements in $\Lambda^4(\Lambda^3 H)$ each with an indication of genera where it is defined. They should correspond to the values of $\dim(\Lambda^4(\Lambda^3 H))^{Sp}$ for $g = 2, 3, 4, 5$ and $g \geq 6$ listed above.

$$\begin{aligned}
\xi_1 =& (x_1 \wedge y_1 \wedge x_2) \wedge (x_1 \wedge y_1 \wedge y_2) \wedge (x_2 \wedge y_2 \wedge x_1) \wedge (x_2 \wedge y_2 \wedge y_1) \quad (g \geq 2) \\
\xi_2 =& (x_1 \wedge x_2 \wedge x_3) \wedge (y_1 \wedge y_2 \wedge y_3) \wedge (x_1 \wedge y_2 \wedge y_3) \wedge (y_1 \wedge x_2 \wedge x_3) \quad (g \geq 3) \\
\xi_3 =& (x_1 \wedge y_1 \wedge x_2) \wedge (y_2 \wedge x_3 \wedge y_3) \wedge (y_3 \wedge x_1 \wedge x_2) \wedge (x_3 \wedge y_1 \wedge y_2) \quad (g \geq 3) \\
\xi_4 =& (x_1 \wedge y_1 \wedge x_2) \wedge (y_2 \wedge x_1 \wedge x_3) \wedge (y_1 \wedge y_3 \wedge x_1) \wedge (y_1 \wedge x_2 \wedge y_2) \quad (g \geq 3) \\
\xi_5 =& (x_1 \wedge x_2 \wedge x_3) \wedge (y_1 \wedge y_2 \wedge y_3) \wedge (x_1 \wedge y_2 \wedge y_4) \wedge (y_1 \wedge x_2 \wedge x_4) \quad (g \geq 4) \\
\xi_6 =& (x_1 \wedge y_1 \wedge x_3) \wedge (x_2 \wedge y_2 \wedge x_4) \wedge (x_1 \wedge x_2 \wedge y_3) \wedge (y_1 \wedge y_2 \wedge y_4) \quad (g \geq 4) \\
\xi_7 =& (x_1 \wedge x_2 \wedge x_5) \wedge (x_3 \wedge x_4 \wedge y_5) \wedge (y_1 \wedge y_2 \wedge y_3) \wedge (y_4 \wedge x_1 \wedge y_1) \quad (g \geq 5) \\
\xi_8 =& (x_1 \wedge x_2 \wedge x_3) \wedge (y_1 \wedge y_2 \wedge y_3) \wedge (x_4 \wedge x_5 \wedge x_6) \wedge (y_4 \wedge y_5 \wedge y_6) \quad (g \geq 6)
\end{aligned}$$

Then the $(8,8)$-matrix whose (i,j)-entry is equal to $24\,\alpha_{\Lambda_j}(\xi_i)$ is described as

ξ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
ξ_1	-32	48	-32	128	-192	288	-192	128
ξ_2	-48	48	0	0	0	288	0	0
ξ_3	-16	24	0	-16	0	0	96	0
ξ_4	0	-24	24	-48	48	0	0	0
ξ_5	-32	24	0	0	0	288	0	0
ξ_6	16	0	-16	16	0	0	0	0
ξ_7	0	-24	8	0	0	0	0	0
ξ_8	0	0	0	0	0	288	0	0

and its determinant is equal to $2^{35} \cdot 3^5$. Here we simply write α_i for α_{Λ_i}.

Now we are ready to compute.

(I) The case of $g = 5$. In this case, $\dim(\Lambda^4(\Lambda^3 H_{\mathbb{Q}}))^{Sp} = 7$ which is just 1 smaller than the stable dimension so that we have a unique relation. More precisely, we must omit the element ξ_8 from the above table so that there arises a linear relation between the elements α_i. By an explicit computation, we find that the element

$$r_1 = 8\alpha_1 + \frac{16}{3}\alpha_2 + 16\alpha_3 + 8\alpha_4 + \frac{8}{3}\alpha_5 + \frac{4}{9}\alpha_6 + \frac{4}{3}\alpha_7 + \alpha_8$$

vanishes for $g = 5$. If we replace each α_i in the above element r_1 by the corresponding polynomials in e, e_1, e_2 for $g = 5$:

$$\begin{aligned} \alpha_1 &= e_2 + 6ee_1 - 2e^2 & \alpha_5 &= e_2 - 27ee_1 - 1432e^2 \\ \alpha_2 &= e_2 + 6ee_1 - 112e^2 & \alpha_6 &= e_1^2 - 60ee_1 + 900e^2 \\ \alpha_3 &= e_2 - 5ee_1 + 328e^2 & \alpha_7 &= e_1^2 + 50ee_1 - 2400e^2 \\ \alpha_4 &= e_2 - 16ee_1 - 552e^2 & \alpha_8 &= e_1^2 + 160ee_1 + 6400e^2, \end{aligned}$$

we obtain

$$r_1 = 40e_2 + \frac{25}{9}e_1^2.$$

We can now conclude that

$$72e_2 + 5e_1^2 = 0 \quad (g = 5)$$

which coincides with the equality

$$\kappa_1^2 = \frac{72}{5}\kappa_2$$

given already in Faber's paper [7] (note that $\kappa_i = (-1)^{i+1}e_i$).

(II) The case of $g = 4$. In this case $\dim(\Lambda^4(\Lambda^3 H_{\mathbb{Q}}))^{Sp} = 6$ so that we have two relations. More precisely, we must omit two elements ξ_7, ξ_8 from the above table. Then by an explicit computation, we find that the following two elements

$$\begin{aligned} r_2 &= 12\alpha_1 + 8\alpha_2 + 18\alpha_3 + 6\alpha_4 + \alpha_5 + 2/3\alpha_6 + \alpha_7 \\ r_3 &= 8\alpha_1 + 16/3\alpha_2 + 8\alpha_3 - 4/3\alpha_5 + 4/9\alpha_6 - \alpha_8 \end{aligned}$$

vanish for $g = 4$. If we replace each α_i in the above elements r_2, r_3 by the corresponding polynomials in e, e_1, e_2 for $g = 4$:

$$\begin{aligned} \alpha_1 &= e_2 + 6ee_1 - 18e^2 & \alpha_5 &= e_2 - 21ee_1 - 666e^2 \\ \alpha_2 &= e_2 + 6ee_1 - 90e^2 & \alpha_6 &= e_1^2 - 48ee_1 + 576e^2 \\ \alpha_3 &= e_2 - 3ee_1 + 198e^2 & \alpha_7 &= e_1^2 + 24ee_1 - 1152e^2 \\ \alpha_4 &= e_2 - 12ee_1 - 234e^2 & \alpha_8 &= e_1^2 + 96ee_1 + 2304e^2, \end{aligned}$$

we obtain two equalities

$$45e_2 + \frac{5}{3}e_1^2 - 35ee_1 - 210e^2 = 0$$
$$20e_2 - \frac{5}{9}e_1^2 - \frac{100}{3}ee_1 - 200e^2 = 0.$$

From these, we can conclude that the following two relations

$$32e_2 + 3e_1^2 = 0$$
$$7e_2 - 9ee_1 - 54e^2 = 0$$

hold for $g = 4$. The first relation coincides with

$$\kappa_1^2 = \frac{32}{3}\kappa_2$$

given in [6] (see also [7]). Here it is amusing to observe that the fiber integral of the second equality above yields a trivial identity $54e_1 - 54e_1 = 0$, while that of

$$e(7e_2 - 9ee_1 - 54e^2) = 0$$

yields

$$-42e_2 - 9e_1^2 - 54e_2 = -3(32e_2 + 3e_1^2) = 0$$

which is the same as the first relation.

(III) The case of $g = 3$. In this case, $\dim(\Lambda^4(\Lambda^3 H_{\mathbb{Q}}))^{Sp} = 4$ so that we have 4 relations. Similar computations as above yield the following relations

$$e_1^2 = 0, \quad ee_1 + 4e^2 = 0.$$

We can also show that $e^3 = 0$ by our method, but we omit the details here. See [5] for the structure of the Chow algebra.

(IV) The case of $g = 2$. In this case, we can obtain the well-known facts

$$e_1 = 0, \quad e^2 = 0.$$

The details are also omitted.

See [57] for further unstable relations in the tautological algebras.

13. Further implications of the main results

In this section, we describe further results which can be obtained by combining our main results (Theorem 1.1 and Theorem 1.3) with Hain's important results in [14][16].

Let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g relative to an embedded disk $D^2 \subset \Sigma_g$ and let $\{\mathcal{M}_{g,1}(k)\}_k$ be the filtration of $\mathcal{M}_{g,1}$ induced by the lower central series of the fundamental group of $\Sigma_g^0 = \Sigma_g \setminus \operatorname{Int} D^2$ as follows. For any group G, we denote by $\Gamma_k(G)$ the k-th term in the lower central series of G defined as $\Gamma_0(G) = G$ and

$\Gamma_k(G) = [G, \Gamma_{k-1}(G)]$ $(k \geq 1)$. Then $\mathcal{M}_{g,1}(k)$ is the subgroup of $\mathcal{M}_{g,1}$ consisting of all elements which act on the k-th nilpotent quotient $\Gamma_{k-1}(\pi_1\Sigma_g^0)/\Gamma_k(\pi_1\Sigma_g^0)$ of $\pi_1\Sigma_g^0$ trivially. In particular, $\mathcal{M}_{g,1}(1)$ is the Torelli group $\mathcal{I}_{g,1} \subset \mathcal{M}_{g,1}$. We also have similar filtrations $\{\mathcal{M}_{g,*}(k)\}_k$ and $\{\mathcal{M}_g(k)\}_k$ for $\mathcal{M}_{g,*}$ and $\mathcal{M}_g$ respectively (see [51][53][55] for details).

There is another filtration $\{\mathcal{M}'_{g,1}(k)\}_k$ of $\mathcal{M}_{g,1}$ where $\mathcal{M}'_{g,1}(1) = \mathcal{I}_{g,1}$ and for $k \geq 2$, $\mathcal{M}'_{g,1}(k)$ is defined to be the $(k-1)$-th term $\Gamma_{k-1}(\mathcal{I}_{g,1})$ in the lower central series of the Torelli group $\mathcal{I}_{g,1}$. Johnson [29] showed that $\mathcal{M}'_{g,1}(k) \subset \mathcal{M}_{g,1}(k)$ for all k and asked whether they coincide after tensoring with $\mathbb{Q}$ or not. It was proved in [48] that $\mathcal{M}'_{g,1}(3)$ has an infinite index in $\mathcal{M}_{g,1}(3)$ and Hain [14] proved that the same is true for all $k \geq 3$. There are similar filtrations $\{\mathcal{M}'_{g,*}(k)\}_k$, $\{\mathcal{M}'_g(k)\}_k$ for $\mathcal{M}_{g,*}$, $\mathcal{M}_g$ and similar results as above also hold for them.

Associated to these filtrations, we can consider the direct limits

$$H^*_c(\mathcal{M}_{g,1};\mathbb{Q}) = \lim_{k\to\infty} H^*(\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k);\mathbb{Q})$$
$$H^*_{c'}(\mathcal{M}_{g,1};\mathbb{Q}) = \lim_{k\to\infty} H^*(\mathcal{M}_{g,1}/\mathcal{M}'_{g,1}(k);\mathbb{Q})$$

of the cohomology of the successive quotients. We call them the *continuous cohomology* of the mapping class group. There are natural *increasing* filtrations on these continuous cohomology groups which are induced by the original (decreasing) filtrations by subgroups of $\mathcal{M}_{g,1}$. We denote by $H^*_c(\mathcal{M}_{g,1};\mathbb{Q})_k$ (resp. $H^*_{c'}(\mathcal{M}_{g,1};\mathbb{Q})_k$) the $(k+1)$-th term in these filtrations. Thus

$$H^*_c(\mathcal{M}_{g,1};\mathbb{Q})_k = \mathrm{Im}(H^*(\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k+1);\mathbb{Q}) \rightarrow H^*_c(\mathcal{M}_{g,1};\mathbb{Q}))$$

and similarly for $H^*_{c'}(\mathcal{M}_{g,1};\mathbb{Q})_k$. We call $H^*_c(\mathcal{M}_{g,1};\mathbb{Q})_k, H^*_{c'}(\mathcal{M}_{g,1};\mathbb{Q})_k$ the continuous cohomology of order k.

For any $k \geq 2$, we have an extension

$$1 \longrightarrow \mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k) \longrightarrow \mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k) \longrightarrow Sp(2g,\mathbb{Z}) \longrightarrow 1 \tag{36}$$

where $\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k)$ turns out to be a nilpotent group. Now it is easy to see that the natural injective homomorphism $\mathcal{M}_{g,1} \subset \mathcal{M}_{g+1,1}$ sends the subgroup $\mathcal{M}_{g,1}(k)$ to $\mathcal{M}_{g+1,1}(k)$. Hence we obtain the following induced morphism of extensions:

$$\begin{array}{ccc} \mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k) & \longrightarrow & \mathcal{I}_{g+1,1}/\mathcal{M}_{g+1,1}(k) \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k) & \longrightarrow & \mathcal{M}_{g+1,1}/\mathcal{M}_{g+1,1}(k) \\ \downarrow & & \downarrow \\ Sp(2g,\mathbb{Z}) & \longrightarrow & Sp(2g+2,\mathbb{Z}) \end{array} \tag{37}$$

Recall from [51][53][55] that the graded module $\oplus_{k\geq 1}(\mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1))$ has a natural structure of a graded Lie algebra over $\mathbb{Z}$. In fact, the Johnson homomorphisms give rise to an embedding of it into the Lie algebra consisting of derivations of the free Lie algebra generated by $H_1(\Sigma_g;\mathbb{Z})$ which kill the symplectic class, as a Lie subalgebra. Now it is an important consequence of the result of Hain in [14] that this Lie subalgebra tensored by $\mathbb{Q}$ is generated by the degree 1 summand which is isomorphic to $\Lambda^3 H_{\mathbb{Q}}$ as a representation of the algebraic group $Sp(2g,\mathbb{Q})$. In fact, Hain proved in the above cited paper that the natural homomorphism

$$\mathcal{T}_{g,1} \longrightarrow \mathcal{U}_{g,1} \tag{38}$$

from the Malcev completion $\mathcal{T}_{g,1}$ of the Torelli group $\mathcal{I}_{g,1}$ to the prounipotent radical $\mathcal{U}_{g,1}$ of his relative completion of the mapping class group $\mathcal{M}_{g,1}$ is surjective. By the universality of the relative completion, there is a natural homomorphism

$$\mathcal{U}_{g,1} \longrightarrow \varprojlim (\mathcal{I}_{g,1}/\mathcal{I}_{g,1}(k)) \otimes \mathbb{Q} \tag{39}$$

which is also surjective. Hence the composition of the two homomorphisms (38) and (39) is surjective. Then the above fact follows by considering the induced homomorphism between their Lie algebras and then their associated graded Lie algebras, because the graded Lie algebra associated to the Lie algebra of $\mathcal{T}_{g,1}$ is generated by elements of degree 1 (for any $g \geq 3$).

It follows by induction on k that the natural injection

$$(\mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1)) \otimes \mathbb{Q} \to (\mathcal{M}_{g+1,1}(k)/\mathcal{M}_{g+1,1}(k+1)) \otimes \mathbb{Q}$$

stabilizes as rational representations of the symplectic groups $Sp(2g, \mathbb{Q})$ for sufficiently large g. Then if we consider the spectral sequence of the rational cohomology of the morphism (37) and use the spectral sequence comparison theorem together with the Borel vanishing theorem [3][4] concerning the stable cohomology of $Sp(2g, \mathbb{Z})$ with non-trivial rational representations as coefficients, we can deduce for any k that $H^*(\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k); \mathbb{Q})$ stabilizes. More precisely, the E_2 term of the spectral sequence of the rational cohomology of the extension (36) is equal to

$$H^*(Sp(2g, \mathbb{Z}); \mathbb{Q}) \otimes H^*(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k); \mathbb{Q})^{Sp}$$

in a suitable stable range. On the other hand, Hain also proved in [16] that the extension (36) splits over the rationals. Hence the argument in [53] implies that $H^*(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k); \mathbb{Q})^{Sp}$ survives to the E_∞ term so that it can be considered as a subgroup of $H^*(\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k); \mathbb{Q})$. Hence the spectral sequence collapses and we have an isomorphism

$$H^*(\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k); \mathbb{Q}) \cong H^*(Sp(2g, \mathbb{Z}); \mathbb{Q}) \otimes H^*(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k); \mathbb{Q})^{Sp} \tag{40}$$

in the same stable range. Thus we obtain a natural homomorphism

$$\lim_{k \to \infty} H^*(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k); \mathbb{Q})^{Sp} \longrightarrow H^*(\mathcal{M}_{g,1}; \mathbb{Q}). \tag{41}$$

It also follows from Hain's general theory in [17] as is mentioned in [18]. It follows easily from the above facts that the continuous cohomology $H_c^*(\mathcal{M}_{g,1}; \mathbb{Q})_k$ of order k also stabilizes (with respect to the genus). We call the limit

$$\lim_{g \to \infty} H_c^*(\mathcal{M}_{g,1}; \mathbb{Q})_k$$

with respect to g the *continuous cohomology of order* k of the mapping class group and denote it by $H_c^*(\mathcal{M}_{\infty,1}; \mathbb{Q})_k$. We call the union of increasing groups

$$H_c^*(\mathcal{M}_{\infty,1}; \mathbb{Q}) = \bigcup_{k \geq 1} H_c^*(\mathcal{M}_{\infty,1}; \mathbb{Q})_k$$

the stable continuous cohomology of the mapping class group.

Remark 13.1. (i) The natural surjection

$$\lim_{g \to \infty} H^*(\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k+1); \mathbb{Q}) \longrightarrow H_c^*(\mathcal{M}_{\infty,1}; \mathbb{Q})_k$$

is not injective in general. For example, if $k=1$ the former is isomorphic to

$$\mathbb{Q}[c_1, c_3, \cdots] \otimes \mathbb{Q}[\varGamma; \varGamma \in \mathcal{G}]$$

while the latter is isomorphic to

$$\mathbb{Q}[c_1, c_3, \cdots] \otimes \mathbb{Q}[b_1, b_2, \cdots]$$

as will be shown below.
(ii) Although, for any k, $H^*_c(\mathcal{M}_{g,1};\mathbb{Q})_k$ does stabilizes in each degree with respect to g, it is unclear whether $H^*_c(\mathcal{M}_{g,1};\mathbb{Q})$ actually stabilizes in each degree or not. This is because the stable range depends on k so that for a fixed degree r, $H^r_c(\mathcal{M}_{g,1};\mathbb{Q}) = \bigcup_k H^r_c(\mathcal{M}_{g,1};\mathbb{Q})_k$ may not stabilize.

Now recall that Miller proved in [43] that the (ordinary) stable cohomology $H^*(\mathcal{M}_{\infty,1};\mathbb{Q})$ of the mapping class group has a natural stucture of a commutative and cocommutaive graded Hopf algebra so that it is the tensor product of the polynomial algebra generated by primitive elements of even degrees and the exterior algebra generated by primitive elements of odd degrees. Here the coalgebra structure is induced by natural homomorphisms

$$\mathcal{M}_{g,1} \times \mathcal{M}_{g',1} \longrightarrow \mathcal{M}_{g+g',1}$$

which are induced by boundary connected sum of surfaces with one boundary component. Observe that the subgroup $\mathcal{M}_{g,1}(k) \times \mathcal{M}_{g',1}(k)$ goes to $\mathcal{M}_{g+g',1}(k)$ under the above homomorphism. Using this fact, we can modify Miller's argument in the context of the stable continuous cohomology of $\mathcal{M}_{g,1}$ and conclude that $H^*_c(\mathcal{M}_{\infty,1};\mathbb{Q})_k$ also has a natural structure of a commutative and cocommutative graded Hopf algebra for any k.

Borel [3][4] proved that $\lim_{g\to\infty} H^*(Sp(2g,\mathbb{Z});\mathbb{Q}) = \mathbb{Q}[c_1, c_3, \dots]$ where c_i denotes the i-th Chern class of the *dual* of the universal bundle over the classifying space of $Sp(2g,\mathbb{Z})$. Here, instead of c_i, we consider the Newton classes s_i which are certain polynomials in the Chern classes, because c_i is not primitive (except for c_1) while the Newton classes are. Then clearly $\lim_{g\to\infty} H^*(Sp(2g,\mathbb{Z});\mathbb{Q}) = \mathbb{Q}[s_1, s_3, \dots]$ and s_i are all primitive elements.

Now for each i, we consider the cohomology class $\alpha_\varGamma$ where $\varGamma$ is a *connected* trivalent graph with $2i$ vertices (see §3). In view of Theorem 1.3, (i), the pullback of $\alpha_\varGamma$ to $H^*(\mathcal{M}_{g,1};\mathbb{Q})$ does not depend on the choice of the graph and it represents the class $(-1)^i e_i$ there. Clearly this cohomology class can be considered as an element of the stable continuous cohomology of $\mathcal{M}_{g,1}$, in fact as an element of $H^{2i}_c(\mathcal{M}_{\infty,1};\mathbb{Q})_1$, for it comes from $H^{2i}(\mathcal{I}_{g,1}(1)/\mathcal{M}_{g,1}(2);\mathbb{Q})^{Sp}$ and is stable with respect to g. Furthermore, by the argument in the proof of Proposition 3.2, the class $\alpha_\varGamma$ does not depend on the choice of the graph even as an element of the continuous cohomology $H^*_c(\mathcal{M}_{g,1};\mathbb{Q})$. We denote it by

$$b_i \in H^{2i}_c(\mathcal{M}_{\infty,1};\mathbb{Q})_1 \subset H^{2i}_c(\mathcal{M}_{\infty,1};\mathbb{Q}).$$

It is easy to deduce from the definition of $\alpha_\varGamma$ that b_i is primitive (this explains why it is a multiple of e_i and does not contain decomposable terms as a polynomial of e_j's). It was proved in [58][44][43] that $e_{2i-1} = \frac{2i}{B_i} s_{2i-1}$ where B_i denotes the i-th Bernoulli number. Hence the images of b_{2i-1} and s_{2i-1} in the ordinary stable cohomology of $\mathcal{M}_{g,1}$ are linearly dependent. However, we will see in a moment that they are linearly independent as elements of the stable continuous cohomology of $\mathcal{M}_{g,1}$.

With these terminologies, we have the following result which was obtained with Hain and Looijenga (cf. [18], Theorem 9.11 and Theorem 10.7, for a somewhat different formulation).

Theorem 13.2. (i) *The continuous cohomology* $H^*_c(\mathcal{M}_{g,1};\mathbb{Q})_k$ *of order* k *of* $\mathcal{M}_{g,1}$ *with respect to the filtration* $\{\mathcal{M}_{g,1}(k)\}_k$ *stabilizes for any* k *and the stable continuous cohomology*

$$H^*_c(\mathcal{M}_{\infty,1};\mathbb{Q}) = \bigcup_k H^*_c(\mathcal{M}_{\infty,1};\mathbb{Q})_k$$

has a natural structure of a commutative and cocommutative graded Hopf algebra. Furthermore the first term $H^*_c(\mathcal{M}_{\infty,1};\mathbb{Q})_1$ *is isomorphic to the polynomial algebra*

$$\mathbb{Q}[s_1, s_3, \cdots, b_1, b_2, \cdots]$$

where s_{2i-1} *and* b_i *are all primitive elements.*
(ii) *The image of the natural homomorphism*

$$H^*_c(\mathcal{M}_{\infty,1};\mathbb{Q}) \longrightarrow H^*(\mathcal{M}_{\infty,1};\mathbb{Q})$$

is equal to the subalgebra $\mathbb{Q}[e_1, e_2, \cdots]$ *generated by the Mumford-Morita-Miller classes and the ideal of the left hand side generated by the classes* $e_{2i-1} - \frac{2i}{B_i} s_{2i-1}$ $(i = 1, 2, \cdots)$ *goes to zero under the above homomorphism.*

Proof. We first prove (i). By the above discussion, we have only to prove that the two classes s_{2i-1} and b_{2i-1} are linearly independent as elements of $H^*_c(\mathcal{M}_{\infty,1};\mathbb{Q})$ for any i. Assume the contrary. Then, for sufficiently large k and g, s_{2i-1} and b_{2i-1} must be linearly dependent as elements of $H^*(\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k);\mathbb{Q})$. Since the restriction of s_{2i-1} to

$$H^{4i-2}(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k);\mathbb{Q})^{Sp} \subset H^{4i-2}(\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k);\mathbb{Q})$$

(cf. (40)) is trivial, so is the restriction of b_{2i-1}. But then the image of b_{2i-1} in $H^*(\mathcal{M}_{g,1};\mathbb{Q})$ must be zero which is a contradiction.

Next we prove (ii). Hain proved in [16] that, associated to any complex structure on Σ_g, there is defined a mixed Hodge structure on $\lim_{k\to\infty} H^*(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k);\mathbb{Q})^{Sp}$ and each of the following two homomorphisms

$$H^*(\Lambda^3 H_{\mathbb{Q}})^{Sp} \longrightarrow \lim_{k\to\infty} H^*(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k);\mathbb{Q})^{Sp} \longrightarrow H^*(\mathcal{M}_{g,1};\mathbb{Q}) \tag{42}$$

is a morphism of mixed Hodge structures. Furthermore the first homomorphism in (42) surjects onto the lowest weight subalgebra of the middle term. On the other hand, Pikaart [61] proved that the mixed Hodge structure on $H^k(\mathcal{M}_{g,1};\mathbb{Q})$ is pure of weight k for $2k + 1 \leq g$. If we combine these results with Theorem 1.1, which shows that the image of the composite of the two maps in (42) is exactly the subalgebra generated by the classes e_i, we obtain the required result. □

Remark 13.3. We may say that the second statement of the above theorem serves as a supporting evidence for the well-known conjecture that the stable rational cohomology of the mapping class group is isomorphic to the polynomial algebra generated by the classes e_i. We mention that this conjecture has been proved to be true for degrees ≤ 4 by Harer [20][22][24] and for degree 5 by Arbarello and Cornalba [1]. See also [31][32] for another evidence for the conjecture.

In the above discussion, we can replace $\{\mathcal{M}_{g,1}(k)\}_k$ by the other filtration $\{\mathcal{M}'_{g,1}(k)\}_k$ and we have the following result concerning it.

Theorem 13.4. *The continuous cohomology of order k $H^*_{c'}(\mathcal{M}_{g,1};\mathbb{Q})_k$ of $\mathcal{M}_{g,1}$ with respect to the filtration $\{\mathcal{M}'_{g,1}(k)\}_k$ stabilizes for any k and the stable continuous cohomology*

$$H^*_{c'}(\mathcal{M}_{\infty,1};\mathbb{Q}) = \bigcup_k H^*_{c'}(\mathcal{M}_{\infty,1};\mathbb{Q})_k$$

has a natural structure of a commutative and cocommutative graded Hopf algebra. Furthermore the first term $H^*_{c'}(\mathcal{M}_{\infty,1};\mathbb{Q})_1$ is isomorphic to the polynomial algebra

$$\mathbb{Q}[s_1, s_3, \cdots, b_2, b_3, \cdots].$$

Proof. The proof is similar to that of Theorem 13.2. We only mention the different points. This time we use the fundamental theorem of Hain [16] which gives an explicit finite presentation of the Lie algebra associated to the Malcev completion of the Torelli group (with any decoration). Clearly the natural injection $\mathcal{I}_{g,1}\to\mathcal{I}_{g+1,1}$ sends the subgroup $\Gamma_k(\mathcal{I}_{g,1})$ to $\Gamma_k(\mathcal{I}_{g+1,1})$ for any k. Hence we obtain the induced homomorphism

$$(\Gamma_k(\mathcal{I}_{g,1})/\Gamma_{k+1}(\mathcal{I}_{g,1}))\otimes\mathbb{Q}\longrightarrow(\Gamma_k(\mathcal{I}_{g+1,1})/\Gamma_{k+1}(\mathcal{I}_{g+1,1}))\otimes\mathbb{Q}.$$

Hain's theorem mentioned above implies that the above homomorphism stabilizes as representations of $Sp(2g,\mathbb{Q})$ in the stable range. Then a similar argument as before applied to (37) with $\mathcal{M}_{g,1}(k)$ replaced by $\mathcal{M}'_{g,1}(k)$ proves the first half of the theorem.

For the latter half, we again use Hain's theory of relative Malcev completion of the mapping class group in [14]. It is a consequence of the result of [48][49] that the two classes b_1 and $-12s_1$ coincide in $H^*_{c'}(\mathcal{M}_{\infty,1};\mathbb{Q})_1$. It remains to prove that b_{2i-1} and s_{2i-1} are linearly independent in it for any $i>1$. As mentioned before (see (38)(39)), Hain constructed a sequence of natural homomorphisms

$$\mathcal{T}_{g,1} = \varprojlim(\mathcal{I}_{g,1}/\mathcal{M}'_{g,1}(k))\otimes\mathbb{Q}\longrightarrow\mathcal{U}_{g,1}\longrightarrow\varprojlim(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k))\otimes\mathbb{Q}$$

of prounipotent groups and proved that the former is a central extension by $\mathbb{Q}$ whose extension class is the pullback of b_1 in $H^2_c(\mathcal{U}_{g,1})$ (for any $g\geq 3$). Furthermore it was shown that the natural homomorphism (41) factors as

$$\lim_{k\to\infty} H^*(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k);\mathbb{Q})^{Sp}\longrightarrow H^*_c(\mathcal{U}_{g,1};\mathbb{Q})^{Sp}\longrightarrow H^*(\mathcal{M}_{g,1};\mathbb{Q}). \tag{43}$$

Now assume that b_{2i-1} and s_{2i-1} are linearly dependent in $H^*_{c'}(\mathcal{M}_{\infty,1};\mathbb{Q})$. Then b_{2i-1} must vanish in $\lim_{k\to\infty} H^*(\mathcal{I}_{g,1}/\mathcal{M}'_{g,1}(k);\mathbb{Q})^{Sp}$ because s_{2i-1} is clearly zero there. Hence the image of it in $H^*_c(\mathcal{U}_{g,1};\mathbb{Q})$ lies in the kernel of the homomorphism

$$H^*_c(\mathcal{U}_{g,1})\longrightarrow\lim_{k\to\infty} H^*(\mathcal{I}_{g,1}/\mathcal{M}'_{g,1}(k);\mathbb{Q}).$$

On the other hand, the Gysin exact sequence of the central extension

$$0\longrightarrow\mathbb{Q}\longrightarrow\varprojlim(\mathcal{I}_{g,1}/\mathcal{M}'_{g,1}(k))\otimes\mathbb{Q}\longrightarrow\mathcal{U}_{g,1}\longrightarrow 1$$

implies that the kernel is precisely the ideal generated by the class b_1. Hence we can write $b_{2i-1}=b_1x$ for some $x\in H^*_c(\mathcal{U}_{g,1})$. But this contradicts the fact that the image of b_{2i-1} under the latter homomorphism of (43) in $H^*(\mathcal{M}_{g,1};\mathbb{Q})$, namely $-e_{2i-1}$, is primitive. This completes the proof. □

Let us restrict Theorem 13.4 to the Torelli group. Then the above discussion implies that, for any k, the natural homomorphism

$$H_c^*(\mathcal{I}_{g+1,1};\mathbb{Q})_k^{Sp}\longrightarrow H_c^*(\mathcal{I}_{g,1};\mathbb{Q})_k^{Sp}$$

stabilizes in each degree with respect to g. Here $H_c^*(\mathcal{I}_{g,1};\mathbb{Q})_k$ denotes the continuous cohomology of $\mathcal{I}_{g,1}$ of order k with respect to its own lower central series (see [16] for general facts concerning the continuous cohomology). Hence we can consider the limit

$$H_c^*(\mathcal{I}_{\infty,1};\mathbb{Q})_k^{Sp} = \lim_{g\to\infty} H_c^*(\mathcal{I}_{g,1};\mathbb{Q})_k^{Sp}$$

and we call the union

$$H_c^*(\mathcal{I}_{\infty,1};\mathbb{Q})^{Sp} = \bigcup_k H_c^*(\mathcal{I}_{\infty,1};\mathbb{Q})_k^{Sp}$$

the Sp-invariant stable continuous cohomology of the Torelli group. Obviously the restriction induces a homomorphism

$$\mathbb{Q}[b_1,b_2,\cdots]\longrightarrow H_c^*(\mathcal{I}_{\infty,1};\mathbb{Q})_1^{Sp}$$

and we know by a result in [48][49] that b_1 goes to zero.

Theorem 13.5. (i) *The Sp-invariant part $H_c^*(\mathcal{I}_{g,1};\mathbb{Q})_k^{Sp}$ of the continuous cohomology $H_c^*(\mathcal{I}_{g,1};\mathbb{Q})_k$ of order k of the Torelli groups $\mathcal{I}_{g,1}$ stabilizes for any k and the Sp-invariant stable continuous cohomology*

$$H_c^*(\mathcal{I}_{\infty,1};\mathbb{Q})^{Sp}$$

has a natural structure of a commutative and cocommutative graded Hopf algebra. Furthermore the first term is given by

$$H_c^*(\mathcal{I}_{\infty,1};\mathbb{Q})_1^{Sp} \cong \mathbb{Q}[b_2,b_3,\cdots].$$

(ii) *In the continuous cohomology $H_c^*(\mathcal{I}_{g,1};\mathbb{Q})$, the class b_i does not vanish for any $i>1$ in degrees $\leq \frac{2}{3}g$. Hence the natural homomorphism*

$$H_c^*(\mathcal{I}_{g,1};\mathbb{Q})\longrightarrow H^*(\mathcal{I}_{g,1};\mathbb{Q})$$

from the continuous cohomology to the ordinary cohomology has a big kernel because it contains the ideal generated by all odd classes $b_3,b_5,\cdots$ except for the first one b_1.

Remark 13.6. The above result suggests (but not prove) that the secondary characterstic classes of surface bundles introduced in [54] are non-trivial.

Remark 13.7. If we compare Hain's presentation of the Lie algebras associated to the Malcev completions of $\mathcal{I}_{g,1}$ and $\mathcal{I}_g$, it is not difficult to show that the natural homomorphism

$$H_c^*(\mathcal{I}_g;\mathbb{Q})_k^{Sp}\longrightarrow H_c^*(\mathcal{I}_{g,1};\mathbb{Q})_k^{Sp}$$

induced by the projection $\mathcal{I}_{g,1}\to\mathcal{I}_g$ is an isomorphism in a suitable stable range. Hence the above Theorem 13.5 with $\mathcal{I}_{g,1}$ replaced by $\mathcal{I}_g$ also holds.

Conjecture 13.8. The Sp-invariant part $H^*(\mathcal{I}_g;\mathbb{Q})^{Sp}$ of the rational cohomology of the Torelli groups stabilizes with respect to g so that the Sp-invariant stable cohomology $\lim_{g\to\infty} H^*(\mathcal{I}_g;\mathbb{Q})^{Sp}$ is defined. Furthermore we have an isomorphism

$$\lim_{g\to\infty} H^*(\mathcal{I}_g;\mathbb{Q})^{Sp} \cong \mathbb{Q}[e_2,e_4,\cdots].$$

Remark 13.9. It is an important open problem to determine whether the even classes e_{2i} are non-trivial in $H^*(\mathcal{I}_g;\mathbb{Q})$ or not.

14. Concluding remarks

Remark 14.1. In this paper, we considered cocycles of the mapping class group which are derived from Sp-invariant elements of the cohomology of the abelianization of the Torelli group and completely clarified their properties. It is an important problem to generalize these results to other cohomology classes which may be obtained from the homomorphism

$$\lim_{k\to\infty} H^*(\mathcal{I}_{g,1}/\mathcal{M}_{g,1}(k);\mathbb{Q})^{Sp} \longrightarrow H^*(\mathcal{M}_{g,1};\mathbb{Q}).$$

For example, we may ask whether Looijenga's unstable cohomology class for genus 3 moduli space given in [40] can be detected by the above homomorphism or not.

Remark 14.2. Garoufalidis and Levine [9] proved that the filtration, introduced by Ohtsuki [59] (see also [60]), on the vector space generated by oriented homology 3-spheres can be described by the lower central series of the Torelli group $\mathcal{I}_g$. Moreover they showed in [10] a relation between the finite type invariants of homology 3-spheres and the graded module associated to the lower central series of $\mathcal{I}_g$. Also we learned from J. Murakami that the restriction to $\mathcal{I}_g$ of the projective representation of the mapping class group associated to the universal perturbative 3-manifolds invariants due to Le, Murakami and Ohtsuki [39] is a unipotent representation after taking canonical truncations. Hence it should be described in terms of the Malcev completion of the Torelli group, though explicit description is far from being understood.

On the other hand, Theorem 13.5 shows that the Torelli group has a deeper structure than what is reflected in its Malcev completion because the natural homomorphism from the continuous cohomology to the ordinary cohomology of the Torelli group has a big kernel. In view of the well-known close connection between the structure of the Torelli group and the set of homology 3-spheres, it might be natural to expect that there exist some unknown invariants of these manifolds which reflect the structure of the Torelli group other than its nilpotent quotients.

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