UTMS 2001–11 May 1, 2001

Spectral properties of non-symmetric systems of ordinary differential operators

by

Igor Trooshin and Masahiro Yamamoto

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

SPECTRAL PROPERTIES OF NON-SYMMETRIC SYSTEMS OF ORDINARY DIFFERENTIAL OPERATORS

[1]Igor Trooshin and [2]Masahiro Yamamoto

[1] Institute for Problems of Precision Mechanics and Control (IPTMU)
Russian Academy of Sciences, Saratov, Russia
lisu@cocoa.plala.or.jp
[2] Department of Mathematical Sciences, The University of Tokyo
3-8-1 Komaba, Meguro, Tokyo 153 Japan
myama@ms.u-tokyo.ac.jp

Abstract. We consider a nonsymmetric first-order differential operator $(\mathcal{A}u)(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{du}{dx}(x) + P(x)u(x)$, $0 < x < 1$, where P is a 2×2 matrix whose components are in $L^2(0,1)$. We study an eigenvalue problem for $\mathcal{A}$ with boundary conditions at $x = 0, 1$. We establish an asymptotic form of the eigenvalues and prove that the set of the root vectors forms a Riesz basis in $\{L^2(0,1)\}^2$. The key is a transformation formula.

§1. Introduction and the main result for the eigenvalue problem.

We consider a nonsymmetric first-order differential operator in $(0,1)$:

$$(\mathcal{A}u)(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{du}{dx}(x) + P(x)u(x). \quad 0 < x < 1, \tag{1.1}$$

where $u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$ and

$$P(x) = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix},$$

where $p_{k\ell} \in L^2(0,1)$, $1 \le k, \ell \le 2$, are complex-valued functions. We define an operator A in $\{L^2(0,1)\}^2$ by

$$(Au)(x) = (\mathcal{A}u)(x), \quad 0 < x < 1, \tag{1.2}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$$\mathcal{D}(A) = \{u \in \{H^1(0,1)\}^2; u_2(0)\cosh\mu - u_1(0)\sinh\mu = 0,$$

$$u_2(1)\cosh\nu + u_1(1)\sinh\nu = 0\}. \tag{1.3}$$

Throughout this paper, we set $i = \sqrt{-1}$, $\mu, \nu \in \mathbb{C}$, and $L^2(0,1)$ and $H^1(0,1)$ are the Lebesgue space and the Sobolev space of complex-valued functions.

The eigenvalue problem for A can describe proper vibrations with damping both in the medium and at the boundary points:

$$\begin{cases} \dfrac{d^2u}{dx^2}(x) + \lambda p_1(x)u(x) + p_2(x)\dfrac{du}{dx}(x) = \lambda^2 u(x), & 0 < x < 1 \\ \dfrac{du}{dx}(0) + \lambda h u(0) = \dfrac{du}{dx}(1) + \lambda H u(1) = 0, \end{cases}$$

where $h \neq \pm 1, H \neq \pm 1$. In fact, setting $U(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} \lambda u(x) \\ \frac{du}{dx}(x) \end{pmatrix}$, we rewrite the system as

$$\begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \dfrac{dU}{dx}(x) + \begin{pmatrix} p_1(x) & p_2(x) \\ 0 & 0 \end{pmatrix} U(x) = \lambda U(x), & 0 < x < 1 \\ u_2(0) + h u_1(0) = u_2(1) + H u_1(1) = 0, \end{cases}$$

which is an eigenvalue problem for A with $P(x) = \begin{pmatrix} p_1(x) & p_2(x) \\ 0 & 0 \end{pmatrix}$.

In this paper, we establish an asymptotic form of the eigenvalues of A and prove that the set of the root vectors of A forms a Riesz basis in $\{L^2(0,1)\}^2$. Such spectral properties are essential for control problems (e.g. Russell [6]) and inverse problems (e.g. Cox and Knobel [1], Yamamoto [13]) and our result admits the generalisation of those results for A with L^2-coefficients. Here a Riesz basis means a basis equivalent to an orthonormal basis (e.g., Gohberg and Kreĭn [3]), and we call $u \neq 0$ a *root vector* of an operator A for λ if $(A - \lambda)^m u = 0$ for some $m \in \mathbb{N}$. Moreover $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a *Riesz basis* in $\{L^2(0,1)\}^2$ if and only if each $u \in \{L^2(0,1)\}^2$ has a unique expansion $u = \sum_{n=-\infty}^{\infty} c_n\varphi_n$ with $c_n \in \mathbb{C}$, $n \in \mathbb{Z}$ and

$$M^{-1} \sum_{n=-\infty}^{\infty} |c_n|^2 \leq \|u\|^2_{\{L^2(0,1)\}^2} \leq M \sum_{n=-\infty}^{\infty} |c_n|^2,$$

where a constant $M > 0$ is independent of u.

Henceforth we set

$$(1.4) \qquad \theta_1(x) = \frac{1}{2}\int_0^x (p_{12}(y) + p_{21}(y))dy, \quad \theta_2(x) = \frac{1}{2}\int_0^x (p_{11}(y) + p_{22}(y))dy,$$

for $0 \le x \le 1$. We are ready to state our main result:

Theorem.

(i) The spectrum $\sigma(A)$ of the operator defined by (1.1) - (1.3) consists entirely of geometrically simple eigenvalues with finite algebraic multiplicities.

There exist $N \in \mathbb{N}$ and $\Sigma_1, \Sigma_2 \subset \sigma(A)$, such that $\sigma(A) = \Sigma_1 \cup \Sigma_2$, $\Sigma_1 \cap \Sigma_2 = \emptyset$ and the following properties hold:

(1)

$$\Sigma_1 \subset \left\{\lambda; Im\,|(\lambda - \theta_2(1) + \mu + \nu)| \le \left(N - \frac{1}{2}\right)\pi\right\}$$

and the sum of the algebraic multiplicities of the eigenvalues in Σ_1 is $2N-1$.

(2) *Σ_2 consists of eigenvalues $\{\lambda_n\}_{|n|\ge N}$ with algebraic multiplicty 1 and λ_n is in a neighbourhood of $\theta_2(1) - \mu - \nu + n\pi i$ for every $|n| \ge N$.*

Moreover with a suitable numbering $\{\lambda_n\}_{n\in\mathbb{Z}}$ of $\sigma(A)$, the eigenvalues have an asymptotic form

$$(1.5) \qquad \lambda_n = \theta_2(1) - \mu - \nu + n\pi i + \delta_n \quad \textit{where} \sum_{n=-\infty}^{\infty} |\delta_n|^2 < \infty.$$

(ii) The set of all the root vectors $\{\varphi_n\}_{n\in\mathbb{Z}}$ of A is a Riesz basis in $\{L^2(0,1)\}^2$.

Our theorem generalises the previous results by Cox and Knobel [1], Russell [6] and Trooshin and Yamamoto [11]. More precisely, in the case of $P \in \{C^1[0,1]\}^4$, Russell [6] shows our theorem without proof. The proof for $P \in \{C^1[0,1]\}^4$ is found in [11]. In [1], the asymptotic behaviour of the eigenvalues and the Riesz basis are proved in the case of Lipschitz continuous P and special boundary conditions, that is, $\mu = \nu = 0$ or $\pi i/2$.

For the completeness of the eigenvectors of a nonsymmetric system and closely related eigenvalue problems for pencils of ordinary differential operators, we can further refer to Cox and Zuazua [2], Rykhlov [7], Shkalikov [8], Shubov [9], Shubov, Martin, Dauer and Belinskiy [10], Vagabov [12].

This paper is composed of four sections. In Section 2, we show ingredients for the proof of the theorem. In Sections 3 and 4, we prove the first and the second parts of the theorem, respectively.

§2. Transformation formula.

Let

$$\Omega = \{(x, y); 0 < y < x < 1\} \tag{2.1}$$

and

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

We put

$$R(x) = e^{-\theta_1(x)} \begin{pmatrix} \cosh\theta_2(x) & -\sinh\theta_2(x) \\ -\sinh\theta_2(x) & \cosh\theta_2(x) \end{pmatrix}, \quad 0 \le x \le 1, \tag{2.3}$$

where $\theta_1(x),\ \theta_2(x)$ are defined by (1.4).

Lemma 2.1. *For any $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}$ and $P \in \{L^2[0,1]\}^4$, let $\varphi = \varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$ satisfy*

$$\begin{cases} B\dfrac{d\varphi}{dx}(x) + P(x)\varphi(x) = \lambda\varphi(x), \quad 0 < x < 1 \\ \varphi_1(0,\lambda) = \cosh\mu, \quad \varphi_2(0,\lambda) = \sinh\mu. \end{cases} \tag{2.4}$$

Then

$$\varphi(x,\lambda) = R(x)\begin{pmatrix}\cosh(\lambda x+\mu)\\ \sinh(\lambda x+\mu)\end{pmatrix} + \int_0^x K(x,y)\begin{pmatrix}\cosh(\lambda y+\mu)\\ \sinh(\lambda y+\mu)\end{pmatrix} dy, \tag{2.5}$$

for $0 \le x \le 1$, *all* $\lambda \in \mathbb{C}$ *and* $\mu \in \mathbb{C}$. *Here* R *is defined by (2.3) and* $K \in \{L^2(\Omega)\}^4$ *is independent of* λ *and dependent on* P *and* μ *such that* $K(1,\cdot) \in \{L^2(0,1)\}^4$.

Remark. This lemma is valid in a general case of $P \in \{L^1(0,1)\}^4$ with $K \in \{L^1(\Omega)\}^4$ and $K(1,\cdot) \in \{L^1(0,1)\}^4$.

Proof. This lemma was already proved in the case of $P \in \{C^1[0,1]\}^4$ by Yamamoto [13].

To prove the lemma for $P \in \{L^2(0,1)\}^4$, let us note that there exist 2×2 matrix functions $P^n \in \{C^1[0,1]\}^4$ such that $\lim_{n\to\infty} \|P - P^n\|_{\{L^2(0,1)\}^4} = 0$.

We directly see that the solution $\varphi(x,\lambda)$ of the Cauchy problem (2.4) satisfies the following integral Volterra equation

$$\varphi(x,\lambda) = \begin{pmatrix}\cosh\mu\\ \sinh\mu\end{pmatrix} + \int_0^x B(\lambda E - P(s))\varphi(s,\lambda)ds. \tag{2.6}$$

It easily follows that for any fixed $\lambda \in \mathbb{C}$, we have

$$\lim_{n\to\infty} \|\varphi(\cdot,\lambda) - \varphi^n(\cdot,\lambda)\|_{\{L^2(0,1)\}^2} = 0, \tag{2.7}$$

where $\varphi^n(x,\lambda) = \begin{pmatrix}\varphi_1^n(x,\lambda)\\ \varphi_2^n(x,\lambda)\end{pmatrix}$ is the solution to the Cauchy problem

$$\begin{cases} B\dfrac{d\varphi^n}{dx}(x) + P^n(x)\varphi^n(x) = \lambda\varphi^n(x), \quad 0<x<1 \\ \varphi_1^n(0,\lambda) = \cosh\mu, \quad \varphi_2^n(0,\lambda) = \sinh\mu. \end{cases} \tag{2.8}$$

Let us denote by $K^n(x,y)$ the kernel of the transformation operator of the problem (2.8) and let R^n be defined by (2.3) for $P^n \in \{C^1[0,1]\}^4$.

It is easy to see that $\|R^n - R\|_{\{L^2(0,1)\}^4} \to 0$ as $n \to \infty$. Moreover, in [13], it is proved that:

$$K_{11}^n(x,y) = \frac{1}{2}[L_2^n(x,y) + L_3^n(x,y)], \quad K_{12}^n(x,y) = \frac{1}{2}[L_1^n(x,y) + L_4^n(x,y)], \tag{2.9}$$

$$K_{21}^n(x,y) = \frac{1}{2}[L_4^n(x,y) - L_1^n(x,y)], \quad K_{22}^n(x,y) = \frac{1}{2}[L_3^n(x,y) - L_2^n(x,y)] \tag{2.10}$$

and $L^n(x,y) = \begin{pmatrix} L_1^n(x,y) \\ L_2^n(x,y) \\ L_3^n(x,y) \\ L_4^n(x,y) \end{pmatrix}$ are the solutions of the Volterra integral equations:

$$\begin{aligned} L^n(x,y) =& r_1^n\left(\frac{x+y}{2}\right) + r_2^n\left(\frac{x-y}{2}\right) + \int_y^{\frac{x+y}{2}} Q_1^n(s)L^n(x+y-s,s)ds \\ &+ \int_0^y Q_2^n(s)L^n(x-y+s,s)ds + \int_0^{\frac{x-y}{2}} Q_3^n(s)L^n(x-y-s,s)ds. \end{aligned}$$

Furthermore, by [13], we see that the elements of 4×4 matrices $Q_j^n(s)$ are linear combinations of the entries of the matrix P^n and $\lim_{n,m\to\infty} \|Q_j^n - Q_j^m\|_{\{L^2(\Omega)\}^{16}} = 0$ for $j = 1,2,3$, and $r_i^n \in \{C^1[0,1]\}^4$ such that $\lim_{n,m\to\infty} \|r_i^n - r_i^m\|_{\{L^2(0,1)\}^4} = 0$. It follows from these observations that $\lim_{n,m\to\infty} \|K^n - K^m\|_{\{L^2(\Omega)\}^4} = 0$.

The completeness of the space $\{L^2(\Omega)\}^4$ implies that there is a limit function of the sequence $\{K^n\}_{n\in\mathbb{N}}$, which we denote by $K(x,y)$. Let us set

$$\widehat{\varphi}(x,\lambda) = R(x)\begin{pmatrix} \cosh(\lambda x+\mu) \\ \sinh(\lambda x+\mu) \end{pmatrix} + \int_0^x K(x,y)\begin{pmatrix} \cosh(\lambda y+\mu) \\ \sinh(\lambda y+\mu) \end{pmatrix} dy. \tag{2.11}$$

It is easy to see that $\lim_{n\to\infty} \|\widehat{\varphi}(\cdot,\lambda) - \varphi^n(\cdot,\lambda)\|_{\{L^2(0,1)\}^2} = 0$. By (2.7), it means that $\widehat{\varphi}(x,\lambda) = \varphi(x,\lambda)$ almost everywhere on $(0,1)$ and consequently, as continuous functions, everywhere on $[0,1]$.

To prove that $K(1,\cdot) \in \{L^2(0,1)\}^4$, we should repeat the above argument for fixed $x = 1$.

§3. Proof of the first part of Theorem.

We divide the proof into five steps.

First Step. We show

Lemma 3.1. *The spectrum $\sigma(A)$ consists entirely of countable isolated eigenvalues with finite algebraic multiplicities.*

Proof of Lemma 3.1. The proof is similar to Lemma 3.1 in [11] and for completeness, we will give it. We define by $U = U(x,\lambda) = (U_{k\ell}(x,\lambda))_{1\le k,\ell\le 2}$ the solution to the Cauchy problem

$$\begin{cases} B\dfrac{dU}{dx}(x) + P(x)U(x) = \lambda U(x), \quad 0 < x < 1 \\ U(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

Here and henceforth we set

$$B_0 = \begin{pmatrix} -\sinh\mu & \cosh\mu \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ \sinh\nu & \cosh\nu \end{pmatrix}.$$

Then we can ditectly show that $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{D}(A)$ if and only if

$$v \in \{H^1(0,1)\}^2, \qquad B_0 v(0) + B_1 v(1) = 0.$$

On the other hand, in view of variation of constants, the general solution to

$$(\mathcal{A} - \lambda)v = \left(B\frac{d}{dx} + P(x) - \lambda\right) v = f$$

with $f \in \{L^2(0,1)\}^2$, is given by

$$v(x) = U(x,\lambda)\eta + U(x,\lambda)\int_0^x U(y,\lambda)^{-1}Bf(y)dy,$$

where $\eta \in \mathbb{C}^2$ is arbitrary. To satisfy $v \in \mathcal{D}(A)$, we have to choose η such that

$$(B_0 + B_1U(1,\lambda))\eta + B_1U(1,\lambda)\int_0^1 U(y,\lambda)^{-1}Bf(y)dy = 0.$$

If $\det(B_0 + B_1U(1,\lambda)) \ne 0$, then such η exists:

$$\eta = -(B_0 + B_1U(1,\lambda))^{-1}B_1U(1,\lambda)\int_0^1 U(y,\lambda)^{-1}Bf(y)dy$$

and we can write

$$v(x) = -U(x,\lambda)(B_0 + B_1U(1,\lambda))^{-1}B_1U(1,\lambda)\int_0^1 U(y,\lambda)^{-1}Bf(y)dy$$
$$+U(x,\lambda)\int_0^x U(y,\lambda)^{-1}Bf(y)dy.$$

Therefore if $\det(B_0 + B_1U(1,\lambda_0)) \ne 0$ for some $\lambda_0 \in \mathbb{C}$, then $(A-\lambda_0)^{-1}$ is a compact operator from $\{L^2(0,1)\}^2$ to itself. This implies that $\sigma(A)$ consists of isolated eigenvalues with finite algebraic multiplicities (e.g. Kato [4]). Hence it is sufficient to verify that there exists $\lambda_0 \in \mathbb{C}$ such that $\det(B_0 + B_1U(1,\lambda_0)) \ne 0$. Setting $\mu = 0$ and $\mu = \frac{\pi}{2}i$ in Lemma 2.1, we obtain

$$\begin{aligned} U(x,\lambda) =& R(x)\begin{pmatrix} \cosh\lambda x & \sinh\lambda x \\ \sinh\lambda x & \cosh\lambda x \end{pmatrix} \\ &+ \int_0^x K(x,y)\begin{pmatrix} \cosh\lambda y & \sinh\lambda y \\ \sinh\lambda y & \cosh\lambda y \end{pmatrix} dy, \quad 0 \le x \le 1,\ \lambda \in \mathbb{C}. \end{aligned}$$

Here we note that $\cosh(\lambda x + \frac{\pi}{2}i) = i\sinh\lambda x$ and $\sinh(\lambda x + \frac{\pi}{2}i) = i\cosh\lambda x$. Set $\lambda = \alpha + 2m\pi i$ with fixed $\alpha \in \mathbb{C}$ and $m \in \mathbb{N}$.

Then, since $K(1,\cdot) \in \{L^2(0,1)\}^4$ by Lemma 2.1, it follows from the Riemann-Lebesgue lemma that

$$(B_0 + B_1U(1,\alpha+2m\pi i))_{k\ell} = \left(B_0 + B_1R(1)\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}\right)_{k\ell} + o(1),$$

as $|m| \to \infty$ for $1 \le k,\ell \le 2$.

Therefore

$$\det(B_0 + B_1U(1,\lambda)) = \det\left(B_0 + B_1R(1)\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}\right) + o(1) \tag{3.1}$$

as $|m| \to \infty$. On the other hand, we directly verify that

$$B_0 + B_1 R(1) \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$
$$= \begin{pmatrix} -\sinh\mu & \cosh\mu \\ e^{-\theta_1(1)}\sinh(\alpha-\theta_2(1)+\nu) & e^{-\theta_1(1)}\cosh(\alpha-\theta_2(1)+\nu) \end{pmatrix},$$

so that

$$\det\left(B_0 + B_1 R(1) \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}\right) = -e^{-\theta_1(1)}\sinh(\alpha-\theta_2(1)+\mu+\nu).$$

Therefore, choosing $\alpha \neq \theta_2(1) - \nu - \mu + \ell\pi i$, for any $\ell \in \mathbb{Z}$, we obtain that

$$\det\left(B_0 + B_1 R(1) \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}\right) \neq 0.$$

In view of (3.1), $\det(B_0 + B_1 U(1,\lambda_0)) \neq 0$ for $\lambda_0 = \alpha + 2m\pi i$ with sufficiently large $m \in \mathbb{N}$. Thus the proof of Lemma 3.1 is complete.

Second Step. In view of Lemma 3.1, we can denote the set of the eigenvalues of A by $\{\lambda_n\}_{n\in\mathbb{Z}}$. The number λ is an eigenvalue of the operator A if and only if

$$\Phi(\lambda) \equiv \left(\varphi(1,\lambda), \begin{pmatrix} \sinh\nu \\ \cosh\nu \end{pmatrix}\right) = 0, \tag{3.2}$$

where $\varphi(x,\lambda)$ is the solution of the Cauchy problem (2.4). Here and henceforth $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^2$. It follows from (2.5) that

$$\Phi(\lambda) = e^{-\theta_1(1)}\sinh(\lambda+\mu+\nu-\theta_2(1)) + \Phi_1(\lambda), \tag{3.3}$$

where

$$\Phi_1(\lambda) = \left(\int_0^1 K(1,y) \begin{pmatrix} \cosh(\lambda y+\mu) \\ \sinh(\lambda y+\mu) \end{pmatrix} dy, \begin{pmatrix} \sinh\nu \\ \cosh\nu \end{pmatrix}\right). \tag{3.4}$$

We set

$$a_0 = -\mu - \nu + \theta_2(1).$$

It is known by the Luzin theorem that for any $\varepsilon > 0$, one can find a bounded step matrix-function $K_\varepsilon(y)$ such that

$$\int_0^1 |K(1,y) - K_\varepsilon(y)|dy \le \varepsilon.$$

Therefore

$$\left|\left(\int_0^1 (K(1,y) - K_\varepsilon(y)) \begin{pmatrix} \cosh(\lambda y + \mu) \\ \sinh(\lambda y + \mu) \end{pmatrix} dy, \begin{pmatrix} \sinh\nu \\ \cosh\nu \end{pmatrix}\right)\right| \le 4\varepsilon e^{|\mathrm{Re}\mu| + |\mathrm{Re}\nu|} e^{|\mathrm{Re}\lambda|}$$

for $\lambda \in \mathbb{C}$. Since K_ε is a bounded step function, there is a constant $C_\varepsilon > 0$ such that for any $\lambda \in \mathbb{C}$

$$\left|\left(\int_0^1 K_\varepsilon(y) \begin{pmatrix} \cosh(\lambda y + \mu) \\ \sinh(\lambda y + \mu) \end{pmatrix} dy, \begin{pmatrix} \sinh\nu \\ \cosh\nu \end{pmatrix}\right)\right| \le \frac{C_\varepsilon}{|\lambda|} \exp(|\mathrm{Re}\lambda|).$$

Thus we have proved an estimate on the whole λ-plane:

$$|\Phi_1(\lambda)| \le (4\varepsilon e^{|\mathrm{Re}\mu| + |\mathrm{Re}\nu|} + \frac{C_\varepsilon}{|\lambda|}) \exp(|\mathrm{Re}\lambda|).$$

Therefore it follows that for an arbitrary $\varepsilon > 0$, we can choose $\Lambda > 0$ such that

$$|\Phi_1(\lambda)| \le \varepsilon \exp(|\mathrm{Re}\lambda|) \tag{3.5}$$

for all $|\lambda| \ge \Lambda$.

Now we will show that there exists a constant $K > 0$ such that

$$|\mathrm{Re}\,(\lambda_n - a_0)| \le K \tag{3.6}$$

for all the eigenvalues λ_n. Let us suppose contrarily. That is, we can take $C_n > 0$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} C_n = \infty$ and for $n \in \mathbb{N}$, there is an eigenvalue λ_n such that $|\mathrm{Re}\,(\lambda_n - a_0)| > C_n$. Then, without loss of generality, we can suppose that there is a countable sequence of eigenvalues λ_{k_n} so that

$$\mathrm{Re}\,(\lambda_{k_n} - a_0) > C_n, \qquad n \in \mathbb{N}. \tag{3.7}$$

In fact, otherwise, there exists a countable sequence of eigenvalues λ_{k_n} so that $\mathrm{Re}\,(\lambda_{k_n} - a_0) < -C_n$ and we can argue similary for that sequence.

It follows from (3.5) that for any $\varepsilon > 0$ we can choose N such that for any $k_n \geq N$

$$|\Phi_1(\lambda_{k_n})| \leq \varepsilon e^{\mathrm{Re}\,\lambda_{k_n}}, \qquad n \in \mathbb{N}. \tag{3.8}$$

On the other hand, for sufficiently large k_n, we see

$$|e^{-\theta_1(1)} \sinh(\lambda_{k_n} - a_0)| \geq \frac{1}{4}|e^{-\theta_1(1)}| e^{-|\mathrm{Re}\,a_0|} \exp(\mathrm{Re}\,\lambda_{k_n}). \tag{3.9}$$

In fact, by (3.7), we have

$$\begin{aligned} &|\sinh(\lambda_{k_n} - a_0)| = \frac{1}{2} e^{-\mathrm{Re}\,(\lambda_{k_n} - a_0)} |e^{2(\lambda_{k_n} - a_0)} - 1| \\ \geq & \frac{1}{2} e^{-\mathrm{Re}\,(\lambda_{k_n} - a_0)} (e^{2\mathrm{Re}\,(\lambda_{k_n} - a_0)} - 1) \geq \frac{1}{2} e^{-\mathrm{Re}\,(\lambda_{k_n} - a_0)} \times \frac{1}{2} e^{2\mathrm{Re}\,(\lambda_{k_n} - a_0)} \end{aligned}$$

for large n.

Hence, by (3.8) and (3.9), for sufficiently large n, the number λ_{k_n} can not be a zero of the function $\Phi(\lambda) = e^{-\theta_1(1)} \sinh(\lambda - a_0) + \Phi_1(\lambda)$. This is a contradiction. Therefore (3.6) is proved.

Third Step. In (3.6), we further choose $K > 0$ large enough, so that

$$|\Phi_1(\lambda)| < |e^{-\theta_1(1)} \sinh(\lambda - a_0)|$$

for all λ with $|\mathrm{Re}\,\lambda| = K$. This is proved similarly to (3.9), in view of (3.5). Then we set

$$\begin{aligned} K_n = \{\lambda;\, &\mathrm{Re}\,a_0 - K - 1 < \mathrm{Re}\,\lambda < \mathrm{Re}\,a_0 + K + 1, \\ &\mathrm{Im}\,a_0 + n\pi - \frac{\pi}{2} < \mathrm{Im}\,\lambda < \mathrm{Im}\,a_0 + n\pi + \frac{\pi}{2}\}. \end{aligned} \tag{3.10}$$

We will show that

$$\text{there is } N \in \mathbb{N} \text{ such that for any } |n| \geq N, \text{ there exists exactly one eigenvalue in } K_n \text{ with algebraic multiplicity one.} \tag{3.11}$$

Noting that $K_n = \{\lambda + n\pi i; \lambda \in K_0\}$, by the definition (3.10) of K_0, we have

$$\min_{\lambda\in\partial K_n} |e^{-\theta_1(1)} \sinh(\lambda - a_0)| = \min_{\lambda\in\partial K_0} |e^{-\theta_1(1)} \sinh(\lambda - a_0)| \equiv L > 0. \tag{3.12}$$

We see from the estimate (3.5) that we can choose $N \in \mathbb{N}$ such that

$$\sup_{\lambda\in\partial K_n} |\Phi_1(\lambda)| < L, \qquad |n| \geq N. \tag{3.13}$$

In view of (3.12) and (3.13), we apply the Rouché theorem to $\Phi(\lambda) = e^{-\theta_1(1)} \sinh(\lambda - a_0) + \Phi_1(\lambda)$ and $e^{-\theta_1(1)} \sinh(\lambda - a_0)$ in K_n, so that the proof of (3.11) is complete.

By the choice of the constants K and N, we obtain

$$|\Phi_1(\lambda)| < |e^{-\theta_1(1)} \sinh(\lambda - a_0)|$$

on the boundary of $\bigcup_{k=-N+1}^{N-1} K_n$. By the Rouché theorem, it means that there are exactly $2N-1$ eigenvalues including algebraic multiplicities inside of $\bigcup_{k=-N+1}^{N-1} K_n$.

Fourth Step. Next we show that the eigenvalues λ_n have an asymptotic form

$$\lambda_n = n\pi i + a_0 + \delta_n \quad \text{where } \delta_n = o(1) \text{ as } |n| \to \infty. \tag{3.14}$$

For this, it is sufficient to prove that for any $r > 0$, there is a constant N such that there is exactly one eigenvalue inside of the circle $C_n = \{\lambda : |n\pi i + a_0 - \lambda| \leq r\}$ if $|n| \geq N$.

Firstly we easily see that

$$\min_{\lambda\in\partial C_n} |e^{-\theta_1(1)} \sinh(\lambda - a_0)| = \min_{\lambda\in\partial C_0} |e^{-\theta_1(1)} \sinh(\lambda - a_0)| \equiv l > 0.$$

In view of (3.5), we can repeat the argument in the third step and apply the Rouché theorem to finish the proof of (3.14).

Fifth Step. Let us show now that eigenvalues λ_n have an asymptotic form $\lambda_n = n\pi i + a_0 + \delta_n$ where $\sum_{n=-\infty}^{\infty} |\delta_n|^2 < \infty$.

Since $\Phi(\lambda_n) = 0$, by (3.2) and (3.3), we have

$$e^{-\theta_1(1)} \sinh(\lambda_n - a_0) = e^{-\theta_1(1)}(-1)^n \sinh\delta_n = -\Delta_n,$$

where

$$\Delta_n = \Phi_1(\lambda_n) = \left(\int_0^1 K(1,y) \begin{pmatrix} \cosh(\lambda_n y + \mu) \\ \sinh(\lambda_n y + \mu) \end{pmatrix} dy, \begin{pmatrix} \sinh\nu \\ \cosh\nu \end{pmatrix} \right).$$

Then we can write:

$$\Delta_n = \int_0^1 g_1(y) e^{\lambda_n y} dy + \int_0^1 g_2(y) e^{-\lambda_n y} dy$$

with some $g_1, g_2 \in L^2(0,1)$.

Therefore, by (3.14), we obtain

$$\Delta_n = \int_{-\pi}^{\pi} f(y) e^{(n+\sigma_n)iy} dy,$$

where $f \in L^2(-\pi,\pi)$ is suitably given and $\sigma_n = o(1)$ as $|n| \to \infty$.

Now we are going to use the following theorem (pp. 108-109 in Paley and Wiener [5]): If $|\gamma_n - n| \le L < \pi^{-2}$, then a system $\{e^{i\gamma_n x}\}_{n\in\mathbb{Z}}$ constitutes a Riesz basis in $L^2(-\pi,\pi)$.

It follows from this theorem and the asymptotics for eigenvalues that there exists a natural number M such that a system $\{e^{inx}\}_{|n|\le M} \cup \{e^{i(n+\sigma_n)x}\}_{|n|>M}$ constitutes a Riesz basis in $L^2(-\pi,\pi)$.

Hence $\sum_{n=-\infty}^{\infty} |\Delta_n|^2 < \infty$ by $f \in L^2(-\pi,\pi)$. Consequently $\sum_{n=-\infty}^{\infty} |\sinh\delta_n|^2 < \infty$. Noting that $\delta_n = \sinh\delta_n[1 + o(1)]$ by $\lim_{\lambda\to 0} \left|\frac{\sinh\lambda}{\lambda}\right| = 1$, it follows that $\sum_{n=-\infty}^{\infty} |\delta_n|^2 < \infty$.

§4. Proof of the second part of Theorem.

By the first part of Theorem, we can number the eigenvalues of A as follows: $\sigma(A) = \{\lambda_n\}_{|n|\ge N} \cup \{\kappa_\ell\}_{1\le\ell\le m}$ with

$$(4.1) \qquad \begin{cases} \lambda_n = a_0 + n\pi i + \delta_n, & |n| \ge N, \quad \sum_{|n|\ge N} |\delta_n|^2 < \infty, \\ |\mathrm{Im}\,(\kappa_\ell - a_0)| \le \left(N - \dfrac{1}{2}\right)\pi, & 1 \le \ell \le m. \end{cases}$$

Here λ_n is an eigenvalue with algebraic multiplicity one for $|n| \ge N$, and κ_ℓ is an eigenvalue with algebraic multiplicity χ_ℓ for $1 \le \ell \le m$. We note that

$$(4.2) \qquad \sum_{\ell=1}^{m} \chi_\ell = 2N - 1.$$

We choose a basis $\{\psi_{\ell k}\}_{1\le k\le\chi_\ell}$ of the generalized eigenspace (i.e. the root subspace) for κ_ℓ, $1 \le \ell \le m$ and number the sum of all the root vectors as $\{\psi_k\}_{1\le k\le 2N-1}$.

On the other hand, by Lemma 2.1, the function

$$(4.3) \qquad \varphi_n(x) = R(x)\begin{pmatrix} \cosh(\lambda_n x + \mu) \\ \sinh(\lambda_n x + \mu) \end{pmatrix} + \int_0^x K(x,y)\begin{pmatrix} \cosh(\lambda_n y + \mu) \\ \sinh(\lambda_n y + \mu) \end{pmatrix} dy,$$

for $0 \le x \le 1$ and $|n| \ge N$, is an eigenfunction of A for the eigenvalue λ_n.

We will prove that $\{\psi_k\}_{1\le k\le 2N-1} \cup \{\varphi_n\}_{|n|\ge N+1}$ is a Riesz basis in $\{L^2(0,1)\}^2$. For this, we will apply the Bari theorem (e.g. Theorem 2.3 of Chapter VI in Gohberg and Kreĭn [3]). We introduce a sequence of functions

$$(4.4) \qquad \begin{aligned} e_n(x) \equiv& R(x)\begin{pmatrix} \cosh(\alpha_n x + \mu) \\ \sinh(\alpha_n x + \mu) \end{pmatrix} \\ =& R(x)\begin{pmatrix} \cosh(a_0 x + \mu) & i\sinh(a_0 x + \mu) \\ \sinh(a_0 x + \mu) & i\cosh(a_0 x + \mu) \end{pmatrix}\begin{pmatrix} \cos n\pi x \\ \sin n\pi x \end{pmatrix}, \quad n \in \mathbb{Z}. \end{aligned}$$

Here we set

$$(4.5) \qquad \alpha_n = a_0 + n\pi i, \quad n \in \mathbb{Z}.$$

Since $\{\cos n\pi x\}_{n\geq 0}$ and $\{\sin n\pi x\}_{n\geq 1}$ are orthogonal bases in $L^2(0,1)$ respectively, we see that $\left\{\begin{pmatrix}\cos n\pi x\\ \sin n\pi x\end{pmatrix}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $\{L^2(0,1)\}^2$. The 2×2 matrix

$$S(x) \equiv R(x)\begin{pmatrix}\cosh(a_0 x+\mu) & i\sinh(a_0 x+\mu)\\ \sinh(a_0 x+\mu) & i\cosh(a_0 x+\mu)\end{pmatrix}$$

is invertible for $0\leq x\leq 1$, so that the map $y = S\varphi$ from $\{L^2(0,1)\}^2$ to $\{L^2(0,1)\}^2$ transforms the orthonormal basis $\left\{\begin{pmatrix}\cos n\pi x\\ \sin n\pi x\end{pmatrix}\right\}_{n\in\mathbb{Z}}$ into $\{e_n\}_{n\in\mathbb{Z}}\subset\{L^2(0,1)\}^2$. Consequently

$$\{e_n\}_{n\in\mathbb{Z}} \text{ is a Riesz basis in } \{L^2(0,1)\}^2 \tag{4.6}$$

(e.g. Chapter VI, Section 2 of [3]).

Next we will show that

$$\sum_{|n|\geq N} \|e_n-\varphi_n\|^2_{\{L^2(0,1)\}^2} < \infty. \tag{4.7}$$

For this, we set

$$f_n(x) = R(x)\begin{pmatrix}\cosh(\lambda_n x+\mu)\\ \sinh(\lambda_n x+\mu)\end{pmatrix}, \quad |n|\geq N. \tag{4.8}$$

By the mean value theorem, we can estimate

$$\begin{aligned}&|\cosh(\alpha_n x+\mu)-\cosh(\lambda_n x+\mu)|\\ =&|\cosh(\alpha_n x+\mu)-\cosh(\alpha_n x+\mu+\delta_n x)|\leq C\delta_n\end{aligned}$$

and

$$|\sinh(\alpha_n x+\mu)-\sinh(\lambda_n x+\mu)|\leq C\delta_n,$$

for any $0\leq x\leq 1$ and $|n|\geq N$. Hence, since $\sum_{|n|\geq N}|\delta_n|^2<\infty$, we see that

$$\sum_{|n|\geq N} \|e_n-f_n\|^2_{\{L^2(0,1)\}^2} < \infty. \tag{4.9}$$

By (4.3), we note that

$$\varphi_n(x) - f_n(x) = \int_0^x K(x,y) \begin{pmatrix} \cosh(\lambda_n y + \mu) \\ \sinh(\lambda_n y + \mu) \end{pmatrix} dy.$$

Then we will prove that

$$\sum_{|n|\ge N} \|\varphi_n - f_n\|^2_{\{L^2(0,1)\}^2} < \infty. \tag{4.10}$$

Repeating the argument of the fifth step in Section 3, we can show that

$$\varphi_n(x) - f_n(x) = \int_{-\pi x}^{\pi x} F(x,y) e^{(n+\sigma_n)iy} dy,$$

where $F \in \{L^2((0,1)\times(-\pi,\pi))\}^2$ and $\sigma_n = o(1)$ as $|n| \to \infty$.

We have already proved in the fifth step in Section 3 (e.g. [5]) that there exists a natural number M such that a system $\{e^{inx}\}_{|n|\le M} \cup \{e^{i(n+\sigma_n)x}\}_{|n|>M}$ constitutes a Riesz basis in $L^2(-\pi,\pi)$. Therefore

$$\sum_{|n|\ge N} |\varphi_n(x_0) - f_n(x_0)|^2 \le C \int_{-\pi x_0}^{\pi x_0} |F(x_0,y)|^2 dy,$$

for any $x_0 \in [0,1]$ and consequentely

$$\sum_{|n|\ge N} \|\varphi_n - f_n\|^2_{\{L^2(0,1)\}^2} \le C \int_0^1 \int_{-\pi}^{\pi} |F(x,y)|^2 dy dx.$$

Thus (4.10) is proved and so (4.9)–(4.10) imply (4.7).

By (4.6) and (4.7), the Bari theorem completes the proof, if we verify

$$\begin{gathered} \sum_{k=1}^{2N-1} \alpha_k \psi_k + \sum_{|n|\ge N} \beta_n \varphi_n = 0, \qquad \alpha_k, \beta_n \in \mathbb{C} \\ \text{implies } \alpha_k = 0,\ 1 \le k \le 2N-1 \text{ and } \beta_n = 0,\ |n| \ge N. \end{gathered} \tag{4.11}$$

Let us define $P_n = -\frac{1}{2\pi i}\int_{\Gamma_n} (A-\lambda)^{-1} d\lambda, \quad |n| \ge N$, where Γ_n, $|n| \ge N$ are sufficiently small circles centred at λ_n including no other points of $\sigma(A)$. By Lemma

3.1, such circles exist. Then $P_n\varphi_n = \varphi_n$, $P_n\varphi_m = 0$, $P_n\psi_k = 0$, $n \neq m$, $|n|, |m| \geq N$, $1 \leq k \leq 2N-1$ (e.g. Kato [4]). Therefore application of P_n, $|n| \geq N$, to (4.11) yields $\beta_n\varphi_n = 0$, $|n| \geq N$, and so $\sum_{k=1}^{2N-1} \alpha_k\psi_k = 0$. Therefore, since ψ_k, $1 \leq k \leq 2N-1$, are linearly independent, we see that $\alpha_k = 0$, $1 \leq k \leq 2N-1$ and $\beta_n = 0$, $|n| \geq N$. Thus the verification of (4.11) is complete.

References

1. S. Cox and R. Knobel, *An inverse spectral problem for a nonnormal first order differential operator*, Integr. Equat. Oper. Th **25** (1996), 147–162.
2. S. Cox and E. Zuazua, *The rate at which energy decays in a damping string*, Commun. in Partial Differential Equations **19** (1994), 213–243.
3. I.C. Gohberg and M.G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, American Mathematical Society, Providence, Rhode Island, 1969.
4. T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, Berlin, 1980.
5. R.E.A.C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society, New York, 1934.
6. D.L. Russell, *Control theory of hyperbolic equations related to certain questions in harmonic analysis and spectral theory*, J. Math. Anal. Appl. **40** (1972), 336–368.
7. V.S. Rykhlov, *On completeness of eigenfunctions of quadratic bundles of ordinary differential operators*, Russian Mathematics (Iz. VUZ) **36 No.3** (1992), 33–42.
8. A.A. Shkalikov, *Boundary value problems for ordinary differential equation with parameter in the boundary conditions*, Trudy semin. im. I.G. Petrovskogo, Izd-vo Mosk. Univ., Moscow **9** (1983), 190–229.
9. M.A. Shubov, *Spectral operators generated by damped hyperbolic equations*, Integr. Equat. Oper. Th. **28** (1997), 358–372.
10. M.A. Shubov, C.F. Martin, J.P. Dauer and B.P. Belinskiy, *Exact controllability of the damped wave equation*, SIAM J. Control Optim. **35** (1997), 1773–1789.
11. I.Trooshin and M. Yamamoto, *Riesz basis of root vectors of a non-symmetric system of first-order ordinary differential operators and application to inverse problems*, Preprint Series UTMS 2000-39, The Graduate School of Mathematical Sciences, The University of Tokyo, 2000.
12. A.I. Vagabov, *Quadratic bundles of ordinary differential bundles*, Math. Zametski **42** (1987), 708–715.
13. M. Yamamoto, *Inverse spectral problem for systems of ordinary differential equations of first order, I.*, Journal of the Faculty of Science, The University of Tokyo, Sec.IA, Math. **35** (1988), 519–546.

UTMS

2000–67 Kazuya Kato: *P-adic Hodge theory and values of zeta functions of modular forms.*

2001–1 André Martinez, Shu Nakamura and Vania Sordoni: *Phase space tunneling in multistate scattering.*

2001–2 Zhi Lü and R.E. Stong: *Classifying involutions fixing* $\mathbb{RP}^{odd} \sqcup P(h,i)$ *up to equivariant cobordism.*

2001–3 Go Yamamoto: *Algebraic structures on quasi-primary states in superconformal algebras.*

2001–4 J. Cheng, G. Nakamura and M. Yamamoto: *A global conformal uniqueness in the anisotropic inverse boundary value problem.*

2001–5 Fabien Trihan: *Cohomologie syntomique des* $F-T$*-cristaux.*

2001–6 Shigeyuki Morita: *Generators for the tautological algebra of the moduli space of curves.*

2001–7 Shigeo Kusuoka: *Nonlinear transformation containing rotation and Gaussian measure.*

2001–8 Kazuya Kato and Takeshi Saito: *Conductor formula of Bloch.*

2001–9 Hiroshige Kajiura, Yutaka Matsuo and Tadashi Takayanagi: *Exact Tachyon condensation on noncommutative torus.*

2001–10 Mikio Furuta and Yukio Kametani: *The Seiberg-Witten equations and equivariant e-invariants.*

2001–11 Igor Trooshin and Masahiro Yamamoto: *Spectral properties of non-symmetric systems of ordinary differential operators.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012