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# The Seiberg-Witten equations and equivariant $e$-invariants

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**Abstract**

We give a partial result toward the 11/8-conjecture concerning the second Betti number and the signature of a closed spin 4-manifold. When the cup product structure on the first cohomology satisfies some additional condition, a somewhat stronger inequality is obtained.

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# 1 Introduction

It is an open problem to determine which unimodular form of even type is realized as the intersection of a closed spin 4-manifold. By using the classification of the unimodular form and Rochlin's theorem, this problem is summarized by an inequality in [14], which is now known as the 11/8-conjecture. The first step toward this conjecture was Donaldson's celebrated works ([3], [4]). The purpose of this paper is to improve the inequality shown by the first author in [6].

Let $X$ be a 4-dimensional closed oriented spin manifold. Rochlin's theorem implies that $sign(X)$, the signature of $X$, is divisible by 16. Then the 11/8-conjecture reads as $b_2^+(X) \geq 3(-\mathrm{sign}(X)/16)$, where $b_2^+(X)$ is the maximal dimension of the positive definite subspace of $H^2(X, \mathbf{R})$ with respect to the intersection form of $X$. When $b_2^+(X) = 0$, Donaldson [3] has shown $\mathrm{sign}(X) = 0$. By using the Seiberg-Witten equations, the first author showed that if $\mathrm{sign}(X)$ is not zero, the inequality

$$b_2^+(X) \geq 2(-\mathrm{sign}(X)/16) + 1$$

holds [6]. Recently optimal estimates are proved in some special cases; $b_2^+(X) \geq 6$ when $\mathrm{sign}(X) = -32$ ([10], [11]) and $b_2^+(X) \geq 9$ when $\mathrm{sign}(X) = -48$ [9]. In this paper we give a proof of the following inequality, which is a generalization of the above results.

**Theorem 1.** *Suppose that $X$ is a 4-dimensional closed oriented spin manifold. If* $\mathrm{sign}(X) < 0$*,*

$$b_2^+(X) \geq \begin{cases} 2(-\mathrm{sign}(X)/16) + 1 & \text{if } -\mathrm{sign}(X)/16 \equiv 0, 1 \mod 4, \\ 2(-\mathrm{sign}(X)/16) + 2 & \text{if } -\mathrm{sign}(X)/16 \equiv 2 \mod 4, \\ 2(-\mathrm{sign}(X)/16) + 3 & \text{if } -\mathrm{sign}(X)/16 \equiv 3 \mod 4. \end{cases}$$

The above inequality was first proved by N. Minami in [15] by using an equivariant join theorem to reduce the inequality to a theorem of Stolz [17]. Minami showed a stabilized version of Stolz's theorem. Our strategy is also to show this stabilized version (see Theorem 21) for a stable range. However our proof is quite different from his argument and allows us the following result.

To show the above theorem, we may suppose $b_1(X) = 0$ by performing such a surgery along nontrivial loops in $X$ that does not change the intersection form. It, however, does change $H^1(X)$ when $b_1(X) > 0$. Motivated by Ruberman and Strle's work [16], the authors and F. Matsue and Minami showed that the map

$$\wedge^4 H^1(X, \mathbf{Z}) \to \mathbf{Z}, \quad \wedge_{i=1}^4 \alpha_i \to \alpha_1\alpha_2\alpha_3\alpha_4[X]$$

concerns the inequality ([10]). Suppose that the image of $\wedge^4 H^1(X, \mathbf{Z}) \to \mathbf{Z}$ contains an odd integer. Then we obtained the inequality $b_2^+(X) \geq 6$ when $\mathrm{sign}(X) = -16$, and $b_2^+(X) \geq 9$ when $\mathrm{sign}(X) = -32$ ([10]). These inequalities are stronger than the 11/8-inequality. Note that these inequalities are optimal since $K3\#T^4$ or $K3\#K3\#T^4$ satisfies the equality. Our second main theorem is the following generalization of these inequalities.

**Theorem 2.** *Let $X$ be a 4-dimensional closed oriented spin manifold. satisfying* $\mathrm{sign}(X) \leq -64$*. Suppose that the image of $\wedge^4 H^1(X, \mathbf{Z}) \to \mathbf{Z}$ contains an odd integer. Then we have*

$$b_2^+(X) \geq \begin{cases} 2(-\mathrm{sign}(X)/16) + 5 & \text{if } -\mathrm{sign}(X)/16 \equiv 0, 1 \mod 4, \\ 2(-\mathrm{sign}(X)/16) + 6 & \text{if } -\mathrm{sign}(X)/16 \equiv 2 \mod 4, \\ 2(-\mathrm{sign}(X)/16) + 7 & \text{if } -\mathrm{sign}(X)/16 \equiv 3 \mod 4. \end{cases}$$

In the remaining case $\mathrm{sign}(X) = -48$ we can only state $b_2^+(X) \geq 10$ as a corollary (see Remark 30).

It may be natural to ask the following conjecture;

**Conjecture 3.** *When $X$ satisfies the assumption of Theorem 2, we have the inequality* $b_2^+(X) \geq 3(-\mathrm{sign}(X)/16) + 3$*.*

To prove the above two theorems, we use a $Pin_2$-symmetry of the Seiberg-Witten equation for the trivial spin$^c$ structure of spin 4-manifolds first exploited by Kronheimer [13] and its finite dimensional approximation as in ([6], [7], [2]). But we do not use K-theory degree, instead we use equivariant framing and an equivariant version of Adams' $e$-invariant for the zero of the equation. In this sense our proof is along Kronheimer's one. The different point is that we only use a $\mathbf{Z}/4$-symmetry of the equation.

In this paper we first introduce the notion of equivariant e-invariant for $\mathbf{Z}_2$-graded $G$-modules to show Theorem 1. Secondly we formulate a family version of the above construction:

the notion of equivariant e-invariant for $\mathbf{Z}_2$-graded $G$-equivariant vector bundles, which allows us to show Theorem 2.

We could presumably show our results without using directly the finite dimensional approximation which describes the equation globally. But we do not pursue it, since we want to avoid setting up gauge theory.

# 2 Equivariant framed bordisms and $e$-invariants

In this section we define some variants of framed bordism groups and associated equivariant $e$-invariants. Next we give formulae to compute them from the fixed-point data.

**Definition 4.** Let $G$ be a finite group and $U$ a $G$-module over $\mathbf{R}$. We call $U$ *a* $G$-spin$^c$ *vector space* [resp. $G$-spin *vector space*] when the following data are given.

1. A $G$-invariant orientation of $U$ is given.
2. A spin$^c$ [resp. spin] structure on $U$ compatible with the orientation is given.
3. A lift of the $G$-action to the spin$^c$ [resp. spin] structure is given.

For $G$-equivariant vector bundles, the notion of $G$-spin$^c$ and $G$-spin are obviously extended. We call a $G$-manifold $M$ $G$-spin$^c$ [resp. $G$-spin] when $TM$ is $G$-spin$^c$ [resp. $G$-spin]

Let $G$ be a finite group and $U_0$, $U_1$ two $G$-spin$^c$ [resp. $G$-spin] vector spaces over $\mathbf{R}$. We assume $\dim U_0 - \dim U_0 \equiv 0 \bmod 2$ [resp. $\dim U_0 - \dim U_1 \equiv 4 \bmod 8$].

Consider a pair $(M, \phi)$ such that

1. $M$ is a closed free $G$-manifold.
2. $\phi$ is a $G$-equivariant isomorphism; $\phi : (TM \oplus \mathbf{R}) \oplus U_1 \cong U_0$, where $\mathbf{R}$ is the 1-dimensional trivial $G$-module.

Note that $\phi$ induces a $G$-spin$^c$ [resp. $G$-spin]- structure on $TM$.

We denote by $\Omega(U_0, U_1)$ the bordism group of such pairs $(M, \phi)$, where $(M_1, \phi_1)$ and $(M_2, \phi_2)$ are bordant if there exists a pair $(W, \phi_W)$ such that

1. $W$ is a compact free $G$-manifold with a $G$-equivariant diffeomorphism $\partial W \cong M_1 \coprod M_2$.
2. $\phi_W$ is a $G$-equivariant isomorphism $\phi_W : TW \oplus U_1 \cong U_0$ which extends $\phi_1$ and $\phi_2$.

Here we identify $TW|M_1$ with $TM_1 \oplus \mathbf{R}$ by using the inward normal direction, and $TW|M_2$ with $TM_2 \oplus \mathbf{R}$ by using the outward normal direction.

Let $\Omega^{\mathrm{b,f}}_{\mathbf{C}}(U_0, U_1)$ [resp. $\Omega^{\mathrm{b,f}}_{\mathbf{R}}(U_0, U_1)$ ] be the subgroup of classes of the pairs $[M, \phi] \in \Omega(U_0, U_1)$ such that the free $G$-spin$^c$ [resp. $G$-spin] manifold $M$ is diffeomorphic to the boundary of some compact free $G$-spin$^c$ [resp. $G$-spin] manifold $W$ by a $G$-equivariant diffeomorphism preserving $G$-spin$^c$ [resp. $G$-spin] structures.

## 2.1 Definition of $e_{\mathbf{C}}(M, \phi)$

We use the following notation. For a based compact Hausdorff space $Z$, let $Pic(Z)$ be the group of isomorphism classes of complex line bundles over $Z$.

The map

$$e^{c/2}\hat{A} - 1 : KO(Z) \times Pic(Z) \to H^*(Z, \mathbf{Q}), \quad (\alpha, \xi) \mapsto e^{c(\xi)/2}\hat{A}(\alpha) - 1$$

induces a map $e^{c/2}\hat{A} - 1 : \tilde{KO}(Z) \times Pic(Z) \to \tilde{H}^*(Z, \mathbf{Q})$. It implies that we have a functorially defined morphism

$$e^{c/2}\hat{A} - 1 : KO(Z_0, Z_1) \times Pic(Z_0/Z_1) \to H^*(Z_0, Z_1)$$

for any compact Hausdorff pair $Z_0 \supset Z_1$. Let $[E_0, E_1, \psi]$ be an element of $KO(Z_0, Z_1)$. Suppose $E_0$ and $E_1$ are given some spin$^{\mathrm{c}}$-structures and a lift of $\psi$ to the spin$^{\mathrm{c}}$-structures is fixed. We define $[L(\phi)] \in Pic(W/M)$ as follows. Let $\det E_0$ and $\det E_1$ be the complex line bundles associated to the spin$^c$ structures on $E_0$ and $E_1$ respectively. Then $\psi$ induces an isomorphic $\det \psi : \det E_0 \mid Z_1 \cong \det E_1 \mid Z_1$ over $Z_1$. It implies that $L(\psi) := \det E_0 \otimes (\det E_1)^*$ has a trivialization over $Z_0$ and hence gives an element of $Pic(Z_0/Z_1)$. Then we have the element

$$(e^{c/2}\hat{A} - 1)([E_0, E_1, \psi], L(\psi)) \in H^*(Z_0, Z_1, \mathbf{Q}).$$

Returning to our setting of the bordism group $\Omega^{\mathrm{b,f}}_{\mathbf{C}}(U_0, U_1)$, we define a homomorphism $e_{\mathbf{C}} : \Omega^{\mathrm{b,f}}_{\mathbf{C}}(U_0, U_1) \to \mathbf{Q}/\mathbf{Z}$ as follows. For $[M, \phi] \in \Omega^{\mathrm{b,f}}_{\mathbf{C}}(U_0, U_1)$, let $W$ be a compact free $G$-spin$^{\mathrm{c}}$ manifold with a $G$-equivariant diffeomorphism $\partial W \cong M$ preserving $G$-spin$^{\mathrm{c}}$ structures.

Now we define:

$$e_{\mathbf{C}}(M, \phi, W) \quad := \quad \frac{1}{\#G} \int_W (e^{c/2}\hat{A} - 1)([TW \oplus U_1, U_0, \phi], L(\phi)) \in \mathbf{Q}.$$

**Lemma 5.** $e_{\mathbf{C}}(M, \phi, W) \bmod \mathbf{Z}$ *does not depend on the choice of* $W$*.*

*Proof.* It suffices to show that $e_{\mathbf{C}}(M, \phi, W)$ is an integer when $M$ is empty. In this case we have

$$\begin{aligned}
e_{\mathbf{C}}(M, \phi, W) &= \frac{1}{\#G} \int_W (e^{c/2}\hat{A} - 1)([TW \oplus U_1, U_0, \phi], c_1(L(\phi))) \\
&= \frac{1}{\#G} \int_W e^{c_1(L(\phi))/2}\hat{A}([TW] + [U_1] - [U_0]) - 1) \\
&= \frac{1}{\#G} \int_W e^{c_1(\det W)/2}\hat{A}(TW) \\
&= \int_{W/G} e^{c_1(\det W/G)/2}\hat{A}(T(W/G))
\end{aligned}$$

The required statement follows from the integrality theorem for the index of the Dirac operator on the closed spin$^c$-manifold $W/G$. ◻

**Definition 6.** $e_{\mathbf{C}}(M, \phi) := e_{\mathbf{C}}(M, \phi, W) \bmod \mathbf{Z} \in \mathbf{Q}/\mathbf{Z}$

We can extend the definition of $e_{\mathbf{C}}(M, \phi, W)$ when the $G$-action on $W$ is not free. For $[M, \phi] \in \Omega^{\mathrm{b,f}}_{\mathbf{C}}(U_0, U_1)$, let $W$ be a $G$-spin$^{\mathrm{c}}$ manifold with a $G$-equivariant diffeomorphism $\partial W \cong M$ preserving $G$-spin$^{\mathrm{c}}$ structures.

We first define $e_{\mathbf{C}}(M, \phi, W)_g \in \mathbf{C}$ for every $g \in G$.

$$e_{\mathbf{C}}(M, \phi, W)_g = \begin{cases} \int_W (e^{c/2}\hat{A} - 1)([TW \oplus U_1, U_0, \phi], L(\phi)) & (g = 1) \\ \mathrm{Spin}^{\mathrm{c}}(g, W) & (g \neq 1), \end{cases}$$

the right-hand-side of where the right-hand-side is the local contribution from the fixed point set $W^g$. More explicitly $\mathrm{Spin}^{\mathrm{c}}(g, W)$ is given by

$$\mathrm{Spin}^{\mathrm{c}}(g, W) = \int_{W^g} \frac{e^{c_1(W^g)/2}\hat{A}(W^g)}{\mathrm{ch}\lambda_{-1}(N^g \otimes \mathbf{C})(g)}$$

where $N^g$ is the normal bundle of $W^g$ in $W$. Then we define $e_{\mathbf{C}}(M, \phi, W)$ by

$$e_{\mathbf{C}}(M, \phi, W) := \frac{1}{\#G} \sum_{g \in G} e_{\mathbf{C}}(M, \phi, W)_g.$$

Now the next lemma is proved by the same argument as in the proof of Lemma 25.

**Lemma 7.** *When the $G$-action on $W$ is not free, we still have $e_{\mathbf{C}}(M,\phi) = e_{\mathbf{C}}(M,\phi,W)$ mod $\mathbf{Z}$.*

**Proposition 8.** *Suppose that $[M,\phi] \in \Omega^{\mathrm{b,f}}_{\mathbf{C}}(U_0,U_1)$ satisfies the following conditions.*

1. *There exists a $G$-$\mathrm{spin}^c$ compact manifold $W$ with a $G$-equivariant diffeomorphism $\partial W \cong M$ preserving $G$-$\mathrm{spin}^c$-structures.*
2. *There exists a $\mathrm{spin}^c$ compact manifold $W'$ with a diffeomorphism $\partial W' \cong -M$.*
3. *When we forget the $G$-action, we can extend the isomorphism $\phi : TM \oplus \mathbf{R} \oplus U_1 \cong U_0$ to an isomorphism $TW' \oplus U_1 \cong U_0$.*

*Then $\tilde{W} := W' \cup_M W$ is a smooth $\mathrm{spin}^c$-manifold and we have the formula*

$$e_{\mathbf{C}}(M,\phi) = \frac{1}{\#G}\int_{\tilde{W}} e^{c_1(\det \tilde{W})/2}\hat{A}(\tilde{W}) + \frac{1}{\#G}\sum_{g\in G, g\neq 1} \mathrm{Spin}^c(g,W) \bmod 1.$$

*Proof.* The existence of the isomorphism $TW' \oplus U_1 \cong U_0$ implies

$$e_{\mathbf{C}}(M,W,\phi_W)_1 = \frac{1}{\#G}\int_{\tilde{W}} e^{c_1(\det \tilde{W})/2}\hat{A}(\tilde{W}). \tag{1}$$

Lemma 13 and these two equalities imply the claim of the proposition. □

## 2.2 Definition of $e_{\mathbf{R}}(M,\phi)$

For $[M,\phi] \in \Omega^{\mathrm{b,f}}_{\mathbf{R}}(U_0,U_1)$, we define $e_{\mathbf{R}}(M,\phi) \in \mathbf{Q}/2\mathbf{Z}$ as follows. Let $W$ be a free $G$-spin manifold with a $G$-equivariant diffeomorphism $\partial W \cong M$ preserving $G$-spin structures.

**Lemma 9.** *$e_{\mathbf{C}}(M,\phi,W)$ mod $2\mathbf{Z}$ does not depend on the choice of $W$.*

*Proof.* It suffices to show that $e_{\mathbf{C}}(M,\phi,W)$ is an even integer when $M$ is empty. As in the proof of Lemma 25, when $M$ is empty, we have

$$e_{\mathbf{C}}(M,\phi,W) = \int_{W/G}\hat{A}(W/G).$$

Since $W/G$ is a $8k+4$-dimensional closed spin manifold, the right-hand-side is an even integer. □

**Definition 10.** $e_{\mathbf{R}}(M,\phi) := e_{\mathbf{C}}(M,\phi,W) \bmod 2\mathbf{Z} \in \mathbf{Q}/2\mathbf{Z}$

**Caution 11.** The above definition does *not* coincide with the usual definition of $e_{\mathbf{R}}$. Actually the above definition is equal to the *twice* of the usual one [18] when $G = \{1\}$.

To calculate $e_{\mathbf{R}}(M,\phi)$, we can use $e_{\mathbf{C}}(M,\phi,W)$ even when the $G$-action on $W$ is not free nor $W$ is not necessarily spin. The precise statement is given by Proposition 15 below.

For $[M,\phi] \in \Omega^{\mathrm{b,f}}_{\mathbf{R}}(U_0,U_1)$, let $W$ be a $G$-$\mathrm{spin}^c$ manifold with a $G$-equivariant diffeomorphism $\partial W \cong M$ preserving $G$-$\mathrm{spin}^c$ structures.

Fix a $G$-invariant Riemannian metric $m_W$ on $W$ which is of the product form near the boundary $\partial W = M$. Note that the $\mathrm{spin}^c$-structure of $W$ is reduced to a spin structure near $\partial W = M$ and hence $\det TW$ is given a trivialization near $M$. Fix a $G$-invariant connection $\theta_W$ on $\det TW$ which respects the trivialization. Let $D(m_W,\theta_W)$ be the $\mathrm{spin}^c$ Dirac operator on $W$ defined by using $m_W$ and $\theta_W$, and let $\mathrm{ind}_G D(m_W,\theta_W) \in R(G)$ be the $G$-equivariant index of $D(m_W,\theta_W)$ with the Atiyah-Patodi-Singer boundary condition.

**Lemma 12.** ([8]) $\mathrm{ind}_G D(m_W, \theta_W) \bmod Rsp(G) \in R(G)/Rsp(G)$ *does not depend on the choice of* $(m_g, \theta_W)$*, where* $Rsp(G)$ *is the Grothendieck group of finite dimensional quaternionic representations of* $G$.

We will give a proof of the above lemma in [8]. Let $\rho_0$ be the one-dimensional trivial representation of $G$. We write $\mathrm{ind}^{\mathbf{Z}/2}_{\rho_0}(W, M) \in \mathbf{Z}/2$ for the $\rho_0$-component of $\mathrm{ind}_G D(m_W, \theta_W) \bmod Rsp(G)$.

**Lemma 13.** $e_{\mathbf{R}}(M, \phi) = e_{\mathbf{C}}(M, \phi, W) - \mathrm{ind}^{\mathbf{Z}/2}_{\rho_0}(W, M) \bmod 2$.

*Proof.* It suffices to show that the right-hand side is zero when $M$ is empty. This is a consequence of the $G$-index theorem for spin$^c$ Dirac operator. ☐

structure is obstruction class

Recall that $\det TW$ has a canonical trivialization near $M = \partial W$, which is given by a nonvanishing section $s_0$ of $\det TW$ near $M$. Let $s$ be a $G$-invariant section of $\det TW$ which is an extension of $s_0$. When $s$ is transverse to the zero section, $Z := s^{-1}(0)$ is a $G$-invariant submanifold of codimension 2.

**Definition 14.** Let $W$ be a compact $G$-spin$^c$ manifold and suppose that a $G$-spin reduction of the $G$-spin$^c$-structure is given near the boundary $M := \partial W$. Then a *$G$-characteristic submanifold of* $(W, M)$ is a codimension 2 $G$-invariant submanifold obtained by using the $G$-invariant section $s$ as above.

**Proposition 15.** *Suppose that* $[M, \phi] \in \Omega^{\mathrm{b,f}}_{\mathbf{R}}(U_0, U_1)$ *satisfies the following conditions.*

1. *There exists a* $G$-spin$^c$ *compact manifold* $W$ *with a* $G$*-equivariant diffeomorphism* $\partial W \cong M$ *preserving* $G$-spin$^c$*-structures.*
2. $(W, M)$ *has a* $G$*-characteristic submanifold* $Z$ *which is contained in the fixed point set* $W^G$.
3. *There exists a* spin *compact manifold* $W'$ *with a diffeomorphism* $\partial W' \cong -M$.
4. *When we forget the* $G$*-action, we can extend the isomorphism* $\phi : TM \oplus \mathbf{R} \oplus U_1 \cong U_0$ *to an isomorphism* $TW' \oplus U_1 \cong U_0$.

*Then* $\tilde{W} := W' \cup_M W$ *is a smooth* spin$^c$*-manifold and we have the formula*

$$e_{\mathbf{R}}(M, \phi) = \left(-1 + \frac{1}{\#G}\right) \int_{\tilde{W}} e^{c_1(\det \tilde{W})/2} \hat{A}(\tilde{W}) + \frac{1}{\#G} \sum_{g \in G, g \neq 1} \mathrm{Spin}^c(g, W) \bmod 2.$$

To prove the above proposition, we need the following lemma.

**Lemma 16.** ([8]) *Let* $W$ *be a compact* $G$-spin$^c$ *manifold. Suppose that a* $G$*-spin reduction of the* $G$-spin$^c$*-structure is given near the boundary* $M := \partial W$ *and that the* $G$*-action is free on* $M$. *If* $(W, M)$ *allow a characteristic submanifold which is contained in the fixed point set* $W^G$*, then* $\mathrm{ind}^{\mathbf{Z}/2}_G(W, M) \in R(G)/Rsp(G)$ *is contained in* $(\mathbf{Z}/2)\rho_0$*, where* $\rho_0$ *is the one-dimensional trivial representation.*

A proof of Lemma 16 will be given in [8].

**Proof of Proposition 15** The above lemma implies that

$$\mathrm{ind}^{\mathbf{Z}/2}_{\rho_0}(W, M) \equiv \mathrm{ind} D(m_W, \theta_W) \bmod 2,$$

where the right-hand side is the non-equivariant index. Let $m_{W'}$ be a Riemannian metric on $W'$ which can be patched together with $m_W$ to get a smooth Riemannian metric $m_{\tilde{W}}$ on $W \cup_M W'$. The Atiyah-Patodi-Singer formula implies the sum formula

$$\mathrm{ind} D(m_{\tilde{W}}, \theta_W) = \mathrm{ind} D(m_W, \theta_W) + \dim \mathrm{Ker} D(m_M) + \mathrm{ind} D(m_{W'}),$$

where $m_M$ is the restriction of $m_W$ ( or $m_{W'}$) on $M$. Since $M$ is a $8k+3$-dimensional spin manifold, the kernel of the Dirac operator $D(m_M)$ have quaternionic structure, and hence $\dim \mathrm{Ker} D(m_M)$ is even. Similarly the index $\mathrm{ind} D(m_{W'})$ is even. ¿From the above two equalities we obtain

$$\begin{aligned}
\mathrm{ind}_G^{\mathbf{Z}/2}(W,M) &\equiv \mathrm{ind} D(m_{\tilde W},\theta_W) \\
&\equiv \int_{\tilde W} e^{c_1(\det \tilde W)/2}\hat A(\tilde W) \bmod 2.
\end{aligned}$$

On the other hand the existence of the isomorphism $TW'\oplus U_1\cong U_0$ implies (1). Lemma 13 and these two equalities imply the claim of the proposition. □

## 2.3 Extension of the definition of $e_{\mathbf{C}}$ and $e_{\mathbf{R}}$

We can define $e_{\mathbf{C}}$ and $e_{\mathbf{R}}$ for wider classes.

**Definition 17.** 1. Suppose that $U_0$ and $U_1$ are two $G$-spin$^c$ vector spaces over $\mathbf{R}$ satisfying $\dim U_0 - \dim U_1 \equiv 0 \bmod 2$. Then let $\Omega^{\mathrm{b}}_{\mathbf{C}}(U_0,U_1)$ be the subgroup of classes of the pairs $[M,\phi]\in\Omega(U_0,U_1)$ such that the free $G$-spin$^c$ manifold $M$ is diffeomorphic to the boundary of some compact $G$-spin$^c$ manifold $W$ by a $G$-equivariant diffeomorphism preserving $G$-spin$^c$ structures.

2. For such $W$, we can define

$$e_{\mathbf{C}}(M,\phi) := e_{\mathbf{C}}(M,\phi,W) \bmod \mathbf{Z} \in \mathbf{Q}/\mathbf{Z}.$$

The well-definedness is shown as in the proof of Lemma 7

**Definition 18.** 1. Suppose that $U_0$ and $U_1$ are two $G$-spin vector spaces over $\mathbf{R}$ satisfying $\dim U_0 - \dim U_1 \equiv 4 \bmod 8$. Then let $\Omega^{\mathrm{b}}_{\mathbf{R}}(U_0,U_1)$ be the subgroup of classes of the pairs $[M,\phi]\in\Omega(U_0,U_1)$ such that the free $G$-spin manifold $M$ is diffeomorphic to the boundary of some compact $G$-spin$^c$ manifold $W$ by a $G$-equivariant diffeomorphism preserving $G$-spin$^c$ structures.

2. For such $W$, we can define

$$e_{\mathbf{R}}(M,\phi) := e_{\mathbf{C}}(M,\phi,W) - \mathrm{ind}_{\rho_0}^{\mathbf{Z}/2}(W,M) \bmod 2\mathbf{Z} \in \mathbf{Q}/2\mathbf{Z}.$$

The well-definedness is shown as in the proof of Lemma 13

**Remark 19.** 1. In both cases $W$ is not necessarily free $G$-manifold. In the latter case $W$ is not necessarily spin.

2. We can further extend our construction to define the maps

$$\begin{aligned}
e^G_{\mathbf{C}} &: \Omega^{\mathrm{b}}_{\mathbf{C}}(U_0,U_1)\to (R(G)\otimes\mathbf{C})/R(G), \\
e^G_{\mathbf{R}} &: \Omega^{\mathrm{b}}_{\mathbf{R}}(U_0,U_1)\to (R(G)\otimes\mathbf{C})/Rsp(G).
\end{aligned}$$

The maps $e_{\mathbf{C}}$ and $e_{\mathbf{R}}$ are nothing but the $\rho_0$-components of $e^G_{\mathbf{C}}$ and $e^G_{\mathbf{R}}$. On the subgroups $\Omega^{\mathrm{b,f}}_{\mathbf{C}}(U_0,U_1)$ and $\Omega^{\mathrm{b,f}}_{\mathbf{R}}(U_0,U_1)$, the images of $e^G_{\mathbf{C}}$ and $e^G_{\mathbf{R}}$ are contained in the multiples of the regular representation. Hence on these subgroups $e^G_{\mathbf{C}}$ and $e^G_{\mathbf{R}}$ are completely determined by $e_{\mathbf{C}}$ and $e_{\mathbf{R}}$.

# 3 A finite dimensional approximation of the Seiberg-Witten equations

In this section we describe a finite dimensional approximation of the Seiberg-Witten equation for a closed spin 4-manifold with the vanishing first Betti number [6]. For general cases we discuss it in the following section.

We use the following notations.

1. $Sp_1 = \{q \in \mathbf{H} \mid |q| = 1\}$
2. $Pin_2 = \{\cos\theta + i\sin\theta\}_{0\leq\theta<2\pi} \cup \{j\cos\phi + k\sin\phi\}_{0\leq\phi<2\pi} \subset Sp_1$
3. We regard $\mathbf{H}$ as a right $Pin_2$ module by the right multiplication.
4. We regard $\operatorname{Im}\mathbf{H}$ as a $Pin_2$ module by the conjugation.
5. Let $\tilde{\mathbf{R}}$ be the non-trivial 1-dimensional real representation of $Pin(2)/S^1 = \{\pm 1\}$.

Note that $\operatorname{Im}\mathbf{H}$ is isomorphic to $\tilde{\mathbf{R}}^3$ as $Pin(2)$-module.

Let $X$ be a smooth closed spin 4-manifold with $b_1(X) = 0$ and $\operatorname{sign}(X) < 0$. In this section we use the notation $k = -\operatorname{sign}(X)/16$ and $l = b_2^+(X)$. Via a finite dimensional approximation of Seiberg-Witten equations for the trivial spin$^c$ structure [6], we obtain a $Pin_2$-equivariant map

$$f_{SW} : S(V_0 \oplus W_0) \to S(V_1 \oplus W_1).$$

for some right $Pin(2)$-modules $V_0, V_1, W_0$ and $W_1$ which satisfy the following conditions.

1. $V_0$ and $V_1$ are isomorphic to the direct sum of a finite number of $\mathbf{H}$'s.
2. $\dim_{\mathbf{H}} V_0 - \dim_{\mathbf{H}} V_1 = k > 0$,
3. $W_0$ and $W_1$ are isomorphic to the direct sum of a finite number of $\tilde{\mathbf{R}}$'s.
4. $\dim_{\mathbf{R}} W_0 - \dim_{\mathbf{R}} W_1 = -l < 0$,

Our problem is to obtain a necessary condition for $l$ and $k$ to allow the $Pin_2$-equivariant map $f_{SW}$. Theorem 1 follows from

**Theorem 20.** *Suppose that there exits a $Pin_2$-equivariant map*

$$f : S(V_0 \oplus W_0) \to S(V_1 \oplus W_1).$$

*Then we have*

$$l \geq \begin{cases} 2k+1 & \text{if } k \equiv 0, 1 \mod 4, \\ 2k+2 & \text{if } k \equiv 2 \mod 4, \\ 2k+3 & \text{if } k \equiv 3 \mod 4. \end{cases} \tag{2}$$

We shall prove Theorem 20 by using the $\mathbf{Z}/4$-action which is just the restriction of the $Pin_2$-action to the subgroup $\mathbf{Z}/4$.

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be the 1-dimensional complex representation space of $\mathbf{Z}/4$ with weight 1 and 2 respectively. For nonnegative integers $a$ and $b$, Let $E_0, E_1, F_0$ and $F_1$ be the four representation spaces of $\mathbf{Z}/4$ which satisfy the following conditions.

1. $E_0$ and $E_1$ are isomorphic to the direct sum of a finite number of $\mathbf{C}_1$'s.
2. $\dim_{\mathbf{C}} E_0 - \dim_{\mathbf{C}} E_1 = a > 0$,
3. $F_0$ and $F_1$ are isomorphic to the direct sum of a finite number of $\mathbf{C}_2$'s.
4. $\dim_{\mathbf{C}} F_0 - \dim_{\mathbf{C}} F_1 = -b < 0$,

Theorem 20 is a consequence of the following theorem.

**Theorem 21.** *Suppose that there exits a $\mathbf{Z}/4$-equivariant map $f : S(E_0 \oplus F_0) \to S(E_1 \oplus F_1)$.*

1. *The inequality $a < 2b - 1$ holds.*
2. *The inequality $a < 2b - 2$ holds if $b \equiv 0 \bmod 4$.*

The above theorem can be thought of a stable version of the non-existence part of Stolz's result [17].

from Theorem 21.

**Proof of Theorem 20 assuming Theorem 21**

Step 1.

We first show $l \geq 2k + 1$ in general.

Suppose that there exits a $Pin_2$-equivariant map $S(V_0 \oplus W_0) \to S(V_1 \oplus W_1)$ for $l = 2k$. It implies the existence of a $\mathbf{Z}/4$-equivariant map $S(E_0 \oplus F_0) \to S(E_1 \oplus F_1)$ for $a = 2k$ and $b = k$. Since $a = 2b$, this contradicts the first part of Theorem 21.

Step 2: the case $k \equiv 3 \bmod 4$.

In this case we show $l \geq 2k + 3$. Suppose that there exits a $Pin_2$-equivariant map $S(V_0 \oplus W_0) \to S(V_1 \oplus W_1)$. for $l = 2k + 2$. It implies the existence of a $\mathbf{Z}/4$-equivariant map $S(E_0 \oplus F_0) \to S(E_1 \oplus F_1)$. for $a = 2k$ and $b = k + 1$. Since $a = 2b - 2$ and $b \equiv 0 \bmod 4$, this contradicts the second part of Theorem 21.

Step 3. the case $k \equiv 2 \bmod 4$.

In this case we show $l \geq 2k + 2$. Suppose that there exits a $Pin_2$-equivariant map $f : S(V_0 \oplus W_0) \to S(V_1 \oplus W_1)$ for $l = 2k + 1$. Let $g : S(\mathbf{H}) \to S(\tilde{\mathbf{R}}^3)$ be the $Pin_2$-equivariant map defined by $g(q) = qi\bar{q}$. Then the join $f * g$ is a $Pin_2$-equivariant map for $k' = k + 1$ and $l' = l + 3$. Since $l' = 2k' + 2$ and $k' \equiv 3 \bmod 4$, we can reduce the proof to the former case.

□

# 4 The equivariant $e$-invariant for a pair of modules

We shall use $\mathbf{Z}/4$-equivariant e-invariants to show Theorem 21.

Let $f_0 : S(E_0 \oplus F_0) \to E_1 \oplus F_1$. be a $\mathbf{Z}/4$-equivariant smooth map. Since $\dim S(F_0) < \dim F_1$, we may assume that, when restricted to $S(0 \oplus F_0)$, $f_0$ is an embedding into $S(0 \oplus F_1)$. Thus the zero set $M = f_0^{-1}(0)$ does not intersect $S(0 \oplus F_0)$, and $\mathbf{Z}/4$ acts freely on $M$. Moreover $df$ induces an $\mathbf{Z}/4$-equivariant trivialization of the normal bundle;

$$\phi : (TM \oplus \mathbf{R}) \oplus (E_1 \oplus F_1) \cong E_0 \oplus F_0$$

where $\mathbf{R}$ is the trivial $\mathbf{Z}/4$-module.

The class $[M, \phi] \in \Omega(E_0 \oplus F_0, E_1 \oplus F_1)$ is independent of the choice of $f_0$, since $\dim S(F_0) + 1 < \dim F_1$ and the zero set $Z$ of a generic $G$-equivariant path of two such maps also does not intersect $S(0 \oplus F_0)$. Moreover the derivative of the path gives an extension of $\phi$;

$$\phi_Z : TZ \oplus (E_1 \oplus F_1) \cong E_0 \oplus F_0.$$

as $\mathbf{Z}/4$-equivariant bundles.

Hence the $\mathbf{Z}/4$-equivariant bordism class $[M, \phi] \in \Omega(E_0 \oplus F_0, E_1 \oplus F_1)$ depends only on the $\mathbf{Z}/4$ modules $E_0 \oplus F_0$, $E_1 \oplus F_1$. We write $c(E_0 \oplus F_0, E_1 \oplus F_1) = [M, \phi]$ for this class.

The following lemma is elementary;

**Lemma 22.** *Let $U$ be the direct sum of a finite number of $\mathbf{C}_1$'s and $\mathbf{C}_2$'s. Then*

$$c(E_0 \oplus F_0 \oplus U, E_1 \oplus F_1 \oplus U) = c(E_0 \oplus F_0, E_1 \oplus F_1).$$

Since $\mathbf{C}_1$ and $\mathbf{C}_2$ are complex representations of $\mathbf{Z}/4$, the $\mathbf{Z}/4$-modules $E_0 \oplus F_0$ and $E_1 \oplus F_1$ are even dimensional over $\mathbf{R}$, and have natural $\mathbf{Z}/4$-spin$^{\mathrm{c}}$ structures. Hence we can define $\Omega^{\mathrm{b,f}}_{\mathbf{C}}(E_0 \oplus F_0, E_1 \oplus F_1)$. When $c(E_0 \oplus F_0, E_1 \oplus F_1)$ sits in $\Omega^{\mathrm{b,f}}_{\mathbf{C}}(E_0 \oplus F_0, E_1 \oplus F_1)$, we define $e_{\mathbf{C}}(E_0 \oplus F_0, E_1 \oplus F_1) \in \mathbf{Q}/\mathbf{Z}$ to be the $e_{\mathbf{C}}$-invariant of $c(E_0 \oplus F_0, E_1 \oplus F_1)$.

When $\dim_{\mathbf{C}} E_0, \dim_{\mathbf{C}} E_1, \dim_{\mathbf{C}} F_0$ and $\dim_{\mathbf{C}} F_1$ are all divisible by 4, they have $\mathbf{Z}/4$-spin structures. Moreover if $a - b \equiv 2 \bmod 4$, then we can define $\Omega^{\mathrm{b,f}}_{\mathbf{R}}(E_0 \oplus F_0, E_1 \oplus F_1)$. When $c(E_0 \oplus F_0, E_1 \oplus F_1)$ sits in $\Omega^{\mathrm{b,f}}_{\mathbf{R}}(E_0 \oplus F_0, E_1 \oplus F_1)$, we define $e_{\mathbf{R}}(E_0 \oplus F_0, E_1 \oplus F_1) \in \mathbf{Q}/2\mathbf{Z}$ to be the $e_{\mathbf{R}}$-invariant of $c(E_0 \oplus F_0, E_1 \oplus F_1)$.

¿From this definition we have the following lemma.

**Proposition 23.** *If there exists a $\mathbf{Z}/4$-equivariant map*

$$f : S(E_0 \oplus F_0) \to S(E_1 \oplus F_1),$$

*we have $c(E_0 \oplus F_0, E_1 \oplus F_1) = 0$ in $\Omega^{\mathrm{b,f}}(E_0 \oplus F_0, E_1 \oplus F_1)$, and hence $e_{\mathbf{C}}(E_0 \oplus F_0, E_1 \oplus F_1) = 0$* mod $\mathbf{Z}$ *or $e_{\mathbf{R}}(E_0 \oplus F_0, E_1 \oplus F_1) = 0 \mod 2\mathbf{Z}$ when they are defined.*

On the other hand we shall prove:

**Theorem 24.** *1. If $a \geq b$, then $c(E_0 \oplus F_0, E_1 \oplus F_1) \in \Omega^{\mathrm{b}}_{\mathbf{C}}(E_0 \oplus F_0, E_1 \oplus F_1)$.*

*2. If $a = 2b - 1$, then $e_{\mathbf{C}}(E_0 \oplus F_0, E_1 \oplus F_1) = 1/2 \mod \mathbf{Z}$*

*3. If $a \geq b$, $a \equiv 0 \bmod 2$ and $b \equiv 0 \bmod 4$, then $c(E_0 \oplus F_0, E_1 \oplus F_1) \in \Omega^{\mathrm{b}}_{\mathbf{R}}(E_0 \oplus F_0, E_1 \oplus F_1)$.*

*4. If $a = 2b - 2$ and $b \equiv 0 \bmod 4$, then $e_{\mathbf{R}}(E_0 \oplus F_0, E_1 \oplus F_1) = 1 \mod 2\mathbf{Z}$*

**Proof of Theorem 21 assuming Theorem 24**
Theorem 21 is an immediate consequence of Proposition 23 and Theorem 24 □

# 5 Brieskorn varieties

By definition we may write $E_0 = \mathbf{C}_1^{p+a}$, $E_1 = \mathbf{C}_1^{p}$, $F_0 = \mathbf{C}_2^{q}$ and $F_1 = \mathbf{C}_2^{q+b}$ for some $p$, $q$. By Lemma 22 it suffices to compute $e_{\mathbf{C}}$ or $e_{\mathbf{R}}$ when $p = q = 0$.

We first prove Theorem— 24 (1) and (3), which enable us to define $e_{\mathbf{C}}$ or $e_{\mathbf{C}}$ for $c(E_0 \oplus F_0, E_1 \oplus F_1))$ if $a \geq b$.

Suppose $a \geq b$.

Let $(a_{i,j})$ be a generic $b \times a$ matrix with complex entries and define the $\mathbf{Z}/4$-equivariant map$f : S(\mathbf{C}_1^a) \to \mathbf{C}_2^b$ to be

$$f(z_1, \cdots, z_a) = (\sum_{j=1}^{a} a_{1,j} z_j^2, \cdots, \sum_{j=1}^{a} a_{b,j} z_j^2).$$

It is a well-known elementary fact that if all the maximal minors of the matrix $(a_{i,j})$ is not 0, then $f$ has 0 as a regular value, so $M = f^{-1}(0)$ is a smooth $\mathbf{Z}/4$-spin$^{\mathrm{c}}$ manifolds with a $\mathbf{Z}/4$-equivariant trivialization

$$\phi = df : TM \oplus \mathbf{R} \oplus \mathbf{C}_2^b \cong \mathbf{C}_1^a.$$

Now the claim of Theorem 24 (1) and (3) follows from the following explicit construction of $W$.

**Construction of** $W$ The map $f$ obviously extends over $\mathbf{C}_1^a$ as a polynomial map $\tilde f$, and the zero set $\tilde f^{-1}(0)$ has only one singularity at the origin. Consider the $\mathbf{Z}/4$-manifold $W^0 = \tilde f^{-1}(0)\setminus D(\mathbf{C}_1^a)$, where $D(\mathbf{C}_1^a)$ is the unit disc around the origin. Let $W$ be the closure of $W^0$ in $P(\mathbf{C}_1^a\oplus\mathbf{C})\supset\mathbf{C}_1^a$. Then $W$ is a smooth compact complex $\mathbf{Z}_4$-manifold with boundary $M$, and has a $\mathbf{Z}/4$-spin$^c$ structure defined by using its $\mathbf{Z}/4$-invariant complex structure. Note that the spin$^c$ structure of $M$ coincides with the restriction of that of $M$.

This completes the proof of Theorem 24 (1),(3).

## 5.1 Calculation of $e_{\mathbf{C}}$ for the case $a = 2b-1$

We prove Theorem 24 (2).

We can use Proposition 8 to calculate $e_{\mathbf{C}}(E_0\oplus F_0, E_1\oplus F_1) = e_{\mathbf{C}}(M,\phi)$.

**Construction of $W'$.** We define $W'$ to be the Milnor fiber of the map $\tilde f$. Let us briefly recall the definition of the Milnor fiber. Take a small nonzero vector $\varepsilon$ in $\mathbf{C}_2^b$ and define $W'$ to be $\tilde f^{-1}(\varepsilon)\cap D(\mathbf{C}^a)$. Then $W'$ is a smooth compact complex manifold whose boundary $\partial W'$ is diffeomorphic to $M$. Then $W'$ has a spin$^c$ structure defined by using its complex structure. Note that the restriction of the spin$^c$ structure of $W'$ on $\partial W'$ is isomorphic to that of $M$, and that $\phi$ extends over $W'$ as $d\tilde f$ through this isomorphism.

**Construction of $\tilde W$.** Let $\tilde W$ be the closed spin$^c$-manifold $W\cup_M W'$. Note that $\tilde W$ is diffeomorphic to the closure of $\tilde f^{-1}(\varepsilon)$ in $P(\mathbf{C}_1^a\oplus\mathbf{C})\supset\mathbf{C}_1^a$. We identify these two manifolds.

The closed complex manifold $\tilde W$ is one of Brieskorn varieties or complete intersections. The Todd-genus of the complete intersection is well known (see e.g. [12]). What we now need is:

$$\int_{\tilde W} e^{c_1(\det\tilde W)/2}\hat A(\tilde W) = \int_{\tilde W}\mathrm{td}(\tilde W) = 1-(-1)^b \quad \text{when } a = 2b-1.$$

**Calculation of $\mathrm{Spin}^c(g,W)$.** The remaining terms we need are $\mathrm{Spin}^c(g,W)$'s for $g\neq 1$. For $g = e^{\sqrt{-1}\pi k/2}$ $(k=1,2,3)$, the fixed point set $W^g$ is equal to $W\cap P(\mathbf{C}_1^a)\subset P(\mathbf{C}_1^a\oplus\mathbf{C})$.

Let $\alpha\in H^2(P(\mathbf{C}_1^a),\mathbf{Z})$ be $c_1(\mathcal{O}(1))$. Then by the defining equation, the homology class $[W^g]$ in $H_{2(a-b-1)}(P(\mathbf{C}_1^a),\mathbf{Z})$ is the Poincaré dual of $(2\alpha)^b$ and the normal bundle of $W^g$ in $P(\mathbf{C}_1^a)$ is $b\mathcal{O}(2)$. In particular we have

$$\mathrm{td}(W^g) = \mathrm{td}(\mathcal{O}(1))^a/\mathrm{td}(\mathcal{O}(2))^b = \left(\frac{\alpha}{1-e^{-\alpha}}\right)^a\left(\frac{1-e^{-2\alpha}}{2\alpha}\right)^b.$$

Since the Dirac operator of $W$ is the Dolbeault operator and thus the local contribution $\mathrm{Spin}^c(g,W)$ is given by (4.4) in [1], we have for $g = e^{\sqrt{-1}\pi k/2}$ $(k=1,2,3)$

$$\begin{aligned}
e_{\mathbf{C}}(M,\phi,W)_g &= \mathrm{Spin}^c(g,W)\\
&= \langle\left(\frac{\alpha}{1-e^{-\alpha}}\right)^a\left(\frac{1-e^{-2\alpha}}{2\alpha}\right)^b\frac{1}{1-g^{-1}e^{-\alpha}}, [\text{P.D. of } 2^b\alpha^b\ ]\rangle\\
&= \mathrm{Res}_{\alpha=0}\frac{\left(1-e^{-2\alpha}\right)^b g\,d\alpha}{\left(1-e^{-\alpha}\right)^a\left(g-e^{-\alpha}\right)}\\
&= \mathrm{Res}_{z=1}\,\omega, \quad \text{where} \quad \omega = \frac{(1-z^2)^b g\,dz}{(1-z)^a(z-g)z},
\end{aligned}$$

We put $z = e^{-\alpha}$ in the last equality.

| $e_{\mathbf{C}}(M,\phi,W)_g$ | $g=1$ | $g=\sqrt{-1}$ | $g=-1$ | $g=-\sqrt{-1}$ |
|---|---|---|---|---|
| $b\equiv 0 \bmod 4$ | 0 | 0 | 2 | 0 |
| $b\equiv 1 \bmod 4$ | 2 | 0 | 0 | 0 |
| $b\equiv 2 \bmod 4$ | 0 | $2-2\sqrt{-1}$ | 2 | $2+2\sqrt{-1}$ |
| $b\equiv 3 \bmod 4$ | 2 | 2 | 0 | 2 |

The poles of $\omega$ are $z=1,0,g$ and $\infty$. Then the residue theorem implies:

$$\begin{aligned}\operatorname{Res}_{z=1}\omega &= -(\operatorname{Res}_{z=0}\omega+\operatorname{Res}_{z=g}\omega+\operatorname{Res}_{z=\infty}\omega) && (3)\\ &= -\{(-1)+\frac{(1-g^2)^b}{(1-g)^a} && (4)\\ &\quad +\operatorname{Res}_{z=\infty}\frac{dz}{-z}(-1)^{b-a-1}g\frac{1}{z^{a-2b-1}}(1+(a+g)\frac{1}{z}+O(\frac{1}{z^2}))\}. && (5)\end{aligned}$$

When $a=2b-1$, we have

$$\operatorname{Res}_{z=1}\omega=-\{(-1)+\frac{(1-g^2)^b}{(1-g)^{2b-1}}+(-1)^b g\},\quad (a=2b-1).$$

**Calculation of $e_{\mathbf{C}}(E_0\oplus F_0,E_1\oplus F_1)$.** The above calculations are summarized by the table below. Here we write $e_{\mathbf{C}}(M,\phi,W)_1$ for $\langle \operatorname{td}(\tilde W),[\tilde W\rangle$.

¿From Proposition 8 we have

$$\begin{aligned}e_{\mathbf{C}}(M,\phi) &= e^{c_1(\det\tilde W)/2}\hat A(\tilde W)+\frac{1}{\#G}\sum_{g\in G}\operatorname{Spin}^c(g,W) \bmod 1.\\ &\equiv 1/2\in\mathbf{Q}/\mathbf{Z}.\end{aligned}$$

This completes the proof of Theorem 21 (2)

## 5.2 Calculation of $e_{\mathbf{R}}$ for the case $a=2b-2$ and $b\equiv 0 \bmod 4$

We prove of Theorem 24 (4).

Suppose $b\equiv 0 \bmod 4$ and $a=2b-2$. We use Proposition 15. to calculate $e_{\mathbf{R}}(E_0\oplus F_0,E_1\oplus F_1)$.

Constructions of $W$, $W'$ and $\tilde W$ are the same as before.

When $a=2b-2$, the Todd genus of $\tilde W$ is given as follows (see e.g. [12]).

$$\int_{\tilde W} e^{c_1(\det\tilde W)/2}\hat A(\tilde W)=\int_{\tilde W}\operatorname{td}(\tilde W)=1+(-1)^b(2b-1). \tag{6}$$

Since $b$ is even, the above expression is equal to $2b$.

**Calculation of** $\operatorname{Spin}^c(g,W)$**.** For $g=e^{\sqrt{-1}\pi k/2}$ $(k=1,2,3)$ we want to calculate $\operatorname{Spin}^c(g,W)$.

When $a=2b-2$. the formula (3) implies

$$\operatorname{Res}_{z=1}\omega=-\{(-1)+\frac{(1-g^2)^b}{(1-g)^a}+(-1)^{t-1}g(2b-2+g)\},\quad (a=2b-2).$$

Then from (3) we have

$$\operatorname{Spin}^c(g,W)=\begin{cases}2\sqrt{-1}b & g=\sqrt{-1},\\ 4-2b & g=-1,\\ -2\sqrt{-1}b & g=-\sqrt{-1}.\end{cases} \tag{7}$$

**$G$-characteristic submanifold.** When $a = 2b-2$, we show that $W \cap P(\mathbf{C}_1^a) \subset P(\mathbf{C}_1^a \oplus \mathbf{C})$ is a $\mathbf{Z}/4$-characteristic submanifold of $(W, M)$.

The isomorphism $T\tilde{f}^{-1}(0) \oplus \mathbf{C}_2^b \cong \mathbf{C}_1^a$ is naturally extended to the isomorphism

$$TW \oplus b\mathcal{O}(2) \cong TP(\mathbf{C}_1^a \oplus \mathbf{C}) \mid W.$$

Taking the determinant, we obtain the isomorphism

$$\det TW \otimes \mathcal{O}(2b) \cong \mathcal{O}(a+1).$$

When $a = 2b-2$, we have a canonical isomorphism $\det TW \cong \mathcal{O}(a+1-2b) = \mathcal{O}(-1)$. Since a spin structure is given on $\partial W = M$, we have a canonical $\mathbf{Z}/4$-invariant nonvanishing section $s_0$ on $\det TW \mid M$. On the other hand $\mathcal{O}(-1) \mid W$ has a $\mathbf{Z}/4$-invariant (anti-holomorphic) section $s$ such that $s$ is transverse to the zero section and $s^{-1}(0) = W \cap P(\mathbf{C}_1^a) \subset P(\mathbf{C}_1^a \oplus \mathbf{C})$. It is easy to see that $s_0$ and $s|M$ is $\mathbf{Z}/4$-equivariantly homotopic to each other. (For instance it follows from $H^1(M/(\mathbf{Z}/4), \mathbf{Z}) = 0$.) It implies that $W \cap P(\mathbf{C}_1^a)$ is a $\mathbf{Z}/4$-characteristic submanifold of $(W, M)$.

**Calculation of $e_{\mathbf{R}}(E_0 \oplus F_0, E_1 \oplus F_1)$.** When $a = 2b-2$, since the $\mathbf{Z}/4$-action on the characteristic submanifold $W \cap P(\mathbf{C}_1^a)$ is trivial, we can apply Proposition 15.

$$\begin{aligned} e_{\mathbf{R}}(M, \phi) &= \left(-1 + \frac{1}{\#G}\right) \int_{\tilde{W}} e^{c_1(\det \tilde{W})/2} \hat{A}(\tilde{W}) + \frac{1}{\#G} \sum_{g \in G, g \neq 1} \mathrm{Spin}^c(g, W) \bmod 2. \\ &\equiv (-1 + \frac{1}{4})2b + \frac{1}{4}\{2\sqrt{-1}b + (4-2b) - 2\sqrt{-1}b\} \\ &\equiv 1 \in \mathbf{Q}/2\mathbf{Z}. \end{aligned}$$

This completes the proof of Theorem 24(4).

$\tilde{\mathbf{R}}^l \subset \tilde{\mathbf{R}}^{l+1}$ implies that

# 6 Equivariant bundles and $e$-invariants

We next show Theorem 2. To this end we need to extend our equivariant bordism group and associated $e$-invariants for a family of modules, that is, $G$-vector bundles. We use the same notation for these objects as before and discuss them briefly, but the different points are emphasized.

## 6.1 Twisted bordism groups and $e$-invariants

Let $G$ be a finite group and $T$ a closed $G$-manifold. Now let $U_0$, $U_1$ be $G$-spin$^c$ [resp. $G$-spin] vector bundles over $T$. We assume $\mathrm{rank} U_0 - \mathrm{rank} U_1 \equiv 0 \bmod 2$ [resp. $\mathrm{rank} U_0 - \mathrm{rank} U_1 \equiv 4 \bmod 8$].

We denote by $\Omega(U_0, U_1)$ the bordism group of the triple $[M, \pi, \phi]$ such that

1. $M$ is a closed free $G$-manifold.
2. $\pi : M \to T$ is a $G$-equivariant map,
3. $\phi$ is a $G$-equivariant isomorphism; $\phi : (TM \oplus \mathbf{R}) \oplus \pi^* U_1 \cong \pi^* U_0$, where $\mathbf{R}$ is the trivial $G$-module,

where $(M_1, \pi_1, \phi_1)$ and $(M_2, \pi_2, \phi_2)$ are bordant if there exists a pair $(W, \pi_W, \phi_W)$ such that

1. $W$ is a compact free $G$-manifold with a $G$-equivariant diffeomorphism $\partial W \cong M_1 \coprod M_2$,
2. $\pi_W : W \to T$ is a $G$-equivariant map which extends $\pi_1$ and $\pi_2$, and

3. $\phi_W$ is a $G$-equivariant isomorphism $\phi_W : TW \oplus \pi_W^* U_1 \cong \pi_W^* U_0$ which extends $\phi_1$ and $\phi_2$.

Here we identify $TW|M_1$ with $TM_2 \oplus \mathbf{R}$ by using the inward normal direction, and $TW|M_1$ with $TM_2 \oplus \mathbf{R}$ by using the outward normal direction.

Note that $\phi$ gives an orientation and a $G$-spin$^c$ [resp. $G$-spin] structure of $M$.

Let $\Omega_{\mathbf{C}}^{\mathrm{b,f}}(U_0, U_1)$ [resp. $\Omega_{\mathbf{R}}^{\mathrm{b,f}}(U_0, U_1)$] be the subgroup of pairs $[M, \phi] \in \Omega(U_0, U_1)$ which have a compact $G$-spin$^c$ [resp. $G$-spin] manifold $W$ and a $G$-equivariant map $\pi_W : W \to T$ such that

1. $G$ acts on $W$ freely,
2. $\partial W = M$, $\pi_W|M = \pi$ and the $G$-spin$^c$ [resp. $G$-spin] structure on $\partial W$ coincides with the one defined by $\phi$.

We define $e_{\mathbf{C}}(M, \phi, \pi, W, \pi_W)$ by:

$$e_{\mathbf{C}}(M, \phi, \pi, W, \pi_W) \quad := \quad \frac{1}{\#G} \int_W (e^{c/2} \hat{A} - 1)([TW \oplus \pi_W^* U_1, \pi_W^* U_0, \phi], L(\phi)) \pi_W^* \hat{A}([U_0] - [U_1]) \in \mathbf{Q}.$$

**Lemma 25.** *Suppose* $\mathrm{rank} U_0 - \mathrm{rank} U_1 < \dim T$. *Then* $e_{\mathbf{C}}(M, \phi, \pi, W, \pi_W) \bmod \mathbf{Z}$ *does not depend on the choice of* $(W, \pi_W)$ [resp. $e_{\mathbf{R}}(M, \phi, \pi, W, \pi_W) \bmod 2\mathbf{Z}$ *does not depend on the choice of* $(W, \pi_W)$ ].

*Proof.* It suffices to show that $e_{\mathbf{C}}(M, \phi, \pi, W, \pi_W)$ is an integer [resp. even integer] when $M$ is empty. In this case we have

$$\begin{aligned} e_{\mathbf{C}}(M, \phi, \pi, W, \pi_W) &= \frac{1}{\#G} \int_W \left( e^{c_1(L(\phi))/2} \hat{A}([TW] + [U_1] - [U_0]) - 1 \right) \pi_W^* \hat{A}([U_0] - [U_1]) \\ &= \int_{W/G} e^{c_1(\det W/G)/2} \hat{A}(T(W/G)) - \frac{1}{\#G} \int_W \pi_W^* \hat{A}([U_0] - [U_1]). \end{aligned}$$

Since we are assuming $\dim W = \mathrm{rank} U_0 - \mathrm{rank} U_1 < \dim T$, the second term is zero. Now the required statement follows from the integrality theorem for the index of the Dirac operator of the spin$^c$-manifold $W/G$ [resp. the Dirac operator of the $8k+4$-dimensional spin manifold $W/G$]. ☐

**Definition 26.** Let $[M, \pi, \phi]$ be an element of $\Omega_{\mathbf{C}}^{\mathrm{b,f}}(U_0, U_1)$ [resp. $\Omega_{\mathbf{R}}^{\mathrm{b,f}}(U_0, U_1)$]. When $\dim_{\mathbf{R}} U_0 - \dim_{\mathbf{R}} U_1 > \dim T$, we define

$$\begin{aligned} e_{\mathbf{C}}([M, \pi, \phi]) &\equiv e_{\mathbf{C}}(M, \pi, \phi, W, \pi_W) \bmod 1 \in \mathbf{Q}/\mathbf{Z} \\ [\text{resp. } e_{\mathbf{R}}([M, \pi, \phi]) &\equiv e_{\mathbf{C}}(M, \pi, \phi, W, \pi_W) \bmod 2 \in \mathbf{Q}/2\mathbf{Z}]. \end{aligned}$$

Moreover we can obviously define $\Omega_{\mathbf{C}}^{\mathrm{b}}(U_0, U_1)$ [resp. $\Omega_{\mathbf{R}}^{\mathrm{b}}(U_0, U_1)$] and the map $e_{\mathbf{C}} : \Omega_{\mathbf{C}}^{\mathrm{b}}(U_0, U_1) \to \mathbf{Q}/\mathbf{Z}$ [resp. $e_{\mathbf{R}} : \Omega_{\mathbf{R}}^{\mathrm{b}}(U_0, U_1) \to \mathbf{Q}/2\mathbf{Z}$] as in Section 2.3.

**Remark 27.**
1. We could further extend the definition to obtain $e_{\mathbf{C}} : \Omega_{\mathbf{C}}^{\mathrm{b}}(U_0, U_1) \to R(G) \otimes (\mathbf{C}/\mathbf{Z})$ [resp. $e_{\mathbf{R}} : \Omega_{\mathbf{R}}^{\mathrm{b}}(U_0, U_1) \to R(G) \otimes (\mathbf{C}/2\mathbf{Z})$] as in Remark 19.
2. The condition that $T$ is a closed $G$-manifold is too strong. In fact to define $e_{\mathbf{C}}$ or $e_{\mathbf{R}}$ it may be sufficient for $T$ to be a compact Hausdorff $G$-space if one define the characteristic classes to be algebraic elements in cohomology. But the dimension condition may not be obvious.

## 6.2 Seiberg-Witten equation : $b_1 > 0$

Now we move to the Seiberg-Witten equation.

Let $X$ be a closed spin 4-manifold with $\mathrm{sign}(X) < 0$.

The moduli space of flat connections on the trivial bundle on $X$ is identified with the Jacobian torus $J = H^1(X;\mathbf{R})/H^1(X;\mathbf{Z})$. We define a $Pin_2$-acton on $J$ by the multiplication of of $Pin_2/S^1 = \{\pm 1\}$. We write $\iota$ for the involutive action of $j \in Pin_2$.

Let $V$ be a $\mathbf{C}$-vector bundle over $J$ with an anti linear lift $\tilde{\iota}_V$ of $\iota$ to $V$. Following Dupont [5] we call $V$ *quaternionic* if the square of $\tilde{\iota}_V$ is $-1$, and denote by $Ksp(J)$ the Grothendieck group generated by quaternionic vector bundles over $J$. A quaternionic vector bundle is nothing but a $Pin_2$-equivariant $\mathbf{R}$-vector bundle over $J$ so that the $S^1$-action on the total space is semifree.

Let $\mathrm{ind}D$ be the index bundle of the Dirac operators twisted by flat connections on $X \times \mathbf{C}$. Then $\mathrm{ind}D$ is an element of $Ksp(L)$ [10].

Now the Seiberg-Witten equation for the spin structure is approximated by a $Pin_2$-equivariant bundle map ([6], [7], [2]);

$$f_{SW} : S(V_0 \oplus W_0) \to S(V_1 \oplus W_1),$$

where $V_0, V_1, W_0$ and $W_1$ are four $Pin_2$-equivariant $\mathbf{R}$-vector bundles over $J$ satisfying the following conditions.

1. $\mathrm{rank}_{\mathbf{R}} V_0 - \mathrm{rank}_{\mathbf{R}} V_1 = 4k > 0$ for $k = -\mathrm{sign}(X)/16$.
2. $\mathrm{rank}_{\mathbf{R}} W_0 - \mathrm{rank}_{\mathbf{R}} W_1 = -l < 0$ for $l = b_2^+(X)$.
3. $V_0$ and $V_1$ have structure of complex vector bundles.
4. $S^1$ acts on $V_0$, $V_1$ as the complex multiplication.
5. $j$ acts on $V_0$, $V_1$ as an anti linear map.
6. $S^1$ acts on $W_0$, $W_1$ trivially.
7. $j$ acts on $W_0$, $W_1$ as a $\mathbf{R}$-linear involution.
8. $[V_0] - [V_1] = \mathrm{ind}D \in Ksp(J)$, where $\mathrm{ind}D$ is the index bundle for the family of Dirac operators of the spin manifold $X$ parameterized by $J$.
9. $[W_0] - [W_1] = -[\tilde{\mathbf{R}}^l] \in KO_{\mathbf{Z}/2}(J)$.

Thus $V_0$ and $V_1$ are quaternionic vector bundles and $W_0$ and $W_1$ are $\mathbf{Z}/2$-equivariant $\mathbf{R}$-vector bundles.

Let $\alpha_1, \alpha_2, \alpha_2$ and $\alpha_4$ are four elements of $H^1(X, \mathbf{Z})$. We write $\tilde{\alpha}$ for $(\alpha_i)_{1 \le i \le 4}$, which gives a homotopy class of a map $\tilde{\alpha} : X \to T^4$. On the other hand we have a map

$$h_{\tilde{\alpha}} : T^4 = (\mathbf{R}/\mathbf{Z})^4 \to J, \quad (x_i) \mapsto \sum x_i \alpha_i.$$

**Lemma 28.** $\int_{T^4} h_{\tilde{\alpha}}^* c_2(\mathrm{ind}D) = \deg\tilde{\alpha} = \int_X \alpha_1\alpha_2\alpha_3\alpha_4$.

*Proof.* The second equality is obvious. The first equality is shown by the index formula for family. Ruberman and Strle [16] gave a proof in the case $b_1(X) = 4$. Their calculation goes through the general case. □

Hence Theorem 2 follows from;

**Theorem 29.** *Suppose that there exits a $\mathbf{Z}/4$-equivariant bundle map*

$$f : S(V_0 \oplus W_0) \to S(V_1 \oplus W_1).$$

*If* $W_1 = W_0 \oplus \tilde{\mathbf{R}}^l$, $k \geq 4$ *and* $\langle c_2(V_0) - c_2(V_1), [T^4]\rangle$ *is odd for some linearly embedded subtorus* $T^4 \subset J$*, then we have*

$$l \geq \begin{cases} 2k+5 & \textit{if } k \equiv 0,1 \mod 4, \\ 2k+6 & \textit{if } k \equiv 2 \mod 4, \\ 2k+7 & \textit{if } k \equiv 3 \mod 4. \end{cases}$$

**Remark 30.** When $k = 3$, we can obtain $l \geq 10$ just in the same way as Section 6.

To prove this theorem we have to make the preceding constructions for a family over $T^4$.
To avoid too many notations we assume that $b_1(X) = 4$ and that $\tilde{\alpha}$ is a basis of $H^1(X, \mathbf{Z})$ so that we can identify $T^4$ with $J$. The proof for the general case goes through quite similarly.
We first construct a $\mathbf{Z}/4$-equivariant bundle map

$$f_0 : S(V_0 \oplus W_0) \to V_1 \oplus W_1.$$

transversal to the zero section. This may be easily constructed as before, since $W_1 = W_0 \oplus \tilde{\mathbf{R}}$ and $\dim S(0 \oplus W_0) < \dim W_1$. Thus the zero set $M$ is a smooth closed $\mathbf{Z}/4$-manifold and the projection $\pi : V_0 \oplus W_0 \to J$ induces the $\mathbf{Z}/4$-equivariant map $\pi : M \to J$. Moreover $df$ induces an $\mathbf{Z}/4$-equivariant trivialization of the normal bundle;

$$\phi : (TM \oplus \mathbf{R}) \oplus \pi^*(V_1 \oplus W_1) \cong \pi^*(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4),$$

where $\tilde{\mathbf{R}}^4$ comes from the tangent bundle of $J$. The class $[M, \pi, \phi]$ is independent of the choice of $f$, thus we denote it by $c(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1)$ and write $e_{\mathbf{C}}[M, \pi, \phi] = e_{\mathbf{C}}(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1)$ [resp. $e_{\mathbf{R}}[M, \pi, \phi] = e_{\mathbf{R}}(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1)$ ] as long as it is defined.
The following proposition and lemma are obvious.

**Proposition 31.** *If there exists a* $\mathbf{Z}/4$*-equivariant map*

$$f : S(V_0 \oplus W_0) \to S(V_1 \oplus W_1),$$

*we have* $c(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1) = 0 \in \Omega^{\mathrm{b,f}}(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1)$.

**Lemma 32.** *Let* $U$ *be the direct sum of a finite number of quaternionic and real vector bundles over* $J$*. Then*

$$c(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4 \oplus U, V_1 \oplus W_1 \oplus U) = c(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1). \oplus U, V_1 \oplus W_1 \oplus U)$$

**Theorem 33.** *Suppose* $b_1(X) = 4$.

1. *When* $l - 4 = 2k$*, we have*

$$e_{\mathbf{C}}(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1) \equiv \frac{1}{2}\int_J (c_2(V_0) - c_2(V_1)) \bmod \mathbf{Z}.$$

2. *When* $l - 4 = 2k + 2$ *and* $k + 1 \equiv 0 \bmod 4$*, we have*

$$e_{\mathbf{R}}(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1) \equiv \int_J (c_2(V_0) - c_2(V_1)) \bmod 2\mathbf{Z}.$$

**Proof of Theorem 2 assuming Theorem 33** Proposition 31 and Lemma 32 and Theorem 33 implies Theorem 2. □

## 6.3 Calculation of $e_{\mathbf{C}}$ and $e_{\mathbf{R}}$

Now we want to compute the equivariant $e$-invariant to show Theorem 33. We first invoke the results in Part I of [10], in which we have investigated quaternionic vector bundle over $J$.

**Proposition 34.** *Any quaternionic vector bundle over $J$ is the direct sum of a rank 2 quaternionic $\mathbf{C}$-vector bundle and the trivial quaternionic vector bundle $\mathbf{H}^{\mathrm{rank}_{\mathbf{C}} V/2-1}$, on which the right multiplication of $j$ acts as the lift of the generator of the involution.*

Thus we can write $V_0 = \hat{V}_0 \oplus \mathbf{H}^{p+k}$ and $V_1 = \hat{V}_1 \oplus \mathbf{H}^p$ for some rank 2 quaternionic vector bundles $\hat{V}_0$, $\hat{V}_1$ over $J$ and a nonnegative integer $p$. By Lemma 32 we may suppose that $p = 0$ and $W_0 = \{0\}$. Thus we have only to consider a $\mathbf{Z}/4$-equivariant map;

$$f : S(\hat{V}_0 \oplus \mathbf{H}^k) \to \hat{V}_1 \oplus \tilde{\mathbf{R}}^l.$$

**Proposition 35.** *For any two quaternionic rank 2 $\mathbf{C}$-vector bundles $E_0$ and $E_1$ over $J$, there exists a $\mathbf{Z}/4$-equivariant $\mathbf{C}$-linear homomorphism $f_E : E_0 \to E_1$ which satisfies;*

1. *$f_E$ is an isomorphism except at a finite set of points $S(f_E)$ on $J$,*
2. *at each point $x$ in $S(f_E)$, there exists a $\mathbf{Z}/4$-equivariant neighborhood $U_x$ of $x$, a diffeomorphism from $U_x$ to a neighborhood $U$ of $0$ in $\tilde{\mathbf{R}}^4$ and trivializations $E_0 \cong U \times \mathbf{H}$ and $E_1 \cong U \times \mathbf{H}$ such that $f_E$ is described as $f_E(v, q) = (v, vqi)$ or $(v, \bar{v}qi)$.*
3. *Let $n^+$ ($n^-$) be the number of the points $x$ of the former (the latter) type respectively. Then $n^+ - n^- = c_2(E_0) - c_2(E_1)$.*

Thus there exists a $\mathbf{Z}/4$-equivariant $\mathbf{C}$-linear homomorphism $f_{\hat{V}} : \hat{V}_0 \to \hat{V}_1$ satisfying the above conditions.

Suppose now that $l - 4 = 2k$ [resp. $l - 4 = 2k + 2$ and $k + 1 \equiv 0 \bmod 4$].

Note that the $e_{\mathbf{C}}$-invariant [resp. $e_{\mathbf{R}}$-invariant ] can be defined, since $k \geq 4$. Here we use Stolz's result [17];

**Stolz' theorem** [17]
*There exits a $\mathbf{Z}/4$-equivariant map $g : S(\mathbf{H}^{k_0}) \to S(\tilde{\mathbf{R}}^{l_0})$ if*

$$l_0 = \begin{cases} 2k_0 + 1 & \text{if } k_0 \equiv 0, 1 \mod 4, \\ 2k_0 + 2 & \text{if } k_0 \equiv 2 \mod 4, \\ 2k_0 + 3 & \text{if } k_0 \equiv 3 \mod 4. \end{cases} \tag{8}$$

Thus there exists a $Z/4$-equivariant map

$$g : S(H^k) \to S(\tilde{\mathbf{R}}^l).$$

Hence we obtain a $\mathbf{Z}/4$-equivariant bundle map

$$f = f_{\hat{V}} * g : S(\hat{V}_0 \oplus H^k) \to S(\hat{V}_1 \oplus \tilde{\mathbf{R}}^l).$$

by the join. Then the zero set of $f$ is contained in fibers of the set $S(f_{\hat{V}})$. Since the $e$-invariant is determined by the neighborhood of the zero, we can separate our computation into the neighborhood of fibers over $S(f)$.

We extend the map $f$ to $\mathbf{Z}/4$-equivariant bundle map

$$\tilde{f} : D(\hat{V}_0 \oplus \mathbf{H}^k) \to \hat{V}_1 \oplus \tilde{\mathbf{R}}^l$$

by the join with the zero at each disc. Then the zero set remains to be unchanged. Thus by restricting $\tilde{f}$ into the boundary of a neighborhood of the fiber at each $x$ of $S(f)$, we obtain $\mathbf{Z}/4$-equivariant maps;

$$\tilde{f}_x : S(\mathbf{H}^{k+1} \oplus \tilde{\mathbf{R}}^4) \to \mathbf{H} \oplus \tilde{\mathbf{R}}^l,$$

and we have

$$\begin{aligned} e_{\mathbf{C}}(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1) &= \sum_{x \in S(f)} e_{\mathbf{C}}(\mathbf{H}^{k+1} \oplus \tilde{\mathbf{R}}^4, \mathbf{H} \oplus \tilde{\mathbf{R}}^l) \\ [\text{resp. } e_{\mathbf{R}}(V_0 \oplus W_0 \oplus \tilde{\mathbf{R}}^4, V_1 \oplus W_1) &= \sum_{x \in S(f)} e_{\mathbf{R}}(\mathbf{H}^{k+1} \oplus \tilde{\mathbf{R}}^4, \mathbf{H} \oplus \tilde{\mathbf{R}}^l)]. \end{aligned}$$

But each term of the right hand side is $1/2 \in \mathbf{Q}/\mathbf{Z}$ [resp. $1 \in \mathbf{Q}/2\mathbf{Z}$] by Theorem 24. This complets the proof of Theorem 33 (1) [resp. Theorem 33 (2)].

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