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Computer vision and quiver varieties

by

Henrique de Campos Affonso

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

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Henrique de Campos Affonso(Graduate School of Mathematical Sciences, the University of Tokyo)

Abstract

We try to provide the initial steps to a new approach to the reconstruction problem for two cameras by relating the essential variety to a subvariety of a quiver variety.

コンピュータービジョンと箙多様体

Henrique de Campos Affonso(東京大学数理科学研究科)

概要

ステレオビジョンの新しいアプローチの初めの一歩として、基本多様体と関係がある箙多様体の部分多様体を作る。

1 Introduction

One of the fundamental problems in the field of computer vision is the so-called reconstruction problem. The problem consists of, given several pictures taken by different cameras, how can we recover from the picture data the real world data of what is being pictured. In other words, we want to recover the original 3d data from pieces of 2d data. Here we are interested in the reconstruction problem for two cameras, in particular.

In the classical approach (called the pinhole model), this is done by modelling the cameras as matrices and using a particular matrix constructed from the camera data, called the essential matrix, to solve the reconstruction problem. Essential matrices are known to be exactly the matrices which have one singular value equal to zero, and the other two singular values equal to each other. And, as in [1, §4], this can be expressed in terms of polynomial equations by

$$\mathcal{E} = \{M \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \mid \det(M) = 0, 2(MM^T)M - \mathrm{tr}(MM^T)M = 0\}.$$

The set $\mathcal{E}$ of all essential matrices can then be given the structure of an algebraic variety, which is called the essential variety. From this point onward, by essential variety, we will actually mean the projectification of the complex solutions of the equations above.

In [2], Fløystad, Kileel and Ottaviani provided an algebro-geometric approach to the reconstruction problem. They computed the Chow form of the essential variety, which they then used to get new insights into the reconstruction problem. One crucial step in this process was the identification of the essential variety with a particular subvariety of the variety of 4x4 symmetric matrices.

Quiver varieties, introduced by Nakajima (cf [3]), are a class of varieties defined by taking a geometric invariant quotient of the space of framed representations of a given quiver. More precisely, after fixing the vector spaces V of the representation and W of the framing, the quiver variety $\mathcal{M}(V,W)$ is defined as

$$\mathcal{M}(V,W) = \mu^{-1}(0) /\!\!/ \mathrm{GL}(V).$$

Where μ is the moment map of the natural Hamiltonian $\mathrm{GL}(V)$-action.

Our goal here is to provide the initial steps for a new approach to the reconstruction problem for two cameras. Namely, we want to move towards showing that we can relate the essential variety $\mathcal{E}$ to a particular subvariety E of a quiver variety in such a way that, by studying E, we can get information about $\mathcal{E}$.

2 Methods and discussion

Consider the quiver variety $\mathcal{M}(V,W)$ associated with

$$\begin{tikzcd} W_1 \arrow[r, shift right, "j_1"'] & V_1 \arrow[l, shift right, "i_1"'] \arrow[r, shift right, "B"'] & V_2 \arrow[l, shift right, "B'"'] \arrow[r, shift right, "j_2"'] & W_2 \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

where $V_1 = V_2 = \mathbb{C}^2$, $W_1 = W_2 = \mathbb{C}^3$, and $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$.
The composite $j_2 B j_1$ can then be viewed as a 3x3 complex matrix of rank 2. Therefore, we can consider the subvariety E of the quiver variety $\mathcal{M}(V,W)$ above, consisting of the points $[(B,i,j)]$ satisfying the condition $j_2 B j_1 \in \mathcal{E}$, where $\mathcal{E}$ denotes again the essential variety. That is, E is the subvariety comprised of the points such that the composite of the bottom maps is in the essential variety.
We then have a surjective morphism

$$\sigma \colon E \to \mathcal{E}$$

given by

$$\sigma([(B,i,j)]) = j_2 B j_1.$$

Let us compute the dimension of $\sigma^{-1}(x)$ for an arbitrary $x \in \mathcal{E}$.
Step 1: Note that we can reduce the problem to the case

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

since all fibers are in the $\mathrm{GL}(W_2)$-orbit of $\sigma^{-1}(x)$, with x as above.
Step 2: Note that, by using the $\mathrm{GL}(V)$-action, every element in the fiber of x can then be made to be such that

$$(i_1, B, j_2) = (\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \mathrm{Id}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}).$$

Step 3: Compute the dimension explicitly by considering the restrictions imposed by the moment map equations

$$i_1 j_1 = B' B,$$
$$i_2 j_2 = B B'.$$

In other words, we note that every element in the fiber has a representative for which i_1, B, j_2 are equal to some particular maps, and then we check how the remaining maps can vary by seeing what the restrictions they have, which are given by the moment map equation, are, in practice.

3 Result

Theorem 1 *Every fiber $\sigma^{-1}(x)$ has dimension equal to 9. In particular, the map σ has equidimensional fibers.*

This theorem is the first step that there is some nice relationship between the varieties E and $\mathcal{E}$, meaning that we can obtain some information about $\mathcal{E}$ by examining E. Our expectation is that, after further examining the algebro-geometric properties of E, the theorem above can be used to show that σ is faithfully flat. This would mean that, for example, normality at a point $y \in E$ implies normality at the point $\sigma(y) \in \mathcal{E}$.

Therefore, our hope is that this result can pave the way for some new insight on the reconstruction problem.

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References

[1] Faugeras O.D. and Maybank S., Motion from point matches: Multiplicity of solutions, nt. J. Comput.Vision 4, no. 3, 225–246, 1990.

[2] Fløystad G., Kileel J. and Ottaviani G., The Chow form of the essential variety in computer vision, Journal of Symbolic Computation, ISSN: 0747-7171, Vol: 86, Page: 97-119, 2018.

[3] Nakajima H., Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras, Duke Math. J. 76 , no. 2, 365–416, 1994.