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The BV space in variational and evolution problems

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Preface

Here is the set of lecture notes for the course I delivered in the Fall of 2016 at the Graduate School of Mathematical Sciences of the University of Tokyo. My goal was to get the audience acquainted with a range of topics where the space of function of bounded variation, BV, plays a significant role. Some well-known results are presented as well as quite recent ones.

Of course, the first thing to do was a presentation of the space itself and its properties. I paid attention to Anzellotti's theory of integration by parts because it is necessary to study variational and evolutionary problems.

The starting point for the calculus of variations is the ROF functional, which is used in image reconstruction. It serves as an introduction of application of BV in the calculus of variations. The lecture ventures into this area to explore the least gradient problem. Here, also recent results are included. A lot of time is spent on the problem of lower semicontinuity of the linear growth functionals and their relaxation.

In the next section, another result related to the calculus of variations is presented. Namely, a well-known result on Γ-convergence of variational functionals is considered. This type of convergence is used as a tool for constructing solutions to elliptic equations which are local minimizers of these functionals. The BV is necessary here to describe the limit object.

Also recent observations on the role of the BV space in obtaining the crucial estimates sufficient to establish stabilization of solutions to a differential equation are shown. Here, the technique of monotone rearrangements is used in an essential way.

The final chapter exploits the results of the variational problems, because of the study of the gradient flow of the total variation. A sample of the richness of results on the gradient flows of the total variation and similar problems is presented. This area has been active for many years. It is taking its motivation from the crystal growth problems and image analysis.

When I was preparing this set I paid attention to provide relevant references presenting the original material and encouraging further independent studies.

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Piotr Rybka

Chapter 1

The definition and basic properties of space $BV(\Omega)$

The space of function of bounded variation, $BV(\Omega)$, over a region $\Omega \subset \mathbb{R}^N$, plays a peculiar role in analysis and applications. For those who study PDE's, it may look like a poor cousin of Sobolev spaces. However, $BV(\Omega)$ links PDE's with the geometric measure theory. A good way to see this is through the analysis of the Rudin-Osher-Fatemi algorithm for TV denoising, see [41], which leads to the minimization of the following functional,

$$E(u) = \int_\Omega |\nabla u(x)|\, dx + \frac{\lambda}{2}(f(x) - u(x))^2\, dx,$$

where f is the blurred image understood as intensity of light, λ is the *bias parameter* (name after [16]). In practice, the minimization (over a finite dimensional space) is repeated a number of times, leading to a finite sequence, $u_0, \ldots, u_n$, where u_{k+1} is obtained from u_k by minimizing E with $f = u_k$.

We notice that E is well-defined over $W^{1,1}(\Omega)$ and if $N = 2$ and Ω is bounded, then $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$. Unfortunately, this space is too narrow to deduce existence of minimizers. Indeed, if u_n is a sequence minimizing E, i.e.

$$\lim_{n\to\infty} E(u_n) = \inf\{E(u) : \ u \in W^{1,1}(\Omega)\},$$

then in particular

$$\int_\Omega |\nabla u_n|\, dx \le M.$$

However, we may not apply Alaoglu-Banach Theorem here, because L^1 is not a reflexive space. Instead, we may use:

Theorem 1.1 ([4], [21]) *Let $\{\mu_k\}_{k=1}^\infty$ be a sequence of Radon measures on $\mathbb{R}^N$ satisfying*

$$\sup_k \{\mu_k(K)\} < \infty$$

for each compact set $K \subset \mathbb{R}^N$. Then, there exists a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ and a Radon measure μ such that $\mu_{k_j} \rightharpoonup \mu$, i.e.

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} f(x)\, d\mu_{k_j} = \int_{\mathbb{R}^N} f(x)\, d\mu$$

for all $f \in C_c(\mathbb{R}^N)$.

Remarks on notation.
The symbol $C_c(\mathbb{R}^N)$ denotes the space of continuous functions with compact support. By $\mathcal{M}(\mathbb{R}^N)$ we mean the space of Radon measures.

Let $(\mathbb{R}^N, \mathfrak{F}, \mu)$ be a measure space. We say that μ is a *Radon measure* iff μ is Borel regular (i.e. for any measurable $A \subset \mathbb{R}^N$ there is a Borel set $B \subset \mathbb{R}^N$, such that $A \subset B$ and $\mu(A) = \mu(B)$) and $\mu(K) < \infty$ for each compact set $K \subset \mathbb{R}^N$.

In order to apply Theorem 1.1 in our context, we consider $\mu_k = |\nabla u_k(x)|\, dx$, so that Theorem 1.1 implies existence of a Radon measure μ such that

$$|\nabla u_k(x)|\, dx \rightharpoonup \mu.$$

Exercise 1.1 *Show that* $\frac{1}{\sqrt{n}} \exp(-x^2/n) \rightharpoonup c\delta_0$ *as* $n \to \infty$. *Find* c.

Theorem 1.1 and this example suggest that minimizers u of E are such that Du is a vector valued measure. For the sake of completeness we recall:

Definition 1.1 Suppose that $(X, \mathcal{F})$ is a measurable space. A set function $\mu : \mathcal{F} \to \mathbb{R}^m$, $m \geq 1$ is a *vector* (*real* or *signed*, if $m = 1$) *measure*, if
(1) $\mu(\emptyset) = 0$ and (2) $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for any sequence $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint elements of $\mathcal{F}$.

Thus, we are prompted to introduce the following definition.

Definition 1.2 (Functions of bounded variation)
Let us suppose that $\Omega \subset \mathbb{R}^N$ is open. We say that $u \in L^1_{loc}(\Omega)$ is a function of *bounded variation*, provided that the (distributional) partial derivatives of u are signed Radon measures with bounded variation.

In order to complete this definition, we have to recall the variation of a vector measure.

Definition 1.3 If μ is a vector measure on Ω, we define its *variation*, $|\mu|$, for every $E \in \mathcal{F}$, by the following formula,

$$|\mu|(E) := \sup \left\{ \sum_{n=0}^{\infty} |\mu(E_n)| : \ E_n \in \mathcal{F}, \text{pairwise disjoint }, \bigcup_{n=0}^{\infty} E_n = E \right\}.$$

If μ is a real measure, then we may set $\mu^+ := (|\mu| + \mu)/2, \quad \mu^- := (|\mu| - \mu)/2.$

Exercise 1.2 *Let X be the linear space of real/vector measures with finite total variation. Show that* $\mu \mapsto |\mu|(\Omega)$ *is a norm on* X.

Exercise 1.3 (Homework problem # 1)
Suppose that Ω is bounded. Show that space X, defined above, is a Banach space.

Exercise 1.4 *If μ is vector measure, then $|\mu|$ is a positive measure.*

By definition $u \in BV(\Omega)$ iff $u \in L^1_{loc}(\Omega)$ and there exist Radon measures μ_1, ..., μ_N with $|\mu_i|(\Omega) < \infty$ and such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi\, d\mu_i \qquad \forall \varphi \in C_c^\infty(\Omega).$$

Writing this in a concise way, we have

$$\int_\Omega u \operatorname{div} \varphi\, dx = -\int_\Omega \varphi \cdot \sigma\, d\mu \qquad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^N),$$

where μ is the variation of the vector measure $(\mu_1, \ldots, \mu_N)$ and $\sigma_i = \frac{d\mu_i}{d\mu}$ is the Radon-Nikodym derivative. Moreover,

$$|\sigma| = 1 \quad |Du|\text{- a.e.} \tag{1.1}$$

We record,

Theorem 1.2 *(see [4, Chap. 3, Section 3.1]) Let $u \in L^1(\Omega)$. Then, $u \in BV(\Omega)$ iff*

$$\|Du\| := \sup\{\int_\Omega u \operatorname{div} \varphi\, dx : \ \varphi \in C_c^\infty(\Omega; \mathbb{R}^N), |\varphi(x)| \le 1, \forall x \in \Omega\} < \infty.$$

Moreover, $BV(\Omega)$ is Banach space with the norm

$$\|u\|_{BV} = \|u\|_{L^1(\Omega)} + \|Du\|.$$

Finally, if $u \in BV(\Omega)$, then $\|Du\| = |Du|(\Omega)$.

Remarks.
Measure Du may be decomposed in the following way, see [21, §1.6, Theorem 3],

$$Du = [Du]_{ac} + [Du]_s.$$

Since $[Du]_{ac}$ is absolutely continuous with respect to Lebesgue measure, then it has a density. This density is called ∇u. Thus,

$$Du = \nabla u\, dx + [Du]_s.$$

We may further decompose $[Du]_s$ into a continuous part and atomic part, e.g. in one dimension we have $[Du]_s = [Du]_c + \sum_{j \in J} a_i \delta_{a_i}$, $a_i \in \mathbb{R}$, where $[Du]_c(x) = 0$ for all $x \in \mathbb{R}$.

Exercise 1.5 *Suppose $u(x) = \chi_{(0,1)}(x)$. Check that $u \in BV(-1, 1)$, but $u \notin W^{1,p}$ for all $p \ge 1$.*

A special case of a locally integrable function is χ_E, where $E \subset \mathbb{R}^N$.

Definition 1.4 *If $E \subset \mathbb{R}^N$ and $\chi_E \in BV(\mathbb{R}^N)$, then we say that E has a finite perimeter. If Ω is open and we have $\chi_E \in BV(\Omega)$, then E has a finite perimeter with respect to Ω.*

If E has the finite perimeter with respect to Ω, then by the definition, we have

$$\int_E \operatorname{div}\varphi\,dx = -\int_\Omega \varphi\cdot\sigma\,d|D\chi_E| = \int_\Omega \varphi\cdot\nu_E\,d|D\chi_E|,$$

where $\nu_E = -\sigma$. This formula is consistent with the Gauss Theorem.

Example 1.1 If E is open, with a smooth boundary and $\mathcal{H}^{N-1}(\Omega\cap\partial E) < \infty$, then $P(E,\Omega) < \infty$ and $P(E,\Omega) = \mathcal{H}^{N-1}(\Omega\cap\partial E)$.

Indeed, by Gauss formula for any $\varphi\in C_c^\infty(\Omega,\mathbb{R}^N)$ we have,

$$\int_E \operatorname{div}\varphi\,dx = \int_{\Omega\cap\partial E} \nu\cdot\varphi\,d\mathcal{H}^{N-1}.$$

In order to maximize the right-hand-side (RHS) of this formula, we take $\varphi = \psi\nabla d$, where d is the distance function from the boundary ∂E and ψ is a cut-off function.

Exercise 1.6 *If $u\in BV(\Omega)$ and $[Du]_s = 0$, then $u\in W^{1,1}(\Omega)$.*

We present the basic properties of functions of bounded variation.

Proposition 1.1 *Function $BV(\Omega)\ni u\mapsto \|Du\|$ is lower semicontinuous with respect to the $L^1(\Omega)$-topology.*

Sketch of proof. Take $u_n\to u$ in $L^1(\Omega)$ and $\epsilon>0$. We can find $\varphi\in C_c^\infty(\Omega)$ with $\|\varphi\|_\infty\le 1$ such that

$$\|Du\| - \epsilon \le \int_\Omega u\operatorname{div}\varphi\,dx.$$

Furthermore,

$$\int_\Omega u\operatorname{div}\varphi\,dx = \lim_{n\to\infty}\int_\Omega u_n\operatorname{div}\varphi\,dx = \varliminf_{n\to\infty}\int_\Omega u_n\operatorname{div}\varphi\,dx.$$

Of course, $\int_\Omega u_n\operatorname{div}\varphi\,dx \le \|Du_n\|$, because $|\varphi(x)|\le 1$. Hence,

$$\|Du\| - \epsilon \le \varliminf_{n\to\infty}\|Du_n\|.$$

Our claim follows. $\square$

We have to answer the basic question about the relationship between the BV and smooth functions. Is it possible to approximate $u\in BV$ by smooth functions? If so, in what sense? The following example shows that we have to be careful.

Example 1.2 Take $u\in BV(\Omega)$, where $\Omega=(0,1)$ such that $[Du]_s\neq 0$. There is no sequence $\varphi_k\in C_c^\infty(\Omega)$ such that $\|u-\varphi_n\|_{BV}\to 0$ as $n\to\infty$. Indeed, if such a sequence existed, then $|D\varphi_n - Du|(\Omega)\to 0$, but

$$|D\varphi_n - Du|(\Omega) = |(\nabla\varphi_n\,dx - \nabla u\,dx) + [Du]_s|(\Omega).$$

However, if $\lambda\perp\nu$, then $|\lambda+\nu| = |\lambda|+|\nu|$. Hence,

$$\|D\varphi_n - Du\| = \|(\nabla\varphi_n\,dx - \nabla u\,dx)\| + \|[Du]_s\| \ge \|[Du]_s\| > 0.$$

Nonetheless, we can prove:

Theorem 1.3 (Approximation by smooth functions) *If $u \in L^1(\Omega)$, then $u \in BV(\Omega)$ iff there exists a sequence $\{u_k\} \subset C_c^\infty(\Omega)$ converging to u in $L^1(\Omega)$ and satisfying*

$$|Du|(\Omega) = \lim_{k\to\infty} \int_\Omega |\nabla u_k(x)|\, dx.$$

The idea of the proof is the same as in the case of Sobolev spaces. We use the standard mollification

$$u * j_n(x) = \int_\Omega u(y) j_n(x-y)\, dy. \qquad \square$$

This Theorem justifies the following definition:

Definition 1.5 Let $u, u_n \in BV(\Omega)$, $n \in \mathbb{N}$. We say that sequence $\{u_n\}$ converges *strictly* to u, if $u_n \to u$ in $L^1(\Omega)$ and

$$|Du_n|(\Omega) \to |Du|(\Omega).$$

The following theorems depend essentially on the possibility of approximating $u \in BV(\Omega)$ by smooth functions in the strict sense.

Theorem 1.4 *(a) Let us suppose that $\Omega \subset \mathbb{R}^N$ is bounded with smooth boundary. Then, there is a constant $C(\Omega)$, such that for all $p \in [1, N/(N-1)]$, we have*

$$\|u\|_{L^p} \le C\|u\|_{BV}.$$

(b) There is a constant C_1 such that

$$\|u\|_{L^{N/(N-1)}(\mathbb{R}^N)} \le C_1\|u\|_{BV(\mathbb{R}^N)}.$$

Theorem 1.5 *Let us suppose that $\Omega \subset \mathbb{R}^N$ is as above. Then, set*

$$\{u \in BV(\Omega) : \ \|u\|_{BV} \le 1\}$$

is compact in $L^1(\Omega)$.

Theorem 1.6 *Let us suppose that $\Omega \subset \mathbb{R}^N$ is as above and $1 \le q < N/(N-1)$. Then, set*

$$\{u \in BV(\Omega) : \ \|u\|_{BV} \le 1\}$$

is compact in $L^q(\Omega)$.

Let us prove Theorem 1.4, part (a). For this purpose let us take any $u \in BV(\Omega)$ and $\{u_n\}_{n=1}^\infty \subset C_c^\infty(\Omega)$ converging in the strict sense. We recall the embedding $W^{1,1} \hookrightarrow L^1(\Omega)$ is continuous. There is $C = C(\Omega)$, such that

$$\|u_n\|_{L^p} \le C(\|u_n\|_{L^1} + \|\nabla u_n\|_{L^1}).$$

Hence, we deduce that u_n converges weekly to u in L^p, $p > 1$. The lower semicontinuity of the norm yields,

$$\|u\|_{L^p} \le \varliminf_{n\to\infty} \|u_n\|_{L^p} \le \varliminf_{n\to\infty} C(\|u_n\|_{L^1} + \|\nabla u_n\|_{L^1}) = C\|u\|_{BV}.$$

If E has finite perimeter, i.e. $\chi_E \in BV(\Omega)$, then by Theorem 1.2 we have

$$|D\chi_E| =: P(E,\Omega) = \sup\{\int_E \operatorname{div}\varphi\, dx :\ \varphi \in C^1_c(\Omega,\mathbb{R}^N), \|\varphi\|_{L^\infty} \le 1\}.$$

Corollary 1.1 *There is a constant C such that for all sets with finite perimeter in $\mathbb{R}^N$, we have*

$$|E|^{(N-1)/N} \le CP(E,\mathbb{R}^N). \tag{1.2}$$

For the *proof*, take $u = \chi_E$ and $p = 1$ and apply part (b) of Theorem 1.4 to u,

$$|E|^{(N-1)/N} = \|u\|_{L^{N/(N-1)}} \le C_1\|Du\| = C_1 P(E,\mathbb{R}^N). \qquad \square$$

We show an important way of representing BV functions, the coarea formula. For this purpose, we need an auxiliary definition. For a given function f, we set $E_t = \{x \in \Omega : f(x) > t\}$. Here, is a preparatory lemma.

Lemma 1.1 *If u is in $BV(\Omega)$, then $t \mapsto P(E_t,\Omega)$ is measurable with respect to Lebesgue measure $\mathcal{L}^1$.*

Sketch of proof. Obviously, function $(x,t) \mapsto \chi_{E_t}(x)$ is measurable with respect to $\mathcal{L}^N \times \mathcal{L}^1$. Thus, by Fubini's Theorem, for any $\varphi \in C^\infty_c(\Omega;\mathbb{R}^N)$ the following function

$$t \mapsto \int_{E_t} \operatorname{div}\varphi\, dx = \int_\Omega \chi_{E_t} \operatorname{div}\varphi\, dx$$

is measurable with respect to Lebesgue measure $\mathcal{L}^1$. Let $\mathcal{D}$ be a countable subset of $C^\infty_c(\Omega;\mathbb{R}^N)$, which is dense in the $W^{1,1}$-topology, then

$$t \mapsto P(E_t,\Omega) = \sup_{\varphi\in\mathcal{D}, \|\varphi\|_{L^\infty}\le 1} \int_{E_t} \operatorname{div}\varphi\, dx$$

is measurable, as desired. $\square$

Here is the advertised statement,

Theorem 1.7 (The coarea formula) *Let $f \in BV(\Omega)$, then:*
(i) E_t has finite perimeter for a.e. $t \in \mathbb{R}$;
(ii) We have

$$|Df|(\Omega) = \int_{-\infty}^{\infty} P(E_t,\Omega)\, dt.$$

(iii) Suppose $f \in L^1(\Omega)$ and $\int_{-\infty}^\infty P(E_t,\Omega)\, dt$ is finite, then $f \in BV(\Omega)$.

The essential part of the proof is for non-negative f from C^∞_c. Then, one reduces the general case to this one and uses the possibility of approximation of BV functions by smooth one in the sense of strict convergence. First, we notice that for $f(x) \ge 0$ we have $f(x) = \int_0^\infty \chi_{E_t}(x)\, dt$. Then, we notice,

$$\int_\Omega f \operatorname{div}\varphi\, dx = \int_0^\infty \left(\int_{E_t} \operatorname{div}\varphi\, dx\right) dt \le \int_0^\infty P(E_t,\Omega)\, dt.$$

Taking supremum yields (iii), showing the reverse inequality requires a further measure theoretic argument.

Part (i) immediately follows from (ii). $\square$

1.1 Traces and Anzellotti's theory

An important part of the presentation is the definition of traces and the theory developed by Anzellotti, see [5], this is very useful for the discussion of the calculus of variations. Here is the first statement:

Theorem 1.8 *Let us suppose that Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary. There exists a bounded linear operator $\gamma : BV(\Omega) \to L^1(\partial\Omega)$, such that*

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega \varphi \, dDu + \int_{\partial\Omega} (\varphi \cdot \nu)\gamma f \, d\mathcal{H}^{N-1}$$

for all $u \in BV(\Omega)$ and $\varphi \in C^1(\Omega; \mathbb{R}^N)$.

Moreover, γ is continuous in the topology of strict convergence, i.e. if $u_n \to u$ in $L^1(\Omega)$ and $|Du_n|(\Omega) \to |Du|(\Omega)$, then $\gamma u_n \to \gamma u$ in $L^1(\partial\Omega)$.

This theorem is first proved for smooth u and a flat boundary. The line of approach is similar to the one used in case of Sobolev spaces: the difference $f(x_1, \dots, x_{n-1}, \epsilon) - f(x_1, \dots, x_{n-1}, \delta)$ is estimated with the help of the fundamental Theorem of the Calculus of variations and integration with respect to the first variables. This leads the Cauchy condition for the trace of a smooth function f. In the end, we use the strictly convergent approximation of u. $\square$

Here is another expected property of the trace operator.

Theorem 1.9 *Suppose Ω is as above and $u \in BV(\Omega)$. Then, for a.e. $x \in \partial\Omega$ with respect to measure $\mathcal{H}^{N-1}$, we have,*

$$\lim_{\rho \to 0} \rho^{-N} \int_{\Omega \cap B(x,\rho)} |u(y) - \gamma u(x)| \, dy = 0.$$

We know that in case of $p > 1$, the image of the Sobolev space $\gamma(W^{1,p}(\Omega))$ is much smaller than $L^p(\partial\Omega)$. In case of $W^{1,1}$ and BV, we have

$$\gamma(BV(\Omega)) = \gamma(W^{1,1}(\Omega)) = L^1(\partial\Omega).$$

Moreover, one can show, see [5, Lemma 5.5]

Lemma 1.2 *Let us suppose that Ω is a bounded region with Lipschitz continuous boundary. For a given $f \in L^1(\partial\Omega)$ and $\epsilon > 0$, there is $w \in W^{1,1}(\Omega) \cap C(\Omega)$, such that $\gamma w = f$ and*

$$\int_\Omega |\nabla w| \, dx \le \int_{\partial\Omega} |f| \, d\mathcal{H}^{N-1} + \epsilon, \qquad w(x) = 0 \text{ if } \operatorname{dist}(x, \partial\Omega) > \epsilon.$$

Moreover, for fixed $q \in [1, \infty)$, one can find w with $\|w\|_{L^q} \le \epsilon$ and for $f \in L^\infty(\partial\Omega)$, we have $\|w\|_{L^\infty} \le \|f\|_{L^\infty}$.

This lemma plays a key role in the development of the trace theory which is necessary for the calculus of variations in the next section. In order to continue the main topic we introduce the following spaces, while assuming that $\Omega \subset \mathbb{R}^N$ is a bounded region with Lipschitz continuous boundary and $\nu(x)$ is the outer normal to Ω,

$$\begin{array}{ll} BV(\Omega)_c = BV(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega), & \\ BV(\Omega)_q = BV(\Omega) \cap L^q(\Omega), & q \geq \frac{N}{N-1}, \\ X_\mu(\Omega) = \{z \in L^\infty(\Omega;\mathbb{R}^n) : \ \operatorname{div} z \text{ is a bounded measure in } \Omega\}, & \\ X_p(\Omega) = \{z \in L^\infty(\Omega;\mathbb{R}^n) : \ \operatorname{div} z \in L^p(\Omega)\}, & 1 \leq p \leq N. \end{array} \tag{1.3}$$

Theorem 1.10 ([5, Theorem 1.1])
There is a bilinear map $\langle z, u\rangle_{\partial\Omega} : X_\mu(\Omega) \times BV(\Omega)_c \to \mathbb{R}$ such that

$$\langle z, u\rangle_{\partial\Omega} = \int_{\partial\Omega} u(x) z(x) \cdot \nu(x)\, d\mathcal{H}^{N-1}, \qquad \textit{if } z \in C^1(\Omega;\mathbb{R}^n)$$

and

$$|\langle z, u\rangle_{\partial\Omega}| \leq \|z\|_{L^\infty} \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1}, \qquad \textit{for all } z \in X_\mu, u \in BV_c.$$

Proof. For all $u \in BV_c \cap W^{1,1}(\Omega)$ and $z \in X_\mu(\Omega)$,

$$\langle z, u\rangle_{\partial\Omega} := \int_\Omega u \operatorname{div} z + \int_\Omega z \cdot \nabla u\, du.$$

The key point is to notice that if $u, v \in BV_c \cap W^{1,1}(\Omega)$ and $\gamma u = \gamma v$, then $\langle z, u\rangle_{\partial\Omega} = \langle z, v\rangle_{\partial\Omega}$, i.e. $\langle z, (u-v)\rangle_{\partial\Omega} = 0$. From this point one may use the approximation technique and Lemma 1.2 to reach the result. □

Theorem 1.11 ([5, Theorem 1.2])
Let Ω be as above. There is a linear operator $T : X_\mu(\Omega) \to L^\infty(\partial\Omega)$ such that $\|Tz\|_{L^\infty} \leq \|z\|_{L^\infty}$ and

$$\langle z, u\rangle_{\partial\Omega} = \int_{\partial\Omega} T(z) u\, d\mathcal{H}^{N-1} \qquad \text{moreover,} \qquad T(z)(x) = z(x)\cdot\nu(x) \quad \forall x \in \partial\Omega \quad \text{if } z \in C^1.$$

We prefer to write $[z \cdot \nu]$ in place of Tz since its meaning is more clear.

Proof. Take $z \in X_\mu(\Omega)$, we consider $F : L^1(\partial\Omega) \to \mathbb{R}$ defined by

$$F(u) := \langle z, w\rangle_{\partial\Omega},$$

where $w \in BV(\Omega)$, $\gamma w = u$ is given by Lemma 1.2. By Theorem 1.10 we have

$$|F(u)| \leq \|z\|_{L^\infty} \|u\|_{L^1}.$$

The Riesz Theorem on representation of continuous functionals over L^1 yields existence of $T(z) \in L^\infty$ such that $F(u) = \int_{\partial\Omega} T(z) \gamma w\, d\mathcal{H}^{N-1}$. □

We finally define (z, Du). In general, when $u \in BV$ and $z \in L^\infty$ this expression does not make much sense, we cannot multiply any measure by an arbitrary function, even Lebesgue measurable. We assume that (z, u) is from one of the following couples,

$$\begin{array}{lll} u \in BV_{p'}, & z \in X_p, & 1 < p \leq N, \\ u \in BV_\infty, & z \in X_1, & \\ u \in BV_c, & z \in X_\mu. & \end{array} \tag{1.4}$$

We introduce:

Definition 1.6 *For any $\varphi \in \mathcal{D}(\Omega)$ we set,*

$$\langle (z, Du), \varphi \rangle := -\int_\Omega u\varphi \operatorname{div} z - \int_\Omega uz \cdot \nabla\varphi \, dx.$$

Here is the final result:

Theorem 1.12 ([5, Theorem 1.5])
We assume that (u, z) satisfies one of the conditions in (1.4). For all open sets $U \subset \Omega$ and all $\varphi \in \mathcal{D}(U)$ we have,

$$|\langle (z, Du), \varphi \rangle| \le \|\varphi\|_{L^\infty} \|z\|_{L^\infty} \int_\Omega |Du|.$$

Hence, (z, Du) is a Radon measure in Ω.

Proof. We take $\varphi_k \in C_c^\infty$ converging strictly to u. We also take $\varphi \in C_c^\infty(U)$ and an open set V such that $\operatorname{supp} \varphi \subset V \Subset U$. Then,

$$|\langle (z, D\varphi_k), \varphi \rangle| \le \|\varphi\|_{L^\infty} \|z\|_{L^\infty} \int_\Omega |D\varphi_k|.$$

Passing to the limit yields the result. □

One can also prove a version of Gauss formula, where $[z \cdot \nu]$ was defined in Theorem 1.11.

Theorem 1.13 ([5, Theorem 1.9])
Let us suppose that Ω has Lipschitz boundary and one of the conditions (1.4) is satisfied. Then,

$$\int_\Omega u \operatorname{div} z + \int_\Omega (z, Du) = \int_{\partial\Omega} [z \cdot \nu] u \, d\mathcal{H}^{N-1}.$$

Notes There is a number of sources on the functions of bounded variations, during the preparation of this lecture notes I used [4] and [21]. The last book is easier to use as a textbook. The material on traces and their generalizations comes from the original paper by Anzellotti, [5], see also a very good exposition of this subject in [2, Appendix C].

Chapter 2

BV and the Calculus of Variations

The ROF algorithm, see §1 and [41], calls for a study of minimizers of E given by

$$E(u) := \int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega (u-f)^2\, dx,$$

where Ω is a rectangle. However, any Ω with Lipschitz continuous boundary is admissible. In the above problem, no explicit boundary conditions were specified, so we consider the so-called natural boundary conditions. It is our intention to impose the Dirichlet data. We set,

$$X_g = \{u \in BV(\Omega) :\ \gamma u = g\} \tag{2.1}$$

For simplicity we consider

$$F(u) = \begin{cases} \int_\Omega |Du| & u \in X_g, \\ +\infty & u \in L^2(\Omega) \setminus X_g. \end{cases} \tag{2.2}$$

From the point of view of the direct methods of the calculus of variations the following questions are important:
1) Is F lower semicontinuous with respect to the L^2-topology?
2) What is the answer to this question if we replace the Euclidean norm in F (and in E) by a function of linear growth, $f(p)$ or $f(x,u,p)$?

Once we address these questions, we can look more closely at solutions to

$$\min\{\int_\Omega |Du| :\ u \in BV(\Omega),\ \gamma u = g\},$$

(called the least gradient problem), which are distinctively different from solutions to

$$\min\{\int_\Omega |\nabla u|^p :\ u \in W^{1,p}(\Omega),\ \gamma u = g\},$$

where $p \in (1,\infty)$.

Proposition 2.1 *Let us supose that $\Omega \subset \mathbb{R}^N$ is bounded. Then, functional*

$$E(u) = \begin{cases} \int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega (u-f)^2\, dx & u \in BV(\Omega), \\ +\infty & u \in L^2(\Omega) \setminus BV(\Omega). \end{cases}$$

is lower semicontinuous with respect to the L^2-topology.

Proof. Let us take $u_n \to u$ in $L^2(\Omega)$. We may assume that $\underline{\lim}_{n\to\infty} \int_\Omega |Du_n| \le M < \infty$. Then, by Proposition 1.1,

$$M \ge \underline{\lim}_{n\to\infty} \int_\Omega |Du_n| \ge \int_\Omega |Du|,$$

hence $u \in BV$. Since we assumed convergence of u_n, then we see,

$$\begin{aligned} \underline{\lim}_{n\to\infty} \Big(\int_\Omega |Du_n| + \frac{\lambda}{2}\int_\Omega (u_n - f)^2\,dx\Big) &= \underline{\lim}_{n\to\infty} \int_\Omega |Du_n| + \frac{\lambda}{2}\lim_{n\to\infty} \int_\Omega (u_n - f)^2\,dx \\ &\ge \int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega (u - f)^2\,dx. \qquad \square \end{aligned}$$

We shall see that the Dirichlet data significantly change the problem. Take F defined by (2.2) with $\Omega = (-1, 1)$ and $g(x) = \operatorname{sgn} x$ in the definition of X_g. We shall see that F is NOT lower semicontinuous. Indeed, let us consider u_n given as follows

$$u_n(x) = \begin{cases} 1, & -1 + 1/n \le x < 1; \\ \frac{2}{n}(x+1) - 1, & x \in (-1, -1 + 1/n). \end{cases}$$

Of course, u_n restricted to $\partial\Omega$ equals g.

Then, $|Du|(\Omega) = 2$, $u_n \to u = 1$ a.e. so $\gamma u = 1$ and $u \notin X_g$. Thus $F(u) = +\infty$, hence the lower semicontinuity is violated.

So, an obvious question arises: what is the lower semicontinuous envelope of F? The lower semicontinuous envelope of F, i.e. the largest lower semicontinuous function less or equal to F, is defined by

$$\bar F(u) = \inf\{\underline{\lim}_{n\to\infty} F(v_n) : \ v_n \to u \text{ in } L^2\}. \tag{2.3}$$

We will find $\bar F$. For this purpose, we introduce the Fenchel transform. If H is a Hilbert space and $F : H \to \mathbb{R} \cup \{+\infty\}$. We set,

$$F^*(p) = \sup_{x\in H}((p, x) - F(x)) \qquad \text{and} \qquad F^{**} := (F^*)^*.$$

We will use the following result, (see [20], [40]):

Theorem 2.1 *Let us suppose that H is a Hilbert space. We assume that $F : H \to \mathbb{R} \cup \{+\infty\}$ is convex. Then, $\bar F = F^{**}$.*

We proceed in a number of steps.

Step 1. It is easy to check from the definition of the Fenchel transform that always F^* is convex and lower semicontinuous.

Step 2. By the previous step, F^{**} will be always convex and lower semicontinuous.

Step 3. We claim that $F^{**} \le F$. The definition of the Fenchel transform implies that for all $x, p \in H$ we have,

$$(p, x) \le F^*(p) + F(x), \qquad (x, p) \le F^*(p) + F^{**}(x). \tag{2.4}$$

For a fixed $x \in H$ and $n \in \mathbb{N}$ there is $p_n \in H$ such that

$$F^{**}(x) \le (x, p_n) - F^*(p_n) + \frac{1}{n}.$$

Combining this with (2.4) yields,

$$(x, p_n) \le F^*(p_n) + F(x) \le F(x) - F^{**}(x) + (x, p_n) + \frac{1}{n}.$$

The claim follows after passing to the limit with $n \to \infty$.

Step 4. Since F^{**} is lower semicontinuous and $F^{**} \le F$, then $F^{**} \le \bar{F}$ too.

Step 5. From the definition of $F^{**}(x)$ we see that

$$(x, p) - F^{**}(x) \le F^*(p).$$

However, applying sup with respect to x yields,

$$F^{***}(p) \le F^*(p).$$

Step 6. We can easily show from the definition of the Fenchel transform, for any convex F and G such that $F \le G$ then we have $F^* \ge G^*$. As a result, we have the following series of inequalities,

$$F^{**} \le \bar{F} \le F,$$

and due to Step 5

$$F^* \ge F^{***} \ge (\bar{F})^* \ge F^*.$$

Hence, $(\bar{F})^* = F^* = F^{***}$ and this implies $(\bar{F})^{**} = F^{**}$.

Step 7. Due to Step 4 it is sufficient to show that $F^{**} \ge \bar{F}$. Let h be an affine functional with $\bar{F} \ge h$. Then,

$$(\bar{F})^{**} \ge h^{**} = h.$$

As a result, we deduce from Step 6 that

$$F^{**} \ge \sup_{\bar{F} \ge h} h = \bar{F}.$$

Our claim follows. □

We will use Theorem 2.1 to find $\bar{F}$ given by (2.2), in one dimension.

Proposition 2.2 *If Ω is an open, bounded subset of $\mathbb{R}^N$ with Lipschitz continuous boundary, g in (2.1) is in $L^1(\partial\Omega)$ and functional F is defined by* (2.2), *then*

$$\bar{F}(u) = \begin{cases} \int_\Omega |Du| + \int_{\partial\Omega} |\gamma u - g| \, d\mathcal{H}^{N-1}, & u \in BV(\Omega), \\ +\infty, & u \in L^2(\Omega) \setminus BV(\Omega). \end{cases}$$

Fenchel transformations gives an easy proof in the one dimensional case, i.e. when $\Omega = (a, b)$, $g : \partial\Omega \to \mathbb{R}$ is given by $g(a) = v_a$, $g(b) = v_b$ and the formula above takes the form,

$$\bar{F}(u) = \begin{cases} \int_\Omega |Du| + |\gamma u(b) - v_b| + |\gamma u(a) - v_a|, & u \in BV(\Omega), \\ +\infty, & u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{2.5}$$

In higher dimensional cases other methods are used to prove this. One is due to Giaquinta-Modica-Souček, [23], the other one is by Anzellotti, [6, Fact 3.4], see also [2, Theorem 6.4].

Let us compute F^*,

$$F^*(w) = \sup_{u\in L^2(\Omega)} ((w,u) - F(u)).$$

We may restrict our attention to $u \in X_g$, otherwise $(\varphi, u) - F(u) = -\infty$. We may also assume that $w = -\phi'$, where $\phi \in W^{1,2}(\Omega)$. We also notice that if $u \in BV[a,b]$, then for such ϕ we have the following integration by parts formula,

$$\int_a^b \phi Du + \int_a^b \phi' u\, dx = \int_a^b D(u\phi) = (u\phi)(b) - (u\phi)(a). \tag{2.6}$$

Hence,

$$\int_a^b wu\, dx - \int_a^b |Du| = -\int_a^b \phi' u\, dx - \int_a^b |Du| = \int_a^b \phi Du - \int_a^b |Du| + v_a\phi(a) - v_b\phi(b). \tag{2.7}$$

Let us first suppose that w is such that ϕ may be so chosen that, for all $x \in [a,b]$ we have $\phi(x) \in [-1,1]$. We claim that (2.7) implies that

$$F^*(w) = v_a\phi(a) - v_b\phi(b). \tag{2.8}$$

Indeed, for this choice of ϕ we notice that

$$\int_a^b \phi D^+u - \int_a^b D^+u - \int_a^b \phi D^-u - \int_a^b D^-u \le 0.$$

Moreover, the equality holds for any u in the domain of F such that $D^-u(\{\phi > -1\}) = 0$, $D^+u(\{\phi < 1\}) = 0$. Hence, (2.8) holds. If on the other hand $\phi \ge -1$ and the set $\{\phi(x) > 1\}$ has positive measure (or $\phi \le 1$ and the set $\{\phi(x) < -1\}$ has positive measure), then it is easy to deduce that $F^*(w) = +\infty$.

Since we can choose ϕ up to a constant, we infer that

$$F^*(w) = \begin{cases} v_a\phi(a) - v_b\phi(b) & \text{if } w = -\phi', \quad \phi \in H^1((a,b);[-1,1]), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.9}$$

Let us calculate F^{**}, for this purpose we take any $u \in L^2(a,b)$ and $w \in D(F^*)$. We consider

$$(u,w)_2 - F^*(w) = -\int_a^b u\phi'\, dx - (v_a\phi(a) - v_b\phi(b)). \tag{2.10}$$

Taking supremum with respect to w implies that $F^{**}(u)$ is finite if and only if $u \in BV(a,b)$. Hence integration by parts in (2.10) yields,

$$(u,w)_2 - F^*(w) = \int_a^b \phi Du - (\gamma u(b)\phi(b) - \gamma u(a)\phi(a)) - (v_a\phi(a) - v_b\phi(b)).$$

It is now easy to see that F^{**} is given by formula (2.5). Our claim follows. □

This technique may be used to find $\bar{E}$ in case of $E(u) = \int_0^1 W(u_x)$, when $\overline{\lim}_{t\to\infty} W(pt)/t \le M < \infty$:

Exercise 2.1 (Homework problem # 2)
Calculate E^*, where

$$E(u) = \begin{cases} \int_0^1 W(u_x), & u \in W^{1,1}(\mathbb{T}^1), \\ +\infty, & u \in L^2(\mathbb{T}^1) \setminus W^{1,1}(\mathbb{T}^1), \end{cases}$$

where $W(p) = \sqrt{1+p^2}$.

Exercise 2.2 *Calculate E^{**}, where E given in the previous Exercise.*

Here, the problem is nonlinearity of W, for example we do not know what is the square of the Dirac delta function. For such functionals we would rather use a direct approach suggested by definition (2.3).

Before stating any general result, we will analyze an oversimplified example. The purpose is to discover, what is the correct relaxation of E, given by

$$E(u) = \begin{cases} \int_0^1 W(u_x), & u \in W^{1,1}(\Omega), \\ +\infty, & u \in L^2(\Omega) \setminus W^{1,1}(\Omega), \end{cases}$$

where $\Omega = (-1, 1)$ and $W(p)$ is convex with linear growth at infinity. Take $u(x) = \operatorname{sgn} x$, then $u \in BV(\Omega)$. We take a special sequence converging to u. It is

$$u_n(x) = \begin{cases} nx, & |x| \le 1/n, \\ \operatorname{sgn} x, & |x| > 1/n. \end{cases}$$

We compute $E(u_n)$,

$$\begin{aligned} E(u_n) &= \int_{-1}^{-1/n} W(0)\,dx + \int_{-1/n}^{1/n} W(n)\,dx + \int_{1/n}^{1} W(0)\,dx \\ &= \int_{|x|>1/n} W(\nabla u) + 2\frac{W(n)}{n}. \end{aligned}$$

Here, we need an additional assumption on f which is existence of the following limit,

$$\lim_{t\to\infty} W(tp)/t =: W^\infty(p), \qquad \forall p \in \mathbb{R}.$$

Function W^∞ is called the *recession function*. We immediately notice that W^∞ is positively one-homogeneous.

If in addition we denote the one-sided limits of u at $x = 0$ by j^+ and j^-, then we see

$$\lim_{n\to\infty} 2\frac{W(n)}{n} = 2W^\infty(1) = (j^+ - j^-)W^\infty(\nu),$$

where $\nu = 1$ is the 'normal' to the jump set $J = \{0\}$. Thus,

$$\lim_{n\to\infty} E(u_n) = \int_\Omega W(\nabla u)\,dx + \int_J (j^+ - j^-) f^\infty(\nu)\,d\mathcal{H}^{N-1}. \tag{2.11}$$

Exercise 2.3 *Perform such a calculation for v_n the sequence of piecewise linear functions approximating $f_{\mathcal{C}} : [0,1] \to [0,1]$ the Cantor function. Compare the result with (2.11).*

We present a theorem encompassing the above computations. For the sake of simplicity we restrict our attention to the scalar case and to a smooth integrand of the form $f = f(x,p)$. In general, f may depend on u and its continuity assumptions may be relaxed, see [3]. The statement below is after [4] in a simplified form.

Theorem 2.2 *Let us suppose that $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is smooth. For all $p \in \mathbb{R}^N$ the limit*

$$\lim_{t\to\infty} f(x,tp)/t =: f^\infty(x,p)$$

exists. Moreover, the following conditions are satisfied:
(H1) Function $f(x,\cdot)$ is convex and $f(x,0) \geq 0$.
(H2) There are positive constants, such that $c_1|p| - c_1' \leq f(x,p) \leq c_2(1+|p|)$;
(H3) For all x_0, $\epsilon > 0$ there is a such $\delta > 0$ that $|x_0 - x| < \delta$ implies $|f(x,p) - f(x_0,p)| \leq \epsilon c_3(1+|p|)$ with c_3 independent of x;
(H4) There is $m > 0$ such that for all $t > 1$ we have $|f(x,tp)/t - f^\infty(x,p)| < c_4/t^m$.
If we define E by the following formula,

$$E(u) = \begin{cases} \int_\Omega f(x,\nabla u(x))\,dx & \text{for } u \in W^{1,1}(\Omega), \\ +\infty & \text{for } u \in L^2(\Omega) \setminus W^{1,1}(\Omega), \end{cases}$$

then the relaxation of E for $u \in BV(\Omega)$ is given below,

$$\bar{E}(u) = \int_\Omega f(x,\nabla u(x))\,dx + \int_\Omega f^\infty(x,[Du]_s).$$

The symbol $\int_\Omega f^\infty(x,\mu)$ denotes

$$\int_\Omega f^\infty(x, \frac{d\mu}{d|\mu|}) d|\mu|,$$

where $\frac{d\mu}{d|\mu|}$ is the Radon-Nikodym derivative.

Before giving any argument it is instructive to look more closely at the case of $u \in BV$, with no Cantor part, i.e. $u \in SBV$,

$$Du = \nabla u\,dx + \nu(j^+ - j^-) J \llcorner \mathcal{H}^{N-1},$$

where J is the jump set of u. Rigorously,

$$J = \{x \in \Omega : \text{ap-}\overline{\lim_{z\to x}}\, u(z) > \text{ap-}\underline{\lim_{z\to x}}\, u(z)\}$$

(here ap-$\overline{\lim}$ and ap-$\underline{\lim}$ denote approximative upper and lower limits) and it is countably rectifiable, hence the normal to J exists $\mathcal{H}^{N-1}$-a.e. see [4] or [21] for more information. In this case,

$$\bar{E}(u) = \int_\Omega f(x,\nabla u(x))\,dx + \int_J (j^+ - j^-) f^\infty(x,\nu)\,\mathcal{H}^{N-1}. \tag{2.12}$$

We will give a proof of the lower estimate of $\bar{E}(u)$ after [22], in case Ω is a cube $\Omega = (-1/2, 1/2)^N$ and $u(x) = a$ if $x_N > 0$ and $u(x) = b$ if $x_N < 0$. It is worth mentioning that foundations for this approach were developed in [7] and further extended in [8].

Proposition 2.3 ([22])
Let us suppose that f and u are as above. Then,

$$\bar{E}(u) \geq \int_\Omega f(x,0)\,dx + \int_{\Omega\cap\{x_n=0\}} (a-b) f^\infty(x,e_n)\,d\mathcal{H}^{N-1},$$

where $e_n = (0,\ldots,0,1)$.

Sketch of proof. By choosing a proper subsequence we may consider $\lim_{n\to\infty} E(u_n)$ in place of $\underline{\lim}_{n\to\infty} E(u_n)$. Hence, $\|\nabla u_n\|_{L^1} \leq M$.

We fix a real number $t_0 > 1$ and a natural number $k > 0$. We have

$$\lim_{n\to\infty} E(u_n) = \lim_{n\to\infty} \left(\int_{\Omega\cap\{|x_N|>1/k\}} f(x,\nabla u_n)\,dx + \int_{\Omega\cap\{|x_N|\leq 1/k\}} f(x,\nabla u_n)\,dx \right)$$

It is not difficult to see that,

$$\lim_{n\to\infty} \int_{\Omega\cap\{|x_N|\leq 1/k\}} f(x,\nabla u_n)\,dx = \int_{\Omega\cap\{|x_N|\leq 1/k\}} f(x,\nabla u)\,dx = \int_\Omega f(x,0)\,dx + O(1/k),$$

where ∇u is the absolutely continuous part of Du.

Let us call by Q cube Ω. With the help of Vitali's Covering Theorem for all ϵ we can find a countable collection of cubes $Q_q^k = x_q^k + \delta(x_q^k)Q$, where $\delta(x_q^k) < 1/k$ and (H3) holds in each cube Q_q^k and $\mathcal{H}^{N-1}(\{x\in\Omega : x_N = 0\} \setminus \bigcup_{q=1}^\infty Q_q^k) = 0$. Hence,

$$\lim_{n\to\infty} \int_{\Omega\cap\{|x_N|>1/k\}} f(x,\nabla u_n)\,dx \geq \underline{\lim}_{n\to\infty} \sum_{q=1}^\infty \int_{Q_q^k} f(x_q^k,\nabla u_n)\,dx + O(1/k).$$

After freezing points x_q^k we separate Q_q^k into a part $F_q^k = \{|\nabla u_n| > t_0\}$ and the rest,

$$\int_{Q_q^k} f(x_q^k,\nabla u_n)\,dx = \int_{Q_q^k\cap F_q^k} f(x_q^k,\nabla u_n)\,dx + \int_{Q_q^k\setminus F_q^k} f(x_q^k,\nabla u_n)\,dx.$$

We apply (H4) to each integral over cubes $Q_q^k \cap F_q^k$,

$$\int_{Q_q^k\cap F_q^k} f(x_q^k,\nabla u_n)\,dx \geq \int_{Q_q^k\cap F_q^k} \left(f^\infty(x_q^k,\nabla u_n) - |\nabla u_n| \left| |\frac{f(x_q^k,|\nabla u_n|\nu_n)}{|\nabla u_n|} - f^\infty(x_q^k,\nu_n) \right| \right) dx,$$

where $\nu_n = \nabla u_n/|\nabla u_n|$. Thus,

$$\int_{Q_q^k\cap F_q^k} f(x_q^k,\nabla u_n)\,dx \geq \int_{Q_q^k\cap F_q^k} [f^\infty(x_q^k,\nabla u_n) - O(t_0^{1-m})]\,dx.$$

So the problem of convergence was reduced to a study of a simple integral over cube Q_q^k. Summing up these contributions will give us

$$\int_{\Omega\cap\{x_N=0\}} f^\infty(x,b-a)\,d\mathcal{H}^{N-1},$$

while the error terms go to zero.

2.1 Least gradient problems

The least gradient problems are of the following types:

$$\min\{\int_\Omega |Du| :\ u \in BV(\Omega),\ \gamma u = f\} \tag{2.13}$$

or

$$\min\{\int_\Omega a(x)\Phi(Du) :\ u \in BV(\Omega),\ \gamma u = f\}, \tag{2.14}$$

where Φ is a norm and a may vanish on a subset of Ω.

In fact, this is an active area of research related to conductivity imaging, [28] or free material design. There is a demand on results on problems like (2.13) or (2.14) despite the lack of lower semicontinuity of the corresponding functional. One may ask the question, in what sense the boundary conditions of (2.13) or (2.14) should be understood in case of general data? What happens if we assume some additional regularity of data f? This question was addressed in [32]. The authors showed that there exists a solution to (2.13) for any data f, however, no uniqueness is expected in general and the boundary values are assumes in the 'viscosity sense', we will see simple clarifying examples in the last lecture. One important obstacle for assuming data in the trace sense is of geometric nature, not analytic. Let us state the basic observation about least gradient functions.

Definition 2.1 We say that $u \in BV(\Omega)$ is a *least gradient function*, if

$$\int_\Omega |Du| \le \int_\Omega |D(u+\varphi)|$$

for all $\varphi \in BV$ with $\operatorname{supp}\varphi \Subset \Omega$.

Proposition 2.4 (see [9])
Let us suppose that $\Omega \subset \mathbb{R}^N$ is open and $N \le 7$. If $u \in BV(\Omega)$ is a least gradient function, then $S := \partial\{u \ge t\}$ is minimal surface. Hence, S is smooth.

We also have a weak maximum principle:

Exercise 2.4 *Let us suppose that u is a solution to* (2.13)*, where the boundary data is assumed in the trace sense. Show that if $f \in C(\partial\Omega)$, then* $\min f \le u \le \max f$.

Another fact is crucial for the idea of construction of solutions to (2.13).

Proposition 2.5 (see [43, Lemma 3.3])
Let us suppose that u is a solution to (2.13)*, where the boundary data is assumed in the trace sense, $\partial\Omega$ is Lipschitz continuous and $f \in C(\partial\Omega)$. Then, for all $t \in f(\partial\Omega)$ we have $\partial\{u \ge t\} \subset f^{-1}(t)$.*

This is an advanced exercise in geometric measure theory. We have to be careful, when we wish to state it for discontinuous f.

Example 2.1 Suppose that $\Omega = (-a,a)\times(-b,b)$ and we define g on the boundary of Ω by formula $g(a,x_2) = \cos(\frac{\pi}{2b}x_2)$ and $g \equiv 0$ on the remaining parts of $\partial\Omega$. Of course $g \in C(\partial\Omega)$, but any attempt to draw the level sets of a possible solution to (2.13), based on Proposition 2.4 and Proposition 2.5, implies that they must lie on the interval with endpoints $(a,-b)$ and (a,b), hence there is no solution to (2.13) with the given data, if the data are to be assumed in the trace sense.

This example shows also that smoothness is not sufficient to get solutions we need to require more from region Ω, its convexity is not enough.

Theorem 2.3 ([43, Theorems 3.6, 3.7, 5.3, Corollary 4.2])
We assume that Ω is strictly convex bounded subset of $\mathbb{R}^N$ and $f \in C(\partial\Omega)$. Then, there is u, a unique solution (2.13). *Moreover, $u \in C(\bar\Omega)$, in addition if $f \in C^\alpha(\partial\Omega)$, then $u \in C^{\alpha/2}(\bar\Omega)$.*

Idea of the *proof* depends of Propositions 2.4 and 2.5. Roughly speaking, we construct candidates for the superlevel sets $A_t \subset \Omega$ in the following way. We extend the data f to a continuous function F over $\mathbb{R}^N\setminus\Omega$, its superlevel sets are $\{x \in \mathbb{R}^N\setminus\Omega :\ F(x) \ge t\} =: \mathcal{L}_t$. Set A_t has to minimize $P(E,\Omega)$ among sets E, such that $E\setminus\Omega = \mathcal{L}_t$. This is a way to say that ∂A_t has to meet $\partial\Omega$ at $f^{-1}(t)$. □

The uniqueness part fails if we drop the continuity assumption. Data discontinuous in just few point may lead to uncountably many solutions:

Example 2.2 (Brothers, see [32])
Let us take $\Omega = B(0,1) \subset \mathbb{R}^2$ and $g(x,y) = x^2 - y^2$ as the boundary data. If we follow the construction of a solution, u, from Theorem 2.3, then we will notice that four points, $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$, where the data vanish, are corner of a square, Q, on which the solution is constant, and the other level sets are parallel to the sides of Q. Of course u is continuous. Now, we may perturb g by introducing a jump discontinuity at those four points, (we assume that $\lambda > 0$),

$$g_\lambda(x,y) = \begin{cases} g(x,y), & \text{if } x^2+y^2 = 1 \text{ and } |x| < \frac{\sqrt{2}}{2}, \\ g(x,y)+\lambda, & \text{if } x^2+y^2 = 1 \text{ and } |x| > \frac{\sqrt{2}}{2}. \end{cases}$$

Following the construction of solutions we draw all the level sets for $t > \lambda$ and $t < 0$. However, we are free to assign any specific value of u on Q.

We also show an example of discontinuous data leading to a unique solution.

Example 2.3 ([27])
Let us assume that $\Omega = B(0,1)$. We select a piecewise constant function on $\partial B(0,1)$ with exactly three jumps at x_1, x_2, x_3. The existence theorem tells us that the corresponding solution has three values and two lines across which the solution jump. Uniqueness of solutions follows here from uniqueness of the level sets, see [27].

It is a good question to ask, what is the space $X \subset L^1(\partial\Omega)$, which is consists of the traces of solutions to (2.13). Examples show that $X \subsetneq L^1(\partial\Omega)$, even maybe $X \subsetneq L^\infty(\partial\Omega)$ e.g. [44]. We also know that $X \supset BV(\partial\Omega)$, see [26].

Chapter 3

An example of Γ convergence

In this lecture we would like to present the notion of Γ convergence and apply it to obtain non-trivial solutions to a non-linear elliptic problem. This approach depends essentially on geometry of the domain $\Omega \subset \mathbb{R}^N$. The results presented here come from [33] and [30]. We recommend also [18] for further application of notions introduced here.

A natural way to prove existence of solutions to elliptic equations, which are Euler-Lagrange equation of a functional F, is to show that F has global or local minimizers. In general nonlinear problems need not have unique solutions. If there is some additional information about the structure of minimizers one can also use the Mountain Pass Lemma.

We study a family of functionals,

$$F_\epsilon(u) = \int_\Omega [\epsilon|\nabla u|^2 + \frac{1}{\epsilon}W(u)]\, dx, \qquad u \in W^{1,2}(\Omega), \tag{3.1}$$

(we consider the natural boundary conditions). Here W has two wells, e.g.

$$W(u) = \frac{1}{4}(1-u^2)^2$$

and $\epsilon > 0$ is a small parameter.

It is a well-known fact, see [14], [15], that if Ω is convex, then the only solutions to

$$\begin{array}{ll} 2\epsilon\Delta u - \frac{1}{\epsilon}f(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{array}$$

where $f(u) = DW(u)$, are constants, $u = \pm 1$ a.e. in Ω.

Here are the questions, which we would like to address here:
(i) are there any other solutions if Ω is no longer convex?
(ii) is there a 'good' notion of convergence of functionals as $\epsilon \to 0$?

We would expect for a metric space X that:

If F_n and $F_0 : X \to \mathbb{R} \cup \{+\infty\}$ and $F_n \to F_0$, then $\inf_X F_n \to \min_X F_0$.
If $u_n \in \operatorname{argmin} F_n$, then $u_n \to u_0 \in \operatorname{argmin} F_0$.

We also hope that the converse statement would be true:
(iii) if u_0 is a minimizer of F_0, then there is a family u_ϵ of minimizers of F_ϵ convergent to u_0.

Here is the answer the first two questions, while the third problem is addressed in Theorem 3.3.

Definition 3.1 *Let (X, d) be a metric space. We have functionals $F_0, F_n : X \to \mathbb{R} \cup \{+\infty\}$. We say that sequence F_n Γ-converges to F_0 iff for all $x \in X$ the following conditions hold:*
(i) for any sequence $x_n \in X$ converging to x we have, $\underline{\lim}_{n\to\infty} F_n(x_n) \geq F_0(x)$;
(ii) there is a sequence $y_n \in X$ converging to x, such that $\overline{\lim}_{n\to\infty} F_n(y_n) \leq F_0(x)$.
We write

$$F_0 = \Gamma - \lim_{n\to\infty} F_n.$$

Exercise 3.1 (Homework problem # 3)
Let us suppose that X is a metric space and $F : X \to \mathbb{R} \cup \{+\infty\}$ is a given functional. We define a sequence of functionals by $F_n = F_0$. Then,

$$\Gamma - \lim_{n\to\infty} F_n = \bar{F},$$

where $\bar{F}$ is the lower semicontinuous envelope of F, see (2.3).

Next theorem shows that this notion should be up to our expectation.

Theorem 3.1 *Let us suppose that $F_0, F_n : X \to \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$ and*

$$\Gamma - \lim_{n\to\infty} F_n = F_0.$$

We assume that $\{x_n\}_{n=1}^{\infty} \in X$ is a relatively compact sequence of almost minimizers, i.e.

$$F_n(x_n) \leq \inf\{F_n(x) : \ x \in X\} + \epsilon_n, \qquad \text{where } \epsilon_n \to 0.$$

If $\bar{x} \in X$ is a limit of a subsequence x_{n_k}, then $\bar{x}$ is a minimizer of F and

$$\lim_{n\to\infty} \inf\{F_n(x) : \ x \in X\} = F(\bar{x}).$$

Proof. Let $\bar{x} = \lim_{k\to\infty} x_{n_k}$. According to part (i) of Definition 3.1 we have,

$$F(\bar{x}) \leq \underline{\lim}_{k\to\infty} F_{n_k}(x_{n_k}) = \underline{\lim}_{k\to\infty} \inf_X F_{n_k}. \tag{3.2}$$

Now, due to part (ii) of Definition 3.1 there is a sequence y_n converging to $\bar{x}$ such that

$$F(\bar{x}) \geq \overline{\lim}_{n\to\infty} F_n(y_n) \geq \overline{\lim}_{k\to\infty} F_{n_k}(y_{n_k}). \tag{3.3}$$

Combining (3.2) and (3.3) yields,

$$F(\bar{x}) \leq \underline{\lim}_{k\to\infty} \inf_X F_{n_k} \leq \overline{\lim}_{k\to\infty} \inf_X F_{n_k} \leq \overline{\lim}_{k\to\infty} F_{n_k}(y_{n_k}) \leq F(x). \qquad \square$$

We shall immediately see that relative compactness of minimizers or almost minimizers of functionals defined by (3.1) is guaranteed.

Proposition 3.1 *Let us suppose that F_ϵ is given by (3.1). If $\{u_\epsilon\}_{\epsilon>0} \subset W^{1,2}$ is such that $F_\epsilon(u_\epsilon) \leq M < \infty$, then the family $\{u_\epsilon\}_{\epsilon>0}$ is relatively compact in $L^1(\Omega)$, i.e. there is $u_{\epsilon_n} \to u_0$ in L^1 as $n \to \infty$.*

Proof. Since

$$M \geq F_\epsilon(u_\epsilon) = \int_\Omega \left[\epsilon|\nabla u_\epsilon|^2 + \frac{1}{4\epsilon}(1-u_\epsilon^2)^2\right] dx,$$

then

$$\int_\Omega (1-u_\epsilon^2)^2\, dx \leq 4\epsilon M \to 0.$$

In other words, $1-u_\epsilon^2 \to 0$ in $L^2(\Omega)$ and a.e. Furthermore, by $2ab \leq a^2+b^2$, we obtain,

$$M \geq \int_\Omega |\nabla u_\epsilon||1-u_\epsilon^2|\, dx. \tag{3.4}$$

We introduce a strictly increasing function $G:\mathbb{R}\to\mathbb{R}$ by formula $G'(u) = |1-u^2|$, e.g.

$$G(t) = \int_{-1}^t (1-u^2)\, du = u - \frac{u^3}{3} + \frac{2}{3}.$$

Then, (3.4) becomes,

$$M \geq \int_\Omega |DG(u_\epsilon)|.$$

Due to the compact embedding of BV into L^1, cf. Theorem 1.5, there is a subsequence $G(u_{\epsilon_k})$ converging in $L^1(\Omega)$ and a.e. to function G^∞. Since G^{-1} is continuous we deduce that $u_{\epsilon_k} \to u_0$ a.e. □

Moreover,

$$u_\epsilon - \frac{u_\epsilon^3}{3} + \frac{2}{3} \to u_0 - \frac{u_0^3}{3} + \frac{2}{3} = \frac{2}{3}u_0 + \frac{2}{3}.$$

So, the limiting function u_0, assuming just two values ± 1 has finite total variation,

$$M \geq \varliminf_{\epsilon\to 0} \int_\Omega |DG(u_\epsilon)| \geq \int_\Omega |DG(u_0)| = c_0|D\chi_{\{u_0=1\}}|,$$

where $c_0 = G(1) - G(-1) = \frac{4}{3}$. In other words,

$$F_0(u) = \begin{cases} c_0 P(\{u=1\},\Omega), & \text{for } |u|=1,\ a.e. \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.5}$$

This is indeed a good candidate for the Γ-limit. Basically, we have already presented the outline of the proof of Definition 3.1 part (i):

Proposition 3.2 *Let us suppose that $\widetilde{F}_\epsilon : L^2(\Omega) \to (-\infty,+\infty]$ is given by*

$$\widetilde{F}_\epsilon(u) = \begin{cases} F_\epsilon(u) & u\in W^{1,2}(\Omega), \\ +\infty & u \in L^2(\Omega)\setminus W^{1,2}(\Omega), \end{cases}$$

and $u_\epsilon \in W^{1,2}(\Omega)$ is such that $u_\epsilon \to u_0$ in L^1 as $\epsilon \to 0$. Then,

$$\varliminf_{n\to\infty} \widetilde{F}_{\epsilon_n}(u_{\epsilon_n}) \geq F_0(u_0).$$ □

The other one is more difficult and we will only outline the idea of the proof.

Theorem 3.2 (see [33] and [34], [35])
Let us suppose that $\partial\Omega \in C^2$ and $u_0 \in L^1(\Omega)$. Then there is a sequence $v_n \to u_0$ in L^1 such that $\overline{\lim}_{n\to\infty} F_n(v_n) \le F_0(u_0)$.

Outline of the proof. Of course it is sufficient to consider $u_0 = \chi_A - \chi_{\Omega\setminus A}$. We may assume for simplicity that $\partial A \in C^2$. We look for u_ϵ having a special form $u_\epsilon(x) = h(d(x)/\epsilon)$, where $h : \mathbb{R} \to \mathbb{R}$ is a profile function and d is the signed distance from A. In this way, we emphasize the role of the normal direction to ∂A, the tangential ones are less important.

Taking into account that we used $a^2 + b^2 \ge 2ab$, we come to conclusion that we need $|a| = |b|$ to turn the inequality into the equality. Since we used such bounds while looking for F_0 we infer that $\epsilon \int_\Omega |\nabla h\left(\frac{d(x)}{\epsilon}\right)|^2$ and $\frac{1}{\epsilon}\int_\Omega W\left(h(\frac{d(x)}{\epsilon})\right)$ should balance. This happens at minimizers. Moreover, h should connect -1 and 1.

We notice that for our choice of u_ϵ we have

$$\epsilon|\nabla u_\epsilon|^2 + \frac{1}{4\epsilon}(1-u_\epsilon^2)^2 = \frac{1}{\epsilon}(|h'|^2 + (1-h^2)^2),$$

so ϵ^{-1} factors out and our problem becomes one-dimension, indeed. Thus, we obtain an ODE for the profile function h,

$$\frac{1}{4}\frac{d}{d\,h}(1-h^2)^2 - 2\frac{d^2h}{dt^2} = 0, \qquad h(+\infty) = 1, \quad h(-\infty) = -1$$

so $u_\epsilon = h(d/\epsilon)$ is the right choice. □

Here is our advertised application.

Theorem 3.3 *Let us assume that $\Omega \subset \mathbb{R}^N$ is a bounded, open set with C^2 boundary and u_0 is an isolated local minimizer (in the L^1 topology) of F_0 given by (3.5). Then, there is $\epsilon_0 > 0$ and a family $\{u_\epsilon\} \subset W^{1,2}(\Omega)$, $0 < \epsilon < \epsilon_0$, such that:*
(1a) u_ϵ is a local minimizer in the L^1 topology of F_ϵ, given by (3.1);
(2b) $\|u_\epsilon - u_0\|_{L^1} \to 0$ as $\epsilon \to 0$.

Remark. There are no isolated local minimizers in L^1 if Ω is convex. We have to find a sufficient condition for their existence. Here it is:

Proposition 3.3 *Let us suppose that $\Omega \subset \mathbb{R}^2$ is open, bounded and its boundary is of class C^2. We further assume that there is a finite number of disjoint line segments $\{\ell_i\}_{i=1}^m$ such that $\ell_i \cap \ell_j = \emptyset$ for all $i \ne j$ and for all $i = 1, \dots, m$ we have:*
(2a) $\ell_i \subset \Omega$ and the endpoints of ℓ_i belong to $\partial\Omega$;
(2b) ℓ_i is orthogonal to $\partial\Omega$ at each endpoint;
(2c) $\partial\Omega$ is strictly concave near each endpoint of ℓ_i.
Let u_0 be locally taking values ± 1 on each component of $\Omega \setminus \bigcup_{i=1}^m \ell_i$, then u_0 is a local minimizer of F_0, which is isolated in the L^1 topology.

We will prove the theorem first. Since u_0 is isolated, it means that $F_0(u_0) < F_0(v)$ for all $v \in L^1(\Omega)$ such that $0 < \|u - v\|_{L^1} < \delta$, for sufficiently small δ. The lower semicontinuity implies existence of u_ϵ, a minimizer of F_ϵ, in $B = \{u \in L^1(\Omega) : \|u_0 - u\|_{L^1} \leq \delta\}$. We would like to establish that for sufficiently small ϵ the family u_ϵ is relatively compact in L^1. This follows from Theorem 3.2, which gives existence of a family v_ϵ converging to u_0 in L^1 such that $\overline{\lim}_{\epsilon\to 0} F_\epsilon(v_\epsilon) \leq F_0(u_0)$. Thus,

$$\underline{\lim}_{\epsilon\to 0} F_\epsilon(u_\epsilon) \leq \overline{\lim}_{\epsilon\to 0} F_\epsilon(v_\epsilon) \leq F_0(u_0). \tag{3.6}$$

We claim that u_ϵ may not be at the boundary of B. If it were, then there would exist $\epsilon_k \to 0$ such that $\|u_0 - u_{\epsilon_k}\|_{L^1} = \delta$, but Proposition 3.1 yields existence of a convergent subsequence $u_{\epsilon_{k_n}}$ with limit u^*. Then, $\|u^* - u_0\| = \delta$ and (3.6) implies that

$$F_0(u^*) \leq \underline{\lim}_{\epsilon\to 0} F_\epsilon(u_\epsilon) \leq F_0(u_0).$$

But this contracts the fact that u_0 is isolated. Essentially, the same argument yields part (2b). □

Remarks.
1) The assumption of the Theorem does not hold if Ω consists of two balls joined in a smooth way by a straight cylinder. There are plenty of local minimizers of F_0 there.
2) Even if we know that u_0 is isolated we do not know if minimizers u_ϵ are also isolated.
3) We reduced solving a PDE to a geometric problem of finding a set with minimal perimeter.

Now, we will provide an argument for Proposition 3.3. We will work with region N as on the picture. It is divided into two parts by a single line segment ℓ satisfying the

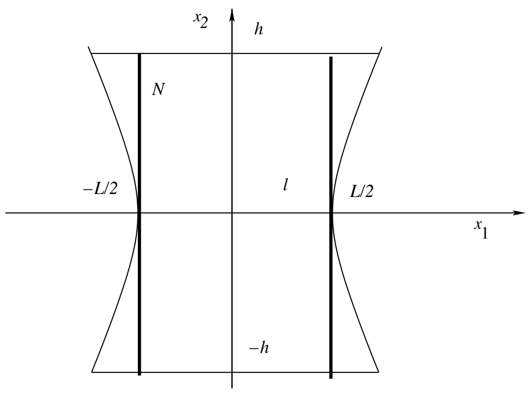

Figure 3.1: region N

assumptions of Proposition 3.3. If v takes only two values, 1 or -1 in $N \setminus \ell$, then

$$\int_N |Dv| = 2|D\chi_{\{v=1\}}|(N) = 2P(\{v = 1\}, N).$$

We will show that for a suitable $\delta > 0$,

$$v(x) = \pm 1, \ a.e. \text{ in } N \text{ and } 0 < \int_N |v - u_0| \, dx < \delta \tag{3.7}$$

implies

$$F_0(v) = \int_N |Dv| > \int_N |Du_0| = 2L = F_0(u_0) \tag{3.8}$$

We stress that it is important that $\|v - u_0\|_{L^1} > 0$.

In order to proceed we have to state the fundamental theorem of calculus for BV functions:

Exercise 3.2 *If $u \in BV(a,b)$, then $\int_a^b Du = u(b) - u(a)$. Here, the boundary values are understood as traces.*

In order to prove (3.8) we assume that $v = u_0$ a.e. along segments

$$-L/2 < x_1 < L/2, \qquad x_2 = \pm a \tag{3.9}$$

for an a such that $h/2 < a < h$. Then,

$$\int_{-a}^{a} \int_{-L/2}^{L/2} \frac{\partial v}{\partial x_2} = \int_{-L/2}^{L/2} (v(x_1, a) - v(x_1, -a)) \, dx_1 = 2L.$$

Thus,

$$\int_N |\nabla v| \geq \int_N |\frac{\partial v}{\partial x_2}| \geq 2L. \tag{3.10}$$

In other words we obtained (3.8) with $\geq$ instead of $>$.

However, equality holds in (3.10) only if $\frac{\partial v}{\partial x_1} = 0$ in N a.e., then v is a function of only one variable – x_2. By the concavity of the boundary of Ω (and N) we have

$$\int_N |\nabla v| > 2L \qquad \text{unless } v = u_0 \text{ in } N.$$

Suppose now that $v \neq u_0$ a.e. on $-L/2 < x_1 < L/2$, $x_2 = \pm a$ for a.e. $a \in (h/2, h)$. Then, we notice that for a.e. $a \in (h/2, h)$ exactly one of the following conditions holds:

$$v \text{ is not constant along } x_2 = a, \ -L/2 < x_1 < L/2, \tag{3.11}$$

$$v \text{ is not constant along } x_2 = -a, \ -L/2 < x_1 < L/2, \tag{3.12}$$

$$v = -u_0 \text{ along both segments.} \tag{3.13}$$

Let us consider set $\mathcal{A} \subset (0, h)$, defined as

$$a \in \mathcal{A} \Leftrightarrow \int_{-L/2}^{L/2} [|v - u_0|(x_1, a) + |v - u_0|(x_1, -a)] \, dx_1 > \frac{4\delta}{h}.$$

By (3.7) we deduce that $|\mathcal{A}| < \frac{h}{4}$. Indeed,

$$\delta > \int_N |v - u_0| \, dx \geq \int_{\mathcal{A}} \int_{-L/2}^{L/2} [|v - u_0|(x_1, a) + |v - u_0|(x_1, -a)] \, dx_1 dx_2 \geq |\mathcal{A}| \frac{4\delta}{h}.$$

If we choose δ so that $\delta < hL$, then condition (3.13) at $x_2 = a$ implies that $a \in \mathcal{A}$. It is so, because of

$$\int_{-L/2}^{L/2} [|v - u_0|(x_1, a) + |v - u_0|(x_1, -a)]\, dx_1 = 4L.$$

Therefore, for a.e. $a \in (h/2, h) \setminus \mathcal{A}$ either (3.11) or (3.12) holds. In either case, v jumps at least one from -1 to 1 or vice versa. So,

$$\int_{-L/2}^{L/2} \left(\left| \frac{\partial v}{\partial x_1} \right| (x_1, a) + \left| \frac{\partial v}{\partial x_1} \right| (x_1, -a) \right) \geq 2.$$

Integrating this over $(h/2, h) \setminus \mathcal{A}$ with respect to a and using $|\mathcal{A}| < h/4$ give us

$$\int_{h/2}^{h} \int_{-L/2}^{L/2} |\nabla v| + \int_{-h}^{-h/2} \int_{-L/2}^{L/2} |\nabla v| \geq \frac{h}{2}. \tag{3.14}$$

On the other hand for any $a \in (0, h/2)$ we have

$$|v(x_1, a) - v(x_1, -a)| \geq 2 - |v(x_1, a) - u_0(x_1, a)| - |v(x_1, -a) - u_0(x_1, -a)|.$$

By $|\mathcal{A}| < h/4$ we may choose an element $a \in (0, h/2) \setminus \mathcal{A}$. Then,

$$\int_{-L/2}^{L/2} |v(x_1, a) - v(x_1, -a)|\, dx_1 \geq 2L - \frac{4\delta}{h}.$$

Hence,

$$\int_{-h/2}^{h/2} \int_{-L/2}^{L/2} |\nabla v| \geq \int_{-a}^{a} \int_{-L/2}^{L/2} |\frac{\partial v}{\partial x_2}| \geq 2L - \frac{4\delta}{h}. \tag{3.15}$$

Since we integrate in (3.15) and (3.14) over disjoint sets we come to

$$\int_N |\nabla v| \geq 2L - \frac{4\delta}{h} + \frac{h}{2} > 2L,$$

provided that $\delta < h^2/8$. Our claim follows as soon as $\delta < \min\{hL, h^2/8\}$. □

Notes We recommend [10] for a first reading on Γ convergence. The example presented here is taken from [30]. The Γ-convergence we discussed is the so-called Modica-Mortola theory, see [33], [34], [35]. In [18] steady states of Cahn-Hilliard equations are constructed by the method of Γ-convergence.

Chapter 4

BV estimates for a dynamical problem

In this lecture we present results on a problem loosely related to that studied in §3. There, we considered

$$\begin{array}{ll} 0 = \epsilon\Delta u + \frac{1}{\epsilon} f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{array} \tag{4.1}$$

where $f(u) = u - u^3$. More general nonlinearities will be permitted here, however they will have the qualitative features of $f(u) = u - u^3$. We are interested in the parabolic problem, whose steady state equation is (4.1), i.e.

$$\begin{array}{ll} u_t = \epsilon\Delta u + \frac{1}{\epsilon} f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{array} \tag{4.2}$$

However, this equation does not conserve mass. The mass conservation version of (4.2) is

$$u_t = \epsilon\Delta u + \frac{1}{\epsilon}(f(u) - \langle f(u)\rangle) \qquad \text{in } \Omega, \tag{4.3}$$

where $\langle g\rangle$ denotes the average of g, i.e.,

$$\langle g\rangle = \frac{1}{|\Omega|}\int_\Omega g(x)\,dx.$$

It is it well-known that for initial stages of developing the interfacial layers in solutions to (4.2) the diffusion operator $\epsilon\Delta$ does not play any substantial role, see [1], [17], [37]. One may expect that this is also true for solutions to (4.3), see [25]. Thus, before the interfaces form in (4.3) we look at

$$u_t = f(u) - \langle f(u)\rangle \qquad \text{in } \Omega, \qquad u(0,x) = u_0(x), \tag{4.4}$$

after we rescaled time. Eq. (4.4) is in fact an infinite system of ODE's coupled through the nonlinear term $\langle f(u)\rangle$. At this moment we state the assumptions on f:

$f \in C^0(\mathbb{R})$ and there exist $m < M$ satisfying $f'(m) = 0 = f'(M)$ and

(H) $\qquad f' < 0$ on $(-\infty, m) \cup (M, +\infty)$, $\qquad f' > 0$ on (m, M).

There exist $s_* < s^*$ satisfying

$$s_* < m < M < s^*, \qquad f(s_*) = f(M), \quad f(s^*) = f(m).$$

We also present the basic assumption on u_0:
(H1) For fixed $s_1 < s_2$, such that $s_1, s_2 \notin (s_*, s^*)$ we have $s_1 \leq u_0(x) \leq s_2$.
Now, we state the basic existence result.

Theorem 4.1 (see [24])
Let us assume that f satisfies (H) and u_0 is in $L^\infty(\Omega)$ conforming to (H1). Then, there is a unique solution to (4.4), $u \in C^1([0,\infty); L^\infty(\Omega))$ and

$$s_1 \leq u(x,t) \leq s_2, \qquad a.e.\ x \in \Omega, t > 0.$$

This is not exactly an exercise in the applications of Banach fixed point theorem, but we will skip it completely, because it does not advance the main topic, which is an application of BV estimates to study time asymptotics.

Another comment is in order, one may look at (4.4) in a point-wise manner or one may treat it as an ODE in a Banach space $L^p(\Omega)$, $p \in [1,\infty]$. These two approaches are equivalent.

We turn our attention to the stabilization of solutions. We stated above the uniform bounds on solutions. If we had an equation

$$\dot{x} = F(x), \qquad x(0) = x_0 \in \mathbb{R}^N$$

and if we knew that $\|x\| \leq M$, then we could select a converging subsequence $x(t_n) \to x^\infty$. Accumulation points (in the L^1-topology) of the orbit $\{x(t) :\ t \geq 0\}$ form the omega-limit set, $\omega(u_0)$. Eq. (4.4) has the form

$$u_t = H(u,t) \qquad \text{in } L^p(\Omega).$$

The uniform bounds, which we established, imply only existence of a sequence converging weakly, what is not sufficient for the present purposes. There is no apparent compactness in (4.4), no smoothing. To the contrary, we expect formation of jumps in infinite time.

We need a clever idea to deduce stabilization or at least existence of $\omega(u_0)$. If we think about a source of compactness for a family of bounded functions of one variable, which need not be continuous, then we may recall the following classical result.

Theorem 4.2 (Helly) *If $\{f_n\}_{n=1}^\infty$ is a sequence of real valued function on (a,b), $a,b \in \mathbb{R}$ which are uniformly bounded and $\|f_n\|_{TV} \leq M$, then there exists a subsequence converging everywhere,*

$$\lim_{k\to\infty} f_{n_k}(x) = f^\infty(x) \qquad \forall x \in (a,b).$$

Remark. One should point out, that the notions of a function of bounded variation (BV) in one dimension does not coincide with the notion of finite variation (TV). They may be reconciled, see [4], but we do not need it, because we may use the classical Helly theorem.

If we think of applying Theorem 4.2, we need:
(i) to find a reduction of the original problem to a one-dimensional equation;

(ii) to show a uniform boundedness of the total variation on orbits, so that existence of $\omega(u_0)$ would follow;
(iii) to prove that convergence of solutions to the reduced problem implies convergence of solutions to the original one.

The trick, which makes this program work, is the *theory of monotone rearrangements.* The monotone rearrangement assigns to each function $u \in L^p(\Omega)$ a monotone function $u^\# \in L^p(0, |\Omega|)$. Thus tasks (i) and (ii) will be addressed. Finally, we will have a positive answer to the question (iii) above.

The rearrangement theory is interesting for its own sake, so we will present it to some extent. Let us present the main notions and ideas. Let us suppose that $w : \Omega \to \mathbb{R}$ is measurable, we define the distribution function μ_w as follows,

$$\mu_w(s) := |\{x \in \Omega : w(x) > s\}|.$$

Of course, μ_w is monotone decreasing, possibly with jumps. The rearrangement is defined as the inverse of μ_w, but we have to take care of any level sets of positive measure. We set $w^\# : [0, |\Omega|] \to \mathbb{R}$, by the formula:

$$\begin{aligned} w^\#(0) &:= \text{ess } \sup w \equiv \inf\{a : \ |\{w(x) > a\}| = 0\}, \\ w^\#(y) &:= \inf\{s : \ \mu_w(s) < y\}, \quad y > 0. \end{aligned}$$

We note the basic and useful properties of rearrangements. The definition implies that $w^\#$ is always nonincreasing, moreover $w^\#$ has jumps whenever w has a level set of positive measure. Since the distribution function is defined in terms of measure of superlevel set, then one can see that if $w_1 = w_2$ a.e. in Ω, then $w_1^\#(s) = w_2^\#(s)$ for all $s \in [0, |\Omega|]$.

Example 4.1
1) An explicit example. Function $u : [-2, 2] \to \mathbb{R}$ is given by the following formula,

$$u(x) = \begin{cases} 2(x+2), & \text{for } x \in [-2, -1], \\ 1, & \text{for } x \in (-1, 0], \\ 2x + 1, & \text{for } x \in (0, \frac{1}{2}], \\ -\frac{4}{3}(x - 2), & \text{for } x \in (\frac{1}{2}, 2]. \end{cases}$$

Please work out μ_u and $u^\#$.
2) Check that $u^\# = \mu_u^{-1}$ and if t is a point of discontinuity of μ_w, then $w^\# = t$ on $[\mu_w(t^+), \mu_w(t^-)]$.

Here are the properties, which we will use. A more comprehensive account on the monotone rearrangement can be found in [29]. Proofs of the statements below are taken from this book.

Proposition 4.1
(1) If $u : \Omega \to \mathbb{R}$ and Ω is bounded, then $u^\#$ is left continuous.
(2) For any measurable function w we have, $\mu_w = \mu_{w^\#}$. Thus, we say that w and $w^\#$ are equidistributed.
(3) Mapping $u \mapsto u^\#$ is nondecreasing, i.e. if $u \leq v$, then $u^\# \leq v^\#$.
(4) If $s_1 \leq w \leq s_2$, then $s_1 \leq w^\# \leq s_2$.
(5) If $u \in L^p(\Omega)$, then $u^\# \in L^p(0, |\Omega|)$ and $\|u\|_{L^p} = \|u^\#\|_{L^p}$.

Proofs. (1) Let $s \in (0, |\Omega|)$. By definition of $u^\#$, for a given $\epsilon > 0$, there is a t such that $u^\#(s) \le t \le u^\#(s) + \epsilon$ and $\mu_u(t) < s$. We can choose $h > 0$ such that $\mu_u(t) < s - h < s$. Then, for all $0 < h' \le h$, we have $\mu_u(t) < s - h' < s$ and so $u^\#(s) \le u^\#(s - h') \le t < u^\#(s) + \epsilon$. This proves that $u^\#$ is left continuous. □

(2) If $u^\#(s) > t$, then by definition, $\mu_u(t) > s$. Thus,

$$\{s : \ u^\#(s) > t\} \subset \{s : \ \mu_u(t) > s\}.$$

Since $u^\#$ is decreasing, then

$$\sup\{s : \ u^\#(s) > t\} \equiv \mu_{u^\#}(t) = |\{s : \ u^\#(s) > t\}| \le \mu_u(t). \tag{4.5}$$

On the other hand, let us suppose that $|\{u^\# \ge t\}| = \mu_{u^\#}(t) = s$. By the left continuity of $u^\#$ and its monotonicity it follows that $u^\#(s) = t$. Then, by definition $|\{u > t\}| \le s$. Thus,

$$\mu_u(t) \equiv |\{u > t\}| \le |\{u^\# \ge t\}|. \tag{4.6}$$

Combining (4.5) and (4.6) and replacing t by $t + h$ yields,

$$\mu_{u^\#}(t + h) \le \mu_u(t + h) \le |\{u^\# \ge t + h\}|.$$

The limit passage gives us,

$$\mu_{u^\#}(t) \le \mu_u(t) \le \mu_{u^\#}(t).$$ □

(3) Since $\{u > t\} \subset \{v > t\}$, then $\mu_u(t) \le \mu_v(t)$. We also have

$$\{t : \ |\{v > t\}| < s\} \subset \{t : \ |\{u > t\}| < s\}.$$

Our claim follows. □

(4) Omitted.

(5) If $p = \infty$, then this fact follows from the definition. For a finite p we notice that $\mu_u = \mu_{u^\#}$. Keeping this in mind yields,

$$\|u\|^p_{L^p} = p \int_0^\infty t^{p-1} \mu_u(t)\, dt = p \int_0^\infty t^{p-1} \mu_{u^\#}(t)\, dt = \|u^\#\|^p_{L^p}.$$ □

This is a good time to state a homework problem:

Exercise 4.1 (Homework problem # 4)
Let us suppose that $u : [0, L] \to \mathbb{R}$ is nonincreasing. Show that $u = u^\#$ a.e. in $[0, L]$.

After this warm up we present the following important fact.

Theorem 4.3 *Let us suppose that $u : \Omega \to \mathbb{R}$ is measurable.*
(1) If $F : \mathbb{R} \to \mathbb{R}$ is non-negative and Borel measurable, then

$$\int_\Omega F(u(x))\, dx = \int_0^{|\Omega|} F(u^\#(s))\, ds.$$

(2) The same conclusion holds if F is integrable.

It is sufficient to present the first stage of the *proof.* Let us take $F = \chi_E$, where $E = (t, \infty)$. Then,

$$\int_\Omega F(u(x))\,dx = |\{u > t\}| = |\{u^\# > t\}| = \int_0^{|\Omega|} F(u^\#(s))\,ds.$$

The remaining part of the proof follows along the standard lines of argument. □

We noticed that the rearrangement preserves the L^p norm, but since it is nonlinear we have to check its continuity separately.

Proposition 4.2 *If $u, v \in L^p(\Omega)$ and $p \in [1, \infty)$, then*

$$\|u^\# - v^\#\|_{L^p} \le \|u - v\|_{L^p}.$$

This estimate shows that rearrangement is Lipschitz continuous but it is not an isometry, so we might have trouble deducing convergence of a sequence v_n from convergence of $v_n^\#$. However, some hope comes from the following result.

Proposition 4.3 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be nondecreasing. Then, $(\Phi(w))^\# = \Phi(w^\#)$ a.e. in $(0, |\Omega|)$.*

Our strategy now is to show that the equation for the rearrangement has the properties we imagined. Here the advertised BV estimates play the key role. Next step is to connect the rearrangement and the original problem. This will be achieved with the help of facts stated above.

Here, we consider the equation for rearrangement of the data.

$$v_t = f(v) - \langle f(v) \rangle, \qquad v(0) = u_0^\#. \tag{4.7}$$

Due to Proposition 4.1 (4) we obtain that

$$s_1 \le u_0^\# \le s_2$$

and $u_0^\#$ is nonincreasing. We notice immediately that Theorem 4.1 guarantees existence of solutions to (4.7). However, we need more:

Proposition 4.4 *If (H) holds, then we define $u^\#(y, t) := (u(t, \cdot))^\#(y)$ on $[0, |\Omega|] \times [0, +\infty)$. Then, $u^\#$ is the unique solution to (4.7) and*

$$s_1 \le u^\#(y, t) \le s_2.$$

This Proposition tells us that 'the rearrangement of a solution of the original problem (4.4)' is 'the solution to the rearranged eq. (4.7)'. It is an interesting fact and we will show it as well as tighter link between solutions to (4.4) and (4.7). The proof requires introducing an auxiliary problem.

If $u_0 \in L^\infty$ satisfies (H1), we set $\lambda(t) := \langle f(u) \rangle$ for $t \ge 0$. We study $Y(t, s)$, solutions to

$$\dot{Y} = f(Y) - \lambda(t), \quad t > 0, \qquad Y(0) = s. \tag{4.8}$$

One can check that

$$u(x, t) = Y(t, u_0(x)), \qquad a.e.\ x \in \Omega. \tag{4.9}$$

We record an important property of Y.

Lemma 4.1 *Let $\widetilde{s} < s$ and assume that $Y(\cdot, s), Y(\cdot, \widetilde{s}) \in C^1(0, \infty)$ are solutions to (4.8) with initial conditions s and $\widetilde{s}$, respectively. Then,*

$$Y(t, \widetilde{s}) < Y(t, s), \qquad t \in [0, \infty). \tag{4.10}$$

Proof. This follows immediately from the uniqueness Theorem. □

Now, we can state the follow-up part of Proposition 4.4.

Theorem 4.4 *We assume that the conditions of Proposition 4.4 hold, then*

$$u^{\#}(y, t) = Y(t, u_0^{\#}(y)) \qquad a.e.\ y \in (0, |\Omega|).$$

Proof. We recall (4.9), for all $t \geq 0$ we have,

$$u(x, t) = Y(t, u_0(x)), \qquad a.e.\ x \in \Omega.$$

Since by Lemma 4.1 the function $Y(t, \cdot)$ is increasing, then it follows from Proposition 4.3 that

$$u^{\#}(t) = Y(t, u_0^{\#}), \qquad a.e. \text{ in } (0, |\Omega|).$$

This implies that $u^{\#}$ is the solution to (4.7), because Y is the solution to (4.8). Indeed,

$$\frac{\partial Y}{\partial t}(t, u_0^{\#}) = f(Y(t, u_0^{\#})) - \lambda(t) = f(u^{\#}(t)) - \langle f(u(t))\rangle = f(u^{\#}(t)) - \langle f(u^{\#}(t))\rangle,$$

where we used Theorem 4.3 in the last equality. □

The advantages of (4.7) over (4.4) are simplicity and availability of compactness argument.

Lemma 4.2 *Let $u_0 \in L^\infty(\Omega)$ satisfies (H1), then the orbit $\{u^{\#}(t) :\ t \geq 0\}$ is relatively compact in $L^1(0, |\Omega|)$.*

Proof. Functions from the family $\{u^{\#}(t) :\ t \geq 0\}$ are commonly bounded and decreasing. Due to classical Helly Theorem, see Theorem 4.2, there is a sequence $u^{\#}(t_n)$ converging everywhere and hence in any L^p, $p \in [1, \infty)$ to φ.

This result implies existence of the ω-limit set $\omega(u_0^{\#})$. We want to deduce the same result for eq. (4.4). For this purpose we show:

Lemma 4.3 *Let u be a solution to (4.4) with $u_0 \in L^\infty$ satisfying (H1) and $u^{\#}$ be a solution to (4.7) with $v_0 = u_0^{\#}$. Then, for all $t, \tau \in (0, \infty)$ we have,*

$$\|u^{\#}(t) - u^{\#}(\tau)\|_{L^1(0,|\Omega|)} = \|u(t) - u(\tau)\|_{L^1(\Omega)}.$$

Caveat: cf. Proposition 4.2.

Proof. We will use Theorem 4.3 with $F(s) = |Y(t, s) - Y(\tau, s)|$. Then, we see

$$\begin{aligned}\|u^{\#}(t) - u^{\#}(\tau)\|_{L^1(0,|\Omega|)} &= \int_0^{|\Omega|} |Y(t, u_0^{\#}) - Y(\tau, u_0^{\#})|(y)\, dy = \int_\Omega |Y(t, u_0) - Y(\tau, u_0)|(x)\, dx \\ &= \|u(t) - u(\tau)\|_{L^1(\Omega)}.\end{aligned}$$

We draw several conclusions from this lemma.

Corollary 4.1 *Let us suppose that the sequence t_n converges to $+\infty$ as $n \to +\infty$ and $\psi \in L^1(0, |\Omega|)$. Then, the following statements are equivalent:*
(a) $u^\#(t_n) \to \psi$ in $L^1(0, |\Omega|)$ as $n \to \infty$;
(b) there is $\phi \in L^1(\Omega)$ such that $\lim_{n\to\infty} u(t_n) = \phi$ in $L^1(\Omega)$ and $\phi^\# = \psi$. □

Corollary 4.2 *If $u_0 \in L^\infty$ satisfies $(H1)$ and u is a solution to (4.4), then $\{u(t) : \ t \geq 0\}$ is relatively compact in L^1, hence $\omega(u_0) \neq \emptyset$.*

In fact one can show more, but the proofs of the results below are beyond the scope of this lecture.

Theorem 4.5 *Let us assume that u_0, s_1, s_2 are as in the existence theorem. If $\varphi \in \omega(u_0)$, then $s_1 \leq \varphi \leq s_2$ and φ is a stationary point of (4.4). More precisely,*

$$f(\varphi(x)) = \langle f(\varphi(x)) \rangle = 0.$$

Actually, information on fine structure of $\omega(u_0)$ is available.

Theorem 4.6 *Let us assume that f satisfies (H) and (H1) holds for u_0.*
(1) If in addition

$$\text{(H2)} \qquad \langle u_0 \rangle \notin [s_*, s^*],$$

holds, then $u(t) \to \psi$ in L^∞, where $\psi = \int_\Omega u_0(x)\, dx$. Moreover, the convergence is exponential.
(2) If in addition

$$\text{(H3)} \qquad s_* \leq u_0(x) \leq s^* \qquad \text{and } |\{u_0(x) = s\}| = 0 \quad \forall s \in (m, M),$$

holds, then $u(t) \to \psi$ in L^p for all $p < \infty$, where ψ is a step function. It takes at most two values a_-, a_+ such that $f(a_-) = f(a_+)$ and $f'(a_-) \leq 0$, $f'(a_+) \leq 0$,

$$\varphi = a_- \chi_{\Omega_-} + a_+ \chi_{\Omega_+}.$$

Chapter 5

The total variation flow and related topics

The starting point is again the calculus of variations and the ROF functional. We use it as an introduction to gradient flows. We did not study much the Euler-Lagrange (EL) equation for the functional,

$$E(u) = \int_\Omega |\nabla u| + \frac{\lambda}{2}(u-f)^2\,dx. \tag{5.1}$$

The EL equations are important, because functional minimizers satisfy them. However, since the integrand $p \mapsto |p|$ in E is not differentiable, we will have difficulties writing down the EL equations. This is why we introduce the notion substituting differentiation in case of convex functions. For this purpose we present a few tools from the convex analysis.

Let us suppose that H is Hilbert space and $\Psi : H \to \mathbb{R}\cup\{+\infty\}$ is a convex functional. The *subdifferential* is a substitute of the notion of the derivative. Geometrically, it is the collection of the supporting hyperplanes, analytically it is defined as follows,

$$\partial\Psi(u) = \{p \in H^* :\ \forall h \in H\ \Psi(u+h) - \Psi(u) \geq (p,h)\}.$$

Although we may identify H^* with H, but by writing $p \in H^*$ we emphasize the fact that the definition is correct also for a Banach space in place of H.

It is easy to see and check the following facts.

Proposition 5.1 *If Ψ is convex, then $\partial\Psi(u)$ is closed and convex.* $\square$

Proposition 5.2 *If u_0 is a minimizer of a convex functional Ψ, then $0 \in \partial\Psi(u_0)$.* $\square$

This fact explains the role of subdifferential in constructing the EL equations.

We stress that we consider $\partial\Psi$ as a simple example of a multivalued operator. Strictly speaking A defined in H is called a multivalued operator if $A : D \subset H \to 2^H$. We identify A with its graph $\{(x,y) \in H \times H :\ x \in D,\ y \in A(x)\}$.

The geometric object like $\partial\Psi(u)$ retains an important property of the derivative of convex functions, it is monotone. More precisely, we say that a multivalued operator $A : H \to 2^H$ is *monotone* if for any $(x_i, \xi_i) \in A$, $i = 1, 2$ we have

$$(\xi_2 - \xi_1, x_2 - x_1) \geq 0. \tag{5.2}$$

Let us take $\xi_i \in \partial\Psi(x_i)$, $i = 1, 2$, then by the definition we have

$$\Psi(x_2 + h_2) - \Psi(x_2) \geq (\xi_2, h_2), \qquad \Psi(x_1 + h_1) - \Psi(x_1) \geq (\xi_1, h_1). \tag{5.3}$$

We can take $h_2 = x_1 - x_2$ and $h_1 = x_2 - x_1$. Then, we obtain,

$$\Psi(x_1) - \Psi(x_2) \geq (\xi_2, x_1 - x_2), \qquad \Psi(x_2) - \Psi(x_1) \geq (\xi_1, x_2 - x_1).$$

After adding up these inequalities we reach (5.2).

We state without proof a more substantial fact. We define the domain of Ψ (resp. $\partial\Psi$) by the following formula, $D(\Psi) = \{x \in H : \Psi(x) < \infty\}$, (resp. $D(\partial\Psi) = \{x \in H : \partial\Psi(x) \neq \emptyset\}$). Then, one can prove, see [11].

Theorem 5.1 *If Ψ is convex, proper (i.e. $\Psi \not\equiv +\infty$) and lower semicontinuous, then $\partial\Psi$ is a maximal monotone operator and $\overline{D(\partial\Psi)} = \overline{D(\Psi)}$.*

We explain the new notion which appeared in the above statement. A monotone operator A is *maximal monotone*, if for any monotone B such that $B \supset A$, then $A = B$.

It is good moment to present to compute subdifferentials. We begin with $H = \mathbb{R}^n$ and $\Psi(p)$ equal to the Euclidean norm. Since Φ is differentiable away from $p = 0$, then $\partial\Psi(p) = \{\nabla\Psi(p)\}$, for $p \neq 0$. If $p = 0$, then the condition in the definition of $\Psi(0)$ reads as,

$$|h| \geq (z, h), \tag{5.4}$$

where $z \in \Psi(0)$. Obviously, (5.4) is satisfied if and only if $|z| \leq 1$.

We are now in a position to show how $\partial E(u)$ looks like. At this is point we have to specify the ambient Hilbert space H, this will be $L^2(\Omega)$. We first notice that $E(u)$ is a sum of a convex functional

$$\Psi(u) = \int_\Omega |Du|$$

and a Frechet differentiable one, which is $\frac{\lambda}{2}\int_\Omega (u - f)^2\,dx$. Thus, we can easily see that $\partial E(u) = \partial\Psi(u) + \lambda(u - f)$.

Functional $E(u)$ is well-defined on BV, but for the didactic purpose we restrict our attention to $u \in W^{1,1}$. Later, we shall present the full rigorous statement. If $u \in W^{1,1}(\Omega)$, then the above remark on the subdifferential of $|\cdot|$ implies that

$$E(u + h) - E(u) \geq \int_\Omega z \cdot \nabla h\,dx + \lambda \int_\Omega (u - f)h\,dx,$$

where $z(x)$ is an element of the subdifferential of the Euclidean norm $|\cdot|$ at $\nabla u(x)$. Thus, $|z(x)| \leq 1$, for a. e. $x \in \Omega$ and $z = \frac{\nabla u}{|\nabla u|}$ whenever $\nabla u \neq 0$. In order to integrate by parts we use the trace theory presented in §1, see Theorem 1.13. Thus, we come to

$$E(u + h) - E(u) \geq -\int_\Omega \operatorname{div} zh\,dx + \int_{\partial\Omega} z \cdot \nu h + \lambda \int_\Omega (u - f)h\,dx,$$

We have to justify separately that $z \cdot \nu$ vanishes on $\partial\Omega$, however, we omit the argument here. Then, due to Proposition 5.2, we reach the conclusion that the Euler-Lagrange equation has the following form of a differential inclusion,

$$0 \in -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda(u - f) \equiv -\operatorname{div} z + \lambda(u - f), \tag{5.5}$$

where $\|z\|_{L^\infty} \le 1$ and $z \cdot \nu = 0$, so $-\operatorname{div} z$ belongs to the subdifferential of E at u.

It is good moment to present rigorously the subdifferential of $\Psi(u) = \int_\Omega |Du|$. For the proof we refer the reader to [2, Proposition 1.10].

Proposition 5.3 *Let $u, v \in L^2(\Omega)$ and $u \in BV(\Omega)$. The following assertions are equivalent:*
(a) $v \in \partial\Psi(u)$;
(b) we have

$$\int_\Omega vu\,dx = \Psi(u), \tag{5.6}$$

$$\text{and there is } z \in X_2(\Omega),\ \|z\|_{L^\infty} \le 1,\ \text{such that } v = -\operatorname{div} z \text{ in } \mathcal{D}'(\Omega) \tag{5.7}$$

and

$$[z \cdot \nu] = 0 \text{ on } \partial\Omega; \tag{5.8}$$

(c) (5.7) and (5.8) hold and

$$\int_\Omega (z, Du) = \int_\Omega |Du|. \tag{5.9}$$

Comments.
Part (b) tells us that elements of the subdifferential of Ψ are in the form of the divergence of a special vector field from X_2. Part (c) may be easily understood as an application of the Anzellotti theory, see Theorem 1.13, to the integral,

$$-\int_\Omega \operatorname{div} zu\,dx,$$

where $-\operatorname{div} z = v$ is an element of $\partial\Psi(u)$.

If we assume that λ is big and consider u^{n+1} in place of u and u^n in place of f, then we can see that

$$\frac{u^{n+1} - u^n}{\lambda^{-1}} \approx \frac{\partial u}{\partial t}(\cdot, t_n).$$

Hence, equation (5.5) is the time implicit semidiscretization of the parabolic problem,

$$\begin{array}{ll} \frac{\partial u}{\partial t} \in \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) & (x,t) \in \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} = 0 & (x,t) \in \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x) & x \in \Omega. \end{array} \tag{5.10}$$

We have just seen that (at least formally) the RHS of (5.10) is minus gradient of functional Ψ, hence (5.10) is the gradient flow of Ψ.
Natural questions:
(1) Is inclusion (5.10) solvable for $u_0 \in BV(\Omega)$?
(2) What are the properties of solutions to (5.10)?

We will address this questions. We will also present examples making clear the character of eq. like $u_t = (\operatorname{sgn} u_x)_x$.

Once we explained the notion of the subdifferential of $\Psi(u)$, we may state (5.10) as

$$u_t \in -\partial\Psi(u), \qquad u(0) = u_0. \tag{5.11}$$

If we state the evolution problem in this way, then existence of solutions is well-known due to the Kōmura theory. We state the main results after Brezis, see [11].

Theorem 5.2 *Let us suppose that H is a Hilbert space, $\Psi : H \to (-\infty, +\infty]$ is proper, convex and lower semicontinuous. We also set $\mathcal{A} = \partial\Psi$.*
(a) If $u_0 \in D(\partial\Psi)$, then there is a unique function $u \in C(0,\infty; H)$ such that for all $t > 0$ we have $u(t) \in D(\partial\Psi)$, $u_t \in L^\infty(0,\infty; H)$,

$$-u_t \in \partial\Psi(u) \qquad \textit{for a.e. } t > 0, \tag{5.12}$$

and

$$\|\frac{\partial u}{\partial t}\|_{L^\infty(0,\infty;H)} \le \|\mathcal{A}^0(u_0)\| \quad \textit{for a.e. } t > 0;$$

$$\frac{d^+u}{dt} + \mathcal{A}^0(u) = 0 \quad \textit{for all } t > 0. \tag{5.13}$$

Here, $\mathcal{A}^0(u)$ denotes the canonical selection of $\mathcal{A}(u)$ i.e. the element of the smallest norm.
(b) If $u_0 \in \overline{D(\partial\Psi)}$, then there is a unique function $u \in C(0,\infty; H)$, locally Lipschitz continuous, a solution to (5.12) such that for all $t > 0$ we have $u(t) \in D(\partial\Psi)$ and (5.13) holds.

Remark. Theorem 5.2 (a) is valid if $\mathcal{A}$ is a general maximal monotone operator.

These are nice theorems on the evolution problem, however, they do not address the relationship between (5.5) and (5.10), in other words, we did not address the issue of the discretization error. We could write (5.5) more precisely as

$$\frac{u-f}{h} + \partial\Psi(u) \ni 0,$$

(where $h = 1/\lambda$) or

$$u + h\partial\Psi(u) \ni f.$$

If we use the nonlinear resolvent, then the last inclusion takes the form

$$u = J_h f, \qquad \text{where } J_h = (Id + h\partial\Psi)^{-1}.$$

In fact we are interested in iterating this process k times after setting $f = u_0$, so that we obtain,

$$u^k = J_h^k u_0.$$

We want to know how close u^k is to the true solution $u(kh)$. Indeed, the Crandall-Liggett generation theorem answers this question in a very general setting, but we use it for $\mathcal{A} = \partial\Psi$.

Theorem 5.3 (Crandall-Liggett, [19]) *Let us suppose that $\mathcal{A} = \partial\Psi$, where Ψ is as in Theorem 5.2. Then for all $x_0 \in \overline{D(\mathcal{A})}$ and $t > 0$ we have,*

$$\lim_{h\searrow 0,\ kh\to t} J_h^k x_0 = S(t)x_0,$$

where $S(t)$ is the semigroup generated by $-\mathcal{A}$, given by Theorem 5.2. The convergence is uniform on compact intervals of $[0,\infty)$.

Remarks. A similar statement can be found in [11], see Theorem 4.3.

Crandall-Liggett Theorem contains also an error estimate, which is not optimal. Estimating the error has been studied by several people in the Numerical Analysis community. Here is a prominent result, giving good bounds.

Theorem 5.4 [42, Theorem 5]
We assume that $\mathcal{A} = \partial\Psi$ and Ψ is as in Theorem 5.2. We define $u^h : (0,T) \to L^2(\Omega)$ as a step function, i.e. $u^h(t) := J_h^k u_0$, for $t \in [kh,(k+1)h)$, then

$$\|u^h(T) - u(T)\|_H^2 \le 2h^2 \|\mathcal{A}^0(u_0)\|_H^2.$$

We can strengthen the conclusion by adding positive terms to the LHS. However, the RHS of the above estimate depends on the $\|\mathcal{A}^0(u_0)\|_H$, this may lead to difficulties, since polynomials having max or min in Ω do not belong to $D(\partial\Psi)$. For example, if $\Omega = (-1,1)$, then $v(x) = 1 - x^2$ does not belong to $D(\partial\Psi)$ (if we consider the homogeneous Dirichlet boundary data). Indeed, $(\operatorname{sgn} v_x)_x = 2\delta_0 \notin L^2(\Omega)$.

We would like to study properties of solutions. This is easier in one dimension. We would like also to consider the Dirichlet boundary conditions. In the first approach we would prefer to avoid additional complications. This is indeed possible if we assume sufficient regularity of the initial condition.

Theorem 5.5 (see [38])
If $u_0 \in L^1(0,L)$, $u_{0,x} \in BV(0,L)$, then for any $T > 0$ there exists a unique solution (e.g. in the sense of Theorem 5.2) to

$$\begin{array}{ll} u_t = (\operatorname{sgn} u_x)_x & (x,t) \in (0,L)\times(0,T), \\ u(0,t) = 0 = u(L,t) & t \in (0,L), \qquad\qquad \textit{end} \\ u(x,0) = u_0(x) & x \in (0,L). \end{array} \tag{5.14}$$

Moreover, $u_t \in L^2(0,T;L^2(0,L))$, $u \in L^\infty(0,T;BV(0,L))$ and there is $\alpha > 0$ such that

$$u \in C^{\alpha,\alpha/2}([0,T]\times[0,L]),$$

in particular the boundary conditions are satisfied in the sense of traces of continuous functions.

With the help of this theorem we will analyze (5.14) for nice data, $u_0 \in C^1(0,L)$, $u_0(0) = 0 = u_0(L)$. Since (5.14) is a gradient flow of the total variation, then we know from Theorem 5.2 that $\operatorname{sgn} u_x$ should be understood as $-v$, where v is an element of the subdifferential of $\int_0^L |Du|$ and $v = -z_x$, where $z \in H^1$ and $|z|(x) \le 1$, so (5.14) becomes

$$u_t = z_x. \tag{5.15}$$

We will see how this works in practice. If $u_x > 0$ (or $u_x < 0$) on (a,b), then $(\operatorname{sgn} u_x)_x = \pm\frac{\partial}{\partial x}1 = 0$, so $u_t = 0$ there. If $u_x = 0$ on (a,b), then the conclusion is no longer trivial. In this case, we integrate (5.15) over (a,b), to get

$$\int_a^b u_t\,dx = z(b) - z(a). \tag{5.16}$$

The continuity of z implies that

$$z(b) = \operatorname{sgn} u_x(b^+) \quad \text{and} \quad z(a) = \operatorname{sgn} u_x(a^-). \tag{5.17}$$

Theorem 5.2 tells us also that among many possible v's in the subdifferential we should select the one which minimizes the integral,

$$\int_0^L (z_x)^2\,dx$$

with the constraint $|z| \le 1$ and boundary conditions (5.17). In our case the minimizer is a linear function

$$z(x) = \frac{\operatorname{sgn} u_x(b^+) - \operatorname{sgn} u_x(a^-)}{b-a}(x-a) + \operatorname{sgn} u_x(a^-),$$

so we conclude u_t is constant over (a,b). Hence, eq. (5.16) becomes,

$$V(b-a) = (\operatorname{sgn} u_x(b^+) - \operatorname{sgn} u_x(a^-)) =: \chi \in \{-2, 0, 2\}. \tag{5.18}$$

Here V is the vertical velocity of the graph of u over (a,b). We see that such flat parts of graphs of solutions to (5.14) or more generally (5.10) are special, they evolve without changing shape what is unusual for e.g. the heat equation. Thus, we will call them facets. More precisely, by a *facet* we understand a flat part of the graph of u, a solution to (5.14), (5.10) or even (5.11) if it is maximal (with respect to set inclusion) with the property that the slope of u (or the normal to the graph of u) is a singular point of the RHS of (5.14), (5.10) or (5.11). Here, the singular slope is zero, but other singularities are possible.

Thus, we may reinterpret eq. (5.18). We have just learned that a facet may move up or down or stay in place if $\operatorname{sgn} u_x(b^+) = \operatorname{sgn} u_x(a^-)$, while the neighboring monotone parts of the graph of u do not move. Thus, we may describe the evolution of a facet by specifying its height h over the axis, because the endpoints will be determined from the equation involving the initial condition u_0,

$$h = u_0(a(h)) = u_0(b(h)). \tag{5.19}$$

This view is correct in a neighborhood of a local minimum or maximum. As a result, (5.18) is an ODE for the facet position,

$$\frac{dh}{dt} = \frac{\chi}{b(h) - a(h)}, \qquad h(0) = u_0(a_0), \tag{5.20}$$

where χ is defined in (5.18).

Here is an example of u_0, for which the evolution may be completely worked out. We take, $u_0(x) = 1 - x^2$ in $\Omega = (-1, 1)$. We notice that our u_0 does not belong to the domain of the subdifferential of the total variation. In this case we take $a_0 = 0$ and $h(0) = 1$, then eq. (5.19) becomes,

$$h = 1 - x^2(h),$$

so $a(h) = -\sqrt{1-h}$, $b(h) = \sqrt{1-h}$. Thus, (5.20) takes the following form

$$\dot{h} = \frac{-1}{\sqrt{1-h}}, \qquad h(0) = 1.$$

Hence,

$$h(t) = 1 - \left(\frac{3}{2}t\right)^{2/3}.$$

We also notice from this formula that at $t = T = 2/3$ the solution becomes zero as well as u_t. The evolution stops, we discovered a finite extinction time.

Actually, the finite extinction time is a generic property of solutions to (5.14) or (5.10).

Proposition 5.4
(1) Let us suppose that u is a solution to (5.14) constructed in Theorem 5.5, then there is a constant C_1 such that the evolution stops at $T_{ext} \le \|u_0\|_{L^2}/C_1$.
(2) Let us suppose that u is a solution to (5.10), then there is a constant C_2 such that the evolution stops at $T_{ext} \le \|u_0\|_{L^2}/C_2$.

The proofs are basically the same, the first one depends on the Sobolev inequality, while the second one uses the Poincaré inequality. We will present the last one.

Let us consider $\|u - \langle u \rangle\|_{L^2}^2$, where u is a solution to (5.10). We compute,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u - \langle u \rangle|^2\, dx = \int_\Omega (u - \langle u \rangle) u_t\, dx = \int_\Omega (u - \langle u \rangle) \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) dx,$$

but we should use z, yielding the subdifferential. Thus,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u - \langle u \rangle|^2\, dx = \int_\Omega (u - \langle u \rangle) \operatorname{div} z\, dx = -\int_\Omega (z, Du),$$

where we used Theorem 1.13 to integrate by parts. Finally, we invoke (5.8) from Proposition 5.3, to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u - \langle u \rangle|^2\, dx = -\int_\Omega |Du| \le -C_2 \|u - \langle u \rangle\|_{L^2}.$$

The last inequality follows from the Poincaré inequality. Thus, we arrive at a simple differential inequality,

$$\frac{d}{dt}\|u - \langle u \rangle\|_{L^2} \le -C_2.$$

Its solution is $\|u - \langle u \rangle\|_{L^2}(t) \le \|u - \langle u \rangle\|_{L^2}(0) - C_2 t$. Hence, we deduce an estimate for the stopping time. $\square$

The last examples deal with discontinuous data in BV. Before discussing them, we would like to point to possibility of more complicated functionals. In order to be more specific, we mention $\Psi(u) = \int_\Omega \Phi(Du)$, where Φ is an arbitrary norm on $\mathbb{R}^N$, see [36] and [39] for a special case. However, we would like to explain the meaning of Dirichlet boundary conditions on a simple one-dimensional equation. In this equation $\Psi(u) = \int_\Omega [|Du+1| + |Du-1|]$.

Proposition 5.5 (see [31])
There is a unique solution, in the sense of Theorem 5.2, of the following problem,

$$\begin{aligned} u_t = (\operatorname{sgn}(u_x+1) + \operatorname{sgn}(u_x-1))_x \quad & (x,t) \in (0,L)\times(0,T), \\ u(0,t) = A, \quad u(L,t) = B \quad & t \in (0,L), \\ u(x,0) = u_0(x) \quad & x \in (0,L), \end{aligned} \tag{5.21}$$

when $u_0 \in BV(0,L)$. Moreover, $u_t \in L^2(0,T;L^2(0,L))$, $u \in L^\infty(0,T;BV(0,L))$, in addition the boundary data are assumed in the following sense:
(A) we say that the boundary datum (5.21_2) is satisfied at $x=0$ if either of the following conditions holds:

$$\gamma u(0) = A$$

or

$$\textit{if } \gamma u(0) > A, \quad \textit{then } z(0) = 2,$$

or

$$\textit{if } \gamma u(0) < A, \quad \textit{then } z(0) = -2.$$

(B) we say that the boundary datum (5.21_2) is satisfied at $x=L$ if either of the following conditions holds:

$$\gamma u(L) = B$$

or

$$\textit{if } \gamma u(L) > B, \quad \textit{then } z(0) = -2,$$

or

$$\textit{if } \gamma u(L) < B, \quad \textit{then } z(0) = 2.$$

As a matter of fact, this definition is consistent with the approximation of the solutions to (5.21) by smooth functions.

We saw that in the one dimensional case that is $u_{0,x} \neq 0$, then the solution does not move. This is no longer true in higher dimensions, because is $\nabla u \neq 0$ in a neighborhood of a point, then $\nu = \frac{\nabla u}{|\nabla u|}$ is well defined vector normal to the level set $\{u = t\}$. Hence, $\operatorname{div} \frac{\nabla u}{|\nabla u|} = \operatorname{div} \nu = u_t$, so the level sets move by their curvature. We will not elaborate on it here.

There is also no reason to expect that facet will not change their shape during the evolution. In fact they do, there is a lot of literature on this subject. However, we may ask if new jumps are formed, so that facets could break. The answer to this question is negative:

Proposition 5.6 (see [13])
If u is a solution to (5.10), then no new jumps are formed. Moreover, the size of the jump also decreases with time.

A similar result holds for minimizers of the ROF functional E, defined in (5.1), see [12].

We finish by presenting an example from [2].

Proposition 5.7 (see [2, Lemma 4.2])
Let us suppose that $N \geq 2$ and we consider the equation,

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right), \quad (x,t) \in \mathbb{R}^N \times (0,+\infty), \qquad u_0(x,0) = u_0(x).$$

We take $u_0 = k\chi_{B_r(0)}$. Then, the unique solution to this eq. is given by the formula,

$$u(x,t) = \operatorname{sgn}(k)\frac{N}{r}\left(\frac{|k|r}{N} - t\right)^+ \chi_{B_r(0)}(x).$$

Proof. We may assume that $k > 0$. We will look for a solution of the following form, $\alpha(t)\chi_{B_r(0)}(x)$ on $(0,T)$. We need to find $z \in X_2(\mathbb{R}^N)$ (see (1.3)), $\|z\|_{L^\infty} \leq 1$ and such that

$$u_t = \operatorname{div} z, \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{5.22}$$

$$\int_{\mathbb{R}^N} (z(t), Du(t)) = \int_{\mathbb{R}^N} |Du|,$$

(see the characterization of the subdifferential, Proposition 5.3, part (c). We guess and take $z(t,x) = -\frac{x}{r}$. Then, integrating (5.22) over $B_r(0)$ yields,

$$\alpha'(t)|B_r(0)| = \int_{B_r(0)} \operatorname{div} z\, dx = \int_{\partial B_r(0)} z \cdot \nu \, d\mathcal{H}^{N-1} = -\mathcal{H}^{N-1}(\partial B_r(0)).$$

Thus, $\alpha' = -\frac{N}{r}$, hence $\alpha(t) = k - \frac{N}{r}t$. Thus, $T = \frac{kr}{N}$ is the stopping time.

In order to complete our job we have to find $z \in \mathbb{R}^N \setminus B_r(0)$. We ask the reader to go over the book [2] for this purpose. □

We close with a homework problem.

Exercise 5.1 (Homework problem # 5)
Let us consider equation

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right), \quad (x,t) \in B_R(0), \qquad u_0(x,0) = u_0(x).$$

We impose the Dirichlet data,

$$u(x,y,t) = 0, \quad x^2 + y^2 \leq R.$$

We take $u_0(x) = R^2 - x^2 - y^2$. Work out the exact solution.

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