令和５（2023）年度　東京大学大学院

数理科学研究科　数理科学専攻　修士課程

英　語（筆記試験）

令和４（2022）年８月２９日（月）
１０：４０　～　１２：００

問題は全部で２題ある．２題とも解答すること．

（１）　解答しようとする各問ごとに解答用紙を１枚使用すること．
各解答用紙の所定欄に各自の**氏名**，**受験番号**と解答する**問題の番号**を記入すること．

（２）　草稿用紙の上部に各自の**受験番号**を明記すること．ただし氏名を記入してはならない．

（３）　試験終了後に提出するものは，１題につき１枚，計**２枚の答案**，および**草稿用紙**である．着手した問題数が２題にみたない場合でも，氏名と受験番号のみを記入した白紙の答案を補い，２枚とすること．
指示に反したもの，**答案が２枚でないものは無効**とする．

（４）　解答用紙の裏面を使用する場合は，表面の右下に「裏面使用」と明記すること．

E 第 1 問

(1) 次の英文を和訳せよ．ただし，数学記号はそのまま訳文に用いても良い．

Convexity is a primitive notion, based on nothing but the bare bones of the structure of linear spaces over the reals. Yet some of its basic results are surprisingly deep; furthermore, these results make their appearance in an astonishingly wide variety of topics.

X is a linear space over the reals. For any pair of vectors x, y in X, the *line segment* with endpoints x and y is defined as the set of points in X of form

$$ax + (1 - a)y, \quad 0 \leq a \leq 1.$$

Definition. A set K in X is called *convex* if, whenever x and y belong to K, all points of the line segment with endpoints x,y also belong to K.

Examples of Convex Sets

(a) $K =$ the whole space X.
(b) $K = \emptyset$, the empty set.
(c) $K = \{x\}$, a single point.
(d) $K =$ any line segment.
(e) Let l be a linear function in X; then the sets

$$\begin{aligned} &l(x) = c, \ \text{called a } \textit{hyperplane}, \\ &l(x) < c, \ \text{called an } \textit{open half-space}, \\ &l(x) \leq c, \ \text{called a } \textit{closed half-space}, \end{aligned}$$

are all convex sets.

[出典] P. D. Lax, Linear Algebra and Its Applications, Second Edition, John Wiley & Sons, Inc. 2007, p. 187 (一部改変).

(2) 次の英文の下線部を和訳せよ．ただし，数学記号はそのまま訳文に用いても良い．

Discrete subgroups of the Möbius group

The elements of the Möbius group are fractional linear bijections

$$z \to S(z) = \frac{az+b}{cz+d}, \qquad ad - bc = 1,$$

of the complex plane, ∞ included. The theory of Fuchsian differential equations led Poincaré to a general theory of discrete subgroups of Möbius group, i.e., such subgroups where orbits from a point do not return arbitrarily close to the point. The connection is the following one. When u_1, u_2 form a basis of the solutions of a second order Fuchsian equation, the quotient $d(z) = \frac{u_1(z)}{u_2(z)}$ is transformed as above when z runs through a loop from a base point around a singular point. The inverse function $\varphi(z)$ defined by $\varphi(d(z)) = z$ then has the property that $\varphi(S(z)) = \varphi(z)$ for every element S of the monodromy group G of the equation. When G leaves a circle invariant (we can let it be the unit circle), Poincaré named these functions Fuchsian (against the advice of Klein) and in other cases Kleinean. Poincaré's great papers about automorphic functions were the main attraction that Mittag-Leffler secured for the first volumes of *Acta Mathematica*.

Briefly, Poincaré's general theory runs as follows. When G is a discrete subgroup of the Möbius group, two points are said to be equivalent if they are images of each other under the group. A fundamental domain of G is a maximal part of the plane which only contains inequivalent points. Poincaré proved that there exist fundamental domains bounded by circular arcs. When the group is Fuchsian, there is a fundamental domain in the unit disk bounded by circular arcs orthogonal to the unit circle. Such a fundamental domain is also the analytical and topological image of a Riemannian surface whose universal covering surface is bijective to the unit disk (the other possibilities are the entire plane or the plane plus one point). A Riemann surface with finitely many leaves gives a fundamental region strictly contained in the unit disk. The analytic functions on the Riemann surface correspond to functions which are automorphic under a Fuchsian group.

In general a discrete group G has infinitely many elements and it is therefore not possible to construct automorphic functions by just taking the mean value over the group of a rational function $f(z)$. Instead Poincaré constructed so-called thetafuchsian functions $F(z)$ with the property that

$$F(S(z)) = F(z)(S'(z))^m, \quad S'(z) = \frac{1}{(cz+d)^2}$$

where m is a positive or negative integer, called the weight of F. Since $T'(Sz) = \dfrac{(TS)'(z)}{S'(z)}$, the function

$$F(z) = \sum_{T \in G} f(T(z))(T'(z))^{-m}, \qquad f(z) \text{ rational}$$

is thetafuchsian of weight m when the series converges. If m is large positive, depending on G, we may get uniform convergence and then construct automorphic functions as quotients of two thetafuchsian ones with the same weight.

[注] Möbius: メビウス (人名), Fuchs: フックス (人名), Poincaré: ポアンカレ (人名), Klein: クライン (人名), Riemann: リーマン (人名),

[出典] L. Gårding, Mathematics and Mathematicians, —Mathematics in Sweden before 1950—, History of Mathematics, Volume 13, American Mathematical Society, London Mathematical Society, pp. 173–174 (一部改変).

E 第 2 問

次の和文を英訳せよ．ただし，数学記号はそのまま訳文に用いても良い．

ここで集合に関する用語と記号を確認しておく．点 x が実数の集合 A に属することを $x \in A$，属さないことを $x \notin A$ とかく．$x \in A$ のとき，x を A の元という．元のまったくないものも集合と認め，空集合と称する．

定義　A を実数から成る集合とする．A が上に有界であるとは，ある実数 M をとると，A のすべての元 x に対して $x \leqq M$ が成りたつことである．このような M を A の上界という．M が上界で $M \leqq M'$ なら M' も上界である．A が下に有界だという概念およびそのときの下界も同様に定義される．

A が上にも下にも有界のとき，単に有界であるという．

定理と定義　実数の空でない集合 A が上に有界なら，最小の上界が存在する．これを A の上限と言い，$\sup A$ とかく．同様に，A が下に有界なら最大の下界が存在する．これを A の下限と言い，$\inf A$ とかく．

[出典] 齋藤正彦 『微分積分学』東京図書 (2006)，pp. 91–92　(一部改変)